Recursive Non-parametric Estimation of the Copula Density

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supervised by Geoffrey Decrouez

Higher School of Economics

9 June 2018

Marginal and joint PDFs

What is the joint probability distribution of wage and age?

Let us present both random variables in terms of univariate CDFs

$$F_w(w) := \mathbb{P}(W \leqslant w), \quad \log(w) \sim \mathcal{N}\left(\mu_w, \sigma_w^2\right)$$
$$F_a(a) := \mathbb{P}(A \leqslant a), \quad a \sim \mathcal{N}\left(\mu_a, \sigma_a^2\right).$$

Then, the joint distribution has the following form

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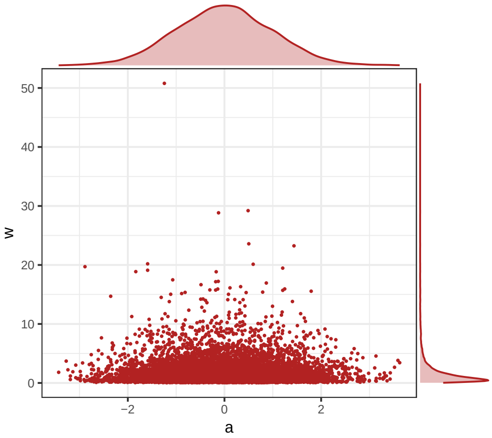
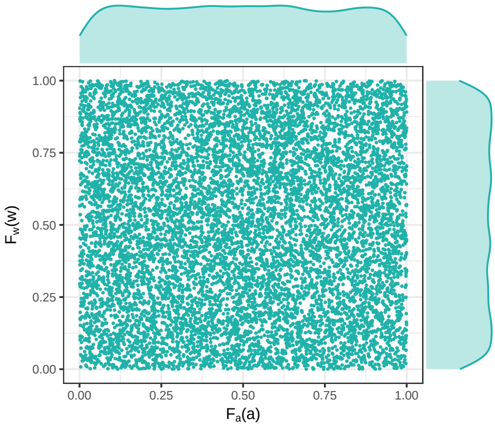
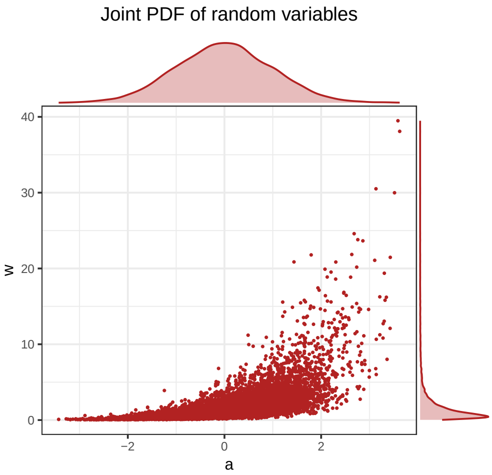
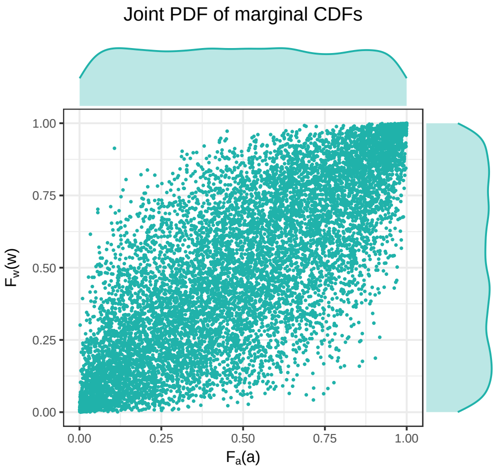
Marginal and joint PDFs of independent samples





Marginal and joint PDFs of dependent samples ($\rho = 0.8$)





How to measure dependence between variables?

We wish to describe dependence structure between w and a. What to do?

1st idea: Pearson correlation coefficient (✗: non-linear dependence);

2nd idea: Spearman's rank correlation coefficient (✗: information is too general);

3rd idea: to consider joint PDF of marginals (✗: family of CDFs are rarely known).

