Hyperbolic problems

These are problems modelling transport phenomena and exhibiting conservation properties.

Note: stark contrast to the diffusion properties we saw for parabolic problems.

[Some slides with typical equations and a few applications.]

Our model problems will be

- the advection equation: $u_t + a u_x = 0$
- the wave equation: $u_{tt} = c^2 u_{xx}$
- nonlinear conservation laws: $u_t + (f(u))_x = 0$.

The advection equation

$$u_t + a u_x = 0 \quad , \quad a \in \mathbb{R}$$

Trivial problem, we know the solution:

If $u(0,x) = g(x)$, some function g,

then

$$u(t,x) = g(x-at)$$

solves the eq.

Proof:
$$u_t(t,x) = g'(x-at) \cdot (-a)$$
$$u_x(t,x) = g'(x-at) \cdot 1 \quad \square$$

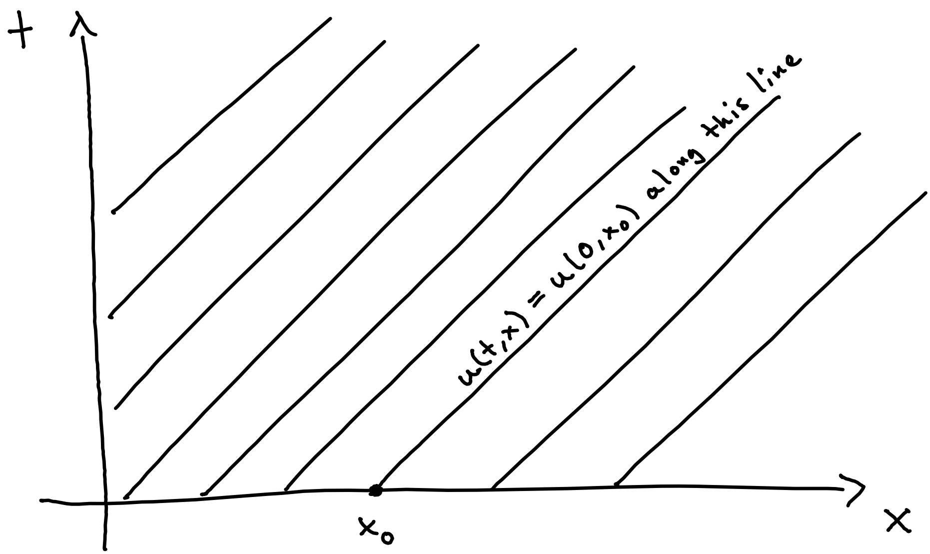
Like linear test eq., good test case for our methods.

Note: solution constant along $x - at = C$.

<u>Def.</u> A curve $(t(s), x(s))$, $s \in [0, \infty)$, is called a <u>characteristic</u> if $u(t(s), x(s))$ is constant $\forall s$.

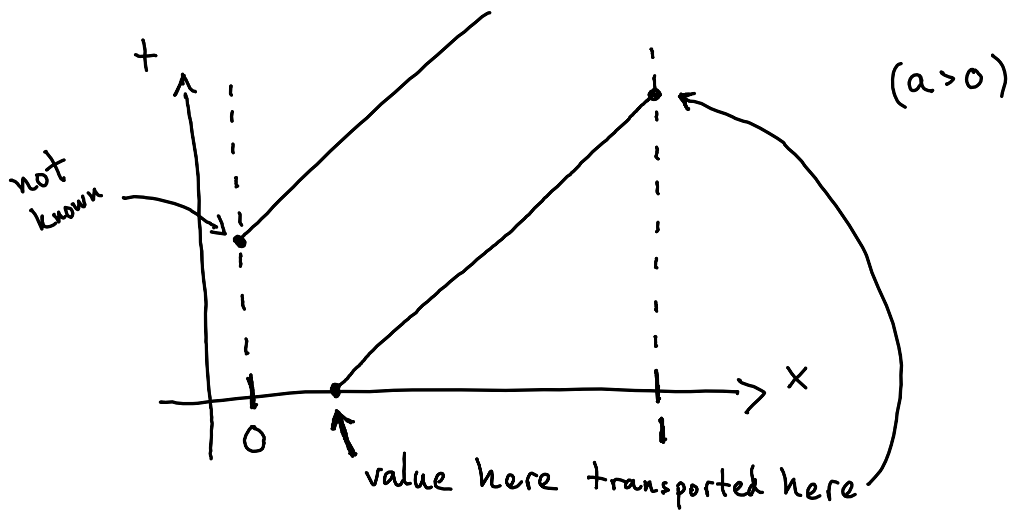
For the advection eq., the characteristics are the straight lines $x - at = C$.

We typically illustrate this by a 2D-plot of the characteristics, rather than a 3D-plot of the solution:



(every line has slope a)

Note that this creates a boundary value issue when working on a finite spatial domain:



$\Rightarrow$ cannot specify BC at $x=1$

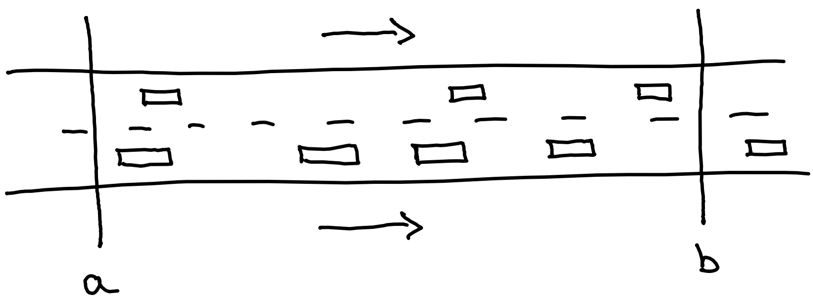
because the values there are already determined by $u(0,x)$ and the equation.

But we must specify a BC at $x=0$.

$\therefore$ Only one BC and IC for the adv. eq. rather than two BC and IC for the parabolic problems.

