1. Exercise 1

1.1 Consider the graph described by the following adjacency matrix:

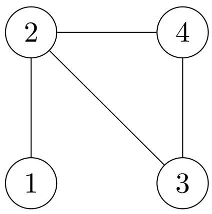
$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

a. Are there any self-loops in this graph? What are the in-degree and the out-degree of node 2? Does the graph have any sinks (i.e., nodes whose out-degree is 0) or sources (i.e., nodes whose in-degree is 0)? List the elements of the out-neighborhood of node 2.

b. Assume the links are directed as follows: $(1, 2), (2, 3), (2, 4)$ and $(3, 4)$. Give the adjacency matrix of the obtained directed graph. What are the in-degree and the out-degree of node 2 now? Does the directed graph have any sinks or sources? List the elements of the out-neighborhood of node 2.

Solution

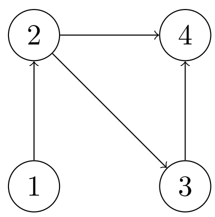
a. Self-loops correspond to non-zero diagonal entries of W, so there are no self-loops. $W_{ij} = 1$ if there is a link from i to j. Hence, the in-degree of node 2 is $\sum_i W_{i,2} = 3$ and its out-degree is $\sum_j W_{2,j} = 3$. This is an undirected graph, since the adjacency matrix is symmetric. There are no sinks or sources. The out-neighborhood of node 2 is $\{1, 3, 4\}$. The graph is depicted in the figure below:



b. The adjacency matrix of the directed version of the graph is

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, the in-degree of node 2 is $\sum_i W_{i,2} = 1$ and the out-degree is $\sum_j W_{2,j} = 2$. This graph has one source, node 1, and one sink, node 4. The out-neighborhood of node 2 is $\{3, 4\}$. The graph is depicted in the figure below:



1.2 Consider the weighted graph in Figure 1.1.

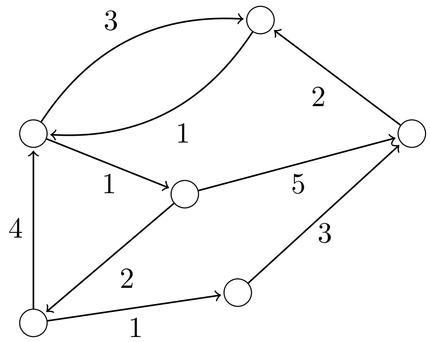


Figure 1.1: The graph for Problem 1.2.

a. Determine its weight matrix.

b. Compute the average degree.

Solution

a. With the following numbering of the nodes

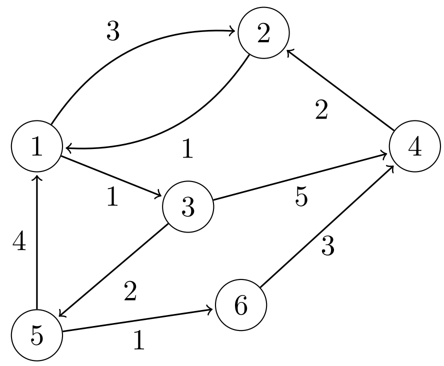


Figure 1.2: The graph for Problem 1.2.

the adjacency matrix can be written as

$$W = \begin{pmatrix} 0 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}.$$

b. The average degree of the graph, computed either based on the out-degree $\bar{w} = \sum_i w_i/n = \mathbb{1}'W\mathbb{1}/n$ or on the in-degree $\bar{w} = \sum_i w_i^-/n = \mathbb{1}'W'\mathbb{1}/n$, where $n = 6$ is the number of nodes, is equal to 3.7 (in both cases).

1.3 a. Give an example of a graph that is balanced but not undirected.

b. Give an example of an undirected graph that is not regular.

Solution

a. See Figure 1.3 below for an example.

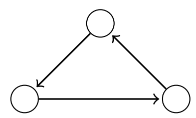


Figure 1.3: Balanced but not undirected

b. See Figure 1.4 for an example.

