Elliptic and parabolic PDEs

Overview of classes of PDEs

We divide PDEs in three different basic types

Elliptic equations

Standard example: $-\Delta u = f$ (Poisson eq.)
such an equation comes with boundary cond.
Important for steady state solutions, no time evolution

Parabolic equations

Standard example: $u_t = \Delta u$ (Diffusion / heat eq)
such an equation comes with boundary cond
+ initial value
Important for energy dissipation, solutions gain regularity

Hyperbolic equations

Standard examples: $u_{tt} = \Delta u$ (Wave eq)
such an equation comes with boundary cond
+ initial value

$u_t + a(u)\, u_x = 0$ (Advection eq)
such an equation comes with boundary cond
+ initial value
Important for energy conservation, no regularity gain.

Approaches to classify equations

1) For a PDE with two independent variables

$$A u_{xx} + 2B u_{xy} + C u_{yy} + L(u_x, u_y, u, x, y) = 0$$

where L is linear in u_x, u_y, u. We then consider

$$\delta = \det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = AC - B^2$$

and then we use δ:

$\delta > 0 \rightarrow$ elliptic eq.
$\delta = 0 \rightarrow$ parabolic eq.
$\delta < 0 \rightarrow$ hyperbolic eq.

2) Another approach is by using the Fourier transform. Let $t, x \mapsto u(t,x)$ and its derivatives tend to zero as $x \to \pm\infty$. For $F(\omega, x) = e^{i\omega x}$, we introduce the Fourier transform $\mathcal{F} : u \mapsto \hat{u}$ by

$$\hat{u}(t,\omega) = \mathcal{F}u(t,\omega) = \langle F, u \rangle (t,\omega)$$
$$= \int_{-\infty}^{\infty} F(\omega, x)\, u(t,x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{-i\omega x}\, u(t,x)\, dx$$

Then, we find that

$$\mathcal{F} u_t = \langle F, u_t \rangle = (\langle F, u \rangle)_t = \frac{d(\mathcal{F}u)}{dt}$$

$$\mathcal{F} u_x = \langle F, u_x \rangle = -\langle F_x, u \rangle = -\langle i\omega F, u \rangle$$
$$= -\overline{i\omega} \langle F, u \rangle = i\omega\, \mathcal{F}u$$

$$\mathcal{F} u_{xx} = \langle F, u_{xx} \rangle = -\langle F_x, u_x \rangle = \langle F_{xx}, u \rangle$$
$$= \langle (i\omega)^2 F, u \rangle = -\omega^2 \langle F, u \rangle = -\omega^2\, \mathcal{F}u$$

Note: The Fourier transform transforms an x-derivative into a multiplication with $i\omega$.

Examples

1) $u_t = u_x \xrightarrow{\mathcal{F}u = \hat{u}} \hat{u}_t = i\omega\, \hat{u}$

The solution of the Fourier transformed equation $\hat{u}(t) = c\, e^{i\omega t}$ has an oscillating behaviour

$\rightarrow$ hyperbolic equation

2) $u_t = u_{xx} \xrightarrow{\mathcal{F}u = \hat{u}} \hat{u}_t = -\omega^2\, \hat{u}$

The solution of the Fourier transformed equation $\hat{u}(t) = c\, e^{-\omega^2 t}$ has a dissipating behaviour

$\rightarrow$ parabolic equation

More general: For $\frac{\partial^p u}{\partial t^p} = \frac{\partial^q u}{\partial x^q}$ the equation is

- hyperbolic if $p+q$ is even
- parabolic if $p+q$ is odd

Examples:

$u_t = u_x$ hyperbolic

$u_t = u_{xx}$ parabolic

$i u_t = u_{xx}$ hyperbolic

we count this as a derivative

$u_t = u_{xxx}$ hyperbolic

$u_t = -u_{xxxx}$ parabolic

$u_{tt} = u_{xx}$ hyperbolic

$u_{tt} = -u_{xxxx}$ hyperbolic

Note: This classification rules are not useful in every case and more a rule of thumb. For nonlinear equations they don't work any longer.

Why the classification?

$\longrightarrow$ Different types of differential equations require different types of methods. To find the right approach, we need to understand what kind of equation we have!

Methods for elliptic problems:

(compare also chapter 3+4)

- ·) simple geometries : FDM, Fourier methods
- ·) complex geometries : FEM
- ·) special problems : Finite volume methods, Boundary element methods

Methods for parabolic problems:

Due to stiffness, we use an A-stable implicit method to approximate the time derivative. For every time step, an elliptic problem has to be solved, mainly FDM, Fourier methods or FEM for complex geometries.

