Network Dynamics 2

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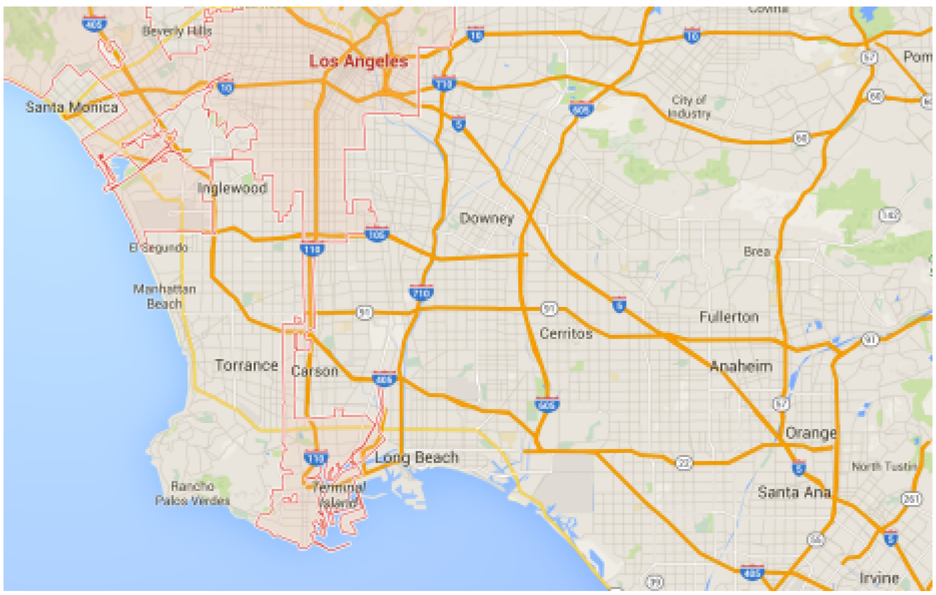

Figure 1: Map of the streets of LA analyzed in this handin, picture taken from provided handout.

1 Introduction

This handin covers an analysis of different traffic flows on a subset of the streets of LA. Optimum flow analysis was done using convex optimization tools. A handout was given with a description of the task and provided data. Firstly, the link node incidence matrix B was given (named traffic in the matlab files). In the $i-th$ row there is a 1 representing the tail of the link and a 0 at the $j-th$ row representing the head of the link. The total traveltime l_e on an empty link was provided in vector *traveltimes*. The max flow i.e. the capacities c_e of the links was provided in a vector *capacities*. Given these datapoints, different characteristics and analysis was done using matlab. Those are described in the sections below.

1.1 Theory

One aspect when analyzing a graph is determining the shortest path. In this case the shortest path is the path between two nodes with the least traffic delay. The shortest path is a convex optimization problem following the problem formulation,

$$\begin{aligned} M(v) := inf \quad & \sum_{e\in\epsilon} \Psi_e(f_e) \\ f &\in R^{\epsilon}_{+} \\ Bf &= \nu, \end{aligned} \tag{1}$$

where $\Psi(f_e)$ is the total cost function depending on the flow on each link f_e. The third line describe the converservation of mass where the total exogenous inflow is equal to the total exogenous outflow. In this specific problem the inflow to node 1 is set to 1, the inflow of node 17 is -1 and the rest of the nodes have no net inflow or outflow. If the shortest path is unique the flow is the unit flow on the links belonging to the shortest path and 0 on all other links.
The maxflow of a system is the maximum throughput between two nodes. It can be calculated by determining the minimal cut. Following theorem 3.2 in the lecture notes provided in the course.
The social optimum of a network is the flow that minimizes the cost function. Here a delay function is introduced which represent the delay on each link as a function of the flow on the link. The social optimum flow is the flow which minimizes the total delay on all links. In this handin the delay is,

$$\tau(f_e) = \frac{l_e}{1 - f_e/c_e}. \tag{2}$$

This delay is used when calculating the system/social optimum.
The wardrop equilibrum is the flow that arises when individuals determine which flow is best for themselves. Defined in the lecture notes as,

$$f^{(0)} = A^{o,d} z \tag{3}$$

where $z > 0$, $1'z = \nu$ and the following condition must be met,

$$z_p > 0 \implies \sum_{e\in\epsilon} A^{o,d}_{ep}\tau(f^{(0)}_e) \quad \leq \quad \sum_{e\in\epsilon} A^{o,d}_{eq}\tau(f^{(0)}_e), \tag{4}$$

