Recursive Non-parametric Estimation of the Copula Density

Vladimir Yashin
supervised by Geoffrey Decrouez

Higher School of Economics

9 June 2018

Marginal and joint PDFs

What is the joint probability distribution of wage and age?

Let us present both random variables in terms of univariate CDFs

$$F_w(w) := \mathbb{P}(W \leqslant w), \quad \log(w) \sim \mathcal{N}\big(\mu_w, \sigma_w^2\big)$$
$$F_a(a) := \mathbb{P}(A \leqslant a), \quad a \sim \mathcal{N}\big(\mu_a, \sigma_a^2\big).$$

Then, the joint distribution has the following form

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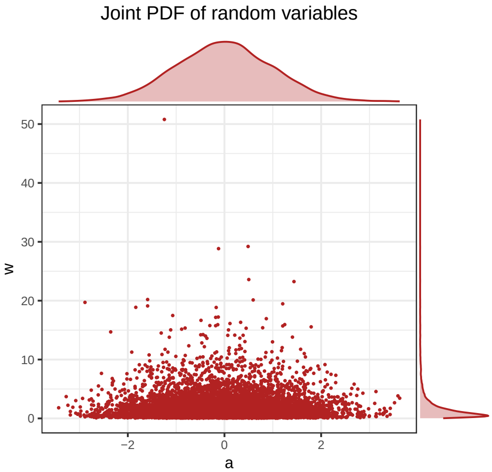
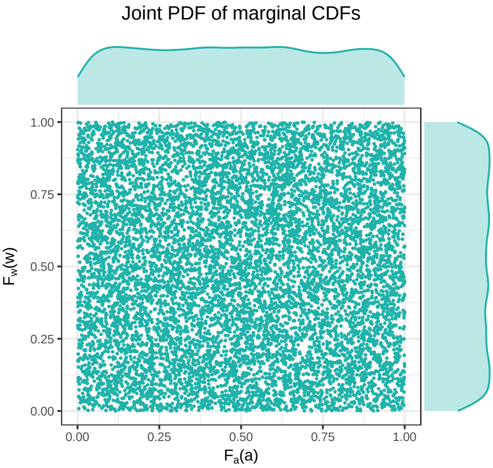
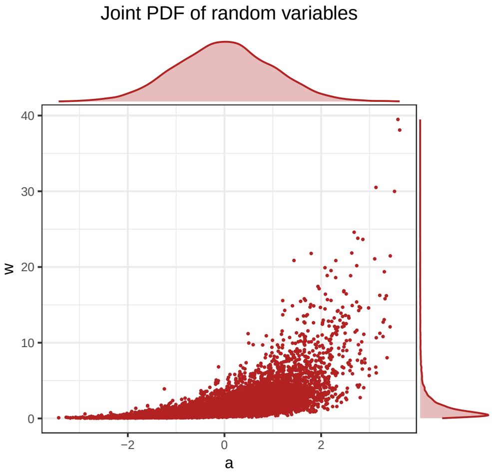
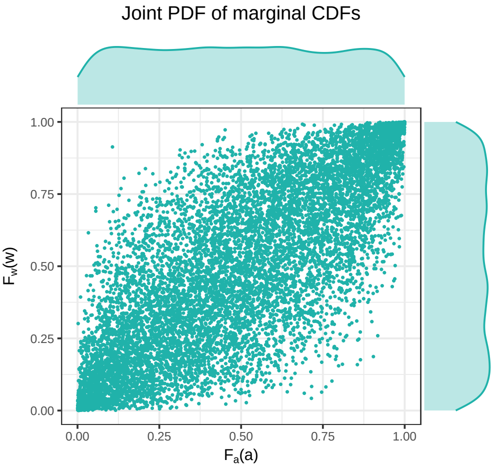
Marginal and joint PDFs of independent samples





Marginal and joint PDFs of dependent samples ($\rho = 0.8$)





How to measure dependence between variables?

We wish to describe dependence structure between w and a. What to do?

1st idea: Pearson correlation coefficient (✗: non-linear dependence);

2nd idea: Spearman's rank correlation coefficient (✗: information is too general);

3rd idea: to consider joint PDF of marginals (✗: family of CDFs are rarely known).