Methods for hyperbolic problems:

Very challenging problems with conservation properties, shocks or multiscale phenonena (eg turbulences). Specialized methods are needed, for discritization in space often FDM, FVM, FEM.

Common denominator:

Numerically this leads to large, sparse systems which can be combined with iterative solvers, like multigrid methods.

In the following, we look at these three classes in more detail.

Elliptic problems:

Important example:

Laplacian (3D) $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Then $\Delta u = 0$ in Ω, $u = u_0$ on $\partial\Omega$ is called Laplace equation. Moreover, $\Delta u = f$ in Ω, $u = u_0$ on $\partial\Omega$ is called Poisson equation.

Remark: Other boundary conditions are also possible.

Applications:

- Equilibrium problems
  - → Structural analysis (strength of materials)
  - → Heat distribution at steady state
- Potential problems
  - → Potential flow (inviscid, subsonic flow)
  - → Electromagnetics (field, radiation)
- Eigenvalue problems
  - → Acoustics
  - → Microphysics

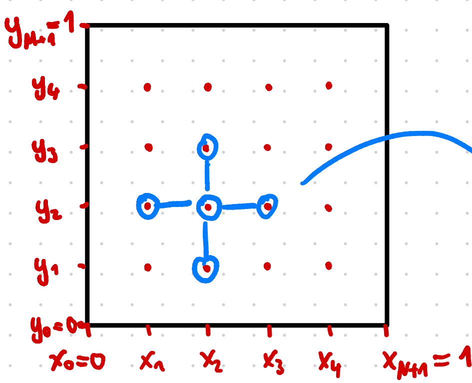
Elliptic model problem with FDM

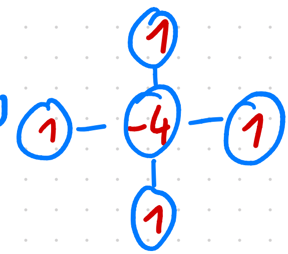
For the unit square $\Omega = [0,1] \times [0,1]$, we consider

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x,y) \;, & (x,y) \in \Omega \\ u(x,y) = 0 \;, & (x,y) \in \partial\Omega \end{cases}$$

We use an equidistant mesh $\{x_i, y_j\}_{i,j=1}^{N,M}$ with $\Delta x = \dfrac{1}{N+1}$, $\Delta y = \dfrac{1}{M+1}$.

Mesh:





"Five-point stencil"

Discretization using finite differences with $u_{i,j} \approx u(x_i, y_j)$

$$-\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} - \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = f(x_i, y_j)$$

for $\Delta x = \Delta y$: $-\dfrac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i,j+1} + u_{j,i+1}}{\Delta x^2}$

Transfer to a linear system of equations

We use a lexicographic order of the solution vector

$$\frac{-1}{\Delta x^2}\begin{pmatrix} T & I & & & & \\ I & T & I & & & \\ & I & T & I & & \\ & & & \ddots & & \\ & & & I & T & I \\ & & & & I & T \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N} \\ u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{N,N} \end{pmatrix} = \begin{pmatrix} f(x_1, y_1) \\ f(x_1, y_2) \\ \vdots \\ f(x_1, y_N) \\ f(x_2, y_1) \\ f(x_2, y_2) \\ \vdots \\ f(x_N, y_N) \end{pmatrix}$$

where I is the identity matrix and $T = \text{tridiag}(1 \ \ -4 \ \ 1)$.
The system is $N^2 \times N^2$ $\rightarrow$ very large and sparse.

Elliptic model problem with FEM

For the PDE $Lu = f$, we use the ansatz $u = \sum c_i \varphi_i$ and get $Lu = \sum c_i L\varphi_i$. To find the coefficients $\{c_i\}$, we use $\langle Lu - f, \varphi_j \rangle = 0$ for all j.

