2. Exercise 2

2.1 **Wisdom of crowds** Suppose that n agents are strongly connected by an unweighted undirected aperiodic network. Each agent observes a common value $\theta \in \mathbb{R}$, with some measurement noise ξ_i, so that

$$x_i(0) = \theta + \xi_i$$

or equivalently,

$$x(0) = \theta \mathbb{1} + \xi$$

where $\xi = (\xi_1, \dots, \xi_n)$. The agents' noises ξ_i have zero expected value, i.e., $\mathbb{E}[\xi_i] = 0$, variance $\mathbb{E}[\xi_i^2] = \sigma^2$ and are mutually independent. Suppose that the agents interact according to French-DeGroot's model of opinion dynamics. The system then converges to the consensus vector $\bar{x} = \alpha \mathbb{1}$ with (random) consensus value α.

a. Prove that the expected value of the consensus value is $\mathbb{E}[\alpha] = \theta$.

b. Prove that the variance of the consensus value, $\sigma_{\bar{x}}^2 := \mathrm{Var}[\alpha]$, is

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \frac{\overline{w^2}}{\overline{w}^2}$$

where

$$\overline{w} = \sum_{k=1}^{\infty} k p_k \quad \text{and} \quad \overline{w^2} = \sum_{k=1}^{\infty} k^2 p_k$$

are the average degree and the average square degree, respectively, and p_k stands for the fraction of nodes with degree k.

c. How does the expression above simplify for a regular graph?

Solution

a. Let π be the invariant probability distribution of the normalize weight matrix P, i.e. π is a nonnegative vector such that $P'\pi = \pi$ and $\mathbb{1}'\pi = 1$. In the French-DeGroot opinion dynamics we then know that

$$P\bar{x} = \bar{x} \quad \text{and} \quad \pi'\bar{x} = \pi' x(0)$$

must hold. Therefore,

$$\begin{aligned}
\mathbb{E}[\alpha] &= \mathbb{E}[\alpha \pi' \mathbb{1}] \\
&= \mathbb{E}[\pi' \bar{x}] \\
&= \mathbb{E}[\pi' x(0)] \\
&= \mathbb{E}[\theta \pi' \mathbb{1} + \pi' \xi] \\
&= \theta + \pi' \mathbb{E}[\xi] \\
&= \theta.
\end{aligned}$$

b. Note that

$$\begin{aligned}
\sigma_{\bar{x}}^2 &= \mathrm{Var}[\alpha] \\
&= \mathrm{Var}\left[\pi' x(0)\right] \\
&= \mathrm{Var}\left[\theta \pi' \mathbb{1} + \pi' \xi\right] \\
&= \mathrm{Var}\left[\sum_{i=1}^{n} \pi_i \xi_i\right] \\
&\overset{(*)}{=} \sum_{i=1}^{n} \pi_i^2 \,\mathrm{Var}\left[\xi_i\right] \\
&= \sum_{i=1}^{n} \pi_i^2 \mathbb{E}\left[\xi_i^2\right] \\
&= \sigma^2 \sum_{i=1}^{n} \pi_i^2
\end{aligned}$$

where $(*)$ follows from the independence of ξ_i and ξ_j when $i \neq j$.

Since $\mathcal{G}$ is undirected, it is balanced. Then we know that $w = W\mathbb{1}$ is an eigenvalue of P' with eigenvalue 1. However, since the network is strongly connected, 1 as an eigenvalue of P' has algebraic and geometric multiplicity 1. Thus, w must be proportional to the the invariant probability distribution π. In particular,

$$\begin{aligned}
\pi &= \frac{w}{\mathbb{1}' w} \\
&= \frac{w}{\mathbb{1}' W \mathbb{1}} \\
&= \frac{w}{n\overline{w}}
\end{aligned}$$

where $\overline{w} := (1/n)\mathbb{1}' W \mathbb{1}$. Using this we get that

$$\begin{aligned}
\sum_{i=1}^{n} \pi_i^2 &= \sum_{i=1}^{n} \frac{w_i^2}{n^2 \overline{w}^2} \\
&= \frac{\overline{w^2}}{n \overline{w}^2}
\end{aligned}$$

if we let $\overline{w^2} := (1/n)\sum_{i=1}^{n} w_i^2$. We conclude that

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \frac{\overline{w^2}}{\overline{w}^2}.$$

