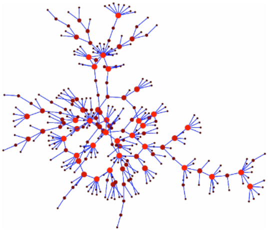
# FRTN30
# Network Dynamics
# Lecture 2
# Algebraic graph theory, network centrality



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Lund, March 20, 2024

# Connected components

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. Recall:

- $j \in \mathcal{V}$ is reachable from $i \in \mathcal{V}$ if there exists a path from $i$ to $j$
- by convection, every $i \in \mathcal{V}$ reachable from itself (length-0 path)
- $j \in \mathcal{V}$ globally reachable if reachable from every $i \in \mathcal{V}$
- $\mathcal{G}$ strongly connected if every node is globally reachable

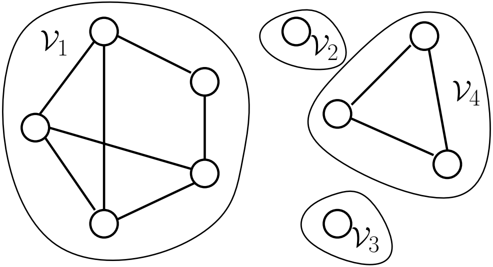
Connected components: largest nodes subsets $\mathcal{V}_h \subseteq \mathcal{V}$ such that if $i, j \in \mathcal{V}_h$ then $i$ reachable from $j$ and $j$ reachable from $i$

- Connected components are equivalence classes w.r.t. relation $v \sim w$ if $v$ is reachable from $w$ and $w$ is reachable from $v$

$$\mathcal{V} = \bigcup_{1 \leq h \leq c_{\mathcal{G}}} \mathcal{V}_h \qquad \mathcal{V}_h \cap \mathcal{V}_k = \emptyset\,, \quad h \neq k$$

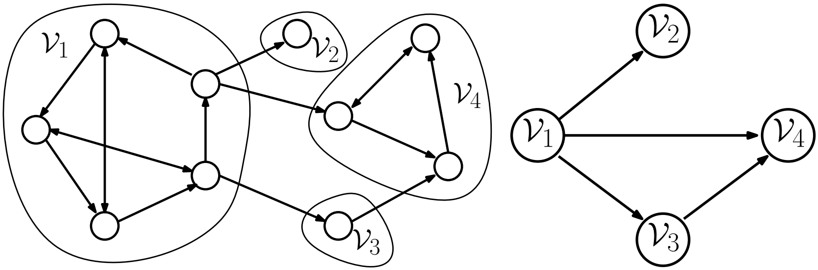
- $c_{\mathcal{G}} = \#\{\text{connected components of } \mathcal{G}\}$
- $\mathcal{G}$ strongly connected $\Leftrightarrow c_{\mathcal{G}} = 1$

# Connected components




In an undirected graph connected components are “islands”

# Condensation graph




- condensation graph $\mathcal{H}_\mathcal{G}$: obtained by collapsing connected components $\mathcal{V}_h$ into single nodes $v_h$ and adding a directed link $(v_h, v_k)$ whenever there exists at least one link in $\mathcal{G}$ from some node $i \in \mathcal{V}_h$ to some node $j \in \mathcal{V}_k$
- $\mathcal{H}_\mathcal{G}$ is circuit-free (DAG)
- $\mathcal{G}$ contains globally reachable node $\Leftrightarrow$ $\mathcal{H}_\mathcal{G}$ has just one sink
- $\mathcal{G}$ undirected $\Longrightarrow$ $\mathcal{H}_\mathcal{G}$ has no links (only isolated nodes)

