Numerical Methods for Differential Equations FMNN10/NUMN32
Tony Stillfjord, Gustaf Söderlind

Review questions and study problems, week 4

1. True or false (justify your answer): *Consider the Sturm-Liouville problem*
$$\frac{d}{dx}\left((1-0.8\sin^2 x)\frac{dy}{dx}\right) - \lambda y = 0, \quad y(0) = y(\pi) = 0.$$
The following discretization
$$\frac{p_{n-1}y_{n-1} - 2p_n y_n + p_{n+1}y_{n+1}}{\Delta x^2} = \lambda_{\Delta x} y_n, \quad n = 1:N$$
$$y_0 = y_{N+1} = 0$$
where $p_n = 1 - 0.8\sin^2 \dfrac{n\pi}{N+1}$ *is of order 2.*

2. Give a 4×4 example of
   - A tridiagonal symmetric Toeplitz matrix
   - A skew-symmetric Toeplitz matrix
   - A lower triangular Toeplitz matrix

3. Solve the linear difference equation
$$\begin{aligned} 6u_{j+2} - 5u_{j+1} + u_j &= 0 \qquad (j = 0 : N-1) \\ u_0 = 1,\ u_{N+1} &= 0. \end{aligned}$$

4. True or false (justify your answer): *If $\lambda[T]$ are the eigenvalues of T and $\lambda[S]$ the eigenvalues of S, then $\lambda[T] + \lambda[S]$ are the eigenvalues of $T + S$.*

5. Let λ be an eigenvalue of the invertible matrix A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

6. True or false: *If $Au = \lambda u$, then e^{tA} has the eigenvalues $\mathrm{e}^{t\lambda}$.*

7. In class we determined the eigenvalues of
$$T_{\Delta x} = \frac{1}{\Delta x^2}\mathrm{tridiag}(1 \quad -2 \quad 1)$$

(a) Sketch the location of the eigenvalues in the complex plane.

(b) Sketch the location of the eigenvalues of $T_{\Delta x}^{-1}$. (Make sure that your sketches have some "reasonable scaling," e.g. by indicating where the eigenvalues are in relation to the unit circle.)

(c) If $\Delta x \to 0$, where will the eigenvalues of $T_{\Delta x}^{-1}$ "cluster"?

8. Let $A \in \mathbb{R}^{N\times N}$ be a normal matrix with the eigenvalues λ_k, $k = 1,\ldots,N$. What are $\|A\|$ and $\|A^{-1}\|$?

9. Consider the discretization $T_{\Delta x}u = f$ of the Poisson equation $y'' = f$, and a discretization $T_{\Delta x}v = f + e$ of the perturbed problem $y'' = f + \epsilon$, where the function ϵ represents measurement errors. Using the previous question and what you know about $\lambda_k[T_{\Delta x}]$, how much can the solutions differ? That is, provide a bound on $\|u - v\|$.

10. Consider the (vector-valued) initial value problem $\frac{\mathrm{d}}{\mathrm{d}t}u = T_{\Delta x}u$ with initial condition $u(0) = v$. Its solution is given by $u(t) = \mathrm{e}^{tT_{\Delta x}}v$.

    (a) Give an upper bound for $\|\mathrm{e}^{tT_{\Delta x}}\|_2$ for $t \geq 0$.

    (b) Sketch the location of the eigenvalues of $\mathrm{e}^{tT_{\Delta x}}$ in the complex plane. (You may consider time t to be a fixed parameter.)

    (c) Where do the eigenvalues of $\mathrm{e}^{tT_{\Delta x}}$ "cluster" as $\Delta x \to 0$ for t fixed?

    (d) Where do they go as $t \to \infty$ for Δx fixed?

    (e) Can you give or suggest an upper bound for the *inverse* $\|\mathrm{e}^{-tT_{\Delta x}}\|_2$ (where $t > 0$)? How does that inverse behave as $\Delta x \to 0$?

    (f) Suppose we solve this initial value problem using the explicit Euler method. What condition on the time step Δt is a minimum requirement for stability?

    (g) Same question for the implicit Euler method.

    (h) Which method is suitable when $\Delta x \to 0$?

    This constitutes a prequel for next week's material on *parabolic* problems; $\frac{\mathrm{d}}{\mathrm{d}t}u = T_{\Delta x}u$ is a spatial discretization of the parabolic partial differential equation $\frac{\mathrm{d}}{\mathrm{d}t}u = \frac{\mathrm{d}^2}{\mathrm{d}x^2}u$ known as the *heat equation.*

11. In class we determined the eigenvalues of symmetric Toeplitz matrix $T = \mathrm{tridiag}(1 \quad 0 \quad 1)$ analytically.

    (Difficult) Determine, with a similar methodology, the eigenvalues of the skew symmetric Toeplitz matrix $S = \mathrm{tridiag}(-1 \quad 0 \quad 1)$.

12. Let y' be approximated by the second order, symmetric difference quotient $S_{\Delta x} = S/(2\Delta x)$. What are the

(a) eigenvalues of $S_{\Delta x}$

(b) Euclidean norm of $S_{\Delta x}$

(c) Euclidean norm of $\mathrm{e}^{tS_{\Delta x}}$, $t \in \mathbb{R}$?

Hint: Every skew-symmetric matrix is *normal*, and the exponential of a skew-symmetric matrix is skew-symmetric.

13. (Difficult) Consider the 2pBVP $-u'' - u' - u = f(x)$ with $u(0) = u(1) = 0$.

    (a) Find a value $\mu > 0$ such that

    $$\Big\langle \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} - 1 \Big) u, u \Big\rangle \geq \mu \|u'\|_2^2.$$

    Does the problem have a unique solution for every right-hand side f? (Use Lax Milgram lemma as presented in the lecture notes.)

    (b) Introduce a suitable grid and discretize the equation above. Use the same techniques as in the previous problem to show that your *discretization* has a unique solution for every right-hand side f.

    (c) Let $u_{\Delta x}$ denote the solution vector on the grid. Give a bound for $\|u_{\Delta x}\|_{\Delta x}$.

14. Consider the 2pBVP $-u'' - \omega^2 y = g(x)$ with homogeneous boundary data $u(0) = u(1) = 0$.

    (a) For what values of the parameter ω can you guarantee that there is a unique solution?

    (b) Let $\omega = \pi$. What happens with the analytical solution? Why?