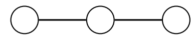


Figure 1.4: Undirected but not regular

1.4 Consider the graph described by the following adjacency matrix:

$$W = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a. Determine the Laplacian matrix L and the normalized weight matrix P of this graph and verify that $L\mathbb{1} = 0$, $P\mathbb{1} = \mathbb{1}$.

b. Check if the graph is balanced and/or regular.

c. Determine the number of connected components in the graph.

Solution

a. The out-degree vector, Laplacian and normalized weight matrix are

$$w = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \quad L = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

b. w is an invariant vector of P' if and only if the graph is balanced. Hence, since in this case $P'w = w$, the considered graph is balanced. Also, since the graph is undirected (because its adjacency matrix is symmetric), it can be immediately concluded that it is also balanced.

$\mathbb{1}$ is an invariant vector of P' if and only if the graph is regular. Hence, since in this case $P'\mathbb{1} \neq \mathbb{1}$, the graph is not regular.

c. The graph is strongly connected, hence the number of connected components is 1.

1.5 Consider two undirected graphs described by the following two adjacency matrices W_1 and W_2:

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

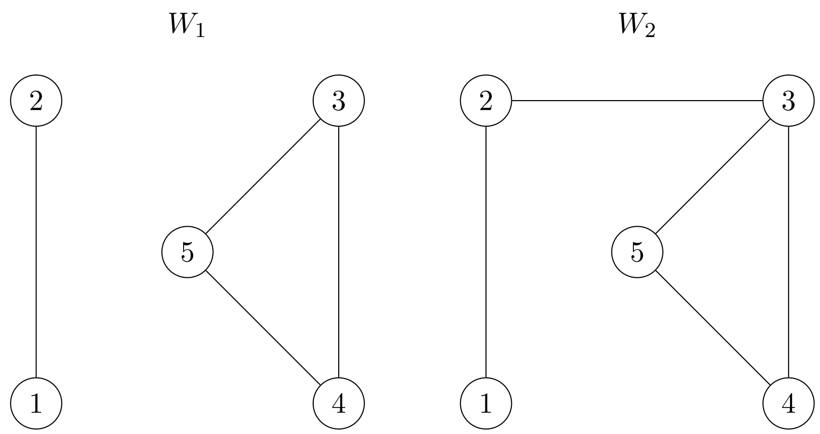
Draw the graphs. For each graph, answer the following questions:

a. Is the graph strongly connected or not?

b. What is the multiplicity of 0 as an eigenvalue of the Laplacian matrix, and of 1 as an eigenvalue of the normalized adjacency matrix?

Solution

a. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with adjacency matrices W_1 and W_2, respectively, are depicted below:



The graph $\mathcal{G}_1$ is not strongly connected and has two connected components, $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{3, 4, 5\}$. The graph $\mathcal{G}_2$ is strongly connected, hence it only has one connected component. The following adjacency matrix, A, gives the same graph as W_1, just with different ordering of the nodes,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

It is not as obvious to determine if the graph is strongly connected or not from A as it was from W_1; indeed, matrix W_1 is in block-diagonal form (from which we can clearly deduce that the graph is not connected), while matrix A is not.

b. If we form the condensation graphs for $\mathcal{G}_1$ and $\mathcal{G}_2$, we see that $\mathcal{G}_1$ is formed by two isolated nodes, while $\mathcal{G}_2$ has a single node. The algebraic and geometric multiplicities of the eigenvalue 1 for P and the eigenvalue 0 for L is the same as the number of sinks in the condensation graph, hence 2 for $\mathcal{G}_1$ and 1 for $\mathcal{G}_2$.

1.6 Consider a k-regular simple graph (where all nodes have degree k).

a. Show that the vector $\mathbb{1}$ is an eigenvector of the adjacency matrix, associated with eigenvalue k.

b. Recall that the eigenvectors of a symmetric matrix are orthogonal. Exploit this property to show that for this graph, no eigenvector x of the adjacency matrix that is associated to an eigenvalue $\lambda \neq k$ can have all nonnegative entries.