# Normalized weight matrix and Laplacian of a graph

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

- $W =$ weight matrix
- $w = W\mathbb{1}$ out-degree vector
- $D = \mathrm{diag}(w)$
- normalized weight matrix[1]

$$P = D^{-1}W$$

$P$ row-stochastic, i.e., nonnegative and such that $P\mathbb{1} = \mathbb{1}$

- Laplacian matrix

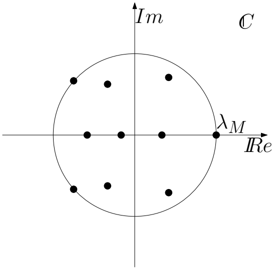
$$L = D - W$$

$-L$ 'Metzler' (nonnegative out of diagonal), $L\mathbb{1} = 0$

- geometric properties of $\mathcal{G} \longleftrightarrow$ spectral properties of $P$ and $L$

[1] We assume that $w_i > 0$ for every $i \in \mathcal{V}$ with no loss of generality

# Perron-Frobenius Theorem

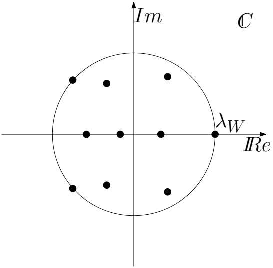



Let $M$ be a nonnegative square matrix. Then, there exist a nonnegative real eigenvalue $\lambda_M \geq 0$ and non-negative right and left eigenvectors $x \neq 0$ and $y \neq 0$ such that:

- $Mx = \lambda_M x, \;\; M'y = \lambda_M y$
- every eigenvalue $\lambda$ of $M$ is such that $|\lambda| \leq \lambda_M$.

$\lambda_M$ dominant eigenvalue, $x$ and $y$ dominant eigenvectors

# Perron-Frobenius applied to graphs

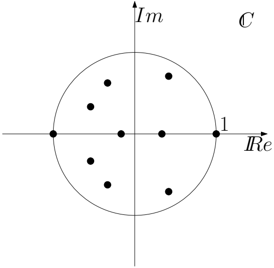



Proposition

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph s.t. $w > 0$. Then,
there exist positive real eigenvalue $\lambda_W \geq 0$ and
dominant right and left eigenvectors

$$x = \lambda_W^{-1} W x \geq 0\,, \qquad y = \lambda_W^{-1} W' y \geq 0$$

# Perron-Frobenius applied to graphs (cont'd)




**Proposition**

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph s.t. $w > 0$, $P = \text{diag}(w)^{-1}W$. Then,

(i) dominant eigenvalue $\lambda_P = 1$;

(ii) there exists invariant distribution $\pi \geq 0$ s.t. $\mathbb{1}'\pi = 1$, $P'\pi = \pi$;

(iii) $\mathcal{G}$ balanced $\Leftrightarrow$ $w$ eigenvector of $P'$ with eigenvalue 1 ;

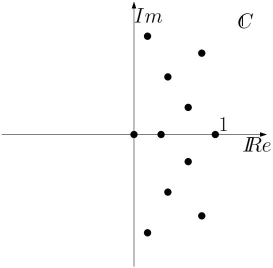
(iv) $\mathcal{G}$ regular $\Rightarrow$ $\mathbb{1}$ eigenvector of $P'$ with eigenvalue 1;

(v) $\mathcal{G}$ is strongly connected $\Rightarrow$ $\lambda_P = 1$ simple and $\pi = P'\pi > 0$;

(vi) $\mathcal{G}$ strongly connected and aperiodic $\Rightarrow$ any eigenvalue $\lambda \neq 1$ of $P$ is such that $|\lambda| < 1$;

(vii) $\mathcal{G}$ is bipartite $\Rightarrow$ $-1$ eigenvalue of $P$.