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Copulas can address these problems

- ✓: can capture non-linear dependence;
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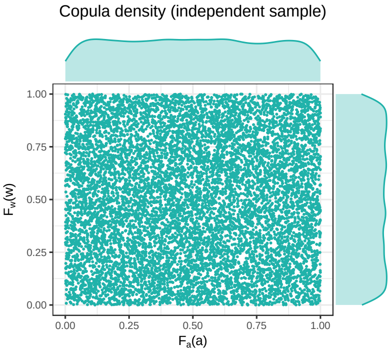

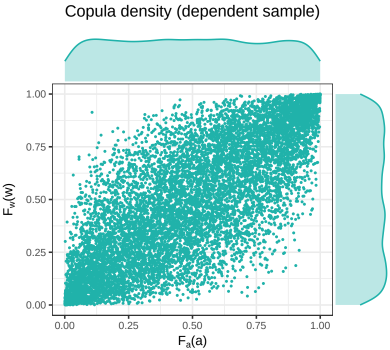

Let's get technical

There exists[1] a unique d-variate copula $C : [0,1]^d \to [0,1]$ such that

$$F(X_1, \dots, X_d) = C\big(F_1(X_1), \dots, F_d(X_d)\big),$$

where $(X_1, \dots, X_d)$ is a set of d random variables like age, wage, etc.

Therefore, the copula density

$$c\big(F_1(X_1), \dots, F_d(X_d)\big) = \frac{f(X_1, \dots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}$$

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Why old methods are bad?

The existing copula estimators are not on-line.

Therefore, it has to be entirely recalculated in case of persistent data flow.

How to estimate copula density?

Recall that

$$c\Big(F_1(X_1), \dots, F_d(X_d)\Big) = \frac{f(X_1, \dots, X_d)}{\prod_{j=d}^{d} f_j(X_j)}.$$

First, we need to recursively estimate joint PDF f and each of marginal PDFs f_j.

Robbins-Monro procedure

A root of unknown function h using only noisy observations can be found as follows[2]

$$z_n = z_{n-1} - \alpha_n H(z_n, \varepsilon_n)$$

where $H(z_n, \varepsilon_n) \equiv (h(z_n) + \varepsilon_n)$ is a noisy observation and α_n is a sequence of positive numbers that converges to 0.

It was proved that z_n converges almost surely to $z^\star$ as $n \to \infty$, given $\mathbb{E}[\varepsilon] = 0$.

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Quantile estimation using Robbins-Monro procedure

To estimate a PDF we first need to estimate quantiles.

An estimator of u-th quantile can be derived from the Robbins-Monro procedure

$$x_n^u = x_{n-1}^u + \alpha_n \big(u - \mathbf{1}(X_{n-1} \leqslant x_{n-1}^u) \big),$$

where x_n^u is a u-th quantile estimator.

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Smoothed recursive non-parametric estimation of marginal PDFs f_j

A smoothed adaptation of the quantile estimator[3]

$$
\begin{aligned}
f_{j,n}^{u_j} &= (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right) \\
a_{j,n}^{u_j} &= \max\left[\mu_{j,n}, \min\left[f_{j,n}^{u_j}, \nu \ln(n+1)\right]\right] \\
x_n^{u_j} &= x_{n-1}^{u_j} + \frac{1}{n a_{j,n}^{u_j}}\left[u_j - \mathbf{1}\left(X_{n,j} \leqslant x_{n-1}^{u_j}\right)\right],
\end{aligned}
$$

where $K_{h_{j,n}}^{u_j}(\cdot)$ is a symmetric, univariate kernel with bandwidth $h_{j,n}$; $\mu_{j,n}$ and ν are some positive constants.

[3]Definition and a proof of convergence in **Amiri2014**

Smoothed recursive non-parametric estimation of joint PDF f

A recursive estimator for the joint PDF defined as follows[4]

$$f_n^{\mathbf{u}} = (1 - 1/n) \cdot f_{n-1}^{\mathbf{u}} + 1/n \cdot \mathbf{K}_{b_n}^{\mathbf{u}}\left(x_{n-1}^{\mathbf{u}} - \mathbf{X}_n\right),$$

where now instead of u_j we have a grid of points $\mathbf{u}$ and multivariate data point $\mathbf{X}_n$; $\mathbf{K}_{b_n}^{\mathbf{u}}(\mathbf{X}) := \prod_{j=1}^{d} K_{b_n}^{u_j}(\mathbf{X}_j)$ where $K_{b_n}^{u_j}(\mathbf{X}_j)$ is a univariate kernel with bandwidth b_n.

