1. Exercise 1

1.1 Consider the graph described by the following adjacency matrix:

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

a. Are there any self-loops in this graph? What are the in-degree and the out-degree of node 2? Does the graph have any sinks (i.e., nodes whose out-degree is 0) or sources (i.e., nodes whose in-degree is 0)? List the elements of the out-neighborhood of node 2.

b. Assume the links are directed as follows: $(1, 2), (2, 3), (2, 4)$ and $(3, 4)$. Give the adjacency matrix of the obtained directed graph. What are the in-degree and the out-degree of node 2 now? Does the directed graph have any sinks or sources? List the elements of the out-neighborhood of node 2.

1.2 Consider the weighted graph in Figure 1.1.

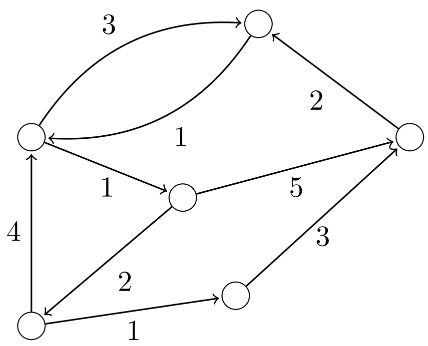


Figure 1.1: The graph for Problem 1.2.

a. Determine its weight matrix.

b. Compute the average degree.

1.3 a. Give an example of a graph that is balanced but not undirected.

b. Give an example of an undirected graph that is not regular.

1.4 Consider the graph described by the following adjacency matrix:

$$W = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a. Determine the Laplacian matrix L and the normalized weight matrix P of this graph and verify that $L\mathbb{1} = 0$, $P\mathbb{1} = \mathbb{1}$.

b. Check if the graph is balanced and/or regular.

c. Determine the number of connected components in the graph.

1.5 Consider two undirected graphs described by the following two adjacency matrices W_1 and W_2:

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Draw the graphs. For each graph, answer the following questions:

a. Is the graph strongly connected or not?

b. What is the multiplicity of 0 as an eigenvalue of the Laplacian matrix, and of 1 as an eigenvalue of the normalized adjacency matrix?

1.6 Consider a k-regular simple graph (where all nodes have degree k).

a. Show that the vector $\mathbb{1}$ is an eigenvector of the adjacency matrix, associated with eigenvalue k.

b. Recall that the eigenvectors of a symmetric matrix are orthogonal. Exploit this property to show that for this graph, no eigenvector x of the adjacency matrix that is associated to an eigenvalue $\lambda \neq k$ can have all nonnegative entries.

1.7 Every tree is a bipartite graph (i.e., a graph such that the node set can be partitioned into two nonempty subsets so that there are no links between nodes in the same subset). Propose a method to split the nodes in two subsets, thus proving the statement.

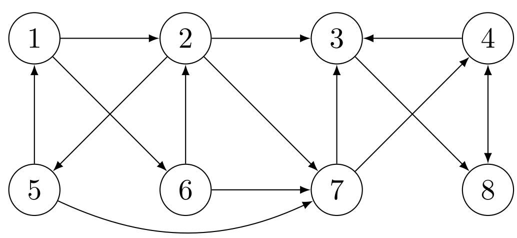
1.8 A graph with $n \geq 5$ nodes has the out-degree distribution $(p_0, p_1, p_2) = (0, \frac{4}{5}, \frac{1}{5})$ and the in-degree distribution $(p_0^-, p_1^-, p_2^-, p_3^-) = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

a. Does the graph have any sink nodes?

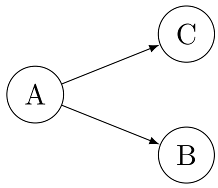
b. Does the graph have any source nodes?

c. Draw a graph with $n = 5$ nodes and with the given degree distribution.

1.9 Consider the graph $\mathcal{G}$ represented in the figure below. Merge the nodes of the connected components and draw the condensation graph.



1.10 Let $\mathcal{H}$ be the directed graph displayed below.



For each of the following statements, say if it is true or false, providing motivation.

a. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is not strongly connected.

b. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is acyclic.

c. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ does not possess a globally reachable node.

d. Every graph $\mathcal{G}$ with condensation graph $\mathcal{H}$ is a tree.

1.11 Let $\mathcal{G}$ be a directed ring graph with n nodes.

a. Find the eigenvector centrality.

b. Find the Katz centrality with $\mu = \mathbb{1}$.

c. Find the Katz centrality with $\mu_1 = 1$ and $\mu_i = 0, \quad \forall i > 1$ and $\beta = 0.5$. Reflect on the difference between **b** and **c**.

1.12 Consider a k-regular undirected graph (i.e., an undirected graph in which every node has degree k). Find the eigenvector centrality, as well as the Katz centrality vector for $\mu = \mathbb{1}$.

1.13 Consider the graph in Figure 1.2.

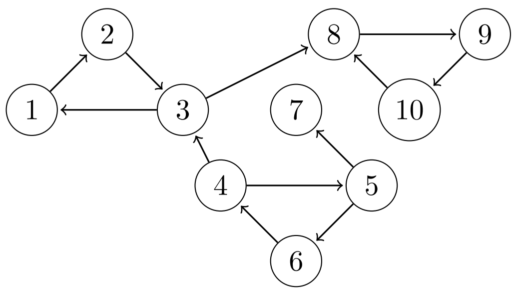


Figure 1.2: Graph in the answer to Problem 1.13.

a. Draw the associated condensation graph. How many sinks does it have?

b. Is the original graph connected? If you were able to delete one node (and all the links connected to it), would your answer change?

c. Determine all possible invariant probability distributions $\pi = P'\pi$ for the given network.