# Recursive Non-parametric Estimation of the Copula Density

Vladimir Yashin
supervised by Geoffrey Decrouez

Higher School of Economics

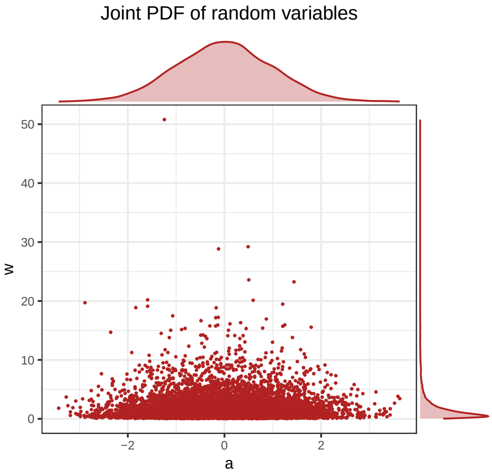
9 June 2018

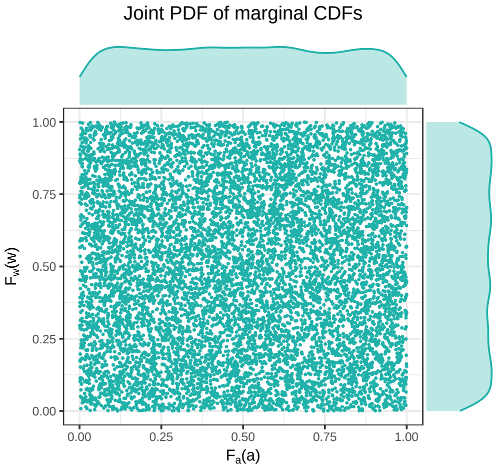
# Marginal and joint PDFs

What is the joint probability distribution of wage and age?

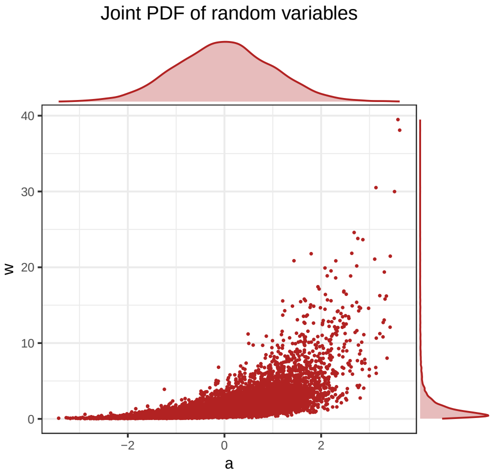
$$\begin{aligned} \log(w) &\sim \mathcal{N}\left(\mu_w, \sigma_w^2\right) \\ a &\sim \mathcal{N}\left(\mu_a, \sigma_a^2\right). \end{aligned}$$

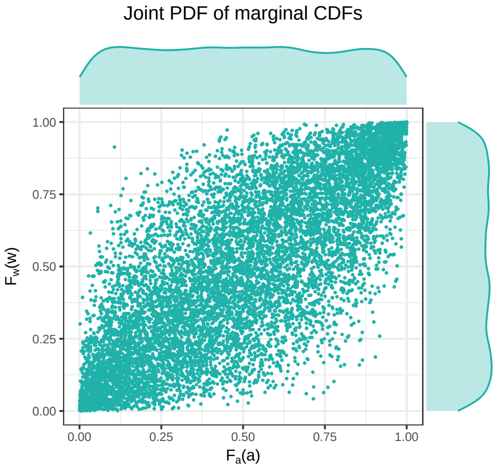
# Marginal and joint PDFs of independent samples



# Marginal and joint PDFs of dependent samples ($\rho = 0.8$)



# How to measure dependence between variables?

We wish to describe dependence structure between $w$ and $a$. What to do?

1$^{\text{nd}}$ idea: Spearman's rank correlation coefficient (✗: information is too general);

2$^{\text{rd}}$ idea: to consider joint PDF of marginals (✗: family of CDFs are rarely known).

# How to measure dependence between variables?

We wish to describe dependence structure between $w$ and $a$. What to do?

1[nd] idea: Spearman's rank correlation coefficient (✗: information is too general);

2[rd] idea: to consider joint PDF of marginals (✗: family of CDFs are rarely known).

# How to measure dependence between variables?

We wish to describe dependence structure between $w$ and $a$. What to do?