[4] Definition and a proof of convergence in **Mokkadem2009**

Some remarks on marginal and joint PDFs estimators

$$f_{j,n}^{u_j} = (1 - 1/n) \cdot f_{j,n-1}^{u_j} + 1/n \cdot K_{h_{j,n}}^{u_j}\left(x_{n-1}^{u_j} - X_{n,j}\right)$$
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Remarks:

- The marginal estimator is just a special case of the joint estimator;
- The authors of the joint estimator didn't provide the estimation for quantiles $x^{\mathbf{u}}$;
- Therefore, the estimation of the quantile x^{u_j} has to be generalized ✓.

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Recursive non-parametric copula density estimator

Given

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$c_n(\cdot)$ defined as follows

$$c_n = \frac{f_n^{\mathbf{u}}}{\prod_{j=1}^{d} f_{j,n}^{u_j}}$$

to the best of our knowdlenge, is the first recursive non-parametric estimator of the copula density $c(\cdot)$

$$c = \frac{f}{\prod_{j=d}^{d} f_j}.$$

How to select the bandwidths?

We have two sets of bandwidths h_j for each of marginals and b for the joint.

Two groups of bandwidths:

- Data-independent;
- Data-dependent.

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Constant bandwidth

The simplest data-independent case: h_j and b are the same[5] and calculated as follows

$$b_n = h_{j,n} = n^{-\tau},$$

where $\tau \in (0, 1/2)$.

[5] As proposed in **Robinson1975** and **Holst1987** for non-smooth recursive algorithms.

Silverman's bandwidth

h_j and b are still the same but let's use aggregated information about data[6]

$$b_n = h_{j,n} = \left(\frac{4}{3} S_{j,n}^5\right)^{\frac{1}{5}} \cdot n^{-\frac{1}{5}},$$

where $S_{j,n}$ is the recursive estimator of the standard deviation.

[6]Adapted from the non-recursive estimator proposed in **Silverman1986**

"Specialized" bandwidths

Another idea is to use different bandwidths for marginal PDFs and the joint PDF.

An optimal bandwidth for the joint can be derived by optimizing $\mathrm{MSE}(f_n^{\mathbf{u}})$. It results in

$$\left[\frac{\widetilde{\eta}^d \cdot f^{\mathbf{u}} \cdot (d+2) \cdot d}{2 \cdot \eta^2 \cdot \left[\sum_{j=1}^d f_{(j,j)}^{\mathbf{u}}\right]^2 \cdot (d+4)}\right]^{\frac{1}{d+4}} \cdot n^{-\frac{1}{d+4}},$$

where

$\widetilde{\eta} \approx 0.28$ and $\eta = 1$ if kernel is standard normal distribution;

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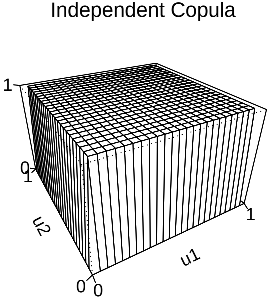
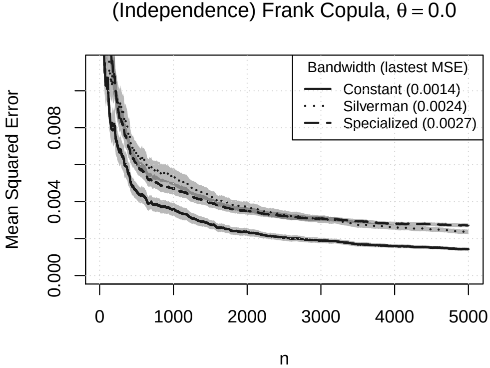
Set-up ($d = 2$)

1. Select a copula which properties are known;
2. Sample data ($\mathbf{X}$) from the selected copula distribution;
3. Choose a grid of points $\mathbf{u}$ to estimate copula on.
   For example, $(0.1, \ldots, 0.9) \times (0.1, \ldots, 0.9)$;
4. Estimate values of the copula density at each point of the selected grid $\mathbf{u}$;
5. Calculate $\mathrm{MSE}_n(c_n(\cdot), c(\cdot))$ at each n.

