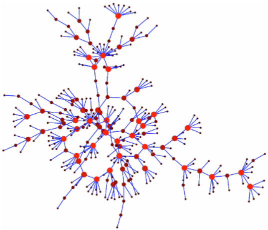
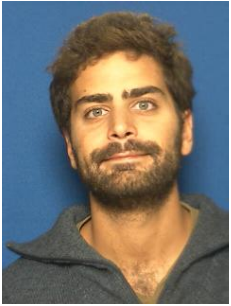
FRTN30
Network Dynamics
Lecture 1
Introduction and basic graph theory



Giacomo Como
Lund, March 19, 2024

We are

Course responsible and lecturer:



Giacomo Como
giacomo.como@control.lth.se

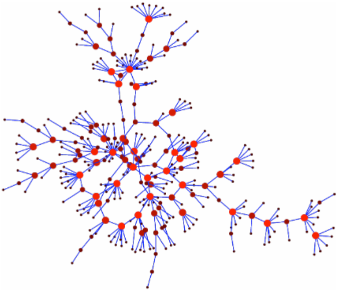
We are

Teaching assistants:

Luka Bakovic David Ohlin

{luka.bakovic, david.ohlin}@control.lth.se

About the course



FRTN30 Network Dynamics

(7.5 hp, VT LP2)

- 14 lectures, most of them in person
- 14 exercise sessions, in person
- Course webpage:

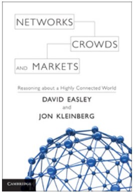
http://www.control.lth.se/Education/EngineeringProgram/FRTN30.html

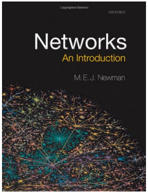
- Canvas platform: registered students should have received email
- all material available through Canvas

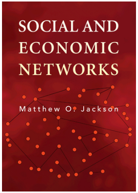
Course material

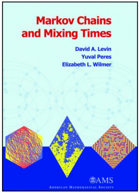
- lecture notes, slides, links to video-lectures, and other material will be posted on course webpage and/or Canvas page

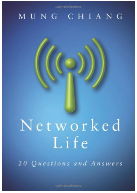
- there are several interesting books on the subject, none of which covers exactly the program. E.g.,

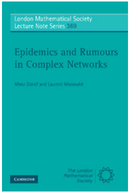












but none of them matches the course contents, lecture notes do

Hand-ins

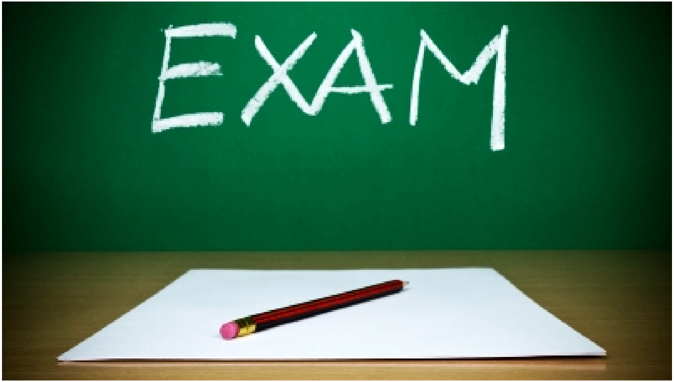
Four compulsory hand-ins with the following deadlines:

- HW1: posted on 25/3; due on 14/4;
- HW2: posted on 15/4; due on 29/4;
- HW3: posted on 3/5; due on 17/5;
- HW4: posted on 8/5; due on 22/5.

- On-time submission is encouraged: 1 point in the exam assigned to each complete on-time submission.

- Collaboration policy: collaboration exchange of ideas among students are encouraged, however, every student has to submit her/his own report (in pdf format) and specify whom she/he has collaborated with and on what particular part of the work.

- Outstanding hand-ins may be rewarded by up to 1 extra point.



- 5-hours written exam on Monday, May 26th, 2024, 8–13.
- Only lecture notes allowed.
- 25 points total (21+4), then quantized to grades F/3/4/5
- Retake Saturday, August 31st, 2024, 8–13.

Course contents (lectures)

- L1 (19/3) Graphs and Networks: basic notions.
- L2 (20/3) Network centrality.
- L3 (21/3) Linear network dynamics: averaging, compartmental.
- L4 (25/3) Connectivity and the max-flow min-cut theorem.
- L5 (26/3) Network flow optimization: system vs user optimum.
- L6 (16/4) Markov chains and random walks, part 1.
- L7 (17/4) Markov chains and random walks, part 2.
- L8 (18/4) Network epidemics: SI, SIR, SIS, voter models. (online)
- L9 (23/4) Basics game theory 1: Nash equilib., potential games.
- L10 (2/5) Basics game theory 2: (noisy) best response dynamics.
- L11 (6/5) Random graphs 1: branching process, Erdos-Renyi.
- L12 (6/5) Random graphs 2: configuration model, pref. attachm.
- L13 (13/5) Seminar Material: Probabilistic Graphical Models.
- L14 (14/5) Course summary. Pointers to further studies.

Course contents (exercise sessions)

E1 (20/3) Graphs and Networks: basic notions.
E2 (22/3) Linear network dynamics.
E3 (25/3) Preparation to HW1.
E4 (27/3) Network connectivity.
E5 (28/3) Network flow optimization
E6 (15/4) Preparation to HW2.
E7 (22/4) Markov chains and random walks.
E8 (24/4) Network epidemics: SI, SIR, SIS models.
E9 (25/4). Game theory 1
E10 (2/5) Game theory 2.
E11 (3/5) Preparation to HW3
E12 (7/5) Random graphs.
E13 (8/5) Preparation to HW4.
E14 (15/5) Summary.
E15 (20/5) Q&A.

(Complex) networks

(Large-scale) systems of (simple) interacting units

- infrastructure networks: transportation, power, gas, and water distribution, sewer, Internet
- informational networks: WWW, citation networks
- social networks: friendships, family ties, Facebook etc.
- economic networks: supply chains, production networks
- financial networks: borrowing-lending nets
- biological networks: neural networks, gene/protein interactions
- ecological networks: food webs, flocks, ...