It remains to verify the expressions for $\overline{w}$ and $\overline{w^2}$. Using that the graph is unweighted, we get that

$$\begin{aligned}
\overline{w} &= \frac{1}{n} \sum_{i=1}^{n} w_i \\
&= \sum_{k=1}^{\infty} k \frac{1}{n} \left|\{i \in \mathcal{V} : w_i = k\}\right| \\
&= \sum_{k=1}^{\infty} k p_k
\end{aligned}$$

and

$$\begin{aligned}\overline{w^2} &= \frac{1}{n}\sum_{i=1}^{n} w_i^2 \\ &= \sum_{k=1}^{\infty} k^2 \frac{1}{n} \left|\{i \in \mathcal{V} : w_i = k\}\right| \\ &= \sum_{k=1}^{\infty} k^2 p_k\end{aligned}$$

as desired.

c. Suppose that the graph is regular, i.e. suppose that there exist some positive integer q such that $w = q\mathbb{1}$. But then

$$\overline{w} = q \quad \text{and} \quad \overline{w^2} = q^2$$

and therefore

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}.$$

2.2 Wisdom of crowds with a misinformed leader Consider a star graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with center node $i = 1$ and $n - 1$ leaves $i = 2, \dots, n$ interacting according to French-DeGroot opinion dynamics

$$x(t+1) = (\beta I + (1-\beta)P)x(t)$$

where $0 < \beta < 1$ is an inertia parameter.

Suppose that for all nodes, it holds that

$$x_i(0) = \theta + \xi_i$$

where $\theta \in \mathbb{R}$ is a constant and the agents' noises ξ_i have

- zero expected value, i.e., $\mathbb{E}[\xi_i] = 0$,
- are mutually independent,
- leaves have variance σ^2, i.e. $\mathbb{E}[\xi_i^2] = \sigma^2$ for $i = 2, \dots, n$, and
- the center node has variance $\gamma\sigma^2$ for some $\gamma > 0$, i.e. $\mathbb{E}[\xi_1^2] = \gamma\sigma^2$.

Find an $\gamma > 0$ such that in the limit as t grows large the leaves will estimate θ better if they do not trust the center node.

Solution

First note that for every $0 < \beta < 1$ the matrix $(\beta I + (1-\beta)P)$ is irreducible (since the star graph is connected) and aperiodic (since its diagonal entries are all positive).

Hence, the French-DeGroot opinion dynamics converges to a consensus value equal to the weighted average $\sum_i \pi_i x_i(0)$ of the initial opinions, where π is the invariant probability distribution of the stochastic matrix $\beta I + (1-\beta)P$, which is the same as that of P.

Given the solution of Problem 2.1, we have

$$\begin{aligned}\sigma_{\bar{x}}^2 &= \sum_i \pi_i^2 \mathbb{E}[\xi_i^2] \\ &= \sigma^2 \left(\gamma \pi_1^2 + \sum_{i=2}^{n} \pi_i^2 \right).\end{aligned}$$

An individual leaf will have a worse estimate compared to the consensus value given by the French-DeGroot opinion dynamics (i.e. compared to relying on the central leader) if

$$\sigma^2 \left(\gamma \pi_1^2 + \sum_{i=2}^{n} \pi_i^2 \right) > \sigma^2,$$

or equivalently

$$\gamma > \pi_1^{-2} \left(1 - \sum_{i=2}^{n} \pi_i^2 \right). \tag{2.1}$$

Since we consider a star graph, we know that the graph is balanced and strongly connected. Therefore, the invariant probability distribution of P (and also $\beta I + (1-\beta)P$) can be computed as

$$\pi = \frac{w}{n\bar{w}}$$

by the same arguments as in the solution of Problem 2.1.

In our star graph, we have that $w_1 = n-1$, $w_i = 1$ for $i = 2, \ldots, n$ and $n\overline{w} = 2(n-1)$. Therefore, $\pi_1 = 1/2$ and $\pi_i = 1/(2(n-1))$ for $i = 2, \ldots, n$. Using this, (2.1) becomes

$$\gamma > 4 - \frac{1}{n-1}.$$

Hence, the leaves will always have a better estimate not using the central leader if $\gamma \geq 4$, independently of the number the nodes n. On the other hand, if $3 < \gamma < 4$, the answer depends on n.