1$^{\text{nd}}$ idea: Spearman's rank correlation coefficient (✗: information is too general);

2$^{\text{rd}}$ idea: to consider joint PDF of marginals (✗: family of CDFs are rarely known).

# Let's use copulas

Copulas can address these problems

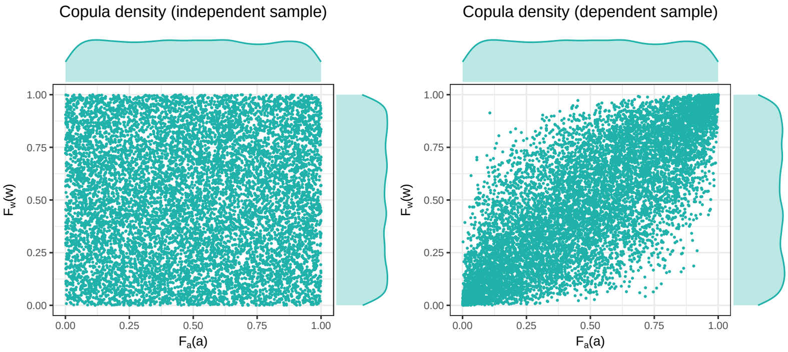
- ✓: can capture non-linear dependence;
- ✓: shows rich visual information of dependence structure;
- ✓: marginals might be of non-parametric families.

# What are copulas?

Copula is a multivariate CDF such that a marginal CDF for each variable is uniform.

# What are copulas?

Copula is a multivariate CDF such that a marginal CDF for each variable is uniform.



# Let's get a bit technical

There exists[1] a unique $d$-variate copula $C : [0,1]^d \to [0,1]$ such that

$$F(X_1, \dots, X_d) = C\big(F_1(X_1), \dots, F_d(X_d)\big),$$

where $(X_1, \dots, X_d)$ is a set of $d$ random variables like age, wage, etc.

Therefore, the copula density

$$c\big(F_1(X_1), \dots, F_d(X_d)\big) = \frac{f(X_1, \dots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}$$

[1] Sklar, 1959.

# Let's get a bit technical

There exists[1] a unique $d$-variate copula $C : [0,1]^d \to [0,1]$ such that

$$F(X_1, \ldots, X_d) = C\big(F_1(X_1), \ldots, F_d(X_d)\big),$$

where $(X_1, \ldots, X_d)$ is a set of $d$ random variables like age, wage, etc.

Therefore, the copula density

$$c\big(F_1(X_1), \ldots, F_d(X_d)\big) = \frac{f(X_1, \ldots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}$$

[1] Sklar, 1959.

# Why old methods are bad?

They have to be entirely recalculated in case of persistent data flow.

# How to estimate copula density?

Recall that

$$c\Big(F_1(X_1), \ldots, F_d(X_d)\Big) = \frac{f(X_1, \ldots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}.$$

First, we need to recursively estimate joint PDF $f$ and each of marginal PDFs $f_j$.

# Smoothed recursive non-parametric estimation of marginal PDFs $f_j$

A smoothed adaptation of the quantile estimator[2]

$$f_{j,n}^{u_j} = (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right)$$

$$a_{j,n}^{u_j} = \max\left[\mu_{j,n}, \min\left[f_{j,n}^{u_j}, \nu \ln(n+1)\right]\right]$$

$$x_n^{u_j} = x_{n-1}^{u_j} + \frac{1}{n a_{j,n}^{u_j}}\left[u_j - \mathbf{1}\left(X_{n,j} \leqslant x_{n-1}^{u_j}\right)\right],$$

where
$x_{n-1}^{u_j}$ is the quantile estimator;
$K_{h_{j,n}}^{u_j}(\cdot)$ is a symmetric, univariate kernel with bandwidth $h_{j,n}$;
$\mu_{j,n}$ and $\nu$ are some positive constants.

[2] Definition and a proof of convergence in Amiri and Thiam, (2014)

# Smoothed recursive non-parametric estimation of marginal PDFs $f_j$

A smoothed adaptation of the quantile estimator[2]

$$
\begin{aligned}
f_{j,n}^{u_j} &= (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right) \\
a_{j,n}^{u_j} &= \max\left[\mu_{j,n}, \min\left[f_{j,n}^{u_j}, \nu \ln(n+1)\right]\right] \\
x_n^{u_j} &= x_{n-1}^{u_j} + \frac{1}{n a_{j,n}^{u_j}}\left[u_j - \mathbf{1}\left(X_{n,j} \leqslant x_{n-1}^{u_j}\right)\right],
\end{aligned}
$$

where
$x_{n-1}^{u_j}$ is the quantile estimator;
$K_{h_{j,n}}^{u_j}(\cdot)$ is a symmetric, univariate kernel with bandwidth $h_{j,n}$;
$\mu_{j,n}$ and $\nu$ are some positive constants.