Network dynamics

network structure + interaction mechanism

$\Downarrow$

emergent behavior

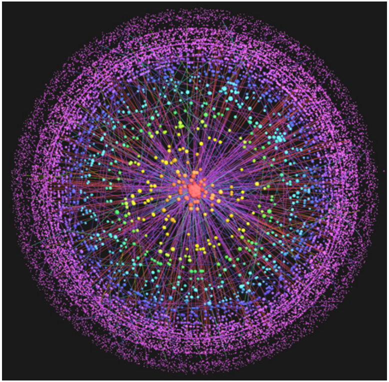
Many applications:

- physical flows in infrastructure networks
- opinion formation, social influence, and learning
- design of distributed optimization/estimation algorithms
- spread of epidemics and innovation
- cascading failures, systemic risk
- ...

Emphasis on common principles:

- network connectivity, fragility and resilience;
- centrality and influence;
- threshold phenomena (the 'tipping point').

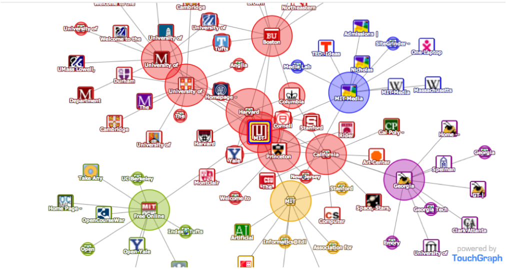
Internet



10 billions connected devices

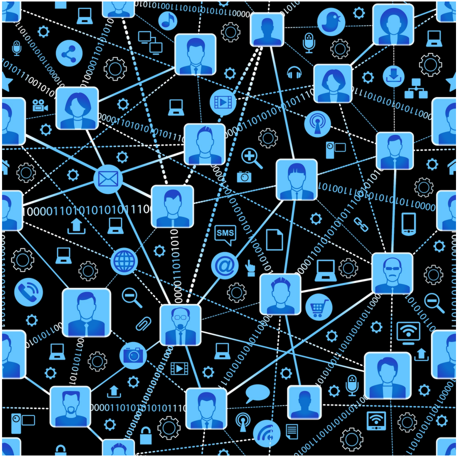
The World-Wide Web

4 billions webpages

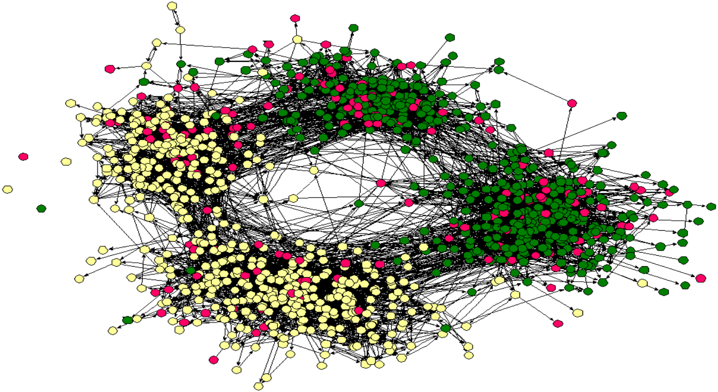


The web link structure centered at http://web.mit.edu

Social networks

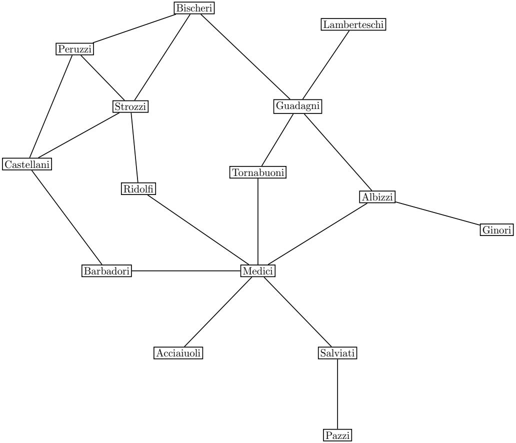


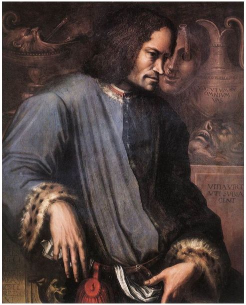
High school friendships



Moody, ‘Race, school integration, and friendship segregation in America’, 2002

Family ties

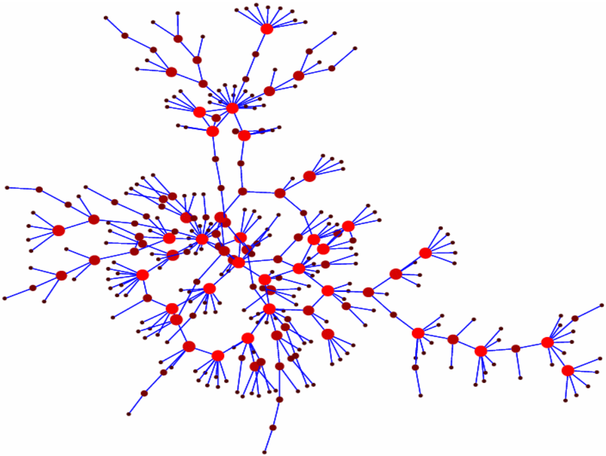




''Lorenzo de' Medici'
G. Vasari, 1534

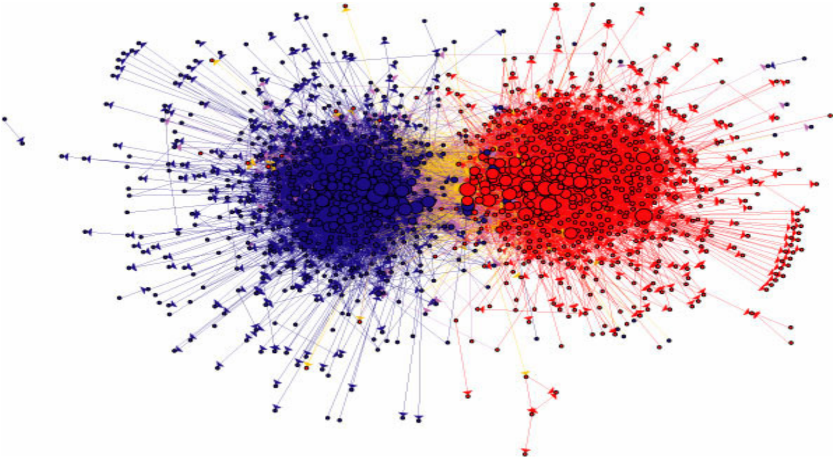
Padgett and Ansell, 'Robust action and the rise of the Medici, 1400-1434', 1993