If $a<0$ we have transport in the opposite direction and need a BC at $x=1$ but cannot have one at $x=0$.

Application: traffic flow



Let $N(t) = \#\{\text{cars in } [a,b] \text{ at time } t\}$

If $u(t,x)$ is the car density, we have

$$N(t) = \int_a^b u(t,x)\,dx.$$

Assume all cars move at constant speed v (for now).

Then $v \cdot u(t,a)$ cars are entering $[a,b]$ at time t and $v \cdot u(t,b)$ are leaving, so $\dot{N} = vu(t,a) - vu(t,b)$

Note that

$$\dot{N}(t) = v \cdot u(t,a) - v \cdot u(t,b) = -\int_a^b v \cdot u_x(t,x)\,dx$$

But we also have

$$\dot{N} = \frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b \frac{d}{dt} u(t,x)\,dx$$

$$= \int_a^b u_t(t,x)\,dx$$

$$\Rightarrow \int_a^b u_t + v \cdot u_x \,dx = 0$$

$[a,b]$ arbitrary $\Rightarrow$ $u_t + v u_x = 0$

Conservation law property: No cars suddenly disappear or are created

Let's make it non-trivial by assuming a non-constant speed $v(u)$, i.e. the speed when there are many cars is different to when there are few. Typically $v(u) \searrow 0$ as $u \nearrow$.

Same reasoning, $\dot{N} = \int u_t$, but

$$\dot{N} = \left(v(u)\,u\right)\Big|_{x=a} - \left(v(u)\,u\right)\Big|_{x=b}$$

$$= -\int_a^b \frac{\partial}{\partial x}\left(v(u)\,u\right)\,dx$$

$$= -\int_a^b v'(u)\,u_x\,u + v(u)\,u_x \qquad \text{(chain rule)}$$

$$\Rightarrow \quad u_t + \left(v'(u)\,u + v(u)\right)u_x = 0$$

<u>Nonlinear</u>, solution values transported at different speeds.

Can model phenomena such as traffic jams etc..

Method of lines

Standard approach:

$u_t + u_x = 0$ $\xrightarrow{\text{discretize in space}}$ $\dot{U} + S_{\Delta x} U = 0$

continuous problem

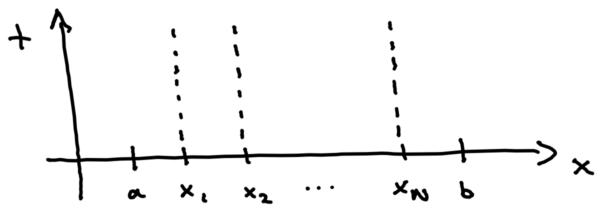
Semi-discretization, vector-valued ODE

discretize in time

$$U^{n+1} = U^n + \Delta t \, S_{\Delta x} U^n$$

Full discretization, algebraic system

"Method of lines", because $U_j(t)$ approximates $u(t, x_j)$ along the line (t, x_j).



Can use different spatial discretizations

and different temporal discretizations.

Adv. eq. $u_t + u_x = 0$, space discr.

Grid $x_j = j\Delta x$, $j = 1, \ldots, N$

Approx. $U(t) \in \mathbb{R}^N$, $U_j(t) \approx u(t, x_j)$

Approx. $u_t(t, x_j) \approx \dot{U}_j(t)$

Approx. $u_x(t, x_j)$?

- Backward diff.: $\dfrac{U_j(t) - U_{j-1}(t)}{\Delta x}$

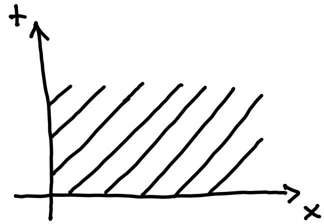
- Forward diff.: $\dfrac{U_{j+1}(t) - U_j(t)}{\Delta x}$

- Central diff.: $\dfrac{U_{j+1}(t) - U_{j-1}(t)}{2\Delta x}$

- Fancy higher-order diff.: $\dfrac{\frac{1}{4}U_{j+1} + \frac{5}{6}U_j - \frac{3}{2}U_{j-1} + \frac{1}{2}U_{j-2} - \frac{1}{12}U_{j-3}}{\Delta x}(t)$

Which to choose?

With $u_t + u_x = 0$, information flows to the right.



$\Rightarrow$ spatial discretization must use information from the left!

With $u_t - u_x = 0$, it must use information from the right.

Def. (informal) A method which looks in the appropriate direction is called an *upwind* method. One that looks in the wrong direction is called a *downwind* method. (Inspired by sailing terminology.)

For $u_t + u_x = 0$, flow to the right:

- Backward diff. : $\dfrac{U_j - \overset{\text{left}}{U_{j-1}}}{\Delta x}$ $\boxed{\text{upwind}}$

- Forward diff. : $\dfrac{\overset{\text{right}}{U_{j+1}} - U_j}{\Delta x}$ downwind

- Symmetric diff.: $\dfrac{\overset{\text{right}}{U_{j+1}} - \overset{\text{left}}{U_{j-1}}}{2\Delta x}$ neither

- Fancy higher-order diff.: $\dfrac{\frac{1}{4}U_{j+1} + \frac{5}{6}U_j - \frac{3}{2}U_{j-1} + \frac{1}{2}U_{j-2} - \frac{1}{12}U_{j-3}}{\Delta x}$

  more left than right : upwind

For $u_t - u_x = 0$ everything is inverted and the bwd diff. is downwind!

> Important: Upwind is necessary for stability.
>
> A downwind method will never work

General semidiscretization (SD)

$$u_x(t, x_j) \approx \frac{1}{\Delta x} \sum_{k=-\ell}^{m} a_k U_{j+k}(t)$$

Determine coefficients a_k by desired order:

Def. The SD method is consistent of order p if

$$\frac{1}{\Delta x} \sum_{k=-\ell}^{m} a_k \, u(t, x + k\Delta x) = u_x(t, x) + \mathcal{O}(\Delta x^p)$$

In practice: expand in Taylor series, match terms

Can state/do this in a fancy way using forward shift operators, see Iserles.

