2. Exercise 2

2.1 **Wisdom of crowds** Suppose that n agents are strongly connected by an unweighted undirected aperiodic network. Each agent observes a common value $\theta \in \mathbb{R}$, with some measurement noise ξ_i, so that

$$x_i(0) = \theta + \xi_i$$

or equivalently,

$$x(0) = \theta \mathbb{1} + \xi$$

where $\xi = (\xi_1, \ldots, \xi_n)$. The agents' noises ξ_i have zero expected value, i.e., $\mathbb{E}[\xi_i] = 0$, variance $\mathbb{E}[\xi_i^2] = \sigma^2$ and are mutually independent. Suppose that the agents interact according to French-DeGroot's model of opinion dynamics. The system then converges to the consensus vector $\bar{x} = \alpha \mathbb{1}$ with (random) consensus value α.

a. Prove that the expected value of the consensus value is $\mathbb{E}[\alpha] = \theta$.

b. Prove that the variance of the consensus value, $\sigma_{\bar{x}}^2 := \mathrm{Var}[\alpha]$, is

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \frac{\overline{w^2}}{\overline{w}^2}$$

where

$$\overline{w} = \sum_{k=1}^{\infty} k p_k \quad \text{and} \quad \overline{w^2} = \sum_{k=1}^{\infty} k^2 p_k$$

are the average degree and the average square degree, respectively, and p_k stands for the fraction of nodes with degree k.

c. How does the expression above simplify for a regular graph?

2.2 **Wisdom of crowds with a misinformed leader** Consider a star graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with center node $i = 1$ and $n - 1$ leaves $i = 2, \ldots, n$ interacting according to French-DeGroot opinion dynamics

$$x(t+1) = (\beta I + (1-\beta)P)x(t)$$

where $0 < \beta < 1$ is an inertia parameter.

Suppose that for all nodes, it holds that

$$x_i(0) = \theta + \xi_i$$

where $\theta \in \mathbb{R}$ is a constant and the agents' noises ξ_i have

- zero expected value, i.e., $\mathbb{E}[\xi_i] = 0$,
- are mutually independent,
- leaves have variance σ^2, i.e. $\mathbb{E}[\xi_i^2] = \sigma^2$ for $i = 2, \ldots, n$, and
- the center node has variance $\gamma\sigma^2$ for some $\gamma > 0$, i.e. $\mathbb{E}[\xi_1^2] = \gamma\sigma^2$.

Find an $\gamma > 0$ such that in the limit as t grows large the leaves will estimate θ better if they do not trust the center node.

2.3 **Placement of stubborn node** Consider the tree in Figure 2.1 and assume that there are two players: player P_A who puts in the tree a stubborn node with value 1; and player P_B who puts in the tree a stubborn node with value 0. Each player wants to maximize the influence of her own stubborn node, namely to keep the average opinion of all the nodes in the tree as close as possible to the opinion of her/his own stubborn node.

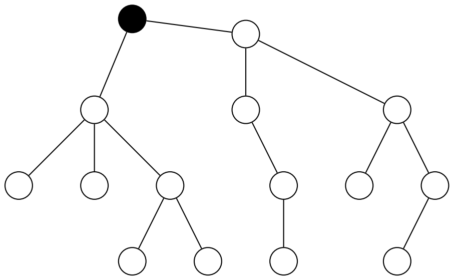

Figure 2.1: The tree for Problem 2.3

a. Suppose that player P_A played first and has chosen the black node in Figure 2.1 to be a stubborn node with value 1. At this stage, where should player P_B put the stubborn node with value 0 to keep the average opinion of the nodes in the graph as close as possible to 0 ?

b. Now, we begin with the tree in Figure 2.1 but without any stubborn nodes. Consider the following scenario: player P_B plays first and assigns one of the nodes as a stubborn with value 0, then player P_A assigns another node as stubborn with value 1. Where should P_B put its stubborn node to maximize its influence on the average opinion of all the nodes?

2.4 **Distributed consensus** Consider the standard discrete-time consensus protocol $x(t+1) = Px(t)$ on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where each agent $i \in \mathcal{V}$ has initial value $x_i(0) \in \mathbb{R}$.

a. Explain why when $\mathcal{G}$ is simple, strongly connected, aperiodic, and regular, we have that

$$x_i(t) \to \frac{1}{n}\sum_{i=1}^{n} x_i(0) \quad \text{as} \quad t \to \infty$$

for each $i \in \mathcal{V}$.

b. If we remove the assuption that $\mathcal{G}$ is regular, the algorithm will instead compute the weighted average $\sum_{i=1}^{n} \pi_i x_i(0)$, where π is the invariant probability distribution of the normalize weight matrix P. One way to get around this is to normalize the initial condition with the degree of each node, i.e.

$$y_i(0) = \frac{x_i(0)}{w_i} \tag{2.1}$$

for each $i \in \mathcal{V}$.

To overcome the lack of knowledge of average degree, we design a parallel consensus algorithm, to estimate the average degree in a distributed way. For this, let $y(t)$ and $z(t)$ be given by

$$y(t+1) = Py(t), \quad y_i(0) = \frac{x_i(0)}{w_i}$$
$$z(t+1) = Pz(t), \quad z_i(0) = \frac{1}{w_i}.$$

Prove that

$$\lim_{t\to\infty} \frac{y_i(t)}{z_i(t)} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$$

for each $i \in \mathcal{V}$.

