Numerical Methods for Differential Equations, FMNN10 / NUMN20

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Solution: Review questions and study problems, week 2

1. True or false (justify your answer): *All explicit Runge-Kutta methods of order 3 are convergent.*

   Solution: Runge–Kutta methods are consistent one-step methods, i.e. y_{n+1} only depending on the previous value y_n and f. As all consistent one-step methods are convergent it follows that all Runge–Kutta methods are convergent.

2. Construct the Butcher tableau for the 3-stage Heun method,

$$\begin{aligned} Y_1' &= f(t_n, y_n) \\ Y_2' &= f(t_n + h/3, y_n + hY_1'/3) \\ Y_3' &= f(t_n + 2h/3, y_n + 2hY_2'/3) \\ y_{n+1} &= y_n + h\,(Y_1' + 3Y_3')/4 \end{aligned}$$

   Solution:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

3. Write the equations for the Runge-Kutta method (RK4) with the Butcher tableau

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

   Is this an explicit or implicit method?

   Solution:

$$\begin{aligned} Y_1' &= f(t_n, y_n) \\ Y_2' &= f(t_n + h/2, y_n + hY_1'/2) \\ Y_3' &= f(t_n + h/2, y_n + hY_2'/2) \\ Y_4' &= f(t_n + h, y_n + hY_3') \\ y_{n+1} &= y_n + h\,(Y_1' + 2Y_2' + 2Y_3' + Y_4')/6 \end{aligned}$$

   This is the classical explicit RK4.

4. Suppose you apply the RK4 method to the linear test equation $y' = \lambda y$. You then get $y_{n+1} = P(h\lambda)y_n$, where the polynomial $P(h\lambda)$ is called the *stability function* of the method. Derive $P(h\lambda)$ for the RK4 method by hand. If you look at the polynomial, you probably recognize it. What does the polynomial approximate? Can you explain this?

   Solution: We apply the Runge–Kutta 4 method to $\dot{y} = \lambda y$ and get

$$
\begin{aligned}
Y_1' &= \lambda y_n \\
Y_2' &= \lambda(y_n + hY_1'/2) = \lambda\Big(y_n + h\frac{\lambda}{2}y_n\Big) = \Big(\lambda + \frac{h\lambda^2}{2}\Big)y_n \\
Y_3' &= \lambda(y_n + hY_2'/2) = \lambda\Big(y_n + \frac{h}{2}\Big(\lambda + \frac{h\lambda^2}{2}\Big)y_n\Big) \\
&= \Big(\lambda + \frac{h\lambda^2}{2} + \frac{h^2\lambda^3}{4}\Big)y_n \\
Y_4' &= \lambda(y_n + hY_3') = \lambda\Big(y_n + h\Big(\lambda + \frac{h\lambda^2}{2} + \frac{h^2\lambda^3}{4}\Big)y_n\Big) \\
&= \Big(\lambda + h\lambda^2 + \frac{h^2\lambda^3}{2} + \frac{h^3\lambda^4}{4}\Big)y_n \\
y_{n+1} &= y_n + \frac{h}{6}(Y_1' + 2Y_2' + 2Y_3' + Y_4') \\
&= y_n + \frac{h}{6}\Big(\lambda + 2\Big(\lambda + \frac{h\lambda^2}{2}\Big) + 2\Big(\lambda + \frac{h\lambda^2}{2} + \frac{h^2\lambda^3}{4}\Big) \\
&\qquad + \Big(\lambda + h\lambda^2 + \frac{h^2\lambda^3}{2} + \frac{h^3\lambda^4}{4}\Big)\Big)y_n \\
&= y_n + \frac{h}{6}\Big(6\lambda + 3h\lambda^2 + h^2\lambda^3 + \frac{h^3\lambda^4}{4}\Big)y_n \\
&= \Big(1 + h\lambda + \frac{(h\lambda)^2}{2!} + \frac{(h\lambda)^3}{3!} + \frac{(h\lambda)^4}{4!}\Big)y_n.
\end{aligned}
$$

   We observe that the stability function is the the forth order Taylor expansion of the function $\exp(h\lambda)$. This suits the theory as $\exp(h\lambda)y_0$ is the solution to the linear test equation $\dot{y} = \lambda y$, $y(0) = y_0$ at the time $t = h$ and the method is a forth order approximation of it.