A <u>full</u> discretization where the spatial discr. is of order p_1 and the temporal discr. is of order p_2 is sometimes said to be of order $p = \min(p_1, p_2)$.

But it is typically more useful to specify the separate orders (p_1, p_2).

The full discr. with $U_j^n \approx u(t_n, x_j)$ will have the form

$$\underbrace{\frac{1}{\Delta t} \sum_i b_i U_j^{n+1-i}}_{\text{time, } u_t} + \underbrace{\frac{1}{\Delta x} \sum_k a_k U_{j+k}^n}_{\text{space, } u_x} = 0$$

$\Rightarrow$ CFL condition for explicit methods will be of the form $\boxed{\frac{\Delta t}{\Delta x} \leq C}$

Note: not problematic! $\Delta t \sim \Delta x$ is fine

(For $u_t = u_{xx}$ we get a $\frac{1}{\Delta x^2}$ from u_{xx} and therefore instead have $\frac{\Delta t}{\Delta x^2} \leq C$.)

Classic methods for $u_t + a u_x = 0$

Approx. $U_j^n \approx u(t_n, x_j)$. Every approx. below at (t_n, x_j).

Use $\mu = \frac{\Delta t}{\Delta x}$.

- Upwind Euler = explicit Euler + bwd diff.

  $u_t \approx \frac{U_j^{n+1} - U_j^n}{\Delta t}$ $\qquad$ $u_x \approx \frac{U_j^n - U_{j-1}^n}{\Delta x}$

$$\to U_j^{n+1} = (1 - a\mu) U_j^n + a\mu U_{j-1}^n$$

- Explicit Euler + fwd diff. = <u>downwind</u> scheme

  Don't use!

- Central difference scheme: explicit Euler + symmetric diff.

$$U_j^{n+1} = U_j^n + \frac{a\mu}{2}\left(U_{j-1}^n - U_{j+1}^n\right)$$

Seems like a good idea, but is always unstable.

Don't use! (proof later)

- Lax-Friedrichs: small modification to central diff. scheme

$$U_j^{n+1} = \underbrace{\frac{U_{j-1}^n + U_{j+1}^n}{2}}_{\text{instead of } U_j^n} + \frac{a\mu}{2}\left(U_{j-1}^n - U_{j+1}^n\right)$$

Convergent, order (1, 2).

- Lax-Wendroff: order 2 in both time and space

$$U_j^{n+1} = \frac{a\mu}{2}(1+a\mu)U_{j-1}^n + (1-a^2\mu^2)U_j^n - \frac{a\mu}{2}(1-a\mu)U_{j+1}^n$$

Auto-upwinding: $a\mu = \frac{1}{2} \Rightarrow U_j^{n+1} = \frac{3}{8}U_{j-1}^n + \frac{3}{4}U_j^n - \frac{1}{8}U_{j+1}^n$

$a\mu = -\frac{1}{2} \Rightarrow U_j^{n+1} = -\frac{1}{8}U_{j-1}^n + \frac{3}{4}U_j^n + \frac{3}{8}U_{j+1}^n$

Weights change depending on flow direction, always upwind.

Let's derive the Lax-Wendroff scheme, by expanding in Taylor series.

(In the time parameter, since we know how to discretize the spatial derivatives.)

We have

everything evaluated at (t,x)

$$u(t+\Delta t, x) = \underbrace{u + \Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt}} + \mathcal{O}(\Delta t^3)$$

If our method exactly reproduces these three terms when we insert the exact solution, the temporal error is of order 2.

To ensure that, we use the equation $u_t + a u_x = 0$ to replace the temporal derivatives with spatial derivatives. These we can approximate using central differences in space, which are exact except for an error $\mathcal{O}(\Delta x^2)$ <u>that does not depend on Δt.</u>

$\longrightarrow$

Since $u_t + a u_x = 0$ we get $u_t = -a u_x$

Then

$$0 = \frac{\partial}{\partial t}\left(u_t + a u_x\right) = u_{tt} + a u_{xt}$$
$$= u_{tt} + a u_{tx}$$
$$= u_{tt} + a \frac{\partial}{\partial x}(u_t)$$
$$= u_{tt} + a \frac{\partial}{\partial x}(-a u_x)$$
$$= u_{tt} - a^2 u_{xx}$$

$$\Rightarrow \quad u_{tt} = a^2 u_{xx}$$

$$\therefore \; u(t+\Delta t, x) = u - a\Delta t\, u_x + \frac{a^2 \Delta t^2}{2} u_{xx} + O(\Delta t^3)$$

Now approx. u_x and u_{xx} with central difference quotients (2nd-order)

$$u_x \approx \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}, \quad u_{xx} \approx \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{\Delta x^2}$$

$$\rightarrow U_j^{n+1} = \frac{a\mu}{2}\left(1 + a\mu\right)U_{j-1}^n + \left(1 - a^2\mu^2\right)U_j^n - \frac{a\mu}{2}\left(1 - a\mu\right)U_{j+1}^n$$

If we insert the exact solution, i.e. replace U_j^n by $u(t_n, x_j)$ etc., we then get

$$u(t_n+\Delta t, x_j) = u + \Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt}$$

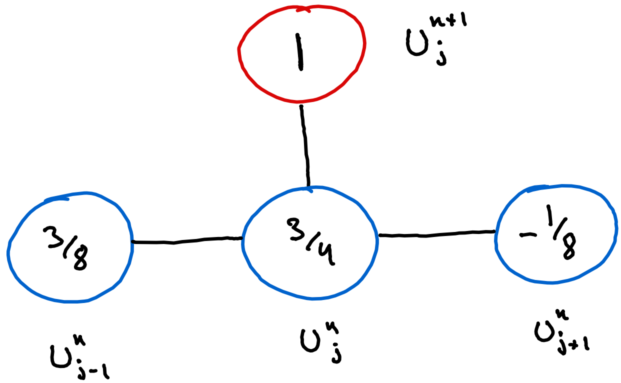
$$+ \underbrace{\mathcal{O}(\Delta t^3)}_{\text{order 2 in time}} + \underbrace{\mathcal{O}(\Delta x^2)}_{\text{order 2 in space}}$$

where u, u_t and u_{tt} are all evaluated at (t_n, x_j).