2.3 **Placement of stubborn node** Consider the tree in Figure 2.1 and assume that there are two players: player P_A who puts in the tree a stubborn node with value 1; and player P_B who puts in the tree a stubborn node with value 0. Each player wants to maximize the influence of her own stubborn node, namely to keep the average opinion of all the nodes in the tree as close as possible to the opinion of her/his own stubborn node.

a. Suppose that player P_A played first and has chosen the black node in Figure 2.1 to be a stubborn node with value 1. At this stage, where should player P_B put the stubborn node with value 0 to keep the average opinion of the nodes in the graph as close as possible to 0 ?

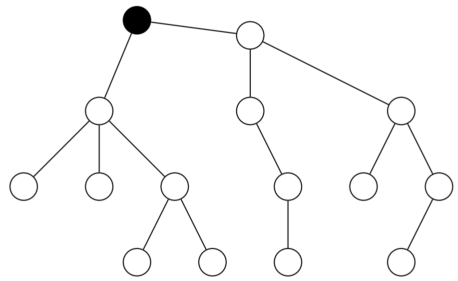

Figure 2.1: The tree for Problem 2.3

b. Now, we begin with the tree in Figure 2.1 but without any stubborn nodes. Consider the following scenario: player P_B plays first and assigns one of the nodes as a stubborn with value 0, then player P_A assigns another node as stubborn with value 1. Where should P_B put its stubborn node to maximize its influence on the average opinion of all the nodes?

Solution

a. If P_B chooses the stubborn node with value 0 denoted by s_0, as a neighbor of the stubborn node with value 1, called s_1, the tree graph will be split into two branches of nodes having a zero or one in the limit. If s_0 is the left-neighbor of s_1, there will be 6 nodes with zero value; while if s_0 is the right-neighbor of s_1, there will be 8 nodes with zero value in the limit, thus the average opinion will be as close as possible to 0. Hence, the best choice for P_B is to choose the right neighbor of the black node as the stubborn node with value 0.

b. The best choice is to put the stubborn node with value 0 in the same node as in Part **a**. In this case P_B, who plays first, needs to minimize the number of nodes that converge to 1, when P_A chooses the stubborn node with value 1 in the optimal way (after the stubborn node with value 0 has been fixed).

2.4 **Distributed consensus** Consider the standard discrete-time consensus protocol $x(t+1) = Px(t)$ on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where each agent $i \in \mathcal{V}$ has initial value $x_i(0) \in \mathbb{R}$.

a. Explain why when $\mathcal{G}$ is simple, strongly connected, aperiodic, and regular, we have that

$$x_i(t) \to \frac{1}{n}\sum_{i=1}^{n} x_i(0) \quad \text{as} \quad t \to \infty$$

for each $i \in \mathcal{V}$.

b. If we remove the assuption that $\mathcal{G}$ is regular, the algorithm will instead compute the weighted average $\sum_{i=1}^{n} \pi_i x_i(0)$, where π is the invariant probability distribution of the normalize weight matrix P. One way to get around this is to normalize the initial condition with the degree of each node, i.e.

$$y_i(0) = \frac{x_i(0)}{w_i} \tag{2.2}$$

for each $i \in \mathcal{V}$.

To overcome the lack of knowledge of average degree, we design a parallel consensus algorithm, to estimate the average degree in a distributed way. For this, let $y(t)$ and $z(t)$ be given by

$$\begin{aligned} y(t+1) &= Py(t), \quad y_i(0) = \frac{x_i(0)}{w_i} \\ z(t+1) &= Pz(t), \quad z_i(0) = \frac{1}{w_i}. \end{aligned}$$

Prove that

$$\lim_{t\to\infty} \frac{y_i(t)}{z_i(t)} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$$

for each $i \in \mathcal{V}$.

Solution

a. Since $\mathcal{G}$ is aperiodic and strongly connected, the network dynamics converge to a consensus. In particular,

$$\lim_{t\to\infty} x(t) = \pi' x(0)\mathbb{1}$$

where π is the invariant probability distribution of the normalize weight matrix P.