2.5 **Construction of a consensus protocol** Consider the undirected graph in Figure 2.2 and suppose that we want to construct a consensus protocol only using the links present in the graph. Let the opinion dynamics be described by

$$x(t+1) = E\,x(t)$$

where $E \in \mathbb{R}_+^{4\times 4}$ is a nonnegative matrix with the properties

$$\begin{aligned} &E\mathbb{1} = \mathbb{1}, \\ &E_{ij} > 0 \Leftrightarrow W_{ij} > 0. \end{aligned} \tag{2.2}$$

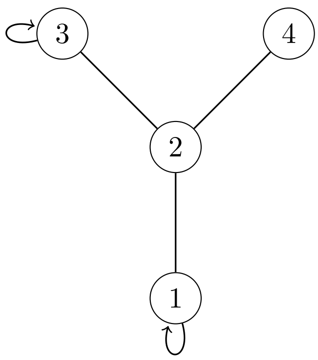

Figure 2.2: The graph for Problem 2.5.

a. Provide a sufficient condition for a vector $c \in \mathbb{R}^4$ and the matrix E, so the agents reach consensus at the value $c'x(0)$.

b. Construct a matrix E, satisfying (2.2), which will allow the agents to reach consensus at $c'x(0) = 0.4x_1(0) + 0.3x_2(0) + 0.2x_3(0) + 0.1x_4(0)$.

2.6 **Discrete vs continous time averaging** Consider the graph in Figure 2.3.

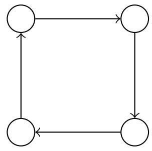

Figure 2.3: The graph for Problem 2.6.

a. Is the discrete-time French-DeGroot opinion dynamics converging? Why?

b. Let the continuous time model be given by

$$\dot{x}_i = \sum_{j \in \mathcal{V}} W_{ij}(x_j - x_i)$$

for each $i \in \mathcal{V}$. Find all equilibria for this dynamical system.

c. Determine which of the equilibria found in **b.** converges to a given initial state $x(0)$.

2.7 **Drug propagation as a compartmental model** A simple model for how a drug propagates through the human body is shown in Figure 2.4.

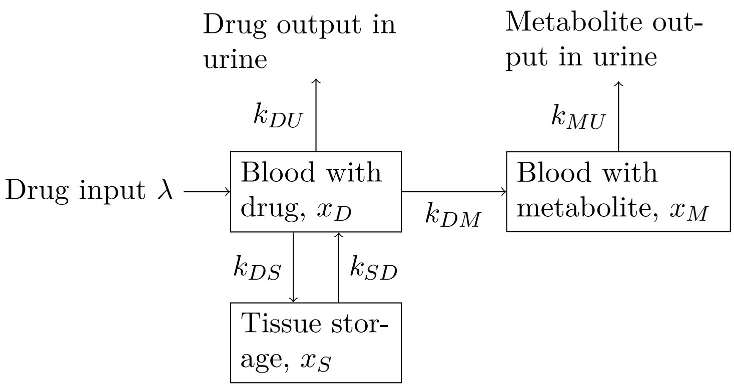

Figure 2.4: A simple model for drug propagation in the human body.

Let x_D, x_S, x_M be the concentration of the drug in each of the three stages. Let the increase in concentration of the drug in stage i caused by a flow from stage j to stage i be proportional to the difference in drug concentration between the two stages, $x_j - x_i$, with proportionality constant k_{ji}.

a. Describe the flow dynamics as a continuous-time compartmental model.

b. Let the inflow $\lambda = 2$ and $k_{DS} = 0.6$, $k_{SD} = 0.3$, $k_{DM} = 0.2$, $k_{DU} = 0.1$ and $k_{MU} = 0.4$. Find the limit densities for each stage.

2.8 Consider the unweighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 2.5.

a. Compute the weight matrix W, out-degree vector w, normalized weight matrix P and Laplacian matrix L.

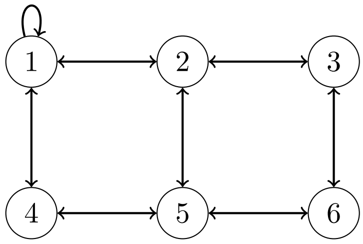

Figure 2.5: The graph for Problem 2.8.

b. Determine the normalized invariant distribution centrality vector π of $\mathcal{G}$.

Now consider the discrete-time French-DeGroot opinion dynamics on $\mathcal{G}$, of the form

$$x(t+1) = Px(t), \qquad t = 0, 1, 2... \tag{2.3}$$

where P is the normalized weight matrix of $\mathcal{G}$ and $x(t)$ is the opinion vector.

c. Does the opinion vector $x(t)$ converge to consensus as t grows large? Why?

d. Which assignment of the list of values $1, 1, 1, 2, 2, 2$ to the nodes (each value in the list has to be assigned only once) as initial opinions in the French-DeGroot opinion dynamics (2.3) maximizes $\lim_{t\to\infty} \sum_{i=1}^{6} x_i(t)$, and which one minimizes it? Motivate your answer and specify the obtained maximum and minimum value for the limit above.

e. Let node 1 be stubborn with value 0 and node 6 be stubborn with value 1. Compute the asymptotic opinion value for the four remaining nodes.