# Perron-Frobenius applied to graphs (cont'd)




Proposition

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, Laplacian $L = \mathrm{diag}(w) - W$. Then,

(i) 0 eigenvalue of $L$ and $L'$, all other eigenv. $\lambda$ have $\mathbb{R}\mathrm{e}(\lambda) > 0$

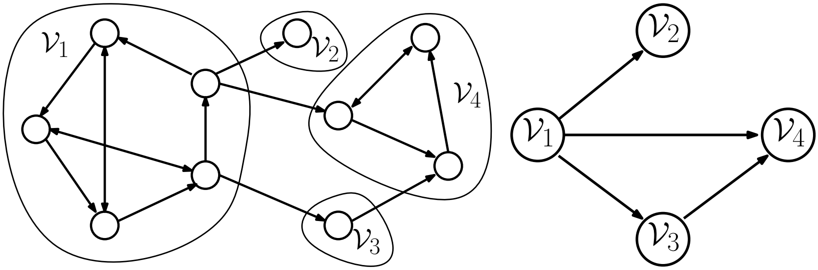
(ii) $\overline{y}$ is in the kernel of $L' \Leftrightarrow \overline{y} = D^{-1}y$ for some $y$ s.t. $P'y = y$;

(iii) there exists Laplace-invariant distribution $\overline{\pi} \geq 0$ such that $\mathbb{1}'\overline{\pi} = 1$ and $L'\overline{\pi} = 0$;

(iv) $\mathbb{1}$ in the kernel of $L' \Leftrightarrow \mathcal{G}$ balanced;

(v) $\mathcal{G}$ is strongly connected, $\Rightarrow$ 0 is simple eigenv. of $L$ and $\overline{\pi} > 0$.

# Perron-Frobenius applied to graphs (cont'd)




**Proposition**

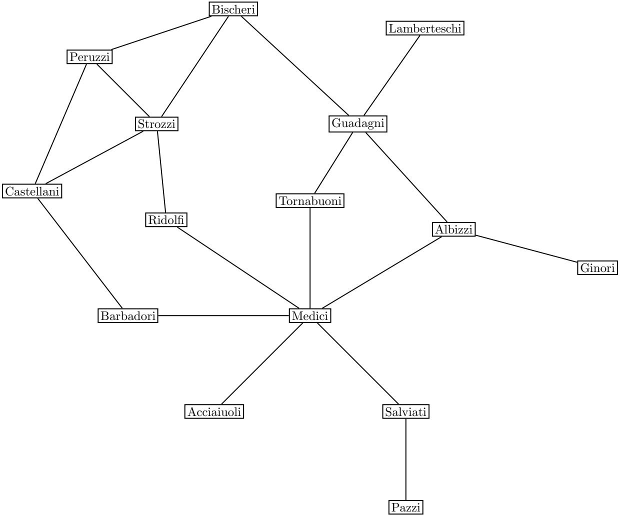
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $\mathcal{H}_{\mathcal{G}}$ condensation graph,

$s_{\mathcal{G}}$ = number of sinks in $\mathcal{H}_{\mathcal{G}}$. Then,

$s_{\mathcal{G}}$ = multip. of 1 as eigenv. of $P$ = # extremal invariant distributions

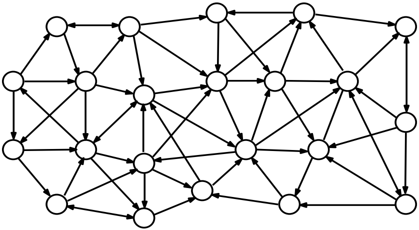
$s_{\mathcal{G}}$ = multip. of 0 as eigenv. of $L$ = # extremal Lap.-inv. distributions

# Network centrality




What is the most central node?

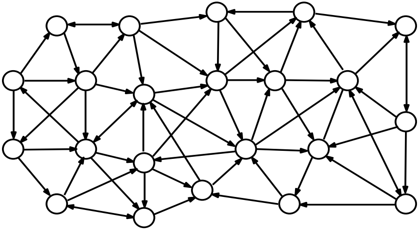
# Network centrality



Most central node in $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$?

- $z_i$ network centrality of node $i$

# Degree centrality

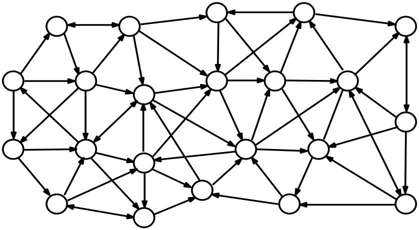


Most central node in $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$?