Sexual contacts



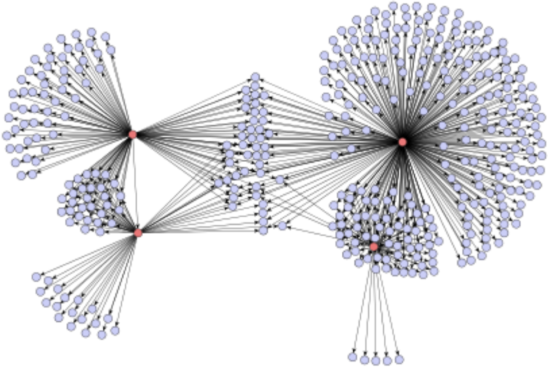
Potterat et al., 'Risk network structure in the early epidemic phase of HIV transmission in Colorado Springs', 2002.

Blog networks



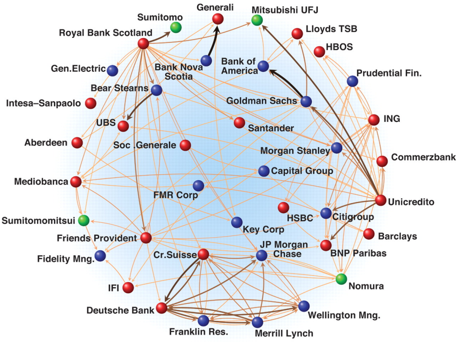
Adamic and Glance, 'The Political Blogosphere and the 2004 U.S. Election: Divided They Blog', 2005

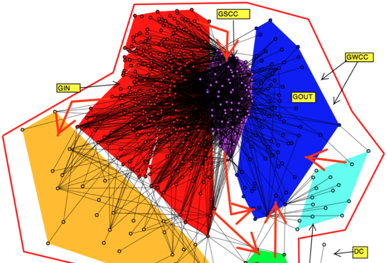
Email chains



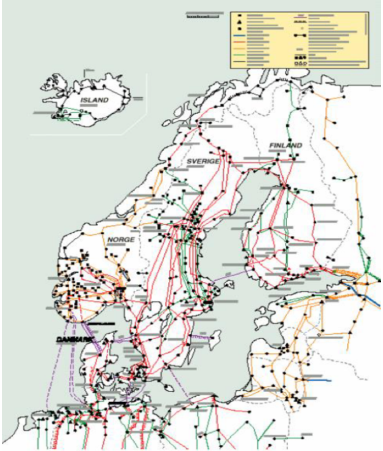
Leskovec, 'E-mail recommendations for a Japanese graphic novel', 2007