Solution

a. In each row, the adjacency matrix W of an unweighted k-regular graph has k entries equal to one, while the other entries are zero. Hence, for every node i, one has that $\sum_j W_{ij} = k$, so that $W\mathbb{1} = \mathbb{1}k$, i.e., $\mathbb{1}$ is an eigenvector of W associated to eigenvalue k.

b. Since W is symmetric, the eigenvector v associated with $\lambda \neq k$ is orthogonal to $\mathbb{1}$ i.e. $v'\mathbb{1} = 0$. Thus $\sum_i v_i = 0$ which makes it clear that not all v_i can be non-negative, apart from the trivial solution $v_i = 0\ \forall i$,

1.7 Every tree is a bipartite graph (i.e., a graph such that the node set can be partitioned into two nonempty subsets so that there are no links between nodes in the same subset). Propose a method to split the nodes in two subsets, thus proving the statement.

Solution

Consider two sets A and B. For any tree graph, pick one node: assign the node itself to set A and its offspring to set B. Further, assign the offspring of the offspring to set A and their offspring to set B. Continue in this manner for the entire graph. In this way, the nodes in set A are only connected to the nodes in set B, and vice versa.

1.8 A graph with $n \geq 5$ nodes has the out-degree distribution $(p_0, p_1, p_2) = (0, \frac{4}{5}, \frac{1}{5})$ and the in-degree distribution $(p_0^-, p_1^-, p_2^-, p_3^-) = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

a. Does the graph have any sink nodes?

b. Does the graph have any source nodes?

c. Draw a graph with $n = 5$ nodes and with the given degree distribution.

Solution

a. No, because the number of nodes with out-degree 0 is $np_0 = 0$.

b. Yes, because the number of nodes with in-degree 0 is $np_0^- = 2n/5 > 0$.

c. A graph with 5 nodes and the given degree distribution is shown in Figure 1.5.

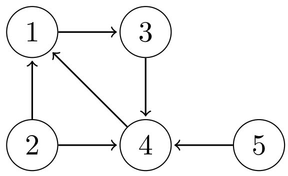
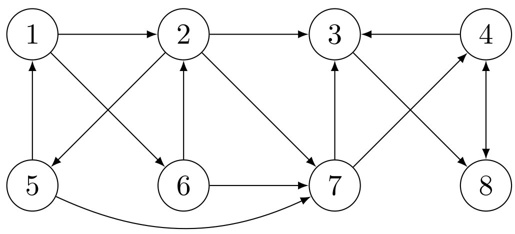


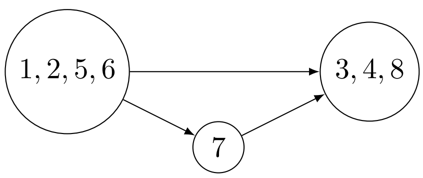
Figure 1.5: Graph in the answer to Problem 1.8.

1.9 Consider the graph $\mathcal{G}$ represented in the figure below. Merge the nodes of the connected components and draw the condensation graph.

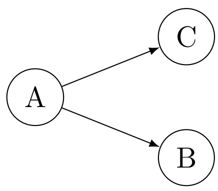


Solution

In the condensation graph, each node corresponds to a connected component of the original graph. A connected component is a set of nodes such that there is a path between each pair of nodes in this set. The condensation graph thus becomes the following:



1.10 Let $\mathcal{H}$ be the directed graph displayed below.



For each of the following statements, say if it is true or false, providing motivation.

a. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is not strongly connected.

b. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is acyclic.

c. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ does not possess a globally reachable node.

d. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is a tree.