<u>Note!</u> Have only proved <u>consistency</u>, not stability. There will be a CFL condition $|a\mu| \leq C$, but unlike e.g. the central diff. scheme, Lax-Wendroff is well-behaved.

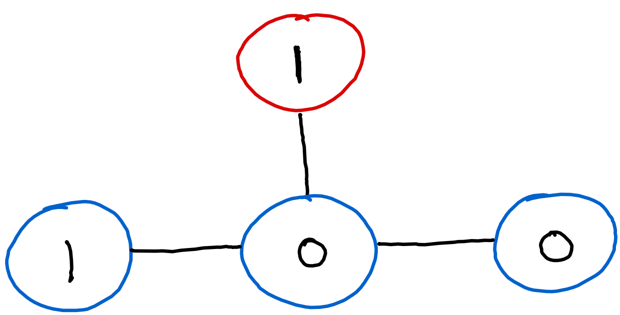
Lax-Wendroff computational stencil

At $a\mu = \frac{1}{2}$:



Note asymmetric coefficients due to auto-upwinding.

At $a\mu = 1$:



Value U_{j-1}^n transported unchanged to U_j^{n+1}.

Exactly matches what exact solution does along characteristic. At this speed $a\mu = 1$, L-W is <u>exact</u>.

Periodic boundary conditions

Boundaries always cause issues, and for hyperbolic problems it's usually worse than for parabolic problems (like how $u_t + u_x = 0$ and $u_t - u_x = 0$ require different setups).

One way to avoid this: consider problem on $\mathbb{R}$

- nice in theory, no boundaries!
- problematic in practice
  - physical phenomena usually limited in size
  - infinite grid?

Another way: consider problem on torus

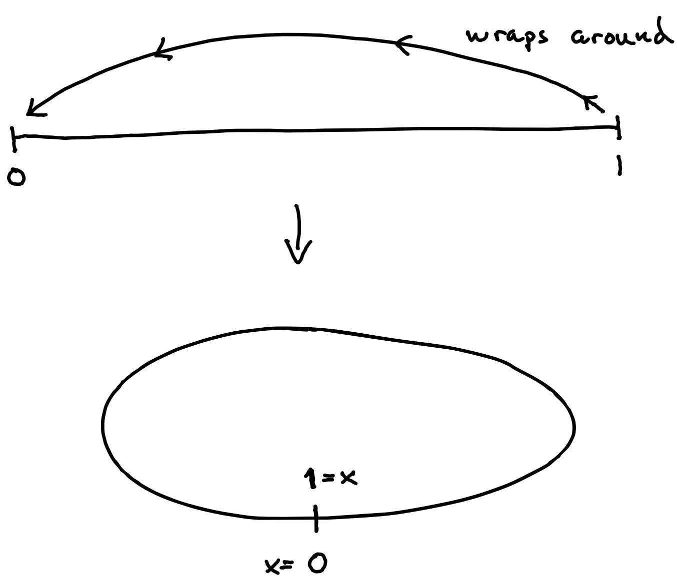
- also nice in theory
- periodicity matches typical physical behaviour

<u>Def.</u> Periodic boundary conditions on $x \in [0,1]$ means that $u(t,0) = u(t,1)$ and also

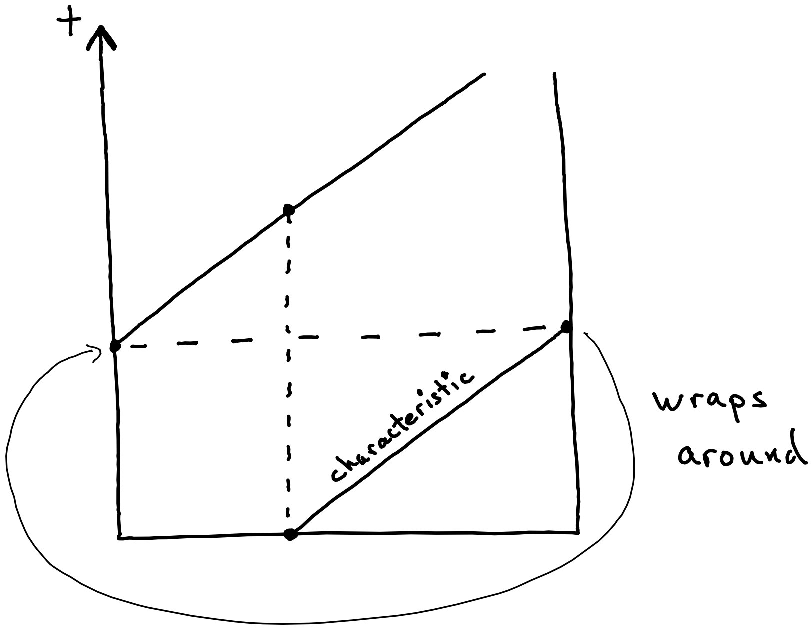
$$u^{(k)}(t,0) = u^{(k)}(t,1), \quad k = 1, 2, \ldots \qquad \left(u^{(k)} = \frac{\partial^k}{\partial t^k} u \right)$$

Usually, we only write out the $u(t,0) = u(t,1)$ part.

