Numerical Methods for Differential Equations FMNN10/NUMN12
Final exam 2020-01-14

Exam duration 08:00 – 13:00. A minimum of 16 points out of 32 are required to pass. At LTH (FMNN10), grade 4 requires 21 points and grade 5 requires 26 points. At the Faculty of Science (NUMN12), grade VG requires 24 points.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** Consider the Fehlberg 3(2) embedded Runge-Kutta pair of orders $p = 2$ and $p = 3$, given by the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ \hline b^1 & 1/2 & 1/2 & 0 \\ \hline b^2 & 1/6 & 1/6 & 2/3 \end{array},$$

where b^1 refers to the "b-coefficients" of the first method and b^2 to those of the second method.

   (a) Write down the formulas for the methods when they are applied to the problem $\dot{y} = f(t, y)$. (1p)

   (b) Find the stability functions of the methods. (2p)

   (c) The embedded pair is used as follows: starting at a point y_n, the first method produces a result $u_{n+1} = M_1(y_n)$, while the second method produces a result $v_{n+1} = M_2(y_n)$. The difference

$$u_{n+1} - v_{n+1} = M_1(y_n) - M_2(y_n)$$

   is used to estimate the error in a single step. Find an expression for this error estimate when the method is applied to the linear test equation $\dot{y} = \lambda y$ and state the order of the error estimate. (1p)

   (d) Is this combination of methods suitable for *stiff* problems? Motivate your answer. (1p)

2. **(6p)** The following formula defines a linear multistep method for the initial value problem $\dot{y} = f(t, y)$:

$$\alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h\beta f(t_{n+2}, y_{n+2}).$$

   (a) Determine the coefficients α_2, α_1, α_0 and β such that the order of consistency is maximal. State the coefficients and the maximal order. (3p)

   (b) Is the method zero-stable? (1p)

   (c) Apply the method to the linear test equation $\dot{y} = \lambda y$ and determine the stability region of the method. It is enough to state the resulting inequality involving $h\lambda$, you do not have to solve it or draw a plot. (1p)

   (d) The method is implicit, so we must solve a possibly nonlinear equation $F(y_{n+2}) = 0$ in each time step. Write down this function F, and the iteration formula we get when applying Newton's method. (1p)

3. **(5p)** Consider the following two-point boundary value problem, where p, q and f are given functions and α, β are given scalars:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right) + q(x)y(x) = f(x), \\ &y(0) = \alpha, \qquad y'(1) = \beta. \end{aligned}$$

   Introduce a suitable grid and discretize the problem with a standard second order method. Give all details about the grid (number of grid points and their location, as well as mesh width Δx) and formulate the discretization. Include the boundary conditions in the equation system and make sure that they are also approximated to second order.

4. **(6p)** Consider the differential operator $\mathcal{L}$ given by

$$\mathcal{L}u = u'' + Ku$$

   for functions u satisfying $u(0) = u(1) = 0$, with a scalar K.

   (a) Show that $\mathcal{L}$ is self-adjoint regardless of the value of K, using the inner product

$$\langle u, v\rangle_2 = \int_0^1 uv \,\mathrm{d}x\,.$$

   (2p)

(b) Give an upper bound for the logarithmic norm $\mu_2[\mathcal{L}]$. (2p)
Hint: You can use Sobolev's lemma: $\|u'\|_2 \geq \pi \|u\|_2$.

(c) For which values of K can we guarantee that there is a unique solution to the associated boundary value problem $\mathcal{L}u = f$? For full points, prove the relevant theorem using the Cauchy–Schwarz inequality $|\langle u, v\rangle_2| \leq \|u\|_2 \|v\|_2$. (2p)

5. **(5p)** Write down the following partial differential equations. For each one of them, state whether they are elliptic, parabolic or hyperbolic. Also give the structure of the CFL condition (if applicable) that will be necessary for stability and convergence, if an explicit time-stepping method is used. It is enough to specify $\Delta t / \Delta x^p \leq C$ with the correct power p, you do not have to specify the value of the constant C.

   (a) The *advection equation*
   (b) The *reaction-diffusion equation*
   (c) The *viscous Burgers' equation*
   (d) The *Poisson equation*
   (e) The *wave equation*.

6. **(5p)** Consider the convection-diffusion equation

$$u_t = u_{xx} + u_x$$

with homogeneous Neumann boundary conditions and the initial value $u(0, x) = g(x)$.

   (a) Write down a standard second-order *semi*-discretization in space (method-of-lines). Give all details about your grid, Δx and how you handle the boundary conditions and the initial value. (3p)
   (b) What would be a good time-stepping method for approximating the solution to this system of ODEs? Motivate your suggestion and write down the full discretization that you get when using this method. (2p)

Good luck!
T.S.