Numerical Methods for Differential Equations FMNN10/NUMN32
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Review questions and study problems, week 4

1. True or false (justify your answer): *Consider the Sturm-Liouville problem*

$$\frac{d}{dx}\left((1-0.8\sin^2 x)\frac{dy}{dx}\right) - \lambda y = 0, \quad y(0) = y(\pi) = 0.$$

The following discretization

$$\frac{p_{n-1}y_{n-1} - 2p_n y_n + p_{n+1}y_{n+1}}{\Delta x^2} = \lambda_{\Delta x} y_n, \quad n = 1:N$$

$$y_0 = y_{N+1} = 0$$

where $p_n = 1 - 0.8\sin^2 \dfrac{n\pi}{N+1}$ *is of order 2.*

Solution: No, this is the 2nd order discretization of another problem,

$$\frac{d^2}{dx^2}(p(x)y(x)) - \lambda y = 0.$$

A correct 2nd order discretization is

$$\frac{p_{n-1/2}y_{n-1} - (p_{n-1/2} + p_{n+1/2})y_n + p_{n+1/2}y_{n+1}}{\Delta x^2} = \lambda_{\Delta x} y_n$$

$$y_0 = y_{N+1} = 0.$$

Note that $(py')'$ is a *self-adjoint* operator. The correct discretization above preserves this property, as the resulting matrix is *symmetric.*

2. Give a 4×4 example of

- A tridiagonal symmetric Toeplitz matrix
- A skew-symmetric Toeplitz matrix
- A lower triangular Toeplitz matrix

Solution: Toeplitz matrices have constant diagonals. The following matrices are in addition tridiagonal and symmetric, skew-symmetric, and lower-triangular, respectively:

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

3. Solve the linear difference equation

$$\begin{aligned} 6u_{j+2} - 5u_{j+1} + u_j &= 0 \qquad (j = 0 : N-1) \\ u_0 = 1,\ u_{N+1} &= 0. \end{aligned}$$

   Solution: We first set up the characteristic equation,

$$6w^2 - 5w + 1 = 0.$$

   The solutions to this are given by $(w - \frac{5}{12})^2 = \frac{25}{144} - \frac{24}{144}$, i.e. $w = \frac{5}{12} \pm \frac{1}{12} \in \{\frac{1}{3}, \frac{1}{2}\}$. Thus $u_j = A \cdot 3^{-j} + B \cdot 2^{-j}$, where the constants A and B are specified by the boundary conditions. From the first we get $1 = u_0 = A + B$, i.e. $A = 1 - B$. From the second, we find

$$\begin{aligned} 0 = u_{N+1} &= A \cdot 3^{-N-1} + B \cdot 2^{-N-1} \\ &= 3^{-N-1} + B\Big(2^{-N-1} - 3^{-N-1}\Big) \end{aligned}$$

   This is equivalent to $0 = 1 + B\Big(\big(\frac{3}{2}\big)^{N+1} - 1\Big)$, i.e.

$$B = \frac{-1}{\big(\frac{3}{2}\big)^{N+1} - 1} = \frac{-2^{N+1}}{3^{N+1} - 2^{N+1}} \quad \text{and} \quad A = \frac{\big(\frac{3}{2}\big)^{N+1}}{\big(\frac{3}{2}\big)^{N+1} - 1} = \frac{3^{N+1}}{3^{N+1} - 2^{N+1}}.$$

4. True or false (justify your answer): *If $\lambda[T]$ are the eigenvalues of T and $\lambda[S]$ the eigenvalues of S, then $\lambda[T] + \lambda[S]$ are the eigenvalues of $T + S$.*

   Solution: Absolutely not! The expression $\lambda[T] + \lambda[S]$ does not even make sense since $\lambda[T]$ and $\lambda[S]$ are both sets of N numbers. How should these sets be added?

   The issue is that the eigenspaces (the eigenvectors) are not necessarily the same. If λ_T is an eigenvalue of T and λ_S is an eigenvalue of S we know that $Tx = \lambda_T x$ and $Sy = \lambda_S y$ for some vectors x and y. But x and y need not be the same vector and we do not get $(T + S)z = (\lambda_T + \lambda_S)z$ for a vector z.