Financial Networks

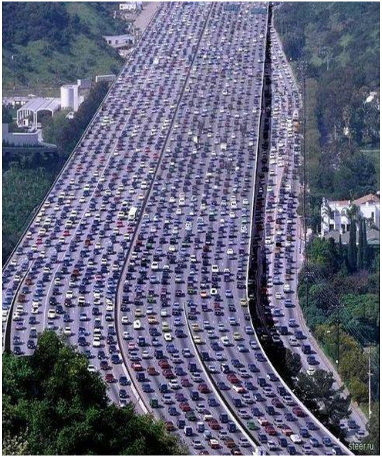


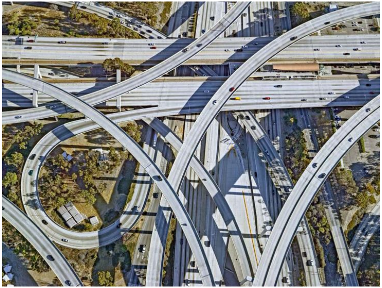


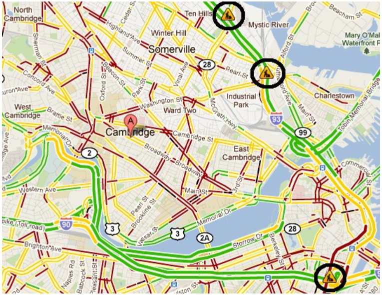
Power transmission grid



Road traffic networks

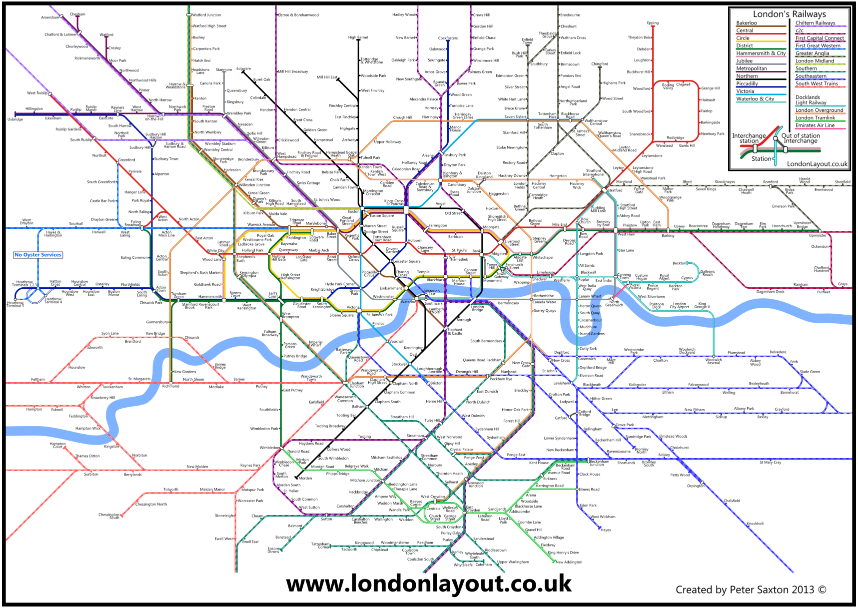




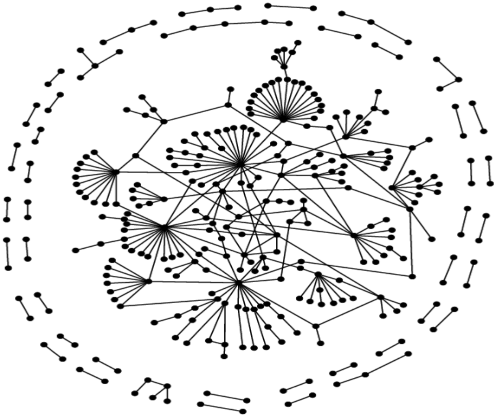


Left.: China Highway 110, August 2010, 100 km-long jam, 10 days
Right: Google Maps, Cambridge (MA, USA), July 11, 2011, 18:30

Public transport

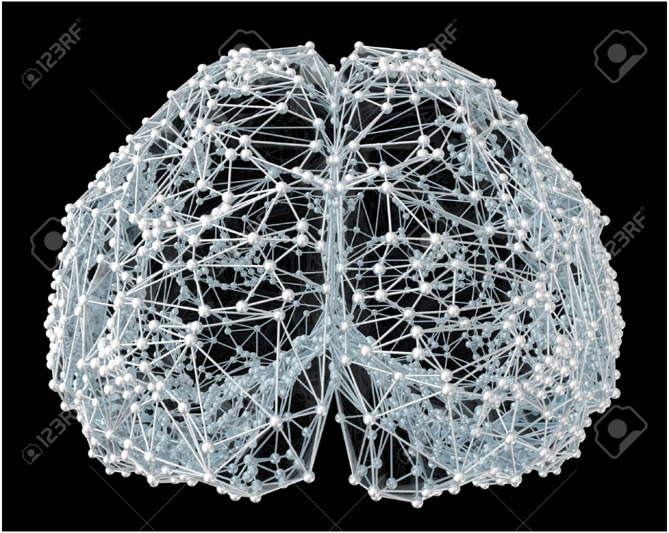


Protein networks



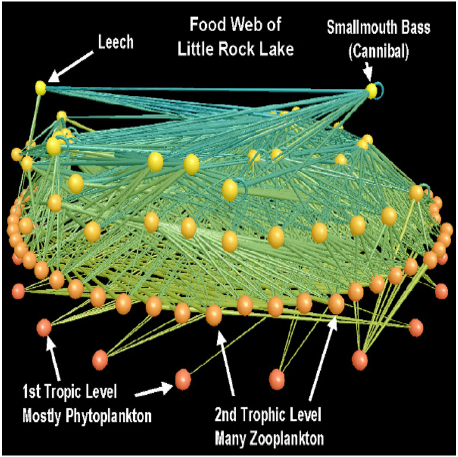
Maslov and Sneppen 'Specificity and stability in topology of protein networks', 2002

Neural Systems



Il cervello

Freshwater food web



Martinez, ‘Artifacts or attributes? Effects of resolution on the Little Rock Lake food web’, 1991

Biological networks




Network dynamics

network structure + interaction mechanism

$\Downarrow$

emergent behavior

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L13 Probabilistic Graphical Models.

L14 Course summary. Pointers to further studies.

Networks as graphs

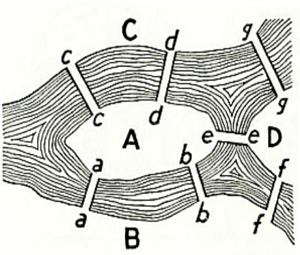
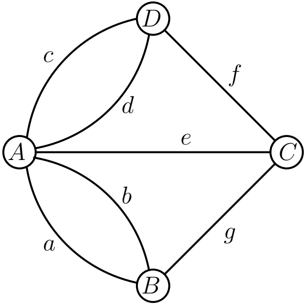
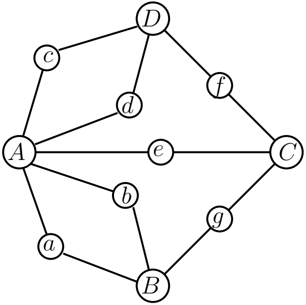


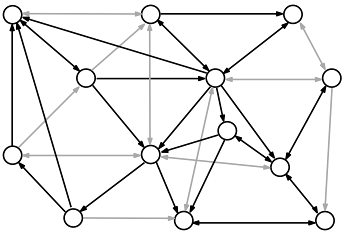
FIGURE 98. *Geographic Map: The Königsberg Bridges.*

L. Euler (1707–1783)





Networks as graphs



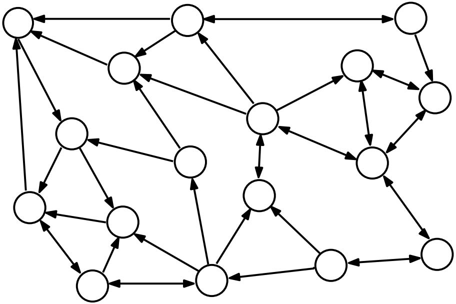
We represent networks as graphs:

- a graph $\mathcal{G}$ has a set $\mathcal{V} = \{1, \ldots, n\}$ of nodes (a.k.a. vertices) and a set $\mathcal{E}$ of links (a.k.a. edges)
- links can be directed ((i, j) points from i to j) or undirected ($\{i, j\} = (i, j)$ and (j, i))
- sometimes convenient to associate to each link a positive weight:

$$\text{weight of link } (i, j) = W_{ij} \begin{cases} > 0 & \text{if} \quad (i, j) \in \mathcal{E} \\ = 0 & \text{if} \quad (i, j) \notin \mathcal{E} \end{cases}$$