[2] Definition and a proof of convergence in Amiri and Thiam, (2014)

# Smoothed recursive non-parametric estimation of joint PDF $f$

A recursive estimator for the joint PDF defined as follows[3]

$$f_n^{\mathbf{u}} = (1 - 1/n) \cdot f_{n-1}^{\mathbf{u}} + 1/n \cdot \mathbf{K}_{b_n}^{\mathbf{u}}\left(x_{n-1}^{\mathbf{u}} - \mathbf{X}_n\right),$$

where now instead of $u_j$ we have a grid of points $\mathbf{u}$ and multivariate data point $\mathbf{X}_n$; $\mathbf{K}_{b_n}^{\mathbf{u}}(\mathbf{X}) := \prod_{j=1}^{d} K_{b_n}^{u_j}(\mathbf{X}_j)$ where $K_{b_n}^{u_j}(\mathbf{X}_j)$ is a univariate kernel with bandwidth $b_n$.

[3] Definition and a proof of convergence in Mokkadem et al., (2009)

# Some remarks on marginal and joint PDFs estimators

$$f^{u_j}_{j,n} = (1 - 1/n) \cdot f^{u_j}_{j,n-1} + 1/n \cdot K^{u_j}_{h_{j,n}}\left(x^{u_j}_{n-1} - X_{n,j}\right)$$
$$f^{\mathbf{u}}_{n} = (1 - 1/n) \cdot f^{\mathbf{u}}_{n-1} + 1/n \cdot \mathbf{K}^{\mathbf{u}}_{b_n}\left(x^{\mathbf{u}}_{n-1} - \mathbf{X}_{n}\right),$$

Remarks:

- The marginal estimator is just a special case of the joint estimator;
- The authors of the joint estimator didn't provide the estimation for quantiles $x^{\mathbf{u}}$;
- Therefore, the estimation of the quantile $x^{u_j}$ has to be generalized ✓.

# Some remarks on marginal and joint PDFs estimators

$$f_{j,n}^{u_j} = (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right)$$
$$f_n^{\mathbf{u}} = (1 - 1/n) \cdot f_{n-1}^{\mathbf{u}} + 1/n \cdot \mathbf{K}_{b_n}^{\mathbf{u}}\left(x_{n-1}^{\mathbf{u}} - \mathbf{X}_n\right),$$

Remarks:

- The marginal estimator is just a special case of the joint estimator;
- The authors of the joint estimator didn't provide the estimation for quantiles $x^{\mathbf{u}}$;
- Therefore, the estimation of the quantile $x^{u_j}$ has to be generalized ✓.

# Recursive non-parametric copula density estimator

Given

$$f_{j,n}^{u_j} = (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right),$$
$$f_n^{\mathbf{u}} = (1 - 1/n) \cdot f_{n-1}^{\mathbf{u}} + 1/n \cdot \mathbf{K}_{b_n}^{\mathbf{u}}\left(x_{n-1}^{\mathbf{u}} - \mathbf{X}_n\right),$$

$c_n(\cdot)$ defined as follows

$$c_n = \frac{f_n^{\mathbf{u}}}{\prod_{j=1}^{d} f_{j,n}^{u_j}}$$

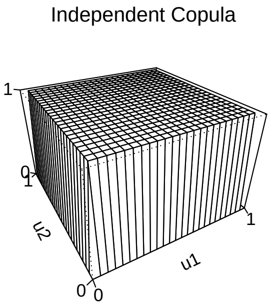
to the best of our knowledge, is the first recursive non-parametric estimator of the copula density $c(\cdot)$.