   However, note that we have $\lambda[\alpha T] = \alpha\lambda[T]$, where the right-hand-side expression means that each eigenvalue should be multiplied with α. Also, since $Ix = x$ for all x we find that $Sx = \lambda_S x$ means that $(I + S)x = Ix + Sx = \lambda_S x + x = (\lambda_S + 1)x$. Thus $\lambda[I + S] = 1 + \lambda[S]$, where the right-hand-side expression means that 1 should be added to each eigenvalue.

5. Let λ be an eigenvalue of the invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

Solution: Since A is nonsingular, $Au = \lambda u$ implies $\lambda \neq 0$ and $u = A^{-1}\lambda u = \lambda A^{-1}u$. Hence, $A^{-1}u = \lambda^{-1}u$. From this we also see that the eigenspaces of A and A^{-1} coincide, since the eigenvector is u in both cases.

More generally, note that $Au = \lambda u$ implies $A^2 u = \lambda A u = \lambda^2 u$. By the same token, $A^p u = \lambda^p u$, i.e. $\lambda[A^p] = (\lambda[A])^p$ for every power p. If A is diagonalized by a matrix U, i.e., $U^{-1}AU = \Lambda$, it follows that $U^{-1}A^pU = \Lambda^p$. In other words, U *diagonalizes every power of* A.

6. True or false: *If* $Au = \lambda u$, *then* e^{tA} *has the eigenvalues* $\mathrm{e}^{t\lambda}$.

Solution: Correct. This means that $\lambda[\mathrm{e}^{tA}] = \mathrm{e}^{t\lambda[A]}$.

This can be seen in several ways. For example, consider $\dot{u} = Au$, with solutions $\mathrm{e}^{tA}u_0$. Assume that $Au_0 = \lambda u_0$ and investigate whether $u(t) = v(t)u_0$, with $v(t)$ scalar and $v(0) = 1$, is a solution to $\dot{u} = Au$.

We get $\dot{u} = \dot{v} \cdot u_0$ and $Au = vAu_0 = (\lambda v)u_0$. Hence $\dot{u} = Au$ provided that $\dot{v} = \lambda v$, so $v(t) = \mathrm{e}^{t\lambda}$. Therefore $u(t) = \mathrm{e}^{t\lambda}u_0$ is a solution to $\dot{u} = Au$. Since $u(t) = \mathrm{e}^{tA}u_0$, we have

$$Au_0 = \lambda u_0 \Rightarrow \mathrm{e}^{tA}u_0 = \mathrm{e}^{t\lambda}u_0.$$

Therefore, $\mathrm{e}^{t\lambda}$ is an eigenvalue of e^{tA}. This holds for every eigenvalue of A.

Alternatively, we can do a direct calculation. If $Au = \lambda u$ then $A^k u = \lambda^k u$ for any integer, so

$$\mathrm{e}^{tA}u = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k u = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k u = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k u = \sum_{k=0}^{\infty} \frac{1}{k!}(t\lambda)^k u = \mathrm{e}^{t\lambda}u.$$

7. In class we determined the eigenvalues of

$$T_{\Delta x} = \frac{1}{\Delta x^2}\mathrm{tridiag}(1 \quad -2 \quad 1)$$

   (a) Sketch the location of the eigenvalues in the complex plane.

   (b) Sketch the location of the eigenvalues of $T_{\Delta x}^{-1}$. (Make sure that your sketches have some "reasonable scaling," e.g. by indicating where the eigenvalues are in relation to the unit circle.)

   (c) If $\Delta x \to 0$, where will the eigenvalues of $T_{\Delta x}^{-1}$ "cluster"?

Solution: We have

$$\lambda_k[T_{\Delta x}] = -\frac{4}{\Delta x^2}\sin^2\frac{k\pi\Delta x}{2},$$

with $\Delta x = 1/(N+1)$. These are real, and located in the interval

$$(-\frac{4}{\Delta x^2}, -\pi^2).$$

The eigenvalues are located *outside* the unit circle. When we invert the matrix, they are still going to be *negative and real*, but located *inside the unit circle*. Because some eigenvalues of $T_{\Delta x}$ are large, the eigenvalues of $T_{\Delta x}^{-1}$ are small, located immediately to the left of the origin. As $\Delta x \to 0$, *they cluster at the origin.*

8. Let $A \in \mathbb{R}^{N\times N}$ be a normal matrix with the eigenvalues λ_k, $k = 1, \ldots, N$. What are $\|A\|$ and $\|A^{-1}\|$?