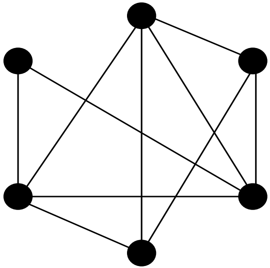
- weight matrix $W = (W_{ij}) \in \mathbb{R}_{+}^{n \times n}$

Examples



- Internet: nodes=routers, edges=direct physical links (und.)
- scientific collab.: nodes=researchers, link=coauthors (und.)
- World Wide Web: nodes=webpages, links=hyperlinks (dir., unw.)
- traffic networks: nodes=junctions, links=roads (directed, weighted)

Weight matrix / Adjacency matrix

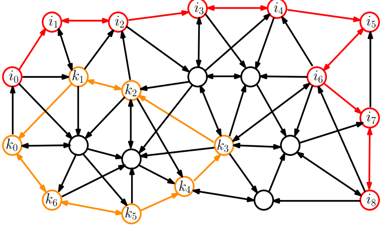


$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

A graph is described by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

- $\mathcal{G}$ unweighted if $W_{ij} \in \{0, 1\}$ for all i, j, i.e., if all existing links have weight 1. If so, W is called adjacency matrix and $(\mathcal{V}, \mathcal{E}) \leftrightarrow W$
- $\mathcal{G}$ undirected if $W' = W$, i.e., if all links are bidirectional with the same weight $W_{ij} = W_{ji}$ in both direction.
- $\mathcal{G}$ simple if
unweighted + undirected + no self-loops ($W_{ii} = 0$ for all $i \in \mathcal{V}$)

Walks, paths, and cycles



In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

- a walk from i to j is a sequence of nodes $i = i_0, i_1, \ldots, i_l = j$ such that $(i_{h-1}, i_h) \in \mathcal{E}$ for $h = 1, \ldots, l$;
- a path from i to j is a walk $i = i_0, i_1, \ldots, i_l = j$ that never visits the same node more than once except for possibly $i = j$;
- a circuit is a walk that starts and ends in the same node $i = j$;
- a cycle is a circuit of length≥ 3 s.t. $i_h \neq i_k$ unless $h, k \in \{0, l\}$;
- l is called the length of the walk, path, or cycle;
- note: length-1 paths = links; length-1 circuits = self-loops;
- $\mathcal{G}$ is called acyclic if it contains no cycles;
- j reachable from i it there exists some a walk ($\Rightarrow$a path) from i to j
- $\mathcal{G}$ is (strongly) connected if every j is reachable from every i

Walks, paths, and cycles

In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

- weight of a length-l walk, path, circuit, or cycle $i_0, i_1, \ldots, i_l$ is

$$W_{i_0 i_1} \cdot W_{i_1 i_2} \cdot \ldots \cdot W_{i_{l-1} i_l} = \prod_{1 \le h \le l} W_{i_{h-1} i_h}$$

- in an unweighted graph all walks, paths and cycles have weight 1

Proposition: In an unweighted graph,

$$\#\{\text{length-}l \text{ walks from } i \text{ to } j\} = (W^l)_{ij}.$$

For a general weighted graph

$$\text{total weight of all length-}l \text{ walks from } i \text{ to } j = (W^l)_{ij}.$$

Neighborhoods and degrees

- out-neighborhood of node i: $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$
- in-neighborhood of node i: $\mathcal{N}_i^- = \{j : (j,i) \in \mathcal{E}\}$
- $\mathcal{G}$ undirected $\Longrightarrow \mathcal{N}_i^- = \mathcal{N}_i$ for all i
- out-degree $w_i = \sum_j W_{ij}$ in-degree $w_i^- = \sum_j W_{ji}$
- degree vectors:

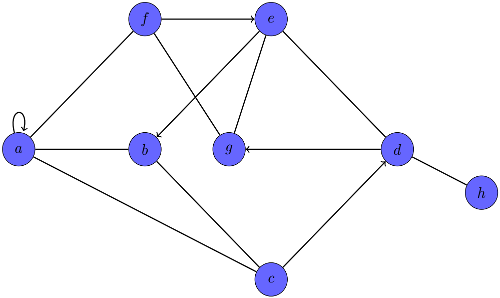
$$w = W\mathbb{1} \qquad w^- = W'\mathbb{1}$$

- average degree:

$$\overline{w} = \frac{1}{n}\mathbb{1}'W\mathbb{1} = \frac{1}{n}\sum_i w_i = \frac{1}{n}\sum_i w_i^-$$

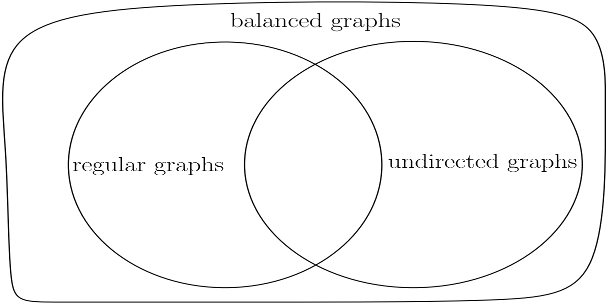
- $\mathcal{G}$ unweighted $\Longrightarrow$ $w_i = |\mathcal{N}_i|$ $w_i^- = |\mathcal{N}_i|$
- i source node if $\mathcal{N}_i^- = \emptyset$ or $\mathcal{N}_i^- = \{i\}$
- i sink node if $\mathcal{N}_i = \emptyset$ or $\mathcal{N}_i = \{i\}$

Example



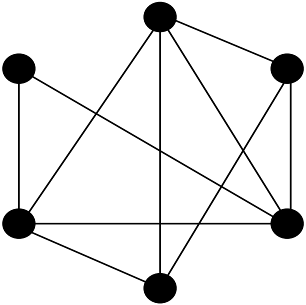
$\mathcal{N}_d^+ = \{e, g, h\}$, $\mathcal{N}_d^- = \{c, e, h\}$.
No sinks, no sources.

Balanced, regular, and undirected graphs



- a graph is called balanced if $w^- = w$
- clearly every undirected graph is balanced, but not vice versa
- a graph is called regular if $w_i = w_i^- = \overline{w}$ for all i
- every regular graph is balanced, but not vice versa; neither regular graphs are necessarily undirected, nor vice versa.