For model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

Weak form

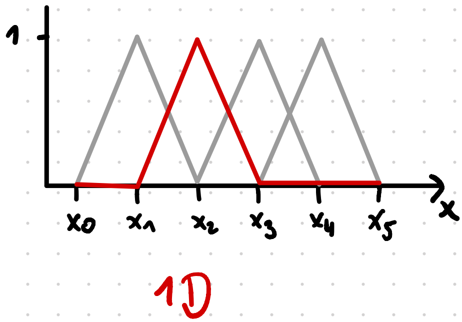
$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx = \langle f, v \rangle$$

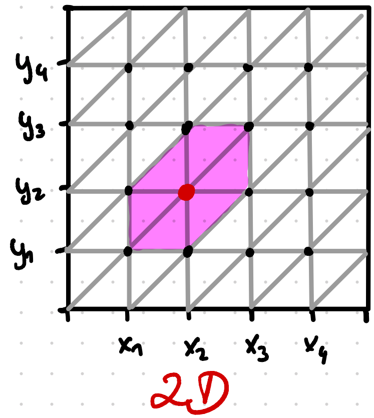
Leads to the linear system

$$Kc = F$$

for $K = \left(a(\varphi_j, \varphi_i)\right)_{i,j}$ and $F = \left(\langle f, \varphi_i \rangle\right)_i$.

Simplest case $\subset G(1)$: Choose $\{\varphi_i\}$ piecewise linear with $\varphi_i(x_j) = \begin{cases} 1, & i=j \\ 0, & \text{else} \end{cases}$.





Parabolic problems

Important example:

Diffusion equation

$$\begin{cases} u_t = \Delta u & \text{for } (t,x) \in (0,T) \times \Omega \\ u_{|\partial\Omega} = 0 & \text{for } (t,x) \in (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0 & \text{for } x \in \Omega \end{cases}$$

BC $\rightarrow$ $u_{|\partial\Omega} = 0$

IC $\rightarrow$ $u(0,\cdot) = u_0$

Numerical approximation:

Combine

- (A-stable) time stepping method
- space discretization (e.g. FDM or FEM)

Applications:

- Diffusion processes
  - Heat conduction $\quad u_t = d \cdot u_{xx}$
- Chemical reactions
  - Reaction-diffusion $\quad u_t = d \cdot u_{xx} + f(u)$
  - Convection-diffusion $\quad u_t = u_x + \frac{1}{Pe} u_{xx}$
- Seismology
  - Parabolic waves $\quad u_t = u u_x + d \cdot u_{xx}$

Note: The problems are often not reversible, e.g.

$u_t = -\Delta u$ does not have a bounded solution.

A parabolic model problem

$$\begin{cases} u_t = u_{xx} \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = g(x) \end{cases}$$

Ansatz: Separation of variables $u(t,x) = X(x)T(t)$

Then $u_t = X\dot{T}$, $u_{xx} = X''T$

$$\Rightarrow \frac{\dot{T}}{T} = \frac{X''}{X} = \lambda$$

$$\Rightarrow T(t) = C e^{\lambda t}, \quad X(x) = A\sin(\sqrt{-\lambda}x) + B\cos(\sqrt{-\lambda}x)$$

Using the boundary condition $X(0) = X(1) = 0$, it follows that

$$X(0) = A\sin(\sqrt{-\lambda}\cdot 0) + B\cos(\sqrt{-\lambda}\cdot 0) = B = 0$$

$$X(1) = A\sin(\sqrt{-\lambda}1) + 0\cos(\sqrt{-\lambda}\ ) = A\sin(\sqrt{-\lambda}) = 0$$

$$\Rightarrow \lambda_k = -(k\pi)^2$$

Then

$$X_k(x) = \sin(k\pi x), \quad T_k(t) = e^{-(k\pi)^2 t}.$$

Assuming that the initial value has a Fourier expansion

$$g(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

the solution takes on the form

$$u(t,x) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

Methods of lines (MOL) discretization

For the parabolic model problem

$$u_t = u_{xx},$$

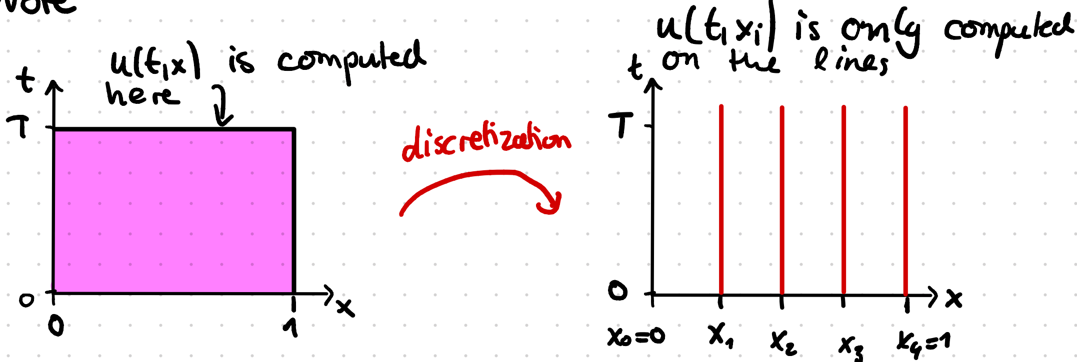
we first discretize $\frac{\partial^2}{\partial x^2}$ by a central quotient