Solution

a. True. That a graph is strongly connected means that there is a path between any pair of nodes i and j in the graph. Since there are no paths between nodes in $\mathcal{G}$ corresponding to nodes C and B in the condensation graph and nodes in $\mathcal{G}$ corresponding to node A in the condensation graph, the graph $\mathcal{G}$ is not strongly connected.

b. False. The graph $\mathcal{G}$ is not necessarily acyclic, since any of the nodes in the condensation graph in general corresponds to several nodes in graph $\mathcal{G}$ between which there can be cycles.

c. True. Any node in C or B can be reached from any node in A, but nodes in A and C cannot be reached from nodes in B, and nodes in A and B cannot be reached from nodes in C. So there is no node that is reachable from every other node in the graph.

d. False. Any node in the condensation graph corresponds to a collection of nodes in $\mathcal{G}$ which are fully connected. Thus, they do not in general have a tree structure.

1.11 Let $\mathcal{G}$ be a directed ring graph with n nodes.

a. Find the eigenvector centrality.

b. Find the Katz centrality with $\mu = \mathbb{1}$.

c. Find the Katz centrality with $\mu_1 = 1$ and $\mu_i = 0, \quad \forall i > 1$ and $\beta = 0.5$. Reflect on the difference between **b** and **c**.

Solution

a. Recall that the eigenvector centrality vector is the eigenvector of W' corresponding to the largest-in-magnitude eigenvalue λ_W, which is real and non-negative. With $\mathcal{G}$ being a directed ring the adjacency matrix is

$$W = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

which implies that $W'\mathbb{1} = \mathbb{1}$. For the normalized weight matrix, $P = D^{-1}W$, it always holds that $\lambda_P = 1$, i.e., that 1 is the largest-in-magnitude eigenvalue. Since, in this case, $W = P$ (the row sums of

W are already 1), we know that $\lambda_W = \lambda_P = 1$. Thus, $\mathbb{1}$ is the sought eigenvector.

b. The Katz centrality vector is the solution of

$$x = \left(\frac{1-\beta}{\lambda}\right) W'x + \beta\mu,$$

where $\beta \in (0, 1]$, $\mu = \mathbb{1}$, and $\lambda = 1$ is again the largest-in-magitude eigenvalue of W'. For the directed ring graph we thus have

$$x_i = (1-\beta)x_{i-1} + \beta, \quad i = 1, \dots, n,$$

where $x_0 = x_n$. From this expression we can make an educated guess that the centrality should be equal in each node. Inserting $x_i = k\ \forall i$ into the system of equations gives

$$k = (1-\beta)k + \beta,$$

which yields $k = 1$ independently of β. As we found a valid solution the Katz centrality, which is unique, is therefore given by $x_i = 1$ for all i.

c. From the previous problem, we know that the Katz centrality for this ring graph fulfills

$$x_i = (1-\beta)x_{i-1} + \beta\mu_i, \quad i = 1, \dots, n,$$

which for $\beta = 0.5$ and $\mu_1 = 1, \mu_i = 0 \forall i > 1$ yields

$$\begin{aligned} x_1 &= 0.5x_N + 0.5 \\ x_i &= 0.5x_{i-1} \quad i > 1 \end{aligned}$$

Let $x_1 = c$, then $x_i = 0.5^{i-1}c\ \forall i > 1$. This gives that

$$\begin{aligned} x_1 = c &= 0.5 \cdot 0.5^{N-1}c + 0.5 \\ c &= \frac{0.5}{1-0.5^N} \end{aligned}$$

Thus the Katz centrality is given by

$$x_i = \frac{0.5^i}{1-0.5^N} \quad \forall i$$

1.12 Consider a k-regular undirected graph (i.e., an undirected graph in which every node has degree k). Find the eigenvector centrality, as well as the Katz centrality vector for $\mu = \mathbb{1}$.

Solution

For a k-regular graph we know that the normalized adjacency matrix $P = \frac{1}{k}W$. We know from Proposition 2.4 in the lecture notes that, for a *regular graph*, the largest-in-magitude eigenvalue for P and P' is 1 with corresponding eigenvector $\mathbb{1}$ for P'. Thus, the dominating eigenvalue of W' is k, with corresponding eigenvector $\mathbb{1}$.