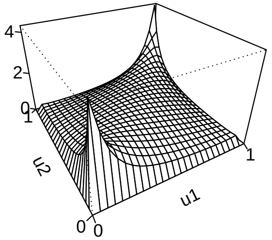
Here we use $n = 5000$ and number of Monte-Carlo simulations is 500.

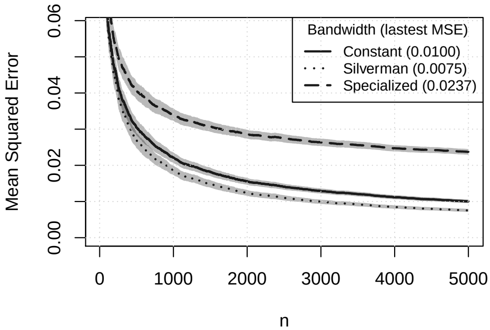
Independent Copula



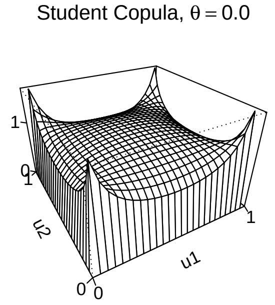


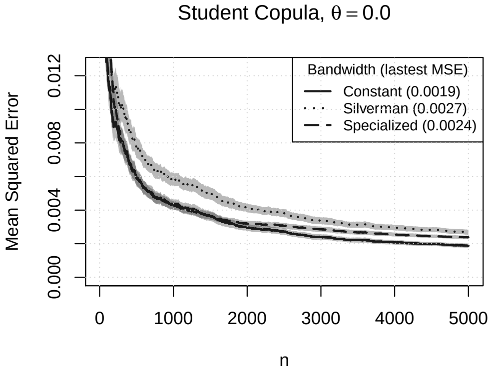
Student Copula ($\theta = 0.5$)



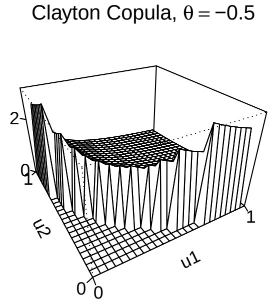


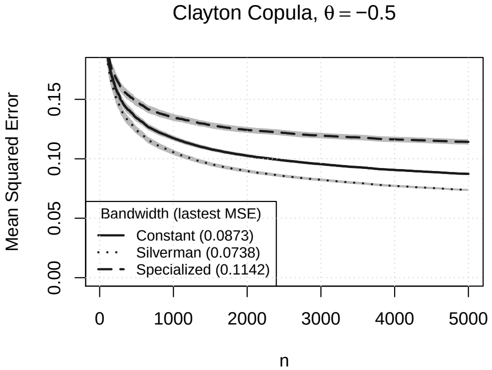
Student Copula ($\theta = 0.0$)



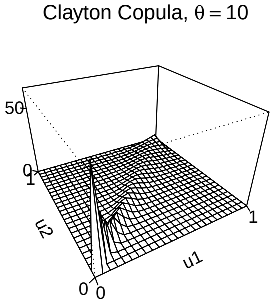


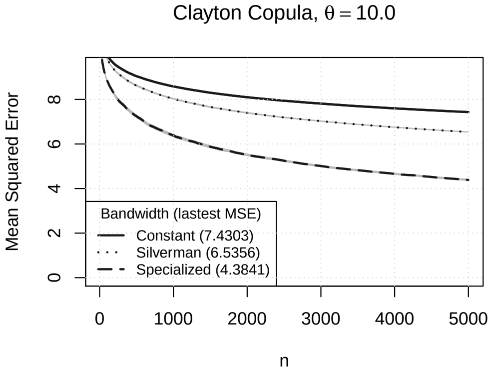
Clayton Copula ($\theta = -0.5$)





Clayton Copula ($\theta = 10$)





Numerical results

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Conclusion

Our contribution consists of the following

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2. In addition, using this result we presented the recursive non-parametric estimator of the copula density $c(\cdot)$;
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