$$u_{xx}(t, x_i) \approx \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{\Delta x^2}$$

for $u_i(t) \approx u(t, x_i)$. Then the PDE can be semi-discretized (discretization in one parameter but not the other). We obtain the ODE in $\mathbb{R}^N$ (N number of discretization points in space)

$$\dot{u}(t) = \underbrace{\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}}_{T_{\Delta x}} u(t)$$

Note



$u_i(t) \approx u(t, x_i)$ along the line where $x = x_i$ in the (t,x) plane

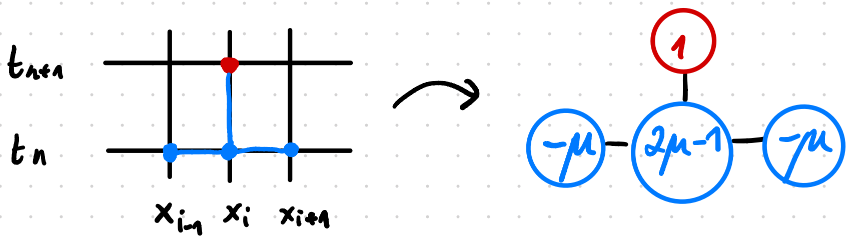
Let's now use the explicit Euler method for a discretization in time

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2}$$

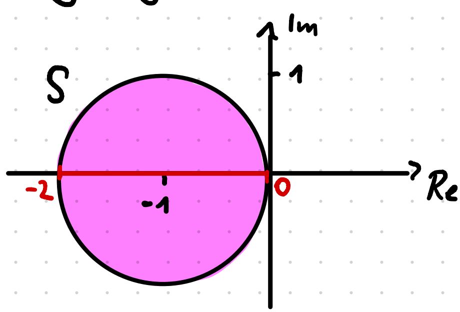
With the Courant number $\mu = \frac{\Delta t}{\Delta x^2}$, we find

$$U_i^{n+1} = U_i^n + \mu\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)$$

We can also represent this scheme through a stencil:



Reminder: The explicit Euler method has a bounded stability region S.



→ How does this affect step size restrictions?

For $u^n = (u_1^n, \dots, u_N^n)$ and $\Delta x = \frac{1}{N+1}$, we find

$$u^{n+1} = u^n + \Delta t \cdot T_{\Delta x} u^n.$$

Previously, we have showed that the eigenvalues of $T_{\Delta x}$ take on the form

$$\lambda_k[T_{\Delta x}] = -4(N+1)^2 \sin^2\left(\frac{k\pi}{2(N+1)}\right) \text{ for } k=1,\dots,N.$$
$$= -\frac{4}{\Delta x^2} \sin^2\left(\frac{\Delta x}{2} k\pi\right)$$

For stability, $\Delta t \cdot \lambda_k \in S$ for all $k = 1, \dots, N$. We observe that

$$\Delta t \, \lambda_k = \underbrace{-4 \frac{\Delta t}{\Delta x^2} \underbrace{\sin^2\left(\frac{\Delta x}{2} k\pi\right)}_{0 \le \quad \le 1}}_{-4\frac{\Delta t}{\Delta x^2} \le \quad \le 0}.$$

Moreover, $\Delta t \, \lambda_N = -4 \frac{\Delta t}{\Delta x^2} \sin^2\Big(\underbrace{\frac{\Delta x}{2} N\pi}_{\frac{N}{2(N+1)}\pi \approx \frac{\pi}{2}}\Big) \approx -4 \frac{\Delta t}{\Delta x^2}.$

And since we need

$$\underbrace{\Delta t \cdot \lambda_k}_{\in \mathbb{R}} \in [-2, 0]$$

for stability, it follows that we need that the lower bound $-4 \frac{\Delta t}{\Delta x^2}$ for the eigenvalues is still larger than the lower bound -2 of the stability region

$$-2 \le -4 \frac{\Delta t}{\Delta x^2} \iff \frac{\Delta t}{\Delta x^2} \le \frac{1}{2}.$$

CFL (Courant, Friedrichs, Lewy 1928) condition

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}.$$

This condition is a heavy restriction on the temporal step size:

$$\Delta t \leq \frac{1}{2} \Delta x^2$$

i.e. if we want/need to use a smaller spatial step size Δx, we also have to reduce the temporal step size Δt. Undesirable!

