**Numerical Methods for Differential Equations, FMNN10 / NUMN32**
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**Review questions and study problems, week 5**

A problem is said to be *well posed* if it has a unique solution which depends *continuously* on the data. For example, in the simple 2p-BVP $u'' = q(x)$ with $u(0) = u(1) = 0$, we showed in the lecture notes of Chapter 4 that with the help of lemma of Lax-Milgram and Sobolev's Lemma that

$$\|u\|_{\Delta x} \leq \frac{\|q\|_{\Delta x}}{\pi^2}.$$

Hence, if the *data* $\|q\|_{\Delta x} \to 0$, then the *solution* $\|u\|_{\Delta x} \to 0$. This implies continuity: a small change in the data $q$ has only a bounded effect on the solution $u$. The problem is *well-posed.*

In a similar way, an initial value problem $\dot{u} = f(u)$ with $u(0) = u_0$ is well posed if the solution depends continuously on the data $u_0$. A small change in the initial value $u_0$ must only have a bounded effect on the solution $u(t)$. Note that here it's enough that the effect is bounded at a finite time $t$. So a problem like

$$\dot{u} = u; \quad u(0) = 1$$

is well posed: if we perturb $u(0)$ by $\varepsilon$ the solution $u(t)$ changes by $\mathrm{e}^t\varepsilon$, which is bounded for any finite $t$.

There are several questions concerning well-posedness below. They cannot be treated in a complete way here, so we will only take a simplified approach, looking at a continuous data dependence. Thus, **you don't have to address existence questions below, only uniqueness**. Moreover, as boundary conditions also affect well-posedness we will for simplicity disregard the influence of boundary conditions below. This corresponds to either having homogeneous boundary conditions, periodic boundary conditions, or "none at all" when the computational domain is infinite, i.e., $x \in (-\infty, \infty)$.

1. Show that the diffusion equation $u_t = u_{xx}$ with boundary conditions $u(t, 0) = u(t, 1) = 0$ and initial value $u(0, x) = g(x)$ is well posed for $t \geq 0$.

   (**Hint:** Using that $-\langle w, w_{xx}\rangle = \langle w_x, w_x\rangle \geq \pi^2\|w\|_2^2$, how large is the influence of a perturbation of $g$? Use the same technique as we used for the initial value problem above.)

2. (Difficult) Show that the diffusion equation $u_t = u_{xx}$ is not well posed in *reverse time*, i.e., for $t < 0$.

   **Hint:** The reversed problem is equivalent to solving $u_t = -u_{xx}$ in forward time. To prove it, we can show by counter-example with a sequence of functions $u^n(t,x)$, $n = 1, 2, \ldots$, that all solve the equation and for which $\|u^n(0,\cdot)\| \to 0$ as $n \to \infty$ but $\|u^n(t,\cdot)\| \to \infty$ for any $t > 0$. Try solutions $u^n$ based on the eigenfunctions to $\partial^2/\partial x^2$ and scale them by $n^r$ for an appropriate value of $r$.

3. Consider the semi-discretization (method of lines) $\dot{U} = T_{\Delta x} U$ of the diffusion equation. If the initial condition is changed by $\varepsilon$, how large is the perturbation at a given time $t > 0$?

4. Same question for the reversed problem, $\dot{U} = -T_{\Delta x} U$. Can you conclude that the semi-discretization is ill posed as $\Delta x \to 0$?

5. Let $Q$ be a rational function of the form

   $$Q(w) = \frac{1 + \alpha w}{\beta + \gamma w}.$$

   Then let $A$ be a matrix with known eigenvalues $\lambda_k[A]$. By the matrix $Q(A)$ we mean

   $$Q(A) = (\beta I + \gamma A)^{-1}(I + \alpha A).$$

   (Assuming, of course, that $\beta I + \gamma A$ is invertible.) Show that the eigenvalues of this matrix are

   $$\lambda_k[Q(A)] = Q(\lambda_k[A]).$$

6. Let the semi-discretization $\dot{U} = T_{\Delta x} U$ be approximated using the time discretization

   $$(I - \frac{\Delta t}{2} T_{\Delta x}) U^{n+1} = (I + \frac{\Delta t}{2} T_{\Delta x}) U^n.$$

   This is the trapezoidal rule, or the Crank–Nicolson method for the diffusion equation $u_t = u_{xx}$. Solving for $U^{n+1}$, it is equivalent to an explicit recursion

   $$U^{n+1} = B(\Delta t, \Delta x) U^n$$

   Use the result from the previous question, and your knowledge of $\lambda_k[T_{\Delta x}]$, to find the eigenvalues of $B(\Delta t, \Delta x)$.