   Solution: We have $\|A\| = \max_k |\lambda_k|$. If some eigenvalue $\lambda_j = 0$ then A is not invertible, so A^{-1} does not exist and "$\|A^{-1}\| = \infty$". Otherwise, we have $\|A^{-1}\| = \max_k \frac{1}{|\lambda_k|} = \frac{1}{\min_k |\lambda_k|}$.

9. Consider the discretization $T_{\Delta x}u = f$ of the Poisson equation $y'' = f$, and a discretization $T_{\Delta x}v = f + e$ of the perturbed problem $y'' = f + \epsilon$, where the function ϵ represents measurement errors. Using the previous question and what you know about $\lambda_k[T_{\Delta x}]$, how much can the solutions differ? That is, provide a bound on $\|u - v\|$.

   Solution: From the equations, we get $T_{\Delta x}(u - v) = -e$, i.e. $u - v = -T_{\Delta x}^{-1}e$. Thus $\|u - v\| \le \|T_{\Delta x}^{-1}\|\|e\|$. We know that $\|T_{\Delta x}^{-1}\|_2 \approx 1/\pi^2$, so $\|u - v\|_2 \le \frac{\|e\|_2}{\pi^2}$. Note that this result does not depend on the number of grid points N. We also have $\|T_{\Delta x}^{-1}\|_2 = \|T_{\Delta x}^{-1}\|_{\mathrm{RMS}}$, so we get the same result in the RMS-norm as in the 2-norm.

10. Consider the (vector-valued) initial value problem $\frac{\mathrm{d}}{\mathrm{d}t}u = T_{\Delta x}u$ with initial condition $u(0) = v$. Its solution is given by $u(t) = \mathrm{e}^{tT_{\Delta x}}v$.

    (a) Give an upper bound for $\|\mathrm{e}^{tT_{\Delta x}}\|_2$ for $t \ge 0$.

    (b) Sketch the location of the eigenvalues of $\mathrm{e}^{tT_{\Delta x}}$ in the complex plane. (You may consider time t to be a fixed parameter.)

    (c) Where do the eigenvalues of $\mathrm{e}^{tT_{\Delta x}}$ "cluster" as $\Delta x \to 0$ for t fixed?

    (d) Where do they go as $t \to \infty$ for Δx fixed?

    (e) Can you give or suggest an upper bound for the *inverse* $\|\mathrm{e}^{-tT_{\Delta x}}\|_2$ (where $t > 0$)? How does that inverse behave as $\Delta x \to 0$?

    (f) Suppose we solve this initial value problem using the explicit Euler method. What condition on the time step Δt is a minimum requirement for stability?

    (g) Same question for the implicit Euler method.

    (h) Which method is suitable when $\Delta x \to 0$?

This constitutes a prequel for next week's material on *parabolic* problems; $\frac{\mathrm{d}}{\mathrm{d}t}u = T_{\Delta x}u$ is a spatial discretization of the parabolic partial differential equation $\frac{\mathrm{d}}{\mathrm{d}t}u = \frac{\mathrm{d}^2}{\mathrm{d}x^2}u$ known as the *heat equation.*

Solution:

(a) Give an upper bound for $\|\mathrm{e}^{tT_{\Delta x}}\|_2$ for $t \geq 0$. Since all the eigenvalues of $T_{\Delta x}$ are negative, with the least negative eigenvalue being $\lambda_1[T_{\Delta x}] \approx -\pi^2$, the largest eigenvalue of $\mathrm{e}^{tT_{\Delta x}}$ is $\mathrm{e}^{t\lambda_1[T_{\Delta x}]} \approx \mathrm{e}^{-t\pi^2}$. Since $T_{\Delta x}$ is symmetric and therefore normal, we thus get the upper bound (in fact, equality, except for the vagueness in $\approx -\pi^2$)

$$\|\mathrm{e}^{tT_{\Delta x}}\|_2 \leq \mathrm{e}^{-\pi^2 t}.$$

(b) Further, because

$$\lambda[\mathrm{e}^{tT_{\Delta x}}] = \mathrm{e}^{t\lambda[T_{\Delta x}]}$$

we can easily sketch the location of the eigenvalues of the exponential. Take some positive t, say $t = 1$, and plot $\mathrm{e}^{\lambda[T_{\Delta x}]}$. Because $\lambda[T_{\Delta x}]$ is negative and real, the exponentials of these are *real, small, and positive.* They are *located immediately to the right of the origin.*

(c) As $t \to \infty$ for Δx fixed, those eigenvalues go to zero.