Way out: The CFL condition can be avoided when using an A-stable time-stepping method, for example the implicit Euler method or the Trapezoidal rule. We will look at them in more detail later.

First: Error analysis of MOL in combination with explicit Euler method.

Equation $\quad u_t = u_{xx}$

Global error $\quad e^n = U^n - u(t_n, \cdot)$, i.e. $\begin{pmatrix} e_1^n \\ \vdots \\ e_N^n \end{pmatrix} = \begin{pmatrix} U_1^n - u(t_n, x_1) \\ \vdots \\ U_N^n - u(t_n, x_N) \end{pmatrix}$

Local error

$$\frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t} = \frac{u(t_n, x_{i-1}) - 2u(t_n, x_i) + u(t_n, x_{i+1})}{\Delta x^2} - \ell_i^n$$

Expanding in Taylor series yields:
(We abbreviate $u = u(t_n, x_i)$, $u_t = u_t(t_n, x_i)$, $u_{tt} = u_{tt}(t_n, x_i), \ldots$)

$$l_i^n = \frac{1}{\Delta x^2}\Big(u - \Delta x\, u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{6}\Delta x^3 u_{xxx} + \frac{1}{24}\Delta x^4 u_{xxxx}$$
$$- \frac{1}{120}\Delta x^5 u_{xxxxx} \qquad - 2u$$
$$+ u + \Delta x\, u_x + \frac{1}{2}\Delta x^2 u_{xx} + \frac{1}{6}\Delta x^3 u_{xxx} + \frac{1}{24}\Delta x^4 u_{xxxx}$$
$$+ \frac{1}{120}\Delta x^5 u_{xxxxx} + O(\Delta x^6) \Big)$$
$$- \frac{1}{\Delta t}\Big(u + \Delta t\, u_t + \frac{1}{2}\Delta t^2 u_{tt} - u + O(\Delta t^3) \Big)$$
$$= u_{xx} + \frac{1}{12}\Delta x^2 u_{xxxx} - u_t - \frac{1}{2}\Delta t\, u_{tt}$$
$$+ O(\Delta x^4) + O(\Delta t^2)$$
$$\underset{u_t = u_{xx}}{=} \frac{\Delta x^2}{12} u_{xxxx} - \frac{\Delta t}{2} u_{tt} + O(\Delta t^2, \Delta x^4)$$

Theorem (Lax Principle)

Consistency + Stability $\Rightarrow$ convergence

Conclusion

We have

•) Consistency : $l_i^n \longrightarrow 0$ as $\Delta t, \Delta x \to 0$

•) Stability : for $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$ (CFL cond)

Thus, with the Lax principle it follows

Convergence : $e_i^n \to 0$ as $\Delta t, \Delta x \to 0$

Note : The choice of norm can make a difference, it is important to choose a suitable one.

We can even get a stronger result:

The local error is of order 2 in Δx and of order 1 in Δt:

$$\| \ell^n \|_{RMS} = O(\Delta t, \Delta x^2)$$

Together with the CFL cond. $\mu = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$, we get the global error

$$\|e^n\|_{RMS} = O(\Delta t, \Delta x^2),$$

i.e. the convergence order in time is 1 and in space is 2. For a fixed $\mu = \frac{1}{2}$, it follows $O(\Delta t) = O(\Delta x^2)$ and therefore

$$\|e^n\|_{RMS} = O(\Delta t) = O(\Delta x^2).$$

Remark: It is possible to use other norms but this can affect the convergence order.

In the following let's look at the Trapezoidal rule in more detail. In a PDE context, it is usually refered to as Crank-Nicolson method.

Crank-Nicolson

Method: $$U^{n+1} = U^n + \frac{\Delta t}{2}\left(\frac{1}{\Delta x^2} T U^n + \frac{1}{\Delta x^2} T U^{n+1}\right)$$
$$= \left(I + \frac{\mu}{2} T\right) U^n + \frac{\mu}{2} T\, U^{n+1}$$

for Courant number $\mu = \frac{\Delta t}{\Delta x^2}$ and $T = \text{tridiag}(1\ {-2}\ 1)$.