   Note that the recursion above can be viewed as a fixed point iteration. State a condition on the eigenvalues for stability (convergence). Given the properties of $\lambda_k[T_{\Delta x}]$, is there a restriction on $\Delta t$ in order to have a contraction? *Hint:* The matrix $B(\Delta t, \Delta x)$ is normal.

7. (A hard problem) Let the skew-symmetric matrix

$$R = \begin{pmatrix} 0 & 1 & 0 & \dots & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & \\ & \dots & 0 & -1 & 0 \end{pmatrix}$$

be given, and note that $R_{\Delta x} = R/(2\Delta x)$ is a second order accurate approximation to $\partial/\partial x$. We saw in the previous study questions that the eigenvalues of $R_{\Delta x}$ are purely imaginary,

$$\lambda_k[R_{\Delta x}] = \frac{\mathrm{i}}{\Delta x} \cos \frac{k\pi}{N+1}, \qquad k = 1, \dots, N.$$

(a) Determine $\mu \in [0, \infty)$ such that

$$\langle R_{\Delta x} v, v \rangle \leq \mu \|v\|_2^2$$

(b) Is the semi-discretization $\dot{u} = R_{\Delta x} u$ well posed in forward time? Is it well posed in reverse time?

8. Find out whether the *convection–diffusion equation* $u_t = u_x + \frac{1}{\mathrm{Pe}} u_{xx}$ is well-posed in forward time. Here $0 < \mathrm{Pe} < \infty$ is the *Peclét number*, which measures the "balance" between convection and diffusion. You can take homogeneous Dirichlet boundary conditions $u(t, 0) = u(t, 1) = 0$. Note that this means that $\langle u, u_x \rangle_{L^2} = -\langle u_x, u \rangle_{L^2} = 0$.

9. Check if the convection–diffusion equation is well posed in reverse time. Which of the two operators, the convection part $\partial/\partial x$, or the diffusion part $\partial^2/\partial x^2$, is decisive for well-posedness? Note that finding an explicit counterexample to well-posedness is not easy, compare to Question 2.

10. Let $\kappa_i$ denote the $i$th eigenvalue of $-\mathrm{d}^2/\mathrm{d}x^2$ on $[0, 1]$ with boundary conditions $X(0) = X(1) = 0$. We know that $\kappa_i = (i\pi)^2$. Now consider the negative Laplacian in 2D,

$$-\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

on the unit square $[0, 1] \times [0, 1]$ with $u = 0$ on the boundary. Assume that $u(x, y) = X(x)Y(y)$ and use this to show that every

$$\lambda_{i,j} = \kappa_i + \kappa_j,$$

for $i, j = 1, 2, \ldots$ is an eigenvalue to the corresponding eigenvalue problem, $-\Delta u = \lambda u$. There are no other eigenvalues, but you don't need to prove this.

What is the smallest eigenvalue of $-\Delta$ on the given domain and with the given boundary conditions?

11. Now consider the discrete eigenvalue problem for the five-point finite difference operator approximating the Laplacian,

$$-\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} - \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = \lambda u_{i,j},$$

using $\Delta x = \Delta y$. Write $u_{i,j} = X_i Y_j$ and try to "separate" the eigenvalue problem like you did in the continuous case. Conclude what the eigenvalues are, using your knowledge of the eigenvalues of standard Toeplitz matrix we use to approximate $\mathrm{d}^2/\mathrm{d}x^2$.

12. In the lecture notes we saw that, using the five-point operator, we can represent the discrete Laplacian (a linear operator) by the matrix

$$L_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix} T & I & 0 & \ldots & & \\ I & T & I & & & \\ & I & T & I & & \\ & & & & & \\ & & & & \ddots & I \\ & \ldots & & 0 & I & T \end{pmatrix},$$

with $T = \mathrm{tridiag}(1 \quad -4 \quad 1)$. Note that the matrix $L_{\Delta x}$ is *symmetric*, but it is not a Toeplitz matrix. Using the information on the eigenvalues of $-L_{\Delta x}$ found in the previous Question 11, find the strong accretivelty/positively constant $\mu$ of $-L_{\Delta x}$.