- $z_i$ network centrality of node $i$
- first attempt: $z_i \propto w_i^-$ in-degree centrality

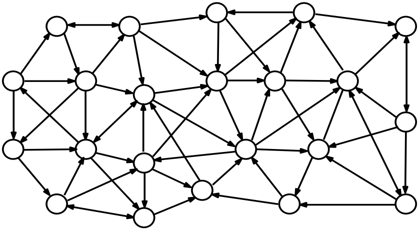
E.g., number of citations of an article, of followers on Twitter

# Eigenvector centrality



- $z_i$ network centrality of node $i$
- a more refined notion: $z_i \propto \sum_j W_{ji} z_j$

# Eigenvector centrality

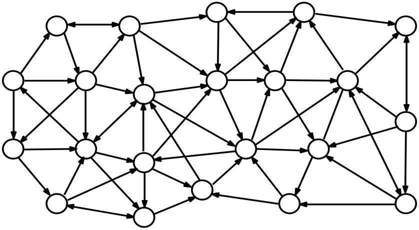


- $z_i$ network centrality of node $i$
- a more refined notion: $z_i \propto \sum_j W_{ji} z_j$

$$z_i = \frac{1}{\lambda} \sum_j W_{ji} z_j \qquad \lambda z = W' z$$

with $\lambda = \lambda_W$: eigenvector centrality

# Eigenvector centrality



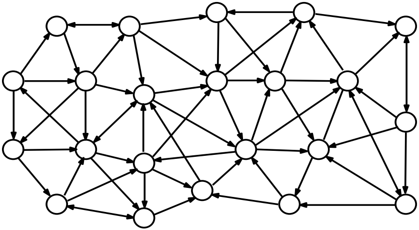
- $z_i$ network centrality of node $i$
- a more refined notion: $z_i \propto \sum_j W_{ji} z_j$

$$z_i = \frac{1}{\lambda} \sum_j W_{ji} z_j \qquad\qquad \lambda z = W' z$$

with $\lambda = \lambda_W$: eigenvector centrality

- drawback: node $j$ contributes proportional to its out-degree $w_j$

# Invariant distribution centrality

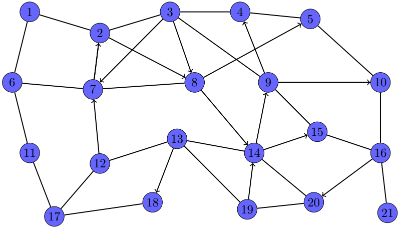


- a fix: normalize by out-degree

$$z_i \propto \sum_j \frac{1}{w_j} W_{ji} z_j = \sum_j P_{ji} z_j$$

- invariant distribution centrality: $z = P'z = \pi$
- Recall: in balanced networks $\pi_i \propto w_i$, not in general

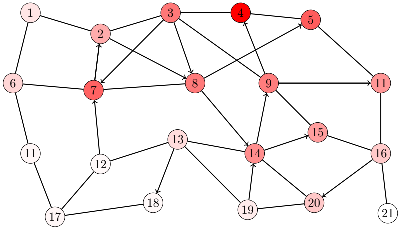
# Degree vs eigenvector vs invariant distribution centralities