Equiv recursion: $$\left(I - \frac{\mu}{2} T\right) U^{n+1} = \left(I + \frac{\mu}{2} T\right) U^n.$$

or $$U^{n+1} = \left(I - \frac{\mu}{2} T\right)^{-1} \left(I + \frac{\mu}{2} T\right) U^n.$$

Note: All matrices are tridiagonal, therefore mult. has lower complexity.

Theorem

The eigenvalues of $A_\mu = \left(I - \frac{\mu}{2} T\right)^{-1}\left(I + \frac{\mu}{2} T\right)$ are
$$\lambda[A_\mu] = \frac{1 + \frac{\mu}{2}\lambda[T]}{1 - \frac{\mu}{2}\lambda[T]}.$$

Proof

Let v be an eigenvector of T to the eigenvalue λ.
Then, it follows that
$$\left(I + \frac{\mu}{2} T\right) v = v + \frac{\mu}{2}\lambda v = \left(1 + \frac{\mu}{2}\lambda\right) v$$

Moreover, we observe that

$$(I - \tfrac{\mu}{2} T)^{-1} v = w$$
$$\Leftrightarrow \quad v = (I - \tfrac{\mu}{2} T) w = (1 - \tfrac{\mu}{2} \lambda) w$$
$$\Leftrightarrow \quad (I - \tfrac{\mu}{2} T)^{-1} v = w = (1 - \tfrac{\mu}{2} \lambda)^{-1} v.$$

Atogether, this implies that

$$A_\mu v = (I - \tfrac{\mu}{2} T)^{-1} (I + \tfrac{\mu}{2} T) v$$
$$= (I - \tfrac{\mu}{2} T)^{-1} (1 + \tfrac{\mu}{2} \lambda) v = \frac{1 + \frac{\mu}{2} \lambda}{1 - \frac{\mu}{2} \lambda} v$$

$\square$

Theorem

Cramk-Nicolson is A-stable, in particular for every temporal step size $\Delta t > 0$, the eigenvalues of A_μ fulfill

$$-1 < \lambda[A_\mu] < 1$$

and therefore $\| A_\mu \|_2 < 1$.

Proof

From a previous lecture, we know that

$$\lambda_k[T_{\Delta x}] = -\frac{4}{\Delta x^2} \sin^2\left(\frac{\Delta x}{2} k\pi\right) \in \left(\frac{-4}{\Delta x^2}, 0\right] \qquad \text{for } k = 1, \dots, N.$$

Therefore, it follows

$$\lambda[A_\mu] = \frac{1 + \frac{\mu}{2} \lambda[T]}{1 - \frac{\mu}{2} \lambda[T]} = \frac{1 + \frac{\Delta t}{2} \lambda[T_{\Delta x}]}{1 - \frac{\Delta t}{2} \lambda[T_{\Delta x}]}$$

Note: $T = \mathrm{tridiag}(1 \ \ -2 \ \ 1)$
$T_{\Delta x} = \frac{1}{\Delta x^2} \mathrm{tridiag}(1 \ \ -2 \ \ 1)$

As $\lambda[T_{\Delta x}] < 0$, it follows that

$$-1+\frac{\Delta t}{2}\lambda[T_{\Delta x}] < 1+\frac{\Delta t}{2}\lambda[T_{\Delta x}] < 1-\frac{\Delta t}{2}\lambda[T_{\Delta x}]$$

And therefore

$$-1 < \frac{1+\frac{\Delta t}{2}\lambda[T_{\Delta x}]}{1-\frac{\Delta t}{2}\lambda[T_{\Delta x}]} < 1.$$

Theorem

The Crank-Nicoloson method is stable for every temporal step size $\Delta t > 0$ and is second order convergent, i.e. the global error fulfills

$$e_i^n = O(\Delta t^2; \Delta x^2).$$

Other example : Convection-diffusion equation

$$\begin{cases} u_t - u_{xx} - \alpha u_x = f \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = g(x) \end{cases}$$

In the following, we look at the space operator

$$L_\alpha(u) = -u_{xx} - \alpha u_x$$

in more detail.