In addition, $\mathcal{G}$ is simple and therefore undirected and thus balanced. In this case we know that

$$\pi = \frac{w}{n\overline{w}}.$$

Since $\mathcal{G}$ is regular, we know that $\overline{w} = w_i$ for each node i. Therefore,

$$\pi = \frac{1}{n}\mathbb{1}$$

and we conclude that

$$\lim_{t\to\infty} x(t) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i(0)\right)\mathbb{1}.$$

b. Since $\mathcal{G}$ no longer is assumed to be regular, we only have that

$$\pi = \frac{w}{n\overline{w}}$$

for above. This implies that

$$\begin{aligned}\lim_{t\to\infty} y_i(t) &= \sum_{i=1}^{n} \pi_i y_i(0)\\ &= \sum_{i=1}^{n} \frac{w_i}{n\overline{w}} \frac{x_i(0)}{w_i}\\ &= \frac{1}{\overline{w}}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i(0)\Big)\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} z_i(t) &= \sum_{i=1}^{n} \pi_i z_i(0)\\ &= \sum_{i=1}^{n} \frac{w_i}{n\overline{w}} \frac{1}{w_i}\\ &= \frac{1}{\overline{w}}\end{aligned}$$

for each $i \in \mathcal{V}$. We conclude that

$$\lim_{t\to\infty} \frac{y_i(t)}{z_i(t)} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$$

for each $i \in \mathcal{V}$.

2.5 **Construction of a consensus protocol** Consider the undirected graph in Figure 2.2 and suppose that we want to construct a consensus protocol only using the links present in the graph. Let the opinion dynamics be described by

$$x(t+1) = E\,x(t)$$

where $E \in \mathbb{R}_+^{4\times 4}$ is a nonnegative matrix with the properties

$$\begin{aligned}&E\mathbb{1} = \mathbb{1},\\ &E_{ij} > 0 \Leftrightarrow W_{ij} > 0.\end{aligned} \tag{2.3}$$

a. Provide a sufficient condition for a vector $c \in \mathbb{R}^4$ and the matrix E, so the agents reach consensus at the value $c'x(0)$.

b. Construct a matrix E, satisfying (2.3), which will allow the agents to reach consensus at $c'x(0) = 0.4x_1(0) + 0.3x_2(0) + 0.2x_3(0) + 0.1x_4(0)$.

Solution

a. Note that E is a stochastic matrices and can be interpreted as the normalized weight matrix P of the graph. Since the graph is undirected, hence balanced and is also connected, with at least one self-loop and hence aperiodic, we have that the agents will converge to a consensus. Thus, if $E'c = c$ and $\mathbb{1}'c = 1$, the agents reach the consensus value $c'x(0)$.

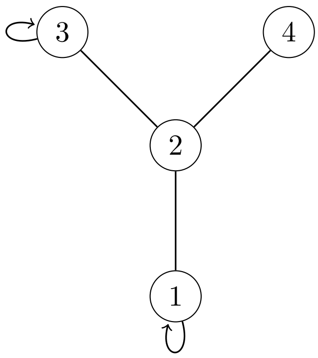

Figure 2.2: The graph for Problem 2.5.

b. We can assign weights to the links so that $E = P$ is the normalized weight matrix of the graph. This leads to the invariant probability distribution

$$\pi = \frac{w}{n\overline{w}}$$

where we want that

$$\pi = c = [\,0.4 \quad 0.3 \quad 0.2 \quad 0.1\,]'.$$

If we assign weight 1 to all the undirected links, a weight 3 to the self-loop on node 1 and a weight 1 to the self-loop on node 3, we get

$$w = [\,4 \quad 3 \quad 2 \quad 1\,]'$$

and therefore $\overline{w} = 2.5$. This gives

$$\pi = c$$

as desired. The corresponding matrix $P = E$ is then given by

$$E = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

2.6 **Discrete vs continous time averaging** Consider the graph in Figure 2.3.

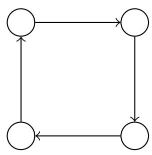

Figure 2.3: The graph for Problem 2.6.

a. Is the discrete-time French-DeGroot opinion dynamics converging? Why?

b. Let the continuous time model be given by

$$\dot{x}_i = \sum_{j \in \mathcal{V}} W_{ij}(x_j - x_i)$$

for each $i \in \mathcal{V}$. Find all equilibria for this dynamical system.

c. Determine which of the equilibria found in **b.** converges to a given initial state $x(0)$.