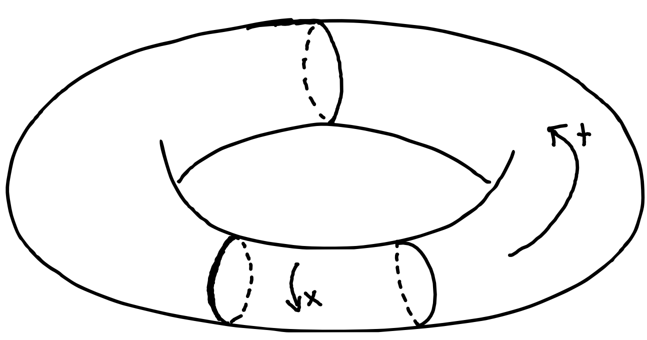
This essentially means that there is no real boundary. The point $x=1$ is <u>the same</u> as the point $x=0$, and the intervals $[1,2], [2,3], \ldots$ are the same as $[0,1]$.



For $u_t + u_x = 0$ the situation will look like

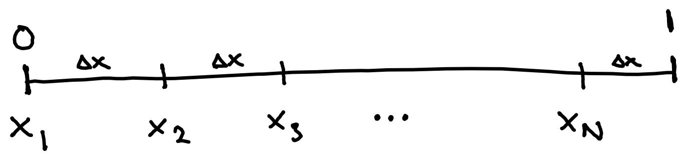


Note how periodicity in space => periodicity in time. We are on a torus:



The way to discretize this properly is with

$$\begin{cases} x_k = (k-1)\Delta x \\ \Delta x = \dfrac{1}{N} \end{cases}.$$



I.e. we have a computational point at $x=0$, because we don't know the value there, but no point at $x=1$ since we do know that the solution there is the same as at $x=0$.

When we need $U_{N+1}(t) \approx u(t, x_{N+1}) = u(t,1)$

we replace it by $U_1(t) \approx u(t, x_1) = u(t,0)$.

Similarly, $U_0(t)$ is replaced by $U_N(t)$.

On matrix-vector form with

$$U^n = [U_1^n, U_2^n, \dots, U_N^n]^T,$$

Lax-Wendroff becomes

$U_0^n \to U_N^n$

$$U^{n+1} = \begin{bmatrix} 1-a^2\mu^2 & \frac{a\mu}{2}(a\mu-1) & & & \frac{a\mu}{2}(a\mu+1) \\ \frac{a\mu}{2}(a\mu+1) & 1-a^2\mu^2 & \frac{a\mu}{2}(a\mu-1) & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ \frac{a\mu}{2}(a\mu-1) & & & \frac{a\mu}{2}(a\mu+1) & 1-a^2\mu^2 \end{bmatrix} U^n,$$

$U_1^n \leftarrow U_{N+1}^n$

i.e. also the rows of the matrix wrap around!

Periodic BC always lead to such <u>circulant matrices</u>. (Note: not "circular" matrices.)

Stability with periodic BC

We can write the scheme as

$$U^{n+1} = A(a\mu)\, U^n$$

and we have stability if $\|A(a\mu)\| \leq 1$.

With periodic BC, $A(a\mu)$ is normal (Not hard, but the proof does not fit in the margin)

and therefore $$\|A(a\mu)\|^2 = \max_k |\lambda_k[A(a\mu)]|$$

where $\lambda_k[A(a\mu)]$ is the k:th eigenvalue of $A(a\mu)$.

Before we tackle the general case and compute eigenvalues, let's consider $a\mu = 1$. Then

$$A(1) = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \quad \left(\text{so } U_j^{n+1} = U_{j-1}^n\right)$$

This is a <u>permutation matrix</u>; $A(1)U$ has the same components as U but in a different order.

Thus $\|A(1)U\|_2 = \|U\|_2$,

so
$$\|A(1)\|_2 = \sup_{U \neq 0} \frac{\|A(1)U\|}{\|U\|} = 1.$$

$\therefore$ Lax-Wendroff is stable at $a\mu = 1$.

With some more work, we could show that $\lambda_k[A(1)] = e^{\frac{2\pi i k}{N}}$, $k = 1, \dots, N$,

but in this simple case we don't need them.

Note that this means that

$$\|U^{n+1}\|_2 = \|U^n\|_2$$

i.e. the norm of the approximation is <u>conserved</u>.

This is true for the exact solution too, since

$$\frac{d}{dt}\|u(t,\cdot)\|_2^2 = \frac{d}{dt}\langle u(t,\cdot), u(t,\cdot)\rangle$$

$$= \langle u_t, u\rangle + \langle u, u_t\rangle$$

$$= 2\langle u, u_t\rangle$$

$$\xrightarrow{u_t + a u_x = 0} \; = -2a\langle u, u_x\rangle .$$

Recall that $\langle u,v\rangle = \int u(x)v(x)\,dx$

and integration by parts

$$\langle u, v_x\rangle = -\langle u_x, v\rangle + \underbrace{u(1)v(1) - u(0)v(0)}_{=0 \text{ if periodic BC}}$$

$$\Rightarrow \quad \langle u, u_x\rangle \underset{\text{int. by parts}}{=} -\langle u_x, u\rangle \underset{\text{just interchange terms in the integral}}{=} -\langle u, u_x\rangle .$$

If $z=-z$ then $z=0$, so $\boxed{\langle u, u_x\rangle = 0}$

$\therefore \frac{d}{dt}\|u(t,\cdot)\|^2 = 0$ and $\|u(t,\cdot)\| = \text{constant}$.

Conservation law!