The Katz centrality is given as the solution x of

$$x = \frac{1-\beta}{k} W' x + \beta \mu.$$

Since every row of W' has the same sum and every element of μ is 1, intuitively we assume that all entries of x are the same, i.e. $x_i = \gamma$ for all $i \in \mathcal{V}$. Inserting this assumption into the equation system gives

$$x = \frac{1-\beta}{k} W' x + \beta \mu \quad \Leftrightarrow \quad \gamma \mathbb{1} = \left(\frac{1-\beta}{k} \right) k \gamma \mathbb{1} + \beta \mathbb{1},$$

which gives $\gamma = 1$. The Katz centrality vector is unique, which makes $x = \mathbb{1}$ the only solution.

1.13 Consider the graph in Figure 1.6.

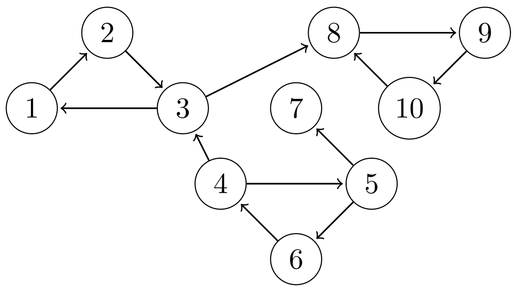


Figure 1.6: Graph in the answer to Problem 1.13.

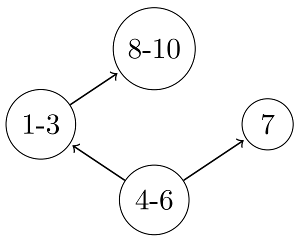
a. Draw the associated condensation graph. How many sinks does it have?

b. Is the original graph connected? If you were able to delete one node (and all the links connected to it), would your answer change?

c. Determine all possible invariant probability distributions $\pi = P'\pi$ for the given network.

Solution

a. The condensation graph is drawn in the figure below.



Hence, the condensation graph has two sinks.

b. The original graph is not connected: indeed, the condensation graph is not formed by a single node. No single node can be deleted in the original graph in order to turn it into a connected graph.

c. Recall that for a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ the following holds: **(i)** Any convex combination of invariant probability distributions of $\mathcal{G}$ is an invariant probability distribution of $\mathcal{G}$. **(ii)** For every sink in the condensation graph $\mathcal{H}$ of $\mathcal{G}$, there exists an invariant probability distribution supported on the connected component of $\mathcal{G}$ corresponding to such a sink of $\mathcal{H}$. These distributions are referred to as *extremals.* **(iii)** Every invariant probability distribution can be obtained as a convex combination of the extremal invariant probability distributions associated to the sinks of the condensation graph.

Since the condensation graph $\mathcal{H}$ of our graph $\mathcal{G}$ has exactly two sinks, there should thus be two extremal probability distributions, $\pi^{(1)}$ and $\pi^{(2)}$. These are supported on the connected components corresponding to the sinks, i.e.,

$$\pi^{(1)} = (0,0,0,0,0,0,0,\pi_8^{(1)},\pi_9^{(1)},\pi_{10}^{(1)})' \quad \text{and}$$

$$\pi^{(2)} = (0,0,0,0,0,0,\pi_7^{(2)},0,0,0)' .$$

The only way that the distribution $\pi^{(1)}$ can be invariant is if the non-zero components are chosen to be the same, since the last three equations of the system $\pi^{(1)} = P'\pi^{(1)}$ read $\pi_8^{(1)} = \pi_9^{(1)}$, $\pi_9^{(1)} = \pi_{10}^{(1)}$, and $\pi_{10}^{(1)} = \pi_8^{(1)}$, respectively. Furthermore, probability distributions require that the elements of the distribution vectors sum up to 1. Thus, the extremal distributions are given by the probability vectors

$$\pi^{(1)} = (0,0,0,0,0,0,0,1/3,1/3,1/3)', \quad \pi^{(2)} = (0,0,0,0,0,0,1,0,0,0)'.$$

Finally, point (iii) gives that any invariant probability distribution is given by

$$\pi = \alpha\pi^{(1)} + (1-\alpha)\pi^{(2)}, \quad \alpha \in [0,1].$$