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Copulas can address these problems

✓: can capture non-linear dependence;

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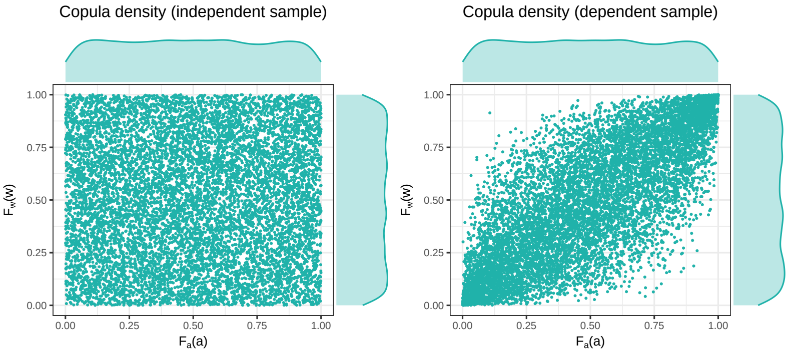

Let's get technical

There exists[1] a unique d-variate copula $C : [0,1]^d \to [0,1]$ such that

$$F(X_1, \ldots, X_d) = C\big(F_1(X_1), \ldots, F_d(X_d)\big),$$

where $(X_1, \ldots, X_d)$ is a set of d random variables like age, wage, etc.

Therefore, the copula density

$$c\big(F_1(X_1), \ldots, F_d(X_d)\big) = \frac{f(X_1, \ldots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}$$

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Why old methods are bad?

The existing copula estimators are not on-line.

Therefore, it has to be entirely recalculated in case of persistent data flow.

How to estimate copula density?

Recall that

$$c\big(F_1(X_1), \ldots, F_d(X_d)\big) = \frac{f(X_1, \ldots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}.$$

First, we need to recursively estimate joint PDF f and each of marginal PDFs f_j.

Robbins-Monro procedure

A root of unknown function h using only noisy observations can be found as follows[2]

$$z_n = z_{n-1} - \alpha_n H(z_n, \varepsilon_n)$$

where $H(z_n, \varepsilon_n) \equiv (h(z_n) + \varepsilon_n)$ is a noisy observation and α_n is a sequence of positive numbers that converges to 0.

It was proved that z_n converges almost surely to $z^\star$ as $n \to \infty$, given $\mathbb{E}[\varepsilon] = 0$.

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Quantile estimation using Robbins-Monro procedure

To estimate a PDF we first need to estimate quantiles.

An estimator of u-th quantile can be derived from the Robbins-Monro procedure

$$x_n^u = x_{n-1}^u + \alpha_n \big(u - \mathbf{1}(X_{n-1} \leqslant x_{n-1}^u)\big),$$

where x_n^u is a u-th quantile estimator.

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Smoothed recursive non-parametric estimation of marginal PDFs f_j

A smoothed adaptation of the quantile estimator[3]

$$
\begin{aligned}
f_{j,n}^{u_j} &= (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right) \\
a_{j,n}^{u_j} &= \max\left[\mu_{j,n}, \min\left[f_{j,n}^{u_j}, \nu \ln(n+1)\right]\right] \\
x_n^{u_j} &= x_{n-1}^{u_j} + \frac{1}{n a_{j,n}^{u_j}}\left[u_j - \mathbf{1}\left(X_{n,j} \leqslant x_{n-1}^{u_j}\right)\right],
\end{aligned}
$$

where $K_{h_{j,n}}^{u_j}(\cdot)$ is a symmetric, univariate kernel with bandwidth $h_{j,n}$; $\mu_{j,n}$ and ν are some positive constants.

[3] Definition and a proof of convergence in Amiri and Thiam, (2014)

Smoothed recursive non-parametric estimation of joint PDF f

A recursive estimator for the joint PDF defined as follows[4]

$$f_n^{\mathbf{u}} = (1 - 1/n) \cdot f_{n-1}^{\mathbf{u}} + 1/n \cdot \mathbf{K}_{b_n}^{\mathbf{u}}\left(x_{n-1}^{\mathbf{u}} - \mathbf{X}_n\right),$$

where now instead of u_j we have a grid of points $\mathbf{u}$ and multivariate data point $\mathbf{X}_n$; $\mathbf{K}_{b_n}^{\mathbf{u}}(\mathbf{X}) := \prod_{j=1}^{d} K_{b_n}^{u_j}(\mathbf{X}_j)$ where $K_{b_n}^{u_j}(\mathbf{X}_j)$ is a univariate kernel with bandwidth b_n.