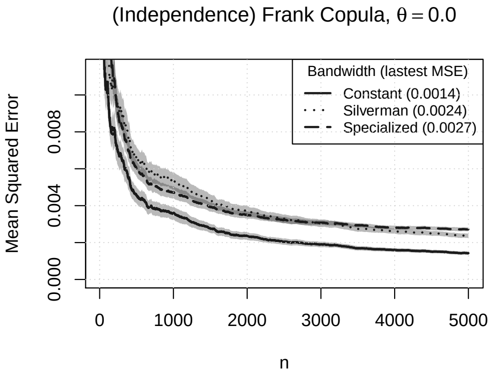
# Set-up ($d = 2$)

1. Select a copula which properties are known;
2. Sample data ($\mathbf{X}$) from the selected copula distribution;
3. Estimate values of the copula density at each point of the selected grid $\mathbf{u}$;
4. Estimate MSE.

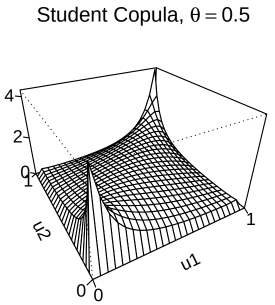
Here we use $n = 5000$ and number of Monte-Carlo simulations is 500.

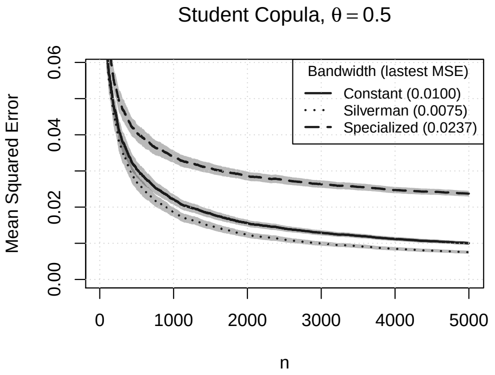
# Independent Copula



# Student Copula ($\theta = 0.5$)



# Numerical results

1. The proposed recursive non-parametric estimator works of copula density works;
2.

# Conclusion

Our contribution consists of the following

1. We introduced the kernel recursive non-parametric estimators of the joint PDF $f^{\mathbf{u}}$;
2. In addition, using this result we presented the recursive non-parametric estimator of the copula density $c(\cdot)$;
3. Github repository: https://github.com/vdyashin/CopulaDensityEstimator.

# Conclusion

Our contribution consists of the following

1. We introduced the kernel recursive non-parametric estimators of the joint PDF $f^{\mathbf{u}}$;
2. In addition, using this result we presented the recursive non-parametric estimator of the copula density $c(\cdot)$;
3. Github repository: https://github.com/vdyashin/CopulaDensityEstimator.

# Conclusion

Our contribution consists of the following

1. We introduced the kernel recursive non-parametric estimators of the joint PDF $f^{\mathbf{u}}$;
2. In addition, using this result we presented the recursive non-parametric estimator of the copula density $c(\cdot)$;
3. Github repository: https://github.com/vdyashin/CopulaDensityEstimator.

# The author sincerely acknowledges

The university Higher School of Economics for providing an opportunity to be an exchange student this semester to meet his professor.

The university staff inc. office managers Maria Neklyudova as well as Ekaterina Pavlova and Ilona Yakovleva for their support and responsiveness.

Special gratitude goes to Felix Camirand (Ph. D.) from University of Melbourne who has provided an early version of his paper.

The uncountable number of contributors of **R** package and the creators of the `copula` library. The contributors of the LaTeX typesetting system are also gratefully acknowledged.

# Recursive Non-parametric Estimation of the Copula Density

Vladimir Yashin
supervised by Geoffrey Decrouez

Higher School of Economics

9 June 2018

- Amiri, A. and B. Thiam (2014). "A smoothing stochastic algorithm for quantile estimation". In: *Statistics & Probability Letters* 93, pp. 116–125. ISSN: 0167-7152.
- Holst, U. (1987). "Recursive estimation of quantitles using recursive kernel density estimators". In: *Sequential Analysis* 6.3, pp. 219–237. ISSN: 0747-4946.
- Mokkadem, A., M. Pelletier, and Y. Slaoui (2009). "The stochastic approximation method for the estimation of a multivariate probability density". In: *Journal of Statistical Planning and Inference* 139.7, pp. 2459–2478. ISSN: 0378-3758.
- Robbins, H. and S. Monro (1951). "A stochastic approximation method". In: *The annals of mathematical statistics*, pp. 400–407. ISSN: 0003-4851.
- Robinson, D. W. (1975). "Non-parametric quantile estimation through stochastic approximation." PhD thesis.
- Silverman, B. W. (1986). *Density estimation for statistics and data analysis*. New-York: Routledge, p. 176. ISBN: 1351456172.
- Sklar, A. (1959). "Fonctions de repartition an dimensions et leurs marges". In: *Publ. Inst. Statist. Univ. Paris* 8, pp. 229–231.

