Numerical Methods for Differential Equations, FMNN10 / NUMN32
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Solution: Review questions and study problems, week 3

1. Consider the two-point boundary value problem

$$y'' = x^2 + y^2 \qquad y(0) = 0,\ y(1) = 0.$$

Approximate y'' by $\dfrac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}$ and write the corresponding discretization for this BVP. Take $N = 4$; write the nonlinear system of equations $F(y) = 0$ for the unknowns y_1, y_2, y_3, y_4.

Solution: For the equation $y''(x) = x^2 + y(x)^2$, $y(0) = y(1) = 0$, we introduce the grid $x_k = k\Delta x$ with $\Delta x = \frac{1}{N+1}$ and the approximation $y_n \approx y(x_n)$. We approximate the second derivative by

$$y''(x_n) \approx \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}, \quad n = 1, \dots, N,$$

which leads to the system

$$\begin{cases} \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = x_n^2 + y_n^2, \quad n = 1, \dots, N \\ y_0 = y_{N+1} = 0. \end{cases}$$

For $N = 4$ we have $\Delta x = \frac{1}{5}$ and the interior points $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$. The nonlinear equation that we need to solve is

$$\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix} + \begin{pmatrix} y_1^2 \\ y_2^2 \\ y_3^2 \\ y_4^2 \end{pmatrix}$$

or in short $A\overline{y} = \overline{x}^2 + \overline{y}^2$. In order to solve the nonlinear equation, we can apply Newton's method. On standard form $F(\overline{y}) = 0$, we get $F(\overline{y}) = A\overline{y} - \overline{x}^2 - \overline{y}^2$.

2. What is the Jacobian for the problem above?

Solution: For the function

$$F(\overline{y}) = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} - \frac{1}{\Delta x^2}\begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix} - \begin{pmatrix} y_1^2 \\ y_2^2 \\ y_3^2 \\ y_4^2 \end{pmatrix}$$

the Jacobian is given by

$$F'(\overline{y})(\overline{z}) = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} - 2\begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & y_4 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

3. Once you have the Jacobian, how do you perform one Newton iteration to solve $F(y) = 0$?

   Solution: Starting from y^j, one Newton iteration is given by

   $$y^{j+1} = y^j - F'(y^j)^{-1}F(y^j) \quad \Leftrightarrow \quad F'(y^j)\big(y^{j+1} - y^j\big) = -F(y^j)$$

   i.e., we must solve

   $$\begin{pmatrix} -\frac{2}{\Delta x^2} - 2y_1^j & \frac{1}{\Delta x^2} & 0 & 0 \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_2^j & \frac{1}{\Delta x^2} & 0 \\ 0 & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_3^j & \frac{1}{\Delta x^2} \\ 0 & 0 & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_4^j \end{pmatrix}\overline{b} = -Ay^j + \begin{pmatrix} \big(y_1^j\big)^2 \\ \big(y_2^j\big)^2 \\ \big(y_3^j\big)^2 \\ \big(y_4^j\big)^2 \end{pmatrix} + \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{pmatrix}$$

   and then set $y^{j+1} = y^j + \overline{b}$.

4. Consider the two-point boundary value problem

   $$y'' = x^2 + y^2 \qquad y(0) = 0,\ y'(1) = 0.$$

   Approximate y'' by $\dfrac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}$ and write the corresponding discretization for this BVP. Take $N = 4$; write the nonlinear system of equations $F(y) = 0$ for the unknowns y_1, y_2, y_3, y_4. Discretize the Neumann boundary condition so that the resulting method is of second order.

   Solution: For the equation $y''(x) = x^2 + y(x)^2$, $y(0) = y'(1) = 0$, we introduce the grid $x_k = k\Delta x$ with $\Delta x = \frac{1}{N+\frac{1}{2}}$ ("approach 2") and the approximation $y_n \approx y(x_n)$. We approximate the second derivative by

   $$y''(x_n) \approx \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}, \quad n = 1, \dots, N-1$$

and the boundary conditions by $y_0 = 0$ and $\frac{y_{N+1}-y_N}{2\Delta x/2} = 0 \Leftrightarrow y_{N+1} = y_N$. This leads to the system

$$\begin{cases} \frac{y_{n-1}-2y_n+y_{n+1}}{\Delta x^2} = x_n^2 + y_n^2, \quad n = 2, \dots, N-1 \\ \frac{y_{N-1}-y_N}{\Delta x^2} = x_N^2 + y_N^2, \\ y_0 = 0. \end{cases}$$