5. Find the stability function of the Runge-Kutta method given by

$$
\begin{array}{c|cc}
1/3 & 1/3 & 0 \\
2/3 & 1/3 & 1/3 \\
\hline
 & 1/2 & 1/2
\end{array}
$$

   Is the method A-stable?

Solution: The method is given by

$$
\begin{aligned}
Y_1' &= f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{3}Y_1'\Big)\\
Y_2' &= f\Big(t_n + \frac{2h}{3}, y_n + \frac{h}{3}(Y_1' + Y_2')\Big)\\
y_{n+1} &= y_n + \frac{h}{2}(Y_1' + Y_2')
\end{aligned}
$$

Thus, for the linear test equation, we get that

$$
\begin{aligned}
&Y_1' = \lambda\Big(y_n + \frac{h}{3}Y_1'\Big)\\
\Leftrightarrow \quad &\Big(1 - \frac{h\lambda}{3}\Big)Y_1' = \lambda y_n\\
\Leftrightarrow \quad &Y_1' = \Big(1 - \frac{h\lambda}{3}\Big)^{-1}\lambda y_n
\end{aligned}
$$

and

$$
\begin{aligned}
Y_2' &= \lambda\Big(y_n + \frac{h}{3}(Y_1' + Y_2')\Big) = \lambda\Big(y_n + \frac{h}{3}\Big(\Big(1 - \frac{h\lambda}{3}\Big)^{-1}\lambda y_n + Y_2'\Big)\Big)\\
\Leftrightarrow \quad &\Big(1 - \frac{h\lambda}{3}\Big)Y_2' = \lambda y_n + \frac{h\lambda}{3}\Big(1 - \frac{h\lambda}{3}\Big)^{-1}\lambda y_n\\
\Leftrightarrow \quad &Y_2' = \Big(1 - \frac{h\lambda}{3}\Big)^{-1}\lambda y_n + \frac{h\lambda}{3}\Big(1 - \frac{h\lambda}{3}\Big)^{-2}\lambda y_n\\
&= \Big(1 - \frac{h\lambda}{3}\Big)^{-2}\Big(1 - \frac{h\lambda}{3} + \frac{h\lambda}{3}\Big)\lambda y_n = \Big(1 - \frac{h\lambda}{3}\Big)^{-2}\lambda y_n.
\end{aligned}
$$

Altogether, we then obtain

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{2}\Big(\Big(1 - \frac{h\lambda}{3}\Big)^{-1}\lambda y_n + \Big(1 - \frac{h\lambda}{3}\Big)^{-2}\lambda y_n\Big)\\
&= y_n + \frac{h\lambda}{2}\Big(1 - \frac{h\lambda}{3}\Big)^{-2}\Big(\Big(1 - \frac{h\lambda}{3}\Big)y_n + y_n\Big)\\
&= \Big(1 - \frac{h\lambda}{3}\Big)^{-2}\Big(\Big(1 - \frac{h\lambda}{3}\Big)^2 + h\lambda - \frac{(h\lambda)^2}{6}\Big)y_n\\
&= \Big(1 - \frac{h\lambda}{3}\Big)^{-2}\Big(1 + \frac{h\lambda}{3} - \frac{(h\lambda)^2}{18}\Big)y_n = R(h\lambda)y_n.
\end{aligned}
$$

The function $R(x) = (1 - \frac{x}{3})^{-2}(1 + \frac{x}{3} - \frac{x^2}{18})$ has a double pole at $x = 3$, which is in the right-half plane. Moreover, on the imaginary axis, we observe

$$
|R(i\phi)|^2 = \frac{\Big|1 + \frac{i\phi}{3} - \frac{(i\phi)^2}{18}\Big|^2}{\Big|1 - \frac{i\phi}{3}\Big|^4} = \frac{1 + \frac{2\phi^2}{9} + \frac{\phi^4}{18^2}}{1 + \frac{2\phi^2}{9} + \frac{\phi^4}{9^2}}.
$$

This shows that $|R(i\phi)|^2 \leq 1$ as the numerator is smaller than the denominator. Altogether, we can now apply the maximum principle which shows that the method is A-stable.