(d) An upper bound for $\|\mathrm{e}^{-tT_{\Delta x}}\|_2$ does not exist, since we would have to take the inverse of those eigenvalues that approach zero. They are positive and become arbitrarily large. This reflects the fact that we are basically dealing with the heat equation in reverse time, which is not a well posed problem.

(e) If we solve $\dot{u} = T_{\Delta x}u$ using the explicit Euler method, we get

$$u^{n+1} = u^n + \Delta t \cdot T_{\Delta x}u^n = (I + \Delta t \cdot T_{\Delta x})u^n.$$

For stability we must require that all eigenvalues of $I + \Delta t \cdot T_{\Delta x}$ are inside the unit circle, i.e.

$$|\lambda[I + \Delta t \cdot T_{\Delta x}]| \leq 1.$$

As $\lambda[I + \Delta t \cdot T_{\Delta x}] = 1 + \Delta t\lambda[T_{\Delta x}]$, and $\lambda[T_{\Delta x}]$ is negative and real, the stability condition is fulfilled if

$$|1 - \frac{4\Delta t}{\Delta x^2}| \leq 1,$$

i.e., if

$$\frac{4\Delta t}{\Delta x^2} \leq 2,$$

which requires that the so-called CFL condition (see Chapter 5)

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

is fulfilled.

(f) If instead we take the implicit Euler method, we get

$$u^{n+1} = u^n + \Delta t \cdot T_{\Delta x} u^{n+1},$$

from which it follows that

$$u^{n+1} = (I - \Delta t \cdot T_{\Delta x})^{-1} u^n.$$

Now we have to require that

$$|\lambda[(I - \Delta t \cdot T_{\Delta x})^{-1}]| \leq 1$$

or

$$\frac{1}{|\lambda[I - \Delta t \cdot T_{\Delta x}]|} \leq 1.$$

This requires that

$$|\lambda[I - \Delta t \cdot T_{\Delta x}]| \geq 1$$

But since

$$\lambda[I - \Delta t \cdot T_{\Delta x}] = 1 - \Delta t \cdot \lambda[T_{\Delta x}],$$

and $\lambda[T_{\Delta x}]$ is strictly negative, we see that it always holds that

$$\lambda[I - \Delta t \cdot T_{\Delta x}] = 1 - \Delta t \cdot \lambda[T_{\Delta x}] > 1,$$

no matter how we choose Δt or Δx, so there is *no condition* on the time step Δt. Therefore this method is usually called *unconditionally stable.*

(g) The implicit Euler method, being unconditionally stable, is *preferable* to the explicit Euler method, which is limited by the CFL condition. However, there's a price to pay. The implicit method requires *equation solving* on every step, and is more expensive to use.

(h) The problem $\frac{\mathrm{d}}{\mathrm{d}t} u = T_{\Delta x} u$ is *stiff* (see Chapter 5), which is why it benefits from being integrated by an implicit method. Give an upper bound for $\|\mathrm{e}^{tT_{\Delta x}}\|_2$ for $t \geq 0$.

11. In class we determined the eigenvalues of symmetric Toeplitz matrix $T = \mathrm{tridiag}(1 \quad 0 \quad 1)$ analytically.

(Difficult) Determine, with a similar methodology, the eigenvalues of the skew symmetric Toeplitz matrix $S = \text{tridiag}(-1 \quad 0 \quad 1)$.

Solution: Consider the eigenvalue problem $Su = \lambda u$. The ith equation reads

$$u_{n+1} - u_{n-1} = \lambda u_n; \quad u_0 = u_{N+1} = 0.$$

This is a difference equation with characteristic equation

$$z^2 - \lambda z - 1 = 0.$$

Note that (compare $(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)z + \alpha\beta$) the product of the two roots is -1 and the sum is λ.

If there is a double root, it therefore satisfies $z^2 = -1$ with the only possible solutions $z = \pm i$. Thus $\lambda = 2z = \pm 2i$ and the general solution to the difference equation is $y_n = (A + Bn)(\pm i)^n$. But this cannot satisfy the boundary conditions $y_0 = 0 =_{N+1}$ unless $A = B = 0$, yielding the trivial solution. Therefore, $\lambda = \pm 2i$ are not actual eigenvalues and we should only consider the single roots of the characteristic equation.