Solution

a. Since the graph is periodic, the dynamics will not converge in discrete-time. Take for instance $x(0) = [1 \quad 0 \quad 0 \quad 0]'$, then the dynamics will not converge.

b. The dynamics

$$\dot{x}_i = \sum_{j \in \mathcal{V}} W_{ij}(x_j - x_i)$$

for each $i \in \mathcal{V}$ can be rewritten as

$$\dot{x} = (Wx - Dx) = -Lx$$

where L is the Laplacian matrix of the graph.

The vector of ones is an equilibrium since $L\mathbb{1} = 0$, and therefore any consensus vector $\alpha\mathbb{1}$, where $\alpha \in \mathbb{R}$, is an equilibrium. Since the graph is strongly connected, we know that 0 as an eigenvalue of L has algebraic and geometric multiplicity 1. In particular, all the equilibria of the system lie on the subspace spanned by consensus vectors.

c. Since the graph is strongly connected, we know that

$$\lim_{t \to \infty} x(t) = \left(\overline{\pi}' x(0)\right) \mathbb{1}$$

where $\overline{\pi}$ is the unique Laplace invariant probability distribution on the graph, i.e. $L'\overline{\pi} = 0$ and $\mathbb{1}'\overline{\pi} = 1$.

However, since the graph is balanced, we know that $L'\mathbb{1} = 0$. In particular, it must hold that

$$\overline{\pi} = \frac{1}{n}\mathbb{1}$$

and therefore we get that

$$\lim_{t \to \infty} x(t) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i(0)\right) \mathbb{1}.$$

2.7 **Drug propagation as a compartmental model** A simple model for how a drug propagates through the human body is shown in Figure 2.4.

Let x_D, x_S, x_M be the concentration of the drug in each of the three stages. Let the increase in concentration of the drug in stage i caused by a flow from stage j to stage i be proportional to the difference in drug concentration between the two stages, $x_j - x_i$, with proportionality constant k_{ji}.

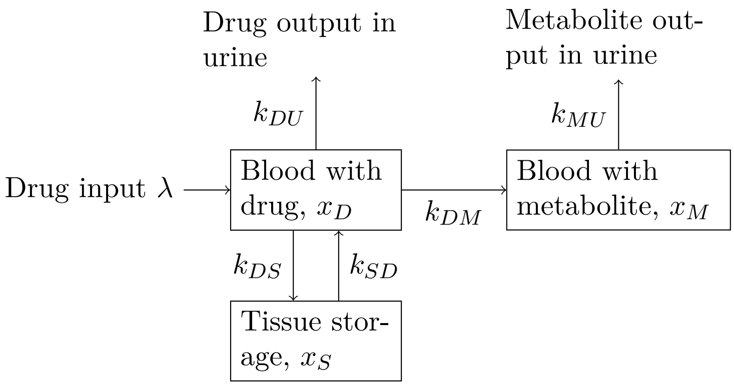

Figure 2.4: A simple model for drug propagation in the human body.

a. Describe the flow dynamics as a continuous-time compartmental model.

b. Let the inflow $\lambda = 2$ and $k_{DS} = 0.6$, $k_{SD} = 0.3$, $k_{DM} = 0.2$, $k_{DU} = 0.1$ and $k_{MU} = 0.4$. Find the limit densities for each stage.

Solution

a. Set nodes $(x_D, x_S, x_M, x_{ou}, x_{mou}, x_i)$. Then

$$W = \begin{pmatrix} 0 & k_{DS} & k_{DM} & k_{DU} & 0 & 0 \\ k_{SD} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{MU} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$-L = -(D-W) = \begin{pmatrix} -k_{DS} - k_{DM} - k_{DU} & k_{DS} & k_{DM} & k_{DU} & 0 & 0 \\ k_{SD} & -k_{SD} & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{MU} & 0 & k_{MU} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We denote by A the upper left 3×3 block matrix of L. The dynamics of state vector $x = (x_D, x_S, x_M)'$ can be written as,

$$\dot{x} = -A'x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \lambda.$$

b. For $\lambda = 2$, $k_{DS} = 0.6$, $k_{SD} = 0.3$, $k_{DM} = 0.2$, $k_{DU} = 0.1$ and $k_{MU} = 0.4$. Then the limit densities are given by

$$\overline{x} = (A')^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \lambda = \begin{bmatrix} 6.7 \\ 13 \\ 3.3 \end{bmatrix}$$

2.8 Consider the unweighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 2.5.

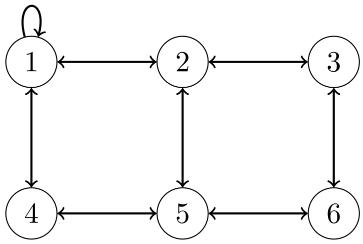

Figure 2.5: The graph for Problem 2.8.

a. Compute the weight matrix W, out-degree vector w, normalized weight matrix P and Laplacian matrix L.

b. Determine the normalized invariant distribution centrality vector π of $\mathcal{G}$.