We observe :

-) the equation is convection dominated for a Péclet number $|\alpha| >> 0$

-) L_α has the eigenvalues and eigenfunctions

$$\lambda_k[L_\alpha] = (k\pi)^2 + \frac{\alpha^2}{4}$$
$$u^k(x) = e^{-\frac{\alpha x}{2}} \sin(k\pi x)$$

$$(u^k(x))' = -\frac{\alpha}{2} e^{-\frac{\alpha x}{2}} \sin(k\pi x) + k\pi\, e^{-\frac{\alpha x}{2}} \cos(k\pi x)$$

$$(u^k(x))'' = \frac{\alpha^2}{4} e^{-\frac{\alpha x}{2}} \sin(k\pi x) - \frac{\alpha}{2} k\pi\, e^{-\frac{\alpha x}{2}} \cos(k\pi x) - \frac{\alpha}{2} k\pi\, e^{-\frac{\alpha x}{2}} \cos(k\pi x) - (k\pi)^2 e^{-\frac{\alpha x}{2}} \sin(k\pi x)$$

$$L_\alpha(u^k(x)) = -(u^k(x))'' - \alpha\,(u^k(x))'$$
$$= \left(-\frac{\alpha^2}{4} + (k\pi)^2\right) e^{-\frac{\alpha x}{2}} \sin(k\pi x) + \alpha k\pi\, e^{-\frac{\alpha x}{2}} \cos(k\pi x) + \frac{\alpha^2}{2} e^{-\frac{\alpha x}{2}} \sin(k\pi x) - \alpha k\pi\, e^{-\frac{\alpha x}{2}} \cos(k\pi x)$$
$$= \left(\frac{\alpha^2}{4} + (k\pi)^2\right) e^{-\frac{\alpha x}{2}} \sin(k\pi x)$$
$$= \left(\frac{\alpha^2}{4} + (k\pi)^2\right) u^k(x)$$

-) L_α is not self-adjoint but it is strongly accretive/positive for every Péclet number α : L_α fulfills $\langle L_\alpha u, u \rangle \geq \mu \|u\|^2$ with

$$\mu = \min_k \left((k\pi)^2 + \frac{\alpha^2}{4}\right)$$
$$= \pi^2 + \frac{\alpha^2}{4} \geq \pi^2 \qquad \text{for every } \alpha$$

Discretization of L_α:

For $\Delta x = \frac{1}{N+1}$, we discretize as follows

$$L_\alpha = -\frac{d^2}{dx^2} - \alpha \frac{d}{dx} \sim \underbrace{\frac{-1}{\Delta x^2} \operatorname{tridiag}(1 \quad -2 \quad 1) - \frac{\alpha}{2\Delta x} \operatorname{tridiag}(-1 \quad 0 \quad 1)}_{=: T_{\Delta x}}$$

Note:

- Eigenvalues of L_α:
$$\lambda_k[L_\alpha] = (k\pi)^2 + \frac{\alpha^2}{4} > 0 \quad \text{and real valued}$$
- Eigenvalues of $T_{\Delta x}$:
$$\lambda_k[T_{\Delta x}] = \frac{2}{\Delta x^2} - \frac{2}{\Delta x^2}\sqrt{1 - \frac{\alpha^2 \Delta x^2}{4}} \cos\left(\frac{k\pi}{N+1}\right)$$

are only positive and real if $\qquad k = 1, \dots, N$

$$1 - \frac{\alpha^2 \Delta x^2}{4} \geq 0 \iff \underbrace{|\alpha \Delta x|}_{\text{mesh Péclet number}} \leq 2$$

This mesh Péclet number is needed to preserve the that the eigenvalues are real (this is not a stability condition)

Note:

Parabolic problems can also be combined with cG(1) FEM.

For the mass matrix

$$M_{\Delta x} = \frac{\Delta x}{6} \operatorname{tridiag}(1 \quad 4 \quad 1)$$

and the stiffness matrix

$$K_{\Delta x} = \frac{1}{\Delta x} \operatorname{tridiag}(-1 \quad 2 \quad -1),$$

the parabolic problem becomes the ODE

$$M_{\Delta x} \dot{c} + K_{\Delta x} c = 0$$

Here, implicit schemes are particularly useful, as

- explicit Euler $M_{\Delta x}(c_{n+1} - c_n) = -\Delta t\, K_{\Delta x} c_n$
- trapezoidal rule $M_{\Delta x}(c_{n+1} - c_n) = -\frac{\Delta t}{2} K_{\Delta x}(c_n + c_{n+1})$

$$\Leftrightarrow \left(M_{\Delta x} + \frac{\Delta t}{2} K_{\Delta x}\right) c_{n+1} = \left(M_{\Delta x} - \frac{\Delta t}{2} K_{\Delta x}\right) c_n$$

have the same cost (for both a linear system containing a tridiag matrix has to be solved).