# Appendix outline

1. Quantile estimation using Robbins-Monro procedure
2. How to select the bandwidths?
3. Other numerical experiments

# Robbins-Monro procedure

A root of unknown function $h$ using only noisy observations can be found as follows[4]

$$z_n = z_{n-1} - \alpha_n H(z_n, \varepsilon_n)$$

where $H(z_n, \varepsilon_n) \equiv (h(z_n) + \varepsilon_n)$ is a noisy observation and $\alpha_n$ is a sequence of positive numbers that converges to 0.

It was proved that $z_n$ converges almost surely to $z^\star$ as $n \to \infty$, given $\mathbb{E}[\varepsilon] = 0$.

[4] Robbins and Monro, 1951.

# Robbins-Monro procedure

A root of unknown function $h$ using only noisy observations can be found as follows[4]

$$z_n = z_{n-1} - \alpha_n H(z_n, \varepsilon_n)$$

where $H(z_n, \varepsilon_n) \equiv (h(z_n) + \varepsilon_n)$ is a noisy observation and $\alpha_n$ is a sequence of positive numbers that converges to 0.

It was proved that $z_n$ converges almost surely to $z^\star$ as $n \to \infty$, given $\mathbb{E}[\varepsilon] = 0$.

[4] Robbins and Monro, 1951.

# Quantile estimation using Robbins-Monro procedure

To estimate a PDF we first need to estimate quantiles.

An estimator of $u$-th quantile can be derived from the Robbins-Monro procedure

$$x_n^u = x_{n-1}^u + \alpha_n\big(u - \mathbf{1}(X_{n-1} \leqslant x_{n-1}^u)\big),$$

where $x_n^u$ is a $u$-th quantile estimator.

# Quantile estimation using Robbins-Monro procedure

To estimate a PDF we first need to estimate quantiles.

An estimator of $u$-th quantile can be derived from the Robbins-Monro procedure

$$x_n^u = x_{n-1}^u + \alpha_n \Big( u - \mathbf{1}(X_{n-1} \leqslant x_{n-1}^u) \Big),$$

where $x_n^u$ is a $u$-th quantile estimator.

# How to select the bandwidths?

We have two sets of bandwidths $h_j$ for each of marginals and $b$ for the joint.

Two groups of bandwidths:

- Data-independent;
- Data-dependent.

# How to select the bandwidths?

We have two sets of bandwidths $h_j$ for each of marginals and $b$ for the joint.

Two groups of bandwidths:

- Data-independent;
- Data-dependent.

# Constant bandwidth

The simplest data-independent case: $h_j$ and $b$ are the same[5] and calculated as follows

$$b_n = h_{j,n} = n^{-\tau},$$

where $\tau \in (0, 1/2)$.

[5] As proposed in Robinson, (1975) and Holst, (1987) for non-smooth recursive algorithms.

# Silverman's bandwidth

$h_j$ and $b$ are still the same but let's use aggregated information about data[6]

$$b_n = h_{j,n} = \left(\frac{4}{3} S_{j,n}^5\right)^{\frac{1}{5}} \cdot n^{-\frac{1}{5}},$$

where $S_{j,n}$ is the recursive estimator of the standard deviation.

[6] Adapted from the non-recursive estimator proposed in Silverman, (1986)

# "Specialized" bandwidths

Another idea is to use different bandwidths for marginal PDFs and the joint PDF.

An optimal bandwidth for the joint can be derived by optimizing $\text{MSE}(f_n^{\mathbf{u}})$. It results in

$$\left[\frac{\widetilde{\eta}^d \cdot f^{\mathbf{u}} \cdot (d+2) \cdot d}{2 \cdot \eta^2 \cdot \left[\sum_{j=1}^{d} f_{(j,j)}^{\mathbf{u}}\right]^2 \cdot (d+4)}\right]^{\frac{1}{d+4}} \cdot n^{-\frac{1}{d+4}},$$

where

$\widetilde{\eta} \approx 0.28$ and $\eta = 1$ if kernel is standard normal distribution;

$f_{(j,j)}^{\mathbf{u}}$ is the 2$^{\text{nd}}$ order derivative on $j$-th component;

In case of marginals $d = 1$.