The general case

Def. A circulant matrix $C \in \mathbb{R}^{N \times N}$ has the form

$$C = \begin{bmatrix} \kappa_0 & \kappa_1 & \cdots & \kappa_{N-1} \\ \kappa_{N-1} & \kappa_0 & \cdots & \kappa_{N-2} \\ \vdots & & & \\ \kappa_1 & \kappa_2 & \cdots & \kappa_0 \end{bmatrix}.$$

Theorem The eigenvalues of C are

$$\lambda_k[C] = \sum_{j=0}^{N-1} \kappa_j \, e^{\frac{2jk\pi i}{N}}, \quad k = 1, \dots, N.$$

We use this on $U^{n+1} = A(a\mu)U^n$ as follows:

- Identify the non-zero κ_j, typically only 3.
- Simplify $\lambda_k[A(a\mu)]$ using trigonometry.
- Identify condition on $a\mu$ such that $|\lambda_k[A(a\mu)]| \leq 1$.
- For those $a\mu$, $\|A(a\mu)\| \leq 1$ and we have stability.

Ex. Lax-Wendroff too complicated for blackboard, let's do Lax-Friedrichs:

$$U_\ell^{n+1} = \frac{U_{\ell-1}^n + U_{\ell+1}^n}{2} + \frac{a\mu}{2}\left(U_{\ell-1}^n - U_{\ell+1}^n\right)$$

i.e.

$$U^{n+1} = \begin{bmatrix} 0 & 1/2 - \frac{a\mu}{2} & & & 1/2 + \frac{a\mu}{2} \\ 1/2 + \frac{a\mu}{2} & 0 & 1/2 - \frac{a\mu}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ 1/2 - \frac{a\mu}{2} & & & 1/2 + \frac{a\mu}{2} & 0 \end{bmatrix} U^n$$

$$\Rightarrow \begin{cases} \kappa_1 = \frac{1}{2} - \frac{a\mu}{2}, \\ \kappa_{N-1} = \frac{1}{2} + \frac{a\mu}{2}, \\ \kappa_j = 0 \text{ otherwise.} \end{cases}$$

$$\Rightarrow \lambda_k\left[A(a\mu)\right] = \frac{1}{2}\left((1-a\mu)\, e^{\frac{2k\pi i}{N}} + (1+a\mu)\, e^{\frac{2k\pi i(N-1)}{N}}\right)$$

Note that $e^{2k\pi i \frac{N-1}{N}} = e^{2k\pi i} \cdot e^{-\frac{2k\pi i}{N}} = e^{-\frac{2k\pi i}{N}}$.

$\frac{N-1}{N} = 1 - \frac{1}{N}$

$\longrightarrow$

$$\Rightarrow$$

$$\lambda_k[A(a\mu)] = \frac{1}{2}\left(\begin{array}{l}(1-a\mu)\left(\cos\left(\frac{2k\pi}{N}\right)+i\sin\left(\frac{2k\pi}{N}\right)\right)\\ +(1+a\mu)\left(\cos\left(\frac{2k\pi}{N}\right)-i\sin\left(\frac{2k\pi}{N}\right)\right)\end{array}\right)$$

$$= \cos\left(\frac{2k\pi}{N}\right) - a\mu\, i\sin\left(\frac{2k\pi}{N}\right).$$

So,

$$\left|\lambda_k[A(a\mu)]\right|^2 = \cos^2\left(\frac{2k\pi}{N}\right) + (a\mu)^2\sin^2\left(\frac{2k\pi}{N}\right)$$

$$= 1 + \left((a\mu)^2 - 1\right)\sin^2\left(\frac{2k\pi}{N}\right).$$

Since $\sin^2\left(\frac{2k\pi}{N}\right) \geq 0$, the eigenvalues are bounded by 1 iff $(a\mu)^2 - 1 \leq 0$.

$\therefore$ the method is stable iff $|a\mu| \leq 1$.

Exercise: follow this line of reasoning to show that the central difference scheme is <u>never</u> stable, for any $a\mu \neq 0$.

The wave equation

$$u_{tt} = c^2 u_{xx}, \quad c \in \mathbb{R},$$

IC: $u(0,x) = g(x)$,
$u_t(0,x) = h(x)$

BC: $u(t,0) = \phi_0(t)$
$u(t,1) = \phi_1(t)$ (for example)

Models e.g. vibrating strings.

Can be studied in terms of advection equations:

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

(operator calculus)

so $u_{tt} = c^2 u_{xx}$ if either $u_t = cu_x$ or $u_t = -cu_x$.

$\Rightarrow$ General solution

$$u(t,x) = g_1(x+ct) + g_2(x-ct),$$

where $u_t(t,x) = cg_1'(x+ct) - cg_2'(x-ct)$ and the initial conditions imply that $\begin{cases} g_1(x) + g_2(x) = g(x) \\ cg_1'(x) - cg_2'(x) = h(x) \end{cases}$.

Solve for g_1 and g_2!

Main point: waves traveling both to the left and to the right.

Discretization:

Can rewrite as 1st-order system

$$z_t + A z_x = 0$$

for $A = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$ and $z = \begin{bmatrix} u \\ v \end{bmatrix}$,

where both u and v solve the wave eq.

Then apply (vector-valued) advection solver.

However, super-confusing to think about initial conditions, etc., and better, direct, methods are possible.

Direct semi-discretization of $u_{tt} = c^2 u_{xx}$

$U_j(t) \approx u(t, x_j)$

$$\rightarrow \ddot{U}_j(t) = \underbrace{c^2 \frac{U_{j-1}(t) - 2U_j(t) + U_{j+1}(t)}{\Delta x^2}}_{\text{order 2 in space, could also use other}} =: f_j(t, U)$$

general discr. $\frac{c^2}{\Delta x^2} \sum_{k=-\ell}^{m} a_k U_{j+k}(t)$

Full discretization

Write as ODE system

$$\begin{cases} \dot{U}_j(t) = V_j(t), & U_j(0) = u(0, x_j), \\ \dot{V}_j(t) = f_j(t, U), & V_j(0) = \dot{u}(0, x_j). \end{cases}$$

Idea: f_j discr. of u_{xx} makes 2nd eq. stiff, should use implicit method.

1st eq. has no u_{xx}, can use explicit method.

This is of course not a proper mathematical argument, but we can use it to create a method and then analyze it properly.