For $N = 4$ we have $\Delta x = \frac{2}{9}$ and the interior points $\{\frac{2}{9}, \frac{4}{9}, \frac{6}{9}, \frac{8}{9}\}$. The nonlinear equation that we need to solve is

$$\frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \frac{1}{81} \begin{pmatrix} 4 \\ 16 \\ 36 \\ 64 \end{pmatrix} + \begin{pmatrix} y_1^2 \\ y_2^2 \\ y_3^2 \\ y_4^2 \end{pmatrix}$$

or in short $A\overline{y} = \overline{x}^2 + \overline{y}^2$. In order to solve the nonlinear equation, we can apply Newton's method. On standard form $F(\overline{y}) = 0$, we get $F(\overline{y}) = A\overline{y} - \overline{x}^2 - \overline{y}^2$.

If we instead used "approach 1", we would have the grid $x_k = k\Delta x$ with $\Delta x = \frac{1}{N}$. In this case, we need to discretize the Neumann boundary condition by $\frac{y_{N+1}-y_{N-1}}{2\Delta x} = 0$, which means that $y_{N+1} = y_{N-1}$. Thus the nonzero elements in the last row of the matrix changes from $\begin{pmatrix} 1 & -1 \end{pmatrix}$ to $\begin{pmatrix} 2 & -2 \end{pmatrix}$. The vector $\bar{x}^2$ in the right-hand-side will also have different values, $\frac{k^2}{16}$ for $k = 1, 2, 3, 4$.

5. What changes in Question 1 and 4 if we instead consider the equation $y'' + y' = x^2 + y^2$?

   Solution: We need to add a term to the left-hand-side which discretizes y'. A second-order discretization is given by the vector with entries $\frac{y_{n+1}-y_{n-1}}{2\Delta x}$. We can write this on matrix-vector form as

$$\frac{1}{2\Delta x} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

   in the case of the Dirichlet boundary conditions $y(0) = 0$, $y(1) = 0$ from Question 1, since $y_0 = y_{N+1} = 0$. In the case of the Dirichlet+Neumann boundary conditions of Question 4, we instead get the matrix

$$\frac{1}{2\Delta x} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   since $y_0 = 0$ and $y_{N+1} = y_{N-1}$.

6. What changes in Question 4 if we instead consider the Dirichlet + Robin boundary conditions $y(0) = \alpha$, $y(1) + y'(1) = \beta$?

   Solution: As always, we need to discretize the boundary conditions to second order. We consider "approach 2" again. The Dirichlet condition simply becomes $y_0 = \alpha$. In the matrix-vector form, this leads to an extra constant vector on the right-hand-side, with a single non-zero element depending on α in the first component. For the Robin condition, we discretize $y'(1)$ by $\frac{y_{N+1}-y_N}{2\Delta x/2}$. For $y(1)$, there is no grid point readily available, we only know $y_N \approx y(1 - \Delta x/2)$ and $y_{N+1} \approx y(1 + \Delta x/2)$. But, like in the lecture notes, we find that $\frac{y_N+y_{N+1}}{2}$ is a second-order approximation to $y(1)$. In total, we get

   $$\frac{y_N + y_{N+1}}{2} + \frac{y_{N+1} - y_N}{2\Delta x/2} = \beta,$$

   from which we deduce

   $$y_{N+1} = \frac{2\Delta x\beta + (2 - \Delta x)y_N}{2 + \Delta x}.$$

   In the matrix-vector form, this will only change the last row of the matrix on the left-hand-side.

   With "approach 1", we would have a grid point $x_N = 1$, which means that the Robin condition would be discretized by $y_N + \frac{y_{N+1}-y_{N-1}}{2\Delta x} = \beta$. Thus y_{N+1} now depends on both y_N and y_{N-1} rather than just y_N as in approach 2.

   Note that the question only asks for modifications to Question 4 and not to Question 1. This is because the grid used in Question 1 is unsuitable for approximating the derivative $y'(1)$ and thus essentially everything in the discretization would need to change.