**Numerical Methods for Differential Equations FMNN10/NUMN12**
**Final exam 2020-08-18**

Exam duration 08:00 – 13:00. A minimum of 15 points out of 30 are required to pass. Higher grades might require an additional brief oral exam.

You may use a pocket calculator and any written notes and books during the exam. Accessing the internet in any form is *not* allowed and neither is getting help from anyone else. The only exception is to use the course webpage for uploading your solutions.

1. **(5p)** A family of implicit Runge–Kutta methods for the initial value problem $\dot{y} = f(t, y)$ is given by the following Butcher tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1-\theta & \theta \\ \hline & 1-\theta & \theta \end{array},$$

   where $\theta$ is a real-valued parameter.

   (a) Introduce suitable notation and write down all formulas for computing stage derivatives and updating the approximation. (1p)

   (b) For a general $\theta$, determine the method's stability function $R(h\lambda)$. (2p)

   (c) Using the maximum principle, determine which $\theta \in [0, 1]$ yield A-stable methods. (2p)

2. **(5p)** The following linear multistep scheme for $\dot{y} = f(t, y)$ is an explicit version of BDF2:

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = h\beta_1 f(y_n) + h\beta_0 f(y_{n-1})$$

   (a) Determine the values of the parameters $\beta_0$ and $\beta_1$ so that the method is of maximal order. What is the maximum order? (3p)

   (b) Determine if the method is zero-stable. (1p)

   (c) Determine if the method is A-stable. (1p)

3. **(5p)** Consider the following nonlinear two-point boundary value problem, where $\alpha$ and $\beta \neq 0$ are given real values and $f$ is a given function:

$$
\begin{aligned}
&y''(x) - (y(x)^2)' + y(x) = f(x), \\
&y(0) = \alpha, \qquad y(1) + \beta y'(1) = 0.
\end{aligned}
$$

   (a) Let $u_i \approx y(x_i)$ with $\Delta x = x_{i+1} - x_i$ for all $i$. Student A discretizes the term $(y(x)^2)'$ at $x_i$ as $u_i \frac{u_{i+1} - u_{i-1}}{\Delta x}$ by using the chain rule. Student B claims that you should not do that and uses instead $\frac{u_{i+1}^2 - u_{i-1}^2}{2\Delta x}$. Use Taylor expansion to determine which, if any, discretization is of second order. (2p)

   (b) The final discrete system will be of the form $F(u) = 0$ with a nonlinear $F$, and will have to be solved by Newton's method. Specify $F$ and its Jacobian $\frac{\partial F}{\partial u}$. Include all necessary details and be careful with the boundary conditions. Use a suitable second-order discretization. (3p)

4. **(5p)** Let $K$ be a given real constant and consider the differential operator $\mathcal{L}$ given by

$$\mathcal{L}u = u'' + Ku'$$

   for functions $u$ satisfying $u(0) = u(1) = 0$.

   (a) Using the inner product

$$\langle u, v \rangle_2 = \int_0^1 uv,$$

   find an expression for the adjoint $\mathcal{L}^*$. (2p)

   (b) Give an upper bound for the logarithmic norm $\mu_2[\mathcal{L}]$. (2p)
   *Hint:* You can use Sobolev's lemma: $\|u'\|_2 \geq \pi \|u\|_2$.

   (c) For which $K$ is there a unique solution to the associated boundary value problem $\mathcal{L}u = f$? (1p)
   *Hint:* The Cauchy-Schwarz inequality $|\langle u, v \rangle| \leq \|u\| \|v\|$ could prove useful here.

5. **(5p)** Consider the linear diffusion equation $u_t = u_{xx}$ on $(t, x) \in [0, 1] \times [0, 1]$ with homogeneous Dirichlet boundary conditions and initial condition $u(0, x) = g(x)$. Let us discretize this problem by combining a standard 2nd-order method-of-lines discretization in space with the implicit Euler method in time.

(a) Introduce a suitable notation and write down this method on matrix-vector form. (2p)

(b) Using the well-known eigenvalues of the $N \times N$ Toeplitz matrix,

$$\lambda_k[T_{\Delta x}] = -4 \cdot (N+1)^2 \cdot \sin^2 \frac{k\pi}{2(N+1)} \qquad k = 1, \dots N.$$

show that the method is *unconditionally stable* for the diffusion equation, i.e., there is no CFL condition and the method is stable for every $\Delta t > 0$. (1p)

(c) Compute the logarithmic norm of $T_{\Delta x}$ and use it to show that the discretized problem has a unique solution. (2p)

6. **(5p)** Consider the advection equation $u_t = u_x$ on $(t, x) \in [0, 1] \times [0, 1]$ with periodic boundary conditions. Combining a standard 2nd-order discretization in space with the trapezoidal rule in time gives the full discretization $U^{j+1} = AU^j$, where $U^j \in \mathbb{R}^N$ denotes the approximation on the spatial grid at time step $j$,

$$A = (I - \Delta t S_{\Delta x})^{-1}(I + \Delta t S_{\Delta x}),$$

and $S_{\Delta x}$ is the standard $N \times N$ circulant matrix that discretizes $u_x$. It is well known that the eigenvalues of this matrix are given by

$$\lambda_k[S_{\Delta x}] = \frac{i}{\Delta x} \sin\left(\frac{2k\pi}{N}\right),$$

for $k = 0, \dots N-1$. We also know that $A$ is a normal matrix.

(a) Show that the norm of the exact solution is preserved, i.e. that $\|u(t, \cdot)\|_2$ is constant for all $t$. (2p)

(b) Show that for this method also the norm of the discrete approximation is preserved, i.e. that $\|U^j\|_2$ is constant for all $j$. (3p)

Good luck!
T.S.