Will not do this here, but it is order 2 in both time and space. Stable if $|c|\frac{\Delta t}{\Delta x} \leq 1$.

Explicit + implicit Euler

$$\rightarrow \begin{cases} U^{n+1} = U^{n} + \Delta t V^{n} \\ V^{n+1} = V^{n} + \Delta t f(t_{n+1}, U^{n+1}). \end{cases}$$

Because of the split system, the method combination is actually explicit!

$$\begin{aligned} U^{n+2} &= U^{n+1} + \Delta t V^{n+1} \\ &= U^{n+1} + \Delta t \left(V^{n} + \Delta t f(t_{n+1}, U^{n+1}) \right) \\ &= U^{n+1} + \Delta t \left(\frac{U^{n+1} - U^{n}}{\Delta t} + \Delta t f(t_{n+1}, U^{n+1}) \right) \end{aligned}$$

The full method is a leapfrog-type scheme often referred to as a <u>Störmer method</u> in this context:

$$U_j^{n+2} - 2U_j^{n+1} + U_j^{n} = \Delta t^2 c^2 \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta x^2}.$$

Nonlinear hyperbolic problems

Are hard problems. There is no general approach.

We will only look at some properties/problems with $u_t + (f(u))_x = 0$,

in particular the inviscid Burgers' equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + u u_x = 0 .$$

Note 1: the apostrophe position; Burgers is a surname.

Note 2: "inviscid" $\Rightarrow$ zero viscosity
$\Rightarrow$ fluid flow with very "thin" fluid.

Viscous Burgers': $u_t + u u_x = \varepsilon u_{xx}$.

Higher ε $\Rightarrow$ higher viscosity $\Rightarrow$ thicker fluid.

Inviscid Burgers'

More specifically, consider

$$u_t + u u_x = 0 \qquad \text{on } (t,x) \in (0,\infty) \times (-\infty,\infty)$$

(i.e. no boundary) and $u(0,x) = g(x)$ with $\|g\|_2 < \infty$.

If we don't think about the details too much, we can write down the solution <u>implicitly</u> as

$$u(t,x) = g(x - ut),$$

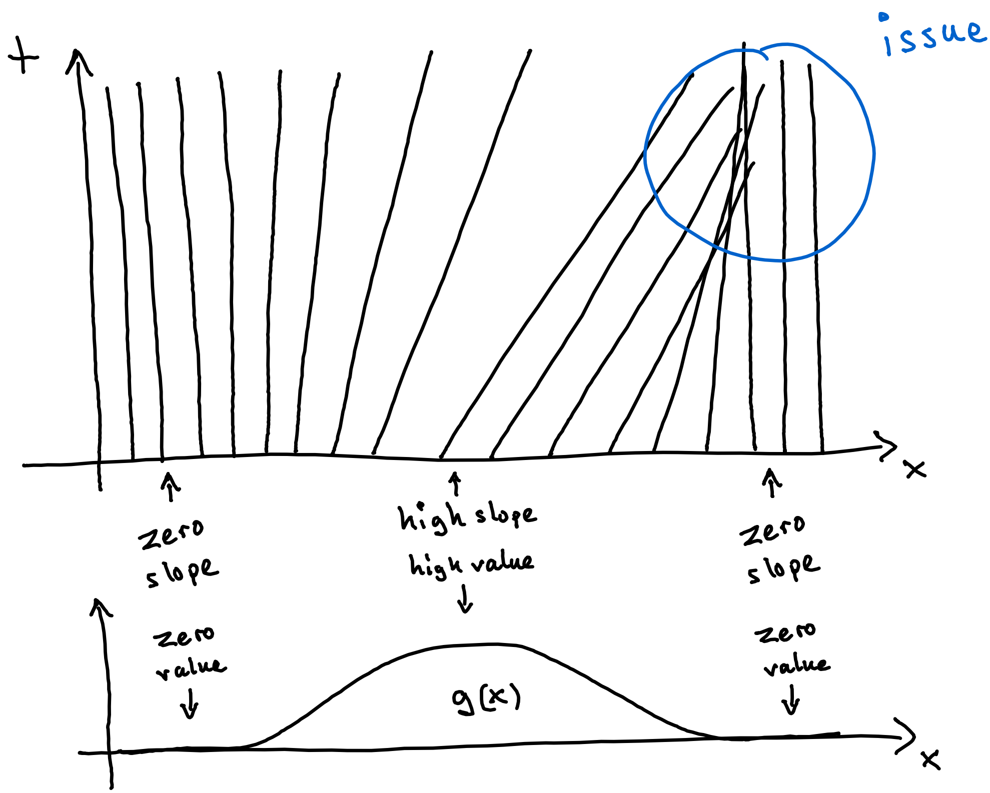
similarly to the advection eq. sol. $g(x-at)$, since then "$u_t = -ug'$ and $u_x = g'$".

The details on how to do this formally are included later in these notes, but the main point is that:

The characteristics (where $u(t,x)$ is constant) are straight lines with slope u:

$$x - ut = C$$

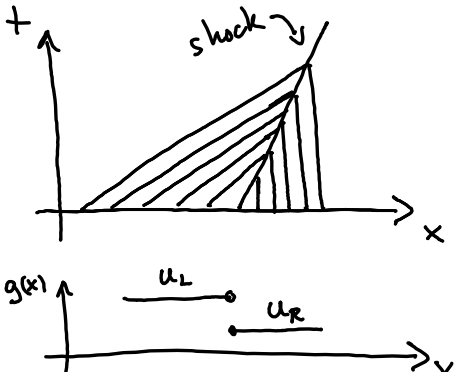
For $u_t + au_x = 0$, the characteristics all have the same slope a, but now they depend on $g(x)$ (the initial condition)

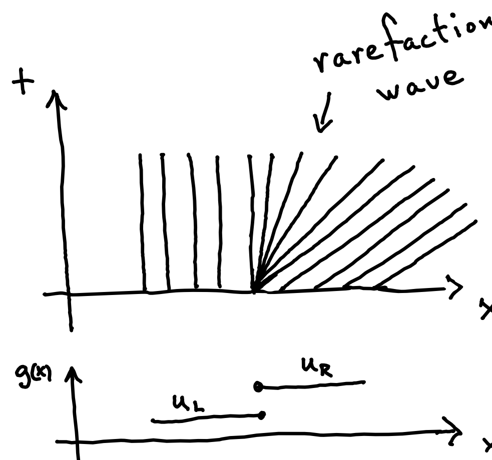


Where the characteristics collide, the solution breaks down, since it cannot take two different values simultaneously.