[4] Definition and a proof of convergence in Mokkadem et al., (2009)

Some remarks on marginal and joint PDFs estimators

$$f_{j,n}^{u_j} = (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right)$$
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Remarks:

- The marginal estimator is just a special case of the joint estimator;
- The authors of the joint estimator didn't provide the estimation for quantiles $x^{\mathbf{u}}$;
- Therefore, the estimation of the quantile x^{u_j} has to be generalized ✓.

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Recursive non-parametric copula density estimator

Given

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$c_n(\cdot)$ defined as follows

$$c_n = \frac{f_n^{\mathbf{u}}}{\prod_{j=1}^{d} f_{j,n}^{u_j}}$$

to the best of our knowdlenge, is the first recursive non-parametric estimator of the copula density $c(\cdot)$

$$c = \frac{f}{\prod_{j=d}^{d} f_j}.$$

How to select the bandwidths?

We have two sets of bandwidths h_j for each of marginals and b for the joint.

Two groups of bandwidths:

- Data-independent;
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Constant bandwidth

The simplest data-independent case: h_j and b are the same[5] and calculated as follows

$$b_n = h_{j,n} = n^{-\tau},$$

where $\tau \in (0, 1/2)$.

[5] As proposed in Robinson, (1975) and Holst, (1987) for non-smooth recursive algorithms.

Silverman's bandwidth

h_j and b are still the same but let's use aggregated information about data[6]

$$b_n = h_{j,n} = \left(\frac{4}{3} S_{j,n}^5\right)^{\frac{1}{5}} \cdot n^{-\frac{1}{5}},$$

where $S_{j,n}$ is the recursive estimator of the standard deviation.

[6] Adapted from the non-recursive estimator proposed in Silverman, (1986)

"Specialized" bandwidths

Another idea is to use different bandwidths for marginal PDFs and the joint PDF.

An optimal bandwidth for the joint can be derived by optimizing $\mathrm{MSE}(f_n^{\mathbf{u}})$. It results in

$$\left[\frac{\widetilde{\eta}^d \cdot f^{\mathbf{u}} \cdot (d+2) \cdot d}{2 \cdot \eta^2 \cdot \left[\sum_{j=1}^{d} f_{(j,j)}^{\mathbf{u}}\right]^2 \cdot (d+4)}\right]^{\frac{1}{d+4}} \cdot n^{-\frac{1}{d+4}},$$

where

$\widetilde{\eta} \approx 0.28$ and $\eta = 1$ if kernel is standard normal distribution;

$f_{(j,j)}^{\mathbf{u}}$ is the 2^{nd} order derivative on j-th component;

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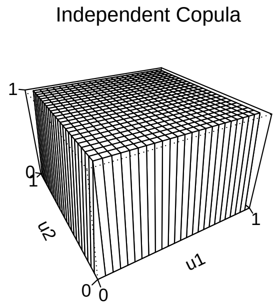
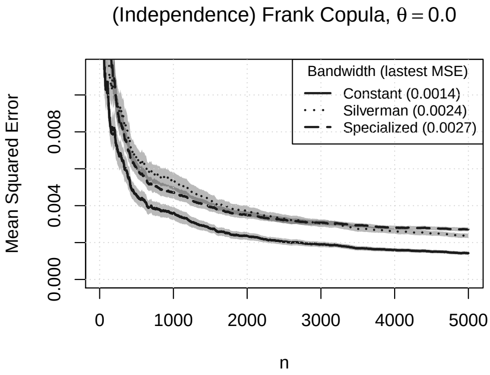
Set-up ($d = 2$)

1. Select a copula which properties are known;
2. Sample data ($\mathbf{X}$) from the selected copula distribution;
3. Choose a grid of points $\mathbf{u}$ to estimate copula on.
   For example, $(0.1, \ldots, 0.9) \times (0.1, \ldots, 0.9)$;
4. Estimate values of the copula density at each point of the selected grid $\mathbf{u}$;
5. Calculate $\mathrm{MSE}_n(c_n(\cdot), c(\cdot))$ at each n.