# "Specialized" bandwidths

Another idea is to use different bandwidths for marginal PDFs and the joint PDF.

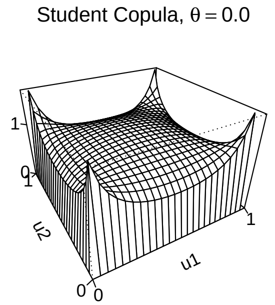
An optimal bandwidth for the joint can be derived by optimizing $\mathrm{MSE}(f_n^{\mathbf{u}})$. It results in

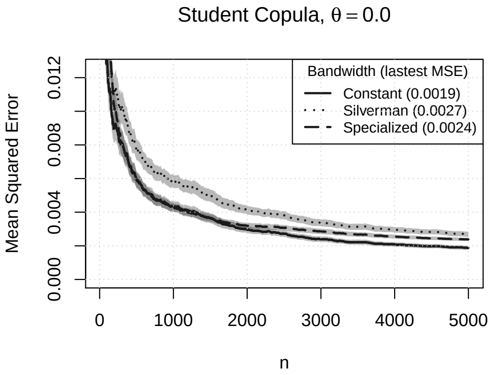
$$\left[\frac{\widetilde{\eta}^d \cdot f^{\mathbf{u}} \cdot (d+2) \cdot d}{2 \cdot \eta^2 \cdot \left[\sum_{j=1}^d f^{\mathbf{u}}_{(j,j)}\right]^2 \cdot (d+4)}\right]^{\frac{1}{d+4}} \cdot n^{-\frac{1}{d+4}},$$

where

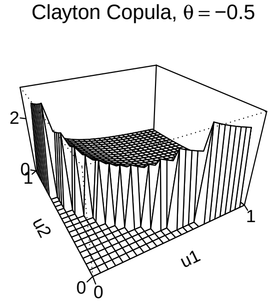
$\widetilde{\eta} \approx 0.28$ and $\eta = 1$ if kernel is standard normal distribution;
$f^{\mathbf{u}}_{(j,j)}$ is the $2^{\text{nd}}$ order derivative on $j$-th component;
In case of marginals $d = 1$.

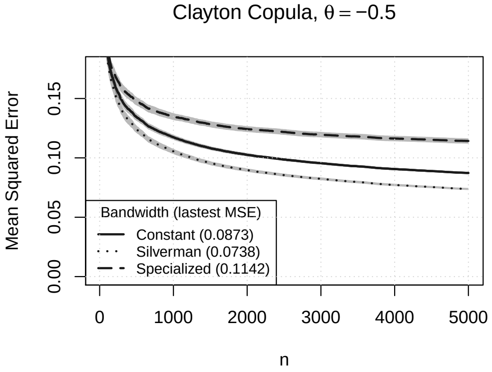
# Student Copula ($\theta = 0.0$)



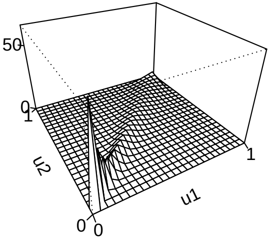
# Clayton Copula ($\theta = -0.5$)



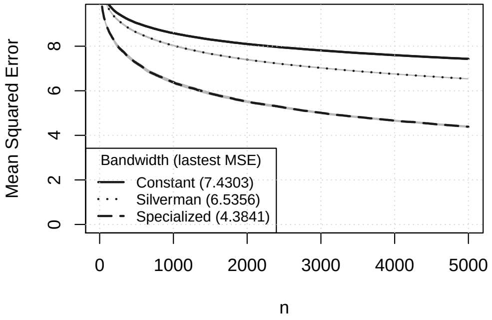



# Clayton Copula ($\theta = 10$)

Clayton Copula, $\theta = 10$




Clayton Copula, $\theta = 10.0$



# Numerical results

1. The proposed recursive non-parametric estimator works of copula density works;
2. The usage of the constant bandwidths is a good choice;
3. Though, data-dependent bandwidths are more reliable in complicated cases.

# Numerical results

1. The proposed recursive non-parametric estimator works of copula density works;
2. The usage of the constant bandwidths is a good choice;
3. Though, data-dependent bandwidths are more reliable in complicated cases.

# Numerical results

1. The proposed recursive non-parametric estimator works of copula density works;
2. The usage of the constant bandwidths is a good choice;
3. Though, data-dependent bandwidths are more reliable in complicated cases.