We call this feature a <u>shock</u>, and can define weaker solution concepts where such discontinuous solutions make sense.

Then two typical situations look like





Shock described by

$$x = \frac{u_L + u_R}{2} t$$

For $u_L t < x < u_R t$,

$$u(t,x) = x/t \, .$$

In Project 3, you will look at viscous Burgers, $u_t + uu_x = \varepsilon u_{xx}$, where the diffusion term smooths out the discontinuities. Then our usual solution concept still works, but as $\varepsilon \to 0$ we get steeper and steeper gradients.

[MATLAB demo of viscous Burgers]

Solving true hyperbolic problems requires highly specialized methods, which we cannot discuss here. See other numerical analysis courses!

We called $u_t + (f(u))_x = 0$ conservation laws, so as the final topic of the course, let's see some conservation.

First, $u_t + uu_x = 0$. We have

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = \frac{d}{dt}\langle u(t,\cdot), u(t,\cdot)\rangle$$

omit $(t,\cdot)$ from now

$$= 2\langle u, u_t\rangle$$

$u_t + uu_x = 0$

$$= -2\langle u, uu_x\rangle .$$

Now

$$\langle u, uu_x\rangle = \int u(t,x)\cdot u(t,x)\,u_x(t,x)\,dx$$

omit (t,x) from now

$$= \int u^2 u_x\,dx$$

int. by parts, assuming suitable BC

$$= -\int (2uu_x)\,u\,dx$$

$$= -2\int u\,uu_x\,dx$$

$$= -2\langle u, uu_x\rangle$$

$$\Rightarrow \langle u, uu_x\rangle = 0$$

$$\Rightarrow \frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = 0 \quad \Rightarrow \|u(t,\cdot)\|_{L^2} = \text{constant}.$$

Norm of solution ("mass") is conserved!

Same approach works for $u_t + u^p u_x = 0$

with integer p, since

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = 2\langle u, u_t\rangle = -2\langle u, u^p u_x\rangle$$

and

$$\langle u, u^p u_x\rangle = \langle u^{p+1}, u_x\rangle$$

int. by parts

$$= -\langle (p+1)u^p u_x, u\rangle$$

$$= -(p+1)\langle u, u^p u_x\rangle$$

$$\Rightarrow \langle u, u^p u_x\rangle = 0, \text{ unless } p = -2.$$

But for $p=-2$ we have

$$\langle u, u^{-2} u_x\rangle = \int \frac{u_x}{u}\,dx = \int \frac{d}{dx}(\log u)\,dx$$

if on $[0,1]$

$$= \log(u(1)) - \log(u(0))$$

with e.g. per. BC

$$= 0.$$

Finally, same approach works for

$$u_t + (f(u))_x = 0 \quad \text{too!}$$

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = 2\langle u, u_t\rangle = -2\langle u, (f(u))_x\rangle$$

assume periodic BC

and $\langle u, (f(u))_x\rangle = -\langle u_x, f(u)\rangle$

assume eq. given on [0,1]

$$= -\int_0^1 f(u)\, u_x \, dx$$

change variables from x to u(x)

$$= -\int_{u(0)}^{u(1)} f(u(x))\, du(x)$$

$$= 0, \quad \text{since } u(0) = u(1)$$

$$\therefore \|u(t,\cdot)\|_{L^2} = \text{constant} \quad \forall t \geq 0.$$

That's all for this course!

Extra: characteristics for $u_t + uu_x = 0$, properly.

Define $\xi(t,x)$ by $x = \xi + g(\xi)t$

for those x, t where a unique solution exists. This puts a limit on t. At e.g. a shock there is no longer a solution.

Then set $u(t,x) = g(\xi(t,x))$.

We get

$$u(t,x) = g(x - g(\xi(t,x))t)$$
$$= g(x - u(t,x)t)$$

and

$$u_x(t,x) = g'(\xi(t,x))\,\xi_x(t,x),$$

where

$$\xi_x(t,x) = 1 - g'(\xi(t,x))\,\xi_x(t,x)\,t$$

$$\Rightarrow \xi_x(t,x) = \frac{1}{1 + g'(\xi(t,x))t}.$$

$\longrightarrow$

Further, $u_t(t,x) = g'(\xi(t,x))\,\xi_t(t,x)$,

where $\xi_t(t,x) = -g'(\xi(t,x))\,\xi_t(t,x)\,t - g(\xi(t,x))$

$$\Rightarrow \quad \xi_t(t,x) = \frac{-g(\xi(t,x))}{1 + t g'(\xi(t,x))} .$$

Thus,

$$u_t + u u_x = -\frac{g'(\xi(t,x))\,g(\xi(t,x))}{1 + t g'(\xi(t,x))}$$

$$+ g(\xi(t,x)) \frac{g'(\xi(t,x))}{1 + t g'(\xi(t,x))}$$

$$= 0 .$$

The equation $x = \xi(t,x) + g(\xi(t,x))t$ has a solution $\begin{cases} \xi(t,x) = C \\ t = \frac{x-C}{g(C)} \end{cases}$ for any constant C.

These straight lines $t = \frac{1}{g(C)} x - \frac{C}{g(C)}$ are the characteristics, along which

$$u(t,x) = g(\xi(t,x)) = g(C) \text{ is constant.}$$