Denote one root by $\mathrm{i}\omega$; then the other root is i/ω. Therefore, the general solution to the difference equation can be written

$$u_n = A(\mathrm{i}\omega)^n + B(\mathrm{i}/\omega)^n.$$

Inserting the boundary condition $u_0 = 0$ shows that $A + B = 0$, and that the solution can be written

$$u_n = A((\mathrm{i}\omega)^n - (\mathrm{i}/\omega)^n).$$

Inserting the other boundary condition then gives

$$0 = A((\mathrm{i}\omega)^{N+1} - (\mathrm{i}/\omega)^{N+1})$$

from which it follows that

$$(\mathrm{i}\omega)^{N+1} = (\mathrm{i}/\omega)^{N+1}.$$

Therefore,

$$\omega^{2(N+1)} = 1 = \mathrm{e}^{2k\mathrm{i}\pi}$$

This equation has $N+1$ solutions

$$\omega_k = \mathrm{e}^{k\mathrm{i}\pi/(N+1)}, \quad k = 0, \ldots, N,$$

but $k = 0$ leads to $\omega_0 = 1$ which yields the double root.

We can now finally determine the eigenvalues; using the fact that $\lambda_k = \mathrm{i}\omega_k + \mathrm{i}/\omega_k$ gives

$$\lambda_k = \mathrm{i}(\mathrm{e}^{k\mathrm{i}\pi/(N+1)} + \mathrm{e}^{-k\mathrm{i}\pi/(N+1)}) = 2\mathrm{i}\cos\frac{k\pi}{N+1}, \quad k = 1, \ldots, N.$$

These were the eigenvalues for $S = \text{tridiag}(-1 \quad 0 \quad 1)$. The eigenvalues of $S_{\Delta x} = (N+1)S/2$ are therefore $(N+1)\lambda_k[S]/2$, i.e.,

$$\lambda_k[S_{\Delta x}] = \mathrm{i}(N+1)\cos\frac{k\pi}{N+1}, \quad k = 1, \dots, N.$$

Hence *the eigenvalues are purely imaginary*, a property shared by all *skew-symmetric matrices* ($S_{\Delta x}^{\mathrm{T}} = -S_{\Delta x}$).

12. Let y' be approximated by the second order, symmetric difference quotient $S_{\Delta x} = S/(2\Delta x)$. What are the

    (a) eigenvalues of $S_{\Delta x}$

    (b) Euclidean norm of $S_{\Delta x}$

    (c) Euclidean norm of $\mathrm{e}^{tS_{\Delta x}}$, $t \in \mathbb{R}$?

    Hint: Every skew-symmetric matrix is *normal*, and the exponential of a skew-symmetric matrix is skew-symmetric.

    Solution:

    (a) For the eigenvalues of $S_{\Delta x}$, see the previous problem. The eigenvalues are located in the interval

    $$\lambda_k[S_{\Delta x}] \in \frac{\mathrm{i}}{\Delta x}(-1, 1)$$

    (b) Because $S_{\Delta x}$ is normal, we have that $\|S_{\Delta x}\|_2 = \max_k |\lambda_k[S_{\Delta x}]|$. Thus

    $$\|S_{\Delta x}\|_2 = \max_k |(N+1)\cos\frac{k\pi}{N+1}| \approx N+1 = \frac{1}{\Delta x}.$$

    This reflects the fact that when we differentiate a function, it generally becomes less smooth and its norm increases a lot. Similarly, when we apply $S_{\Delta x}$ to a vector u, its norm can increase by a factor $\approx N+1$.

    (c) The eigenvalues of the matrix $\mathrm{e}^{tS_{\Delta x}}$ are $\mathrm{e}^{t\lambda_k[S_{\Delta x}]}$. Since the exponential matrix is also normal, we find that

    $$\begin{aligned}\|\mathrm{e}^{tS_{\Delta x}}\|_2 &= \max_k |\mathrm{e}^{t\lambda_k[S_{\Delta x}]}| \\ &= \max_k |\mathrm{e}^{it(N+1)\cos(\frac{k\pi}{N+1})}| \\ &= 1,\end{aligned}$$

    since $|\mathrm{e}^{ix}| = 1$ for any $x \in \mathbb{R}$.