Average degree, hand-shaking lemma



$$4 + 3 + 4 + 3 + 4 + 2 = 2 \cdot 10$$

- Hand-shaking: In a simple graph the total degree is equal to twice the number of undirected links

$$n\overline{w} = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} w_{ij} = 2m$$

Distance and diameter

- distance of j from i:

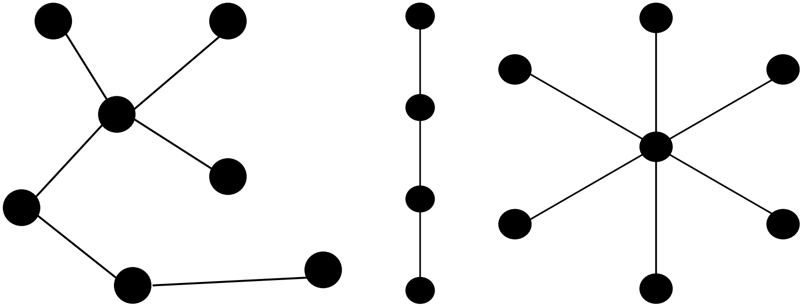
$$\mathrm{dist}(i,j) = \text{length of shortest path from } i \text{ to } j$$

- geodesic path from i to j: path of minimal length
- diameter of graph $\mathcal{G}$:

$$\mathrm{diam}(\mathcal{G}) = \max_{i,j \in \mathcal{G}} \mathrm{dist}(i,j)$$

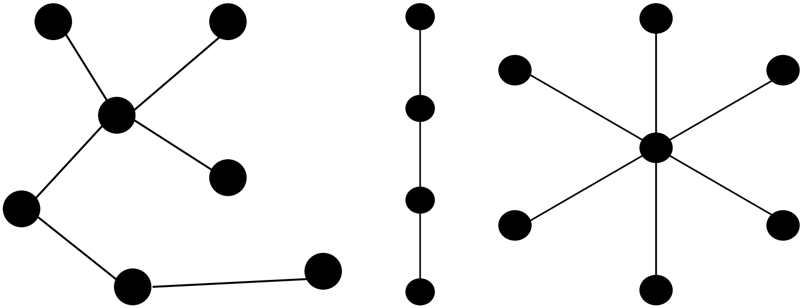
- $\mathcal{G}$ connected if and only if $\mathrm{diam}(\mathcal{G}) < +\infty$

Trees



- a tree is a simple connected graph with no cycles
- degree-1 nodes in a tree are called leaves
- a tree with 2 leaves is called a line
- a tree with $n - 1$ leaves is called a star

Trees

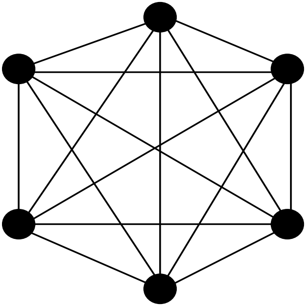


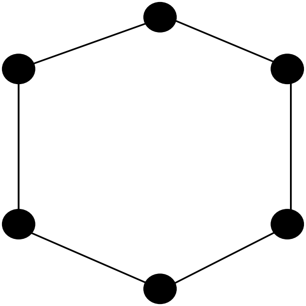
- tree = simple connected graph with no cycles

Theorem

- $\mathcal{G}$ simple + connected $\Longrightarrow m \geq n - 1$
- $\mathcal{G}$ simple + connected. Then, $\mathcal{G}$ tree $\Leftrightarrow m = n - 1$ (prove it!)
- $\mathcal{G}$ tree $\Longleftrightarrow$ connected and every $i, j \in \mathcal{V}$ connected by single path
- tree with $n \geq 3 \Longrightarrow$ number of leaves ≥ 2 and $\leq n - 1$

Other special simple graphs





complete graph K_n

$m = n(n-1)/2$

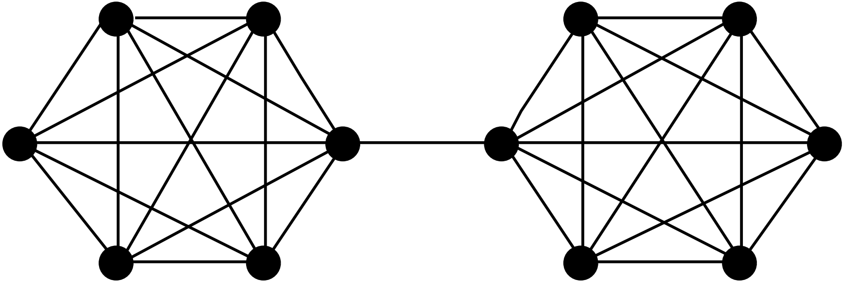
$\mathrm{diam}(\mathcal{G}) = 1$

ring graph C_n

$m = n$

$\mathrm{diam}(\mathcal{G}) = \lfloor n/2 \rfloor$

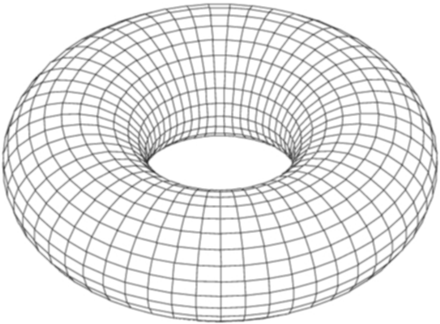
Other special simple graphs



barbell graph: $m = \frac{n}{2}\left(\frac{n}{2} - 1\right) + 1$

Torus graph

$C_h \times C_k$ *Toroidal grid* with $n = h \cdot k$ nodes and $m =?$.