Well-posedness

Definition

A well-posed equation has a solution that

- depends uniformly on the initial value (the "data")
- is uniformly bounded in any compact interval

This means, in particular, that a small change in the initial condition only leads to a small change in the solution.

Example

We consider the linear, homogeneous Dirichlet prob.

$$\begin{cases} u_t = \Delta u \\ u(0,x) = g(x) \\ u(t,0) \equiv 0, \ u(t,1) \equiv 0. \end{cases}$$

We define the solution operator $\mathcal{E}(t)$ that fulfills

$$u(t,x) = \mathcal{E}(t)\, g(x)$$

$\mathcal{E}(t)$ maps the initial value $g(x)$ to the solution $u(t,x)$ at the time t. Then well-posedness is equivalent to:

For every $t^* > 0$ there exists a constant $0 < C(t^*) < \infty$ such that

$$\| \mathcal{E}(t) \| \leq C(t^*)$$

for all $0 \leq t \leq t^*$.

For the heat equation, the solution operator is bounded:

We use the Fourier series expansion

$$g(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

which gives the solution (use ansatz $u(t,x) = T(t)\, X(x)$)

$$u(t,x) = \sum_{k=1}^{\infty} c_k\, e^{-(k\pi)^2 t} \sin(k\pi x).$$

We then find

$$\| \mathcal{E}(t) g \|_2^2 = \int_0^1 |u(t,x)|^2 \, dx$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_k c_j\, e^{-(k^2+j^2)\pi^2 t} \int_0^1 \sin(k\pi x) \sin(j\pi x)\, dx$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} c_k^2 \, e^{-2k^2\pi^2 t}$$

$$\leq \frac{1}{2} \sum_{k=1}^{\infty} c_k^2$$

$$= \sum_{k=1}^{\infty} \int_0^1 c_k^2 \sin^2(k\pi x) \, dx$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_k c_j \int_0^1 \sin(k\pi x) \sin(j\pi x) \, dx = \|g\|_2^2$$

where we use for $k \neq j$

$$\int_0^1 \sin(k\pi x) \sin(j\pi x) \, dx$$

$$= \frac{-1}{j\pi} \sin(k\pi x) \cos(j\pi x) \Big|_0^1 + \frac{k\pi}{j\pi} \int_0^1 \cos(k\pi x) \cos(j\pi x) \, dx$$

$$= \frac{k\pi}{(j\pi)^2} \cos(k\pi x) \sin(j\pi x) \Big|_0^1 + \frac{(k\pi)^2}{(j\pi)^2} \int_0^1 \sin(k\pi x) \sin(j\pi x) \, dx$$

$$\Leftrightarrow \underbrace{\left(1 - \left(\frac{k}{j}\right)^2\right)}_{\neq 0} \int_0^1 \sin(k\pi x) \sin(j\pi x) \, dx = 0$$

$$\Leftrightarrow \int_0^1 \sin(k\pi x) \sin(j\pi x) \, dx = 0 \quad \text{for } k \neq j.$$

and for $k = j$

$$\int_0^1 \sin^2(k\pi x) \, dx = 1 - \int_0^1 \cos^2(k\pi x) \, dx$$

$$= 1 - \Big(\underbrace{\frac{-1}{k\pi} \cos(k\pi x) \sin(k\pi x) \Big|_0^1}_{=0} + \int_0^1 \sin^2(k\pi x) \, dx \Big)$$

$$= 1 - \int_0^1 \sin^2(k\pi x) \, dx$$

$$\Leftrightarrow \int_0^1 \sin^2(k\pi x) \, dx = \frac{1}{2}$$

But the equation

$$\begin{cases} u_t = -\Delta u \\ u(0,x) = g(x) \\ u(t,0) \equiv 0, \; u(t,1) \equiv 0. \end{cases}$$

is not well-posed. For $g_n(x) = \sin(n\pi x)$ we observe

$$\|E(t) g_n\|_2^2 = \int_0^1 | e^{n^2\pi^2 t} \sin(n\pi x)|^2 dx$$

$$= e^{n^2\pi^2 t} \int_0^1 |g_n(x)|^2 = e^{n^2\pi^2 t} \|g_n\|_2^2$$

That means that there exists no constant $C(t^*)$ such that

$$\|E(t)\| \leq C(t^*)$$

as $e^{n\pi t} \to \infty$ as $n \to \infty$.