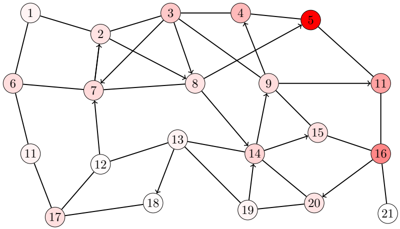

| | Deg | Eig | Inv.Dist. |
|---|---|---|---|
| 1 | 0.0345 | 0.0348 | 0.0313 |
| 2 | 0.0517 | 0.0581 | 0.0451 |
| 3 | 0.0517 | 0.0664 | 0.0613 |
| 4 | 0.0517 | 0.0689 | 0.0680 |
| 5 | 0.0517 | 0.0680 | 0.0869 |
| 6 | 0.0517 | 0.0430 | 0.0490 |
| 7 | 0.0690 | 0.0678 | 0.0514 |
| 8 | 0.0517 | 0.0661 | 0.0444 |
| 9 | 0.0517 | 0.0659 | 0.0491 |
| 10 | 0.0517 | 0.0627 | 0.0761 |
| 11 | 0.0345 | 0.0226 | 0.0324 |
| 12 | 0.0345 | 0.0215 | 0.0240 |
| 13 | 0.0517 | 0.0399 | 0.0317 |
| 14 | 0.0690 | 0.0640 | 0.0548 |
| 15 | 0.0517 | 0.0613 | 0.0464 |
| 16 | 0.0517 | 0.0484 | 0.0817 |
| 17 | 0.0517 | 0.0225 | 0.0481 |
| 18 | 0.0345 | 0.0215 | 0.0240 |
| 19 | 0.0345 | 0.0307 | 0.0300 |
| 20 | 0.0517 | 0.0492 | 0.0441 |
| 21 | 0.0172 | 0.0166 | 0.0204 |

# Degree vs eigenvector vs invariant distribution centralities

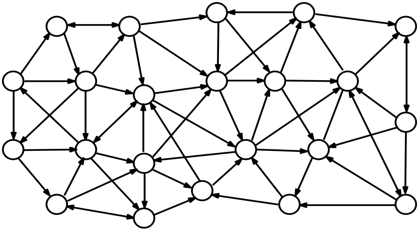
## Eigenvector centrality




## Invariant distribution centrality

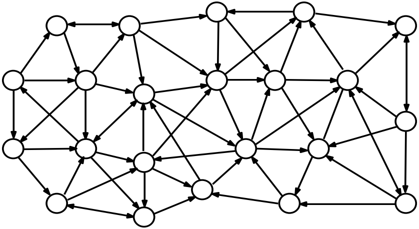


# Drawback of eigenvector and invariant distribution



- single sink node gets all centrality
- single sink connected component gets all centrality
- even if strongly connected, incentive to “cluster”

# Katz centrality

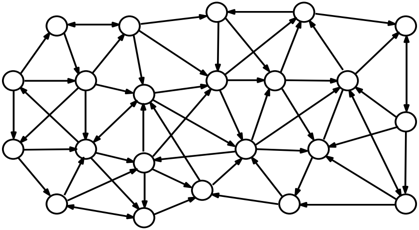


- Fix: convex comb. of intrinsic centrality and network centrality

$$z^{(\beta)} = \left(\frac{1-\beta}{\lambda_W}\right) W' z^{(\beta)} + \beta\mu$$

Katz centrality: $z^{(\beta)} = \left(I - \lambda_W^{-1}(1-\beta)W'\right)^{-1} \beta\mu$

# Bonacich centrality



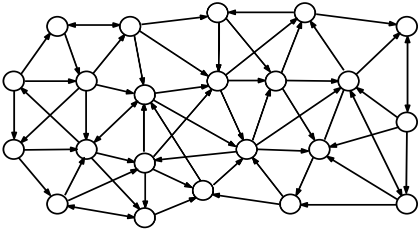
- Fix: convex combination of intrinsic and network centrality

$$z^{(\beta)} = (1-\beta)P' z^{(\beta)} + \beta\mu$$

Bonacich centrality: $\quad z^{(\beta)} = \left(I - (1-\beta)P'\right)^{-1} \beta\mu$