Subgraphs

$\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

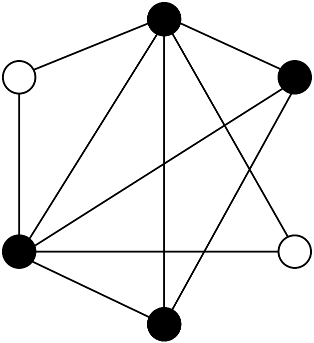
$$\tilde{\mathcal{G}} \subseteq \mathcal{G} \text{ if } \tilde{\mathcal{V}} \subseteq V,\ \tilde{\mathcal{E}} \subseteq \mathcal{E}$$

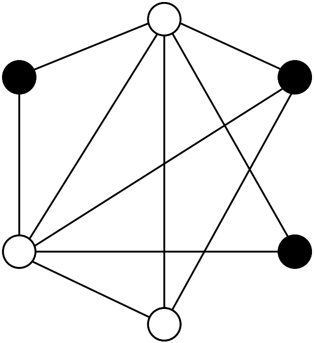
- induced subgraph $\tilde{\mathcal{G}} = \mathcal{G}|_{\tilde{\mathcal{V}}}$: $\tilde{\mathcal{V}} \subseteq \mathcal{V}$, $\tilde{\mathcal{E}} = \mathcal{E} \cap (\tilde{\mathcal{V}} \times \tilde{\mathcal{V}})$
- spanning subgraph: $\tilde{\mathcal{V}} = \mathcal{V}$.

Cliques and independent sets

In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$,

- $\mathcal{U} \subseteq \mathcal{V}$ is a clique if $(i, j) \in \mathcal{E}$ for every $i \neq j$ in $\mathcal{U}$,
  i.e., if the induced subgraph $\mathcal{G}|_{\mathcal{U}}$ contains the complete graph
- $\mathcal{U} \subseteq \mathcal{V}$ is an independent set if $(i, j) \notin \mathcal{E}$ for every $i \neq j$ in $\mathcal{U}$,
  i.e., if the induced subgraph $\mathcal{G}|_{\mathcal{U}}$ contains only isolated nodes





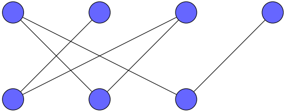
Bipartite graphs

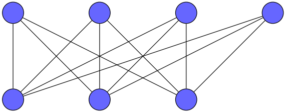
- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ bipartite if the node set can be partitioned as

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \qquad \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$$

  in such a way that both $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent sets, i.e.,

$$\mathcal{E} \subseteq (\mathcal{V}_1 \times \mathcal{V}_2) \cup (\mathcal{V}_2 \times \mathcal{V}_1)$$





- Theorem: A simple graph $\mathcal{G}$ is bipartite if and only if it does not have any odd-length cycle.
- Ex.: trees, rings with even n, d-dimensionale grids, hypercubes

Periodic and aperiodic graphs

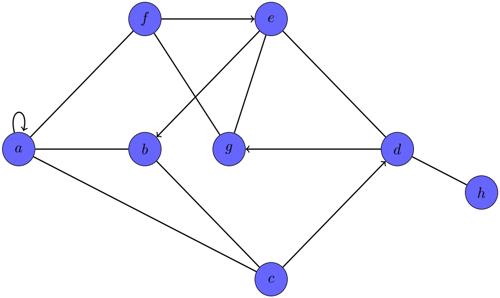
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

- the period of node i in $\mathcal{G}$ is

$$\text{per}(i) := \text{gcd}\{\text{lengths of circuits passing through } i\}$$

- Theorem: if $\mathcal{G}$ strongly connected, $\text{per}(i) = \text{per}(j)$ for all $i, j \in \mathcal{V}$
- $\text{per}_{\mathcal{G}}$ = period of a strongly connected graph $\mathcal{G}$
- strongly connected graph $\mathcal{G}$ is said aperiodic if $\text{per}_{\mathcal{G}} = 1$
- $\mathcal{G}$ contains selfloop $\Longrightarrow$ $\mathcal{G}$ aperiodic
- $\mathcal{G}$ undirected $\Longrightarrow$ $\text{per}_{\mathcal{G}} = 1, 2$

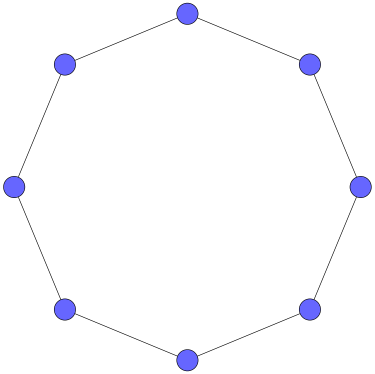
Periodic and aperiodic graphs



Aperiodic

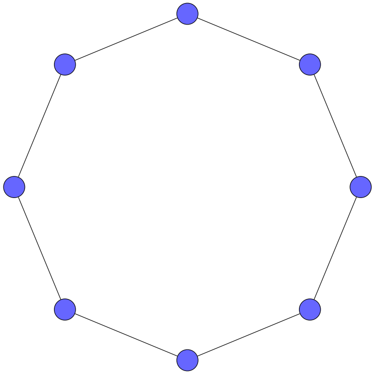
Periodic and aperiodic graphs

C_n *Ring*



Periodic and aperiodic graphs

C_n Ring



$$\mathrm{per}_{C_n} = g.c.d.\{2, n\} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

Relevant statistics for networks

Sometimes too much to describe the network $\mathcal{G}$ completely. Interested in aggregate statistics such as: diameter, degree distribution, clustering coefficient, modularity.