Thus applying $e^{tS_{\Delta x}}$ to a vector does not increase the norm of the vector, regardless of t. We can in fact show that the norm does not change at all, it is *preserved.* This reflects the fact that the norm of the solution to the *advection equation* $\frac{\mathrm{d}}{\mathrm{d}t}u = \frac{\mathrm{d}}{\mathrm{d}x}u$ is constant, in both forward and reverse time (see Chapter 6). It is a desirable property that the discretization also has this property.

13. (Difficult) Consider the 2pBVP $-u''-u'-u = f(x)$ with $u(0) = u(1) = 0$.

    (a) Find a value $\mu > 0$ such that

    $$\Big\langle\Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} - 1\Big)u, u\Big\rangle \geq \mu\|u'\|_2^2.$$

    Does the problem have a unique solution for every right-hand side f? (Use Lax Milgram lemma as presented in the lecture notes.)

    (b) Introduce a suitable grid and discretize the equation above. Use the same techniques as in the previous problem to show that your *discretization* has a unique solution for every right-hand side f.

    (c) Let $u_{\Delta x}$ denote the solution vector on the grid. Give a bound for $\|u_{\Delta x}\|_{\Delta x}$.

Solution:

(a) We abbreviate $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} - 1$ and consider the space $V = \{u \in L^2 : u' \in L^2\}$ and its dual space $V^* = \{f : \langle f, v\rangle \leq \|f\|_{V^*}\|v\|_V\}$ with norms

$$\|v\|_V = \Big(\int_0^1 |v'|^2 dx\Big)^{\frac{1}{2}} = \|v'\|_2,$$
$$\|v\|_{V^*} = \sup_{v\in V} \frac{|\langle f, v\rangle|}{\|v\|_V} = \sup_{\|v\|_V=1} |\langle f, v\rangle|$$

We know that

$$\Big\langle -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u, u\Big\rangle = \|u'\|_2^2 = \|u\|_V^2$$

For $\frac{\mathrm{d}}{\mathrm{d}x}$, we have by integration by parts that $-\langle u, u'\rangle = \langle u', u\rangle = \langle u, u'\rangle$ and therefore $-\langle \frac{\mathrm{d}}{\mathrm{d}x}u, u\rangle = 0$. Also, we obtain by Poincare's inequality that

$$-\langle u, u\rangle = -\|u\|_2^2 \geq -\frac{1}{\pi^2}\|u'\|_2^2 = -\frac{1}{\pi^2}\|u\|_V^2.$$

Thus, we get

$$\langle Au, u\rangle = \Big\langle \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} - 1\Big)u, u\Big\rangle \geq \Big(1 - \frac{1}{\pi^2}\Big)\|u\|_V^2.$$

Moreover, the operator is bounded by

$$\begin{aligned}
\|Au\|_{V^*} &= \sup_{\|v\|_V=1} |\langle Au, v\rangle| = \sup_{\|v\|_V=1} \Big| \int_0^1 (u'v' + uv' - uv)dx \Big| \\
&\leq \sup_{\|v\|_V=1} \Big(\int_0^1 |u'|^2 dx\Big)^{\frac{1}{2}} \Big(\int_0^1 |v'|^2 dx\Big)^{\frac{1}{2}} \\
&\quad + \sup_{\|v\|_V=1} \Big(\int_0^1 |u|^2 dx\Big)^{\frac{1}{2}} \Big(\int_0^1 |v'|^2 dx\Big)^{\frac{1}{2}} \\
&\quad + \sup_{\|v\|_V=1} \Big(\int_0^1 |u|^2 dx\Big)^{\frac{1}{2}} \Big(\int_0^1 |v|^2 dx\Big)^{\frac{1}{2}} \\
&= \sup_{\|v\|_V=1} \|u\|_V\|v\|_V + \sup_{\|v\|_V=1} \|u\|_2\|v\|_V + \sup_{\|v\|_V=1} \|u\|_2\|v\|_2 \\
&\leq \|u\|_V + \frac{1}{\pi}\|u\|_V + \frac{1}{\pi^2} \sup_{\|v\|_V=1} \|u\|_V\|v\|_V \\
&= \Big(1 + \frac{1}{\pi} + \frac{1}{\pi^2}\Big)\|u\|_V
\end{aligned}$$