Now consider the discrete-time French-DeGroot opinion dynamics on $\mathcal{G}$, of the form

$$x(t+1) = Px(t), \qquad t = 0, 1, 2... \tag{2.4}$$

where P is the normalized weight matrix of $\mathcal{G}$ and $x(t)$ is the opinion vector.

c. Does the opinion vector $x(t)$ converge to consensus as t grows large? Why?

d. Which assignment of the list of values $1, 1, 1, 2, 2, 2$ to the nodes (each value in the list has to be assigned only once) as initial opinions in the French-DeGroot opinion dynamics (2.4) maximizes $\lim_{t\to\infty} \sum_{i=1}^{6} x_i(t)$, and which one minimizes it? Motivate your answer and specify the obtained maximum and minimum value for the limit above.

e. Let node 1 be stubborn with value 0 and node 6 be stubborn with value 1. Compute the asymptotic opinion value for the four remaining nodes.

Solution

a.

$$W = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix},$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

b. The invariant probability distribution vector π is the eigenvector that satisfies $\pi = P'\pi$. Since $\mathcal{G}$ is balanced and strongly connected, it can be easily computed because it is proportional to the degree vector, as follows

$$\pi = \frac{w}{\sum_{i=1}^{6} w_i} = \left(\frac{1}{5}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15}, \frac{1}{5}, \frac{2}{15}\right).$$

c. Yes, the opinion vector will converge to a consensus vector, because the graph is strongly connected and aperiodic.

d. Since the graph is strongly connected and aperiodic, we know that

$$\lim_{t\to\infty} \sum_{i=1}^{6} x_i(t) = \sum_{i=1}^{6} \pi_i x_i(0)$$

which is maximized if the largest initial opinion values are assigned to the nodes with the largest centrality and minimized if the largest initial opinion values are assigned to the nodes with the smallest centrality. Hence, the maximizing choice is given by,

$$x(0) = (2, 2, 1, 1, 2, 1)$$

and leading to

$$\lim_{t\to\infty} \sum_{i} x_i(t) = \sum_{i=1}^{6} \pi_i x_i(0) = \frac{24}{15} = 1.6$$

whereas the minimizing choice is

$$x(0) = (1, 1, 2, 2, 1, 2)$$

which leads to

$$\lim_{t\to\infty} \sum_{i} x_i(t) = \sum_{i=1}^{6} \pi_i x_i(0) = \frac{21}{15} = 1.4.$$

e. The opinion vector of the opinion dynamics with stubborn nodes converges to the vector x which is solution of the linear system

$$x_1 = 0\,, \qquad x_6 = 1\,, \qquad x_i = \sum_{j=1}^{6} P_{ij} x_j\,, \quad i = 2, 3, 4, 5\,.$$

Hence, we get

$$x_2 = \frac{x_3}{3} + \frac{x_5}{3}\,, \quad x_3 = \frac{x_2}{2} + \frac{1}{2}\,, \quad x_4 = \frac{x_5}{2}\,, \quad x_5 = \frac{x_4}{3} + \frac{x_2}{3} + \frac{1}{3}\,.$$

Solving the system above (e.g., substituting the second equation into the first one and the third one into the fourth one gives

$$\frac{5}{6}x_2 = \frac{x_5}{3} + \frac{1}{6}\,, \qquad \frac{5}{6}x_5 = \frac{x_2}{3} + \frac{1}{3}\,,$$

then adding the two above gives $x_2 + x_5 = 1$) one gets

$$x_1 = 0\,, \quad x_2 = \frac{3}{7}\,, \quad x_3 = \frac{5}{7}\,, \quad x_4 = \frac{2}{7}\,, \quad x_5 = \frac{4}{7}\,, \quad x_6 = 1\,.$$