- PageRank (Google): $\mu = n^{-1}\mathbb{1}$, $\beta \sim 0.15$

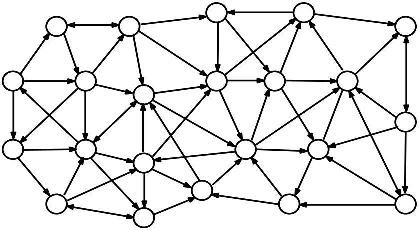
# Bonacich centrality: an interpretation



$$z^{(\beta)} = (1-\beta)P'z^{(\beta)} + \beta\mu$$

$$\begin{aligned} z^{(\beta)} &= \left(I-(1-\beta)P'\right)^{-1}\beta\mu = \beta\sum_{k=0}^{+\infty}(1-\beta)^k(P')^k\mu \\ &= \beta\mu + \beta(1-\beta)P'\mu + \beta(1-\beta)^2(P^2)'\mu + \dots \end{aligned}$$

# Eigenvector centrality: an interpretation



$$z^{(\beta)} = \frac{1-\beta}{\lambda_W} W' z^{(\beta)} + \beta\mu$$

$$\begin{aligned} z^{(\beta)} &= \left(I - \lambda_W^{-1}(1-\beta)W'\right)^{-1} \beta\mu = \beta \sum_{k=0}^{+\infty} \left(\frac{1-\beta}{\lambda_W}\right)^k (W')^k \mu \\ &= \beta\mu + \beta\frac{1-\beta}{\lambda_W} W'\mu + \beta\left(\frac{1-\beta}{\lambda_W}\right)^2 (W^2)'\mu + \dots \end{aligned}$$

# Eigenvector, inv. dist., Katz, and Bonacich centralities

| | |
|---|---|
| Eigenvector centrality<br>$z = \frac{1}{\lambda_W} W' z$ | Invariant distribution<br>$z = P' z$ |
| Katz centrality<br>$z = \frac{1-\beta}{\lambda_W} W' z + \beta \mu$ | Bonacich centrality<br>$z = (1-\beta) P' z + \beta \mu$ |

# Application: a model for production networks

Economy with $n$ sectors producing homogeneous goods.

For each sector $k = 1, \ldots, n$

- $y_k$ = total production
- $c_k$ = consumption
- $p_k$ = unit price
- $w$ = unit cost of labor
- $s_k$ = revenue
- $\pi_k$ = profit
- $l_k$ = employed labor
- $z_{jk}$ = quantity of product $j$ used

$$s_k = p_k y_k$$

$$\pi_k = s_k - \sum_j p_j z_{jk} - w l_k$$

# A model for production networks (cont'd)

Cobb-Douglas production functions

$$y_k = a_k l_k^{\beta} \prod_j z_{jk}^{\alpha G_{jk}}$$

and consumer utility function

$$U = \prod_k c_k^{\mu_k}$$

where
$a_k > 0$ efficiency of sector $k$
$G_{jk} \geq 0$ effectiveness of product $j$ in production of $k$
$\mu_k \geq 0$ consumer preference weight

$$\sum_j G_{jk} = 1 \qquad \sum_k \mu_k = 1$$

# A model for production networks (cont'd)

Assumptions:

- market for goods clears

$$y_k = \sum_j z_{kj} + c_k$$

- market for labor clears

$$\sum_k l_k = 1$$

- budget constraint

$$\sum_k p_k c_k \leq w + \sum_k \pi_k$$

# Walrasian Equilibrium

tuple $(y, c, p, l, z)$ satisfying assumptions and s.t.

- employed labor $l_k$ and intermediate products $z_{jk}$ maximize profit $\pi_k$ given the prices $p_k$
- consumptions $(c_k)_k$ maximize consumer utility $U$ given everything else

# Walrasian Equilibrium

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Theorem[Acemoglu ea.,'12] If $\alpha + \beta = 1$, at Walrasian Equilibrium

- no profits $\pi_k = 0$
- $\log U = (1-\alpha)^{-1} v' \log a + \mathit{Constant}$

where

$$a = \text{efficiency vector}$$

$$v = \beta(I - (1-\beta)G)^{-1}\mu \qquad \text{Bonacich centrality}$$

$$\mathit{Constant} \text{ depends on } \mu, G, \alpha \text{ only}$$