Properties widely observed in empirical studies of large-scale 'complex' networks

1. scale free $\leftrightarrow$ power law degree distribution
2. small world $\leftrightarrow$ diameter $\approx \log n$
3. high clustering $\leftrightarrow$ many triangles

Degree distribution

- in undirected unweighted network

$$p_k := \frac{1}{n}\left|\{i \in \mathcal{V} : w_i = k\}\right|, \qquad k \geq 0$$

fraction of nodes of degree k

- in directed unweighted network distinguish between joint degree,

$$p_{hk} := \frac{1}{n}\left|\{i \in \mathcal{V} : w_i^- = h,\, w_i = k\}\right|, \qquad h, k \geq 0\,,$$

in-degree, and out-degree distributions

$$p_k := \sum_{l \geq 0} p_{lk} = \frac{1}{n}\left|\{i : w_i = k\}\right|, \qquad p_h^- := \sum_{l \geq 0} p_{hl} = \frac{1}{n}\left|\{i : w_i^- = h\}\right|$$

Power law degree distributions

$$w_i := \#\{\text{neighbors of } i\} \qquad p_k := \frac{1}{n}\#\{v : w_i = k\}$$

$$p_k \sim \frac{C_\gamma}{k^\gamma} \qquad C_\gamma = \sum_{k\geq 0} k^{-\gamma} \qquad \gamma > 1$$

Empirical studies:

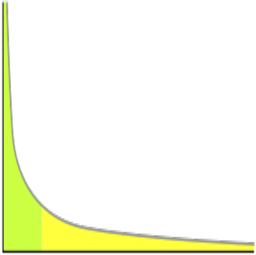
- Barabasi and Albert ('99): WWW has $\gamma_{in} \sim 2.1$, $\gamma_{out} \sim 2.7$
- Faloutos ('99): Internet $\gamma \sim 2.16$ Actor collab.s: $\gamma \sim 2.3$
- Redner ('98): citation network: $\gamma_{in} \sim 2.6$, $p_d^{out} \sim C\exp(-Kd)$
- Liljeros ('01): # sexual partners per year (in Sweden) $\gamma_{male} \sim 3.3$, $\gamma_{female} \sim 3.5$

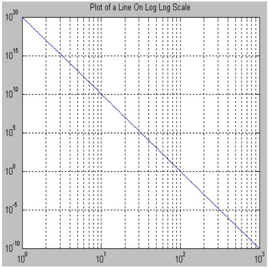
$$2 < \gamma \leq 3 \quad \Longrightarrow \quad \overline{w} = \sum_{k\geq 0} p_k k < +\infty \quad \overline{w^2} = \sum_{k\geq 0} p_k k = +\infty$$

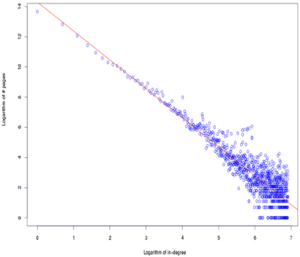
$$\gamma \geq 3 \quad \Longrightarrow \quad \overline{w} = \sum_{k\geq 0} p_k k < +\infty \quad \overline{w^2} = \sum_{k\geq 0} p_k < +\infty$$

power law $\leftrightarrow$ scale free

Power laws



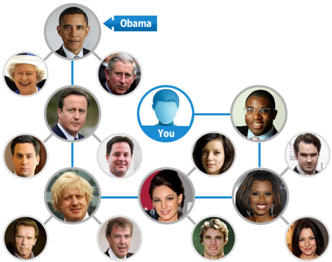


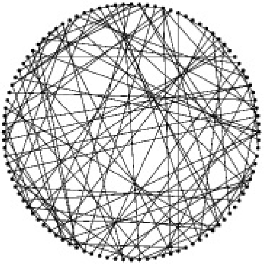


$$p_k \sim C_\gamma / k^\gamma$$

show up in quite different contexts:

Do we live in a small-world?

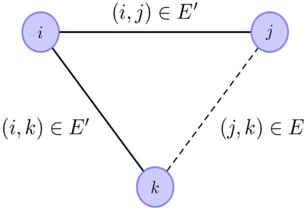




- Milgram's experiment ('67): randomly selected group of few hundreds of people from Omaha (NE). A letter given to each of them to be delivered to a stock broker living in Boston (MA). Letter can only be handed to a person know directly.
  $\Longrightarrow$ 35% letters reach destination, median # of steps: 5.5
  $\Longrightarrow$ "6 degrees of separation"
- Albert, Jeong, and Barabasi ('99): WWW network, $n \sim 800M$ average distance of webpages $\sim 0.35 + 2.06 \log n = 18.59$

Transitivity and clustering

- in social networks, an important property is transitivity: a friend of my friend is more likely to be my friend than a random person.

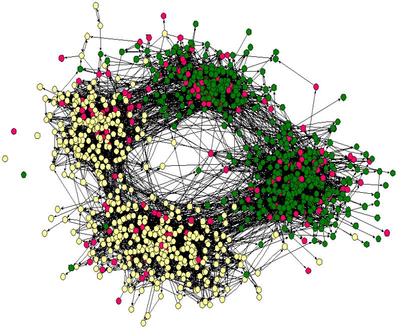


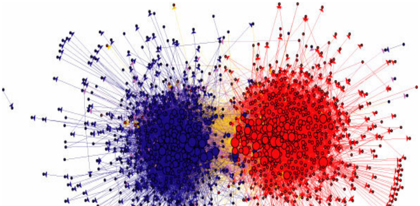
Clustering coefficient (undirected networks):

$$\text{Clust}(\mathcal{G}) = \frac{6 \cdot (\#\text{triangles})}{\#\text{length-2 paths}}$$

Homophily/ assortative mixing

Tendency of people to associate with others whom they perceive to be similar to themselves in some way.





Homophily/ assortative mixing

- partition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_k$ $\qquad$ c_i= class of node i.
- the modularity of an undirected graph is defined as

$$\mathrm{Mod}(\mathcal{G}) = \frac{1}{n\overline{w}} \sum_{i,j} \left(W_{ij} - \frac{w_i w_j}{n\overline{w}} \right) \delta(c_i, c_j)\,,$$

where $\delta(h, l) = 1$ if $h = l$ and $\delta(h, l) = 0$ if $h \neq l$.

$$\mathrm{Mod}(\mathcal{G}) = \left\{ \begin{array}{c} \text{fraction of links} \\ \text{between nodes} \\ \text{of the same class} \\ \text{in graph } \mathcal{G} \end{array} \right\} - \left\{ \begin{array}{c} \text{fraction of links} \\ \text{between nodes} \\ \text{of the same class} \\ \text{if they were drown} \\ \text{`randomly with given degrees'} \end{array} \right\}$$

The rest of this week

L2 Network centrality: Tomorrow (20/3)

L3 Linear Network Dyanmics (21/3)

E1 Tomorrow, 20/3

E2 Friday, 22/3