By Lax Milgram lemma, we find that A^{-1} exists and fulfills

$$\begin{aligned}
\|A^{-1}f\|_V &\leq \Big(1 - \frac{1}{\pi^2}\Big)^{-1}\|f\|_{V^*}, \\
\langle A^{-1}f, f\rangle &\geq \frac{1 - \frac{1}{\pi^2}}{\big(1 + \frac{1}{\pi} + \frac{1}{\pi^2}\big)^2}\|f\|_{V^*} = \frac{\pi^4 - \pi^2}{(\pi^2 + \pi + 1)^2}\|f\|_{V^*}
\end{aligned}$$

for all $f \in V^*$. Applying the Poincare's inequality it follows that

$$\begin{aligned}
\|A^{-1}f\|_2 &\leq \frac{1}{\pi}\|(A^{-1}f)'\|_2 = \frac{1}{\pi}\|A^{-1}f\|_V \leq \frac{1}{\pi}\Big(1 - \frac{1}{\pi^2}\Big)^{-1}\|f\|_{V^*} \\
&\leq \frac{1}{\pi^2}\Big(1 - \frac{1}{\pi^2}\Big)^{-1}\|f\|_2 = \big(\pi^2 - 1\big)^{-1}\|f\|_2
\end{aligned}$$

since

$$\|f\|_{V^*} = \sup_{v\in V} \frac{|\langle f, v\rangle|}{\|v\|_V} \leq \sup_{v\in V} \frac{\|f\|_2\|v\|_2}{\|v\|_V} \leq \sup_{v\in V} \frac{\|f\|_2\frac{1}{\pi}\|v\|_V}{\|v\|_V} = \frac{1}{\pi}\|f\|_2.$$

(b) Introduce the grid $x_k = k\Delta x$, $k = 0, \ldots, N+1$, with $\Delta x = \frac{1}{N+1}$. A second-order discretization of $-u'' - u' - u = f$ is given by $u_k \approx u(x_k)$ and

$$-\frac{u_{k-1} - 2u_k + u_{k+1}}{\Delta x^2} - \frac{u_{k+1} - u_{k-1}}{2\Delta x} - u_k = f(x_k), \quad k = 1, \ldots, N,$$

and $u_0 = u_{N+1} = 0$. On matrix-vector form, we have

$$(-T_{\Delta x} - S_{\Delta x} - I)\bar{u} = \bar{f}$$

with $\bar{u} = [u_1, \ldots, u_N]^T$ and $\bar{f} = [f(x_1), \ldots, f(x_N)]^T$.

(c) We know that $\langle -T_{\Delta x} u, u\rangle \gtrsim \pi^2 \|u\|_2^2$ and by the previous exercise we have $\langle -S_{\Delta x} u, u\rangle = 0$. Thus

$$\Big\langle \big(- T_{\Delta x} - S_{\Delta x} - I\big)u, u\Big\rangle \gtrsim (\pi^2 - 1)\|u\|_2^2,$$

so we have a unique solution $\bar{u}$ satisfying

$$\|\bar{u}\|_2 \leq \frac{1}{\pi^2 - 1}\|\bar{f}\|_2.$$

Note every matrix is bounded but since the upper bound is not relevant for the bound of the inverse, we do not compute this explicitly here.

14. Consider the 2pBVP $-u'' - \omega^2 y = g(x)$ with homogeneous boundary data $u(0) = u(1) = 0$.

(a) For what values of the parameter ω can you guarantee that there is a unique solution?

(b) Let $\omega = \pi$. What happens with the analytical solution? Why?

Solution:

(a) Since

$$\Big\langle -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u - \omega^2 u, u\Big\rangle = \|u'\|_2^2 - \omega^2\|u\|_2^2 \geq \Big(1 - \frac{\omega^2}{\pi^2}\Big)\|u\|_V^2$$

we can only guarantee a unique solution if the differential operator is positive. The left-hand side is positive if and only if $1 - \frac{\omega^2}{\pi^2} > 0$, i.e. if $|\omega| < \pi$.

(b) If $\omega = \pi$, we lose positivity: $\langle -u'' - \pi^2 u, u\rangle = \langle u', u'\rangle + \pi^2\langle u, u\rangle \geq 0$ by Sobolev's lemma, with equality attained for e.g. $u(x) = \sin(\pi x)$.

In fact, if $u_C(x) = C\sin(\pi x)$ we get $-u_C'' + \pi^2 u_C = 0$ for any choice of the constant C. So if u_p is a solution to $-u'' - \pi^2 u = g$ then so is $u_p + u_C$. That is, there are infinitely many solutions when $\omega = \pi$.