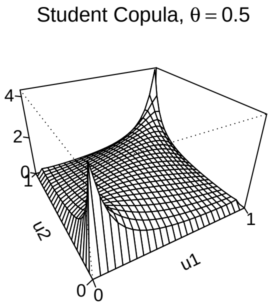
Here we use $n = 5000$ and number of Monte-Carlo simulations is 500.

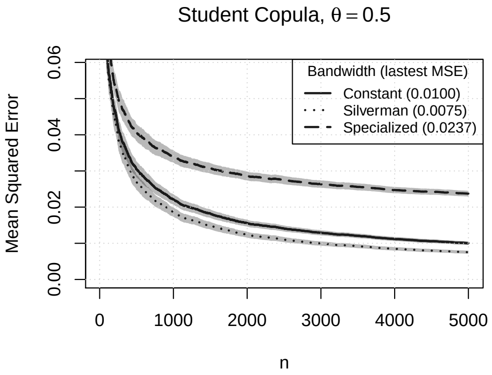
Independent Copula



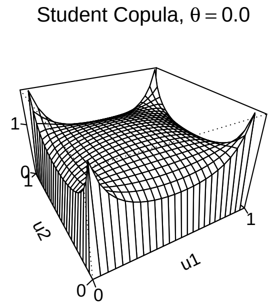


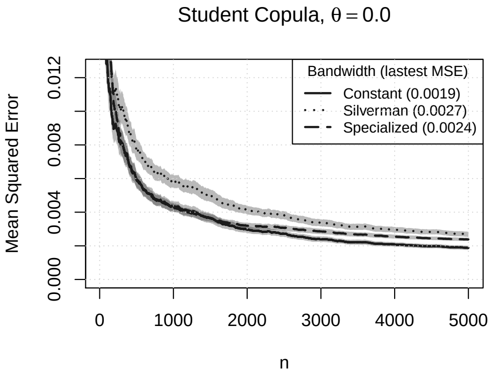
Student Copula ($\theta = 0.5$)



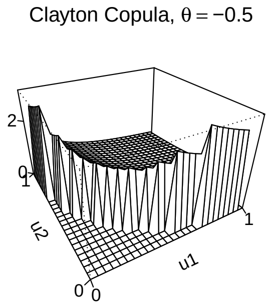


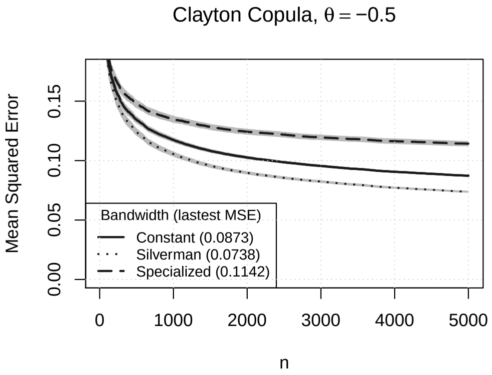
Student Copula ($\theta = 0.0$)



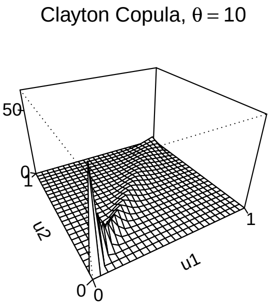


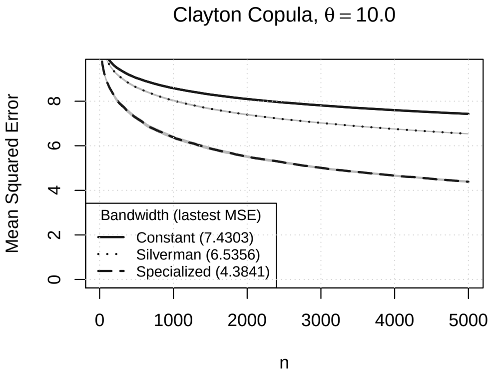
Clayton Copula ($\theta = -0.5$)





Clayton Copula ($\theta = 10$)





Numerical results

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Conclusion

Our contribution consists of the following

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2. In addition, using this result we presented the recursive non-parametric estimator of the copula density $c(\cdot)$;
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