which means that there must be a path p from o to d with less than or equal the delay on path q. The wardrop equilibrium can be manipulated with introduced tolls. Then the total percieved cost becomes,

$$\sum_{e\in\epsilon} A^{o,d}_{ep}(\tau(f^{(0)}_e) + w_e), \tag{5}$$

where w_e is the toll on link e. By introducing a toll that is,

$$w_e^* = f_e^* \tau_e^{'}(f_e^*) \quad e \in \mathcal{E}, \tag{6}$$

(Corollary 4.2 in the lecture notes) the social optimum should coincide with the wardrop equilibrium according to,

$$f^{(w^*)} = f^* \tag{7}$$

Another way to describe the delay is by defining the additional delay over the freeflow on an empty network. The new cost is then,

$$\begin{aligned} c_e &= f_e(d_e(f_e) - l_e), \\ c_e &= f_e(\frac{l_e}{1 - f_e/c_e} - l_e), \\ c_e &= \frac{f_e^2}{c_e - f_e}, \end{aligned} \tag{8}$$

the last rewrite is useful when minimizing the cost function in MATLAB.

1.2 Results and discussion

The shortest path is calculated by matlabs function *shortestpath*. A path traversing the following edges results in the shortest path.

$$\tilde{\mathbf{p}} = \begin{bmatrix} 1 & 2 & 3 & 9 & 13 & 17 \end{bmatrix}$$

. The total time is 32 minutes to go from node 1 to 17 a.k.a Santa Monica to Santa Ana.
The maximum flow for the graph between node 1 and 17 was calculated by the function maximum flow in MATLAB which uses the min-cut max-flow theorem. The maximum flow was 22448.
The flow on each edge for the network with and without tolls are tabulated below. There are initially three different flows.

- Social optimum flow for the traffic network f^*
- Wardrop equilibrium flow without tolls $f^{(w)}$
- Wardrop equilibrium flow with tolls $f^{(w^*)}$

As can be seen in table 1 the wardrop equilibrium when introducing tolls is equal to the free flow. This means that the flow increases by introducing tolls.

When using the additional delay as cost function according to equation 8. The new tolls are given as equation 6 or alternatively by,

$$w_e^* = c_e'(f_e^*) - \tau_e(f_e^*). \tag{9}$$

Table 1: The calculated flows

link $e \in \mathcal{E}$	f^*	$f^{(0)}$	$f^{(w)}$
1	6642	6716	6642
2	6059	6716	6059
3	3132	2367	3132
4	3132	2367	3132
5	10164	10090	10164
6	4638	4645	4638
7	3006	2804	3006
8	2543	2284	2543
9	3132	3418	3132
10	583	0	583
11	0	177	0
12	2927	4171	2927
13	0	0	0
14	3132	2367	3132
15	5525	5445	5525
16	2854	2353	2854
17	4886	4933	4886
18	2215	1842	2215
19	464	697	464
20	2338	3036	2338
21	3318	3050	3318
22	5656	6087	5656
23	2373	2587	2373
24	0	0	0
25	6414	6919	6414
26	5505	4954	5505
27	4886	4933	4886
28	4886	4933	4886

The new Wardrop equilibrium is then given by,

$$f^{(w)} \in \operatorname{argmin} \sum_{e \in \mathcal{E}} (D_e(f_e) + w_e^* f_e) \tag{10}$$

which in this case is equal to,

$$\operatorname{argmin} \sum_{e \in \mathcal{E}} (l_e c_e * (log(c_e) - log(c_e - f^{(w)})) + \frac{f l_e c_e}{(c_e - f)^2}. \tag{11}$$

The constraints are the ones given in equation 1.

$$\begin{aligned} f &>= 0 \\ Bf &= \nu \end{aligned} \tag{12}$$

The resulting flows is presented in table 2. Conclusively, this confirms that when designing tolls in such a way that are presented in the theory, the wardrop equilibrium coincides with the social optimum.

Table 2: Optimal and Wardrop equilibrium flow for the cost function being the additional delay

link $e \in \mathcal{E}$	f^*	$f^{(w^*)}$
1	6653	6653
2	5775	5775
3	3420	3420
4	3420	3420
5	10153	10153
6	4643	4643
7	3106	3106
8	2662	2662
9	3009	3009
10	878	878
11	0	0
12	2355	2355
13	0	0
14	3420	3420
15	5510	5510
16	3044	3044
17	4882	4882
18	2415	2415
19	444	444
20	2008	2008
21	3487	3487
22	5495	5495
23	2204	2204
24	0	0
25	6301	6301
26	5624	5623
27	4882	4882
28	4882	4882