6. What is an *embedded Runge–Kutta method?*

   Solution: An embedded Runge–Kutta method consists of two methods in one. Both methods have the same coefficients $a_{i,j}$ and c_j but different b_j. This leads to different orders of the methods. The different orders can be used for an approximation of the local error of the method in every step.

7. Can you give an example of an A-stable explicit Runge-Kutta method? Can you give an example of an A-stable multistep method of order 3? Motivate the answers.

   Solution: No, there are no A-stable explicit RK methods, because the stability function is a polynomial. For every polynomial, $|P(h\lambda)| \to \infty$ if $h\lambda \to \infty$. Dahlquist's second barrier theorem states that the highest order of an A-stable multistep method is $p = 2$. Thus, there cannot be an A-stable multistep method of order 3.

8. What is the difference between Runge–Kutta and multistep methods?

   Solution: Runge–Kutta methods are one step methods, i.e. y_{n+1} only depends on the previous approximation y_n and the function f. A multistep (s-step) method can depend on $y_{n-s}, y_{n-s+1}, \ldots, y_n$ and the function f.

9. Determine which of the following methods are 0-stable (zero-stable).

   - $y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f_{n+2} + \frac{8}{12}f_{n+1} - \frac{1}{12}f_n\right)$
   - $y_{n+2} = y_n + 2hf_{n+2}$
   - $y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}hf_{n+2}$
   - $y_{n+2} = 3y_{n+1} - 2y_n + hf_{n+2}$

   Solution: A multistep scheme is zero-stable if it solves $\dot{y} = 0$ in a stable way. For this to hold, all zeros of the ρ-polynomial must be inside the unite disc and any roots on the unit circle must be simple.

   - $y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f_{n+2} + \frac{8}{12}f_{n+1} - \frac{1}{12}f_n\right)$ has the polynomial $\rho(w) = w^2 - w^1 = w(w-1)$. The roots are $w_1 = 0$ (in the unit circle) and $w_2 = 1$ (on the unit circle and simple). Thus, the method is zero-stable.
   - $y_{n+2} = y_n + 2hf_{n+2}$ has the polynomial $\rho(w) = w^2 - w^0 = w^2 - 1$. The roots are $w_1 = 1$ (on the unit circle and simple) and $w_2 = -1$ (on the unit circle and simple). Thus, the method is zero-stable.
   - $y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}hf_{n+2}$ has the polynomial $\rho(w) = w^2 - \frac{4}{3}w^1 + \frac{1}{3}w^0$. The roots are $w_1 = \frac{1}{3}$ (in the unit circle) and $w_2 = 1$ (on the unit circle and simple). Thus, the method is zero-stable.

- $y_{n+2} = 3y_{n+1} - 2y_n + hf_{n+2}$ has the polynomial $\rho(w) = w^2 - 3w^1 + 2w^0$. The roots are $w_1 = 1$ (on the unit circle and simple) and $w_2 = 2$ (outside of the unite circle). Thus, the method is not zero-stable.

10. The following is a method of order 4. To what family does it belong?

$$y_{n+3} = y_{n+2} + h\left(\frac{9}{24}f_{n+3} + \frac{19}{24}f_{n+2} - \frac{5}{24}f_{n+1} + \frac{1}{24}f_n\right).$$

Is it explicit or implicit?

Solution: The method is an Adams-Moulton method since $\rho(w) = w^3 - w^2 = w^2(w-1)$. The method is implicit.

11. Find the coefficients of the Adams-Bashforth methods with $k = 1$, 2 and 3 steps by computing

$$b_j^k = \frac{1}{h}\int_{t_{n+k-1}}^{t_{n+k}} \varphi_j^k(\tau)\mathrm{d}\tau,$$

where φ_j^k is the j:th Lagrange basis polynomial when interpolating on a grid with k nodes.

Solution: In the following, we choose $n = 0$ in the integral boundaries. For $k = 1$, the basis polynomial is given by

$$\varphi_1^1(\tau) = 1.$$

Thus, the weight is given by

$$b_1^1 = \frac{1}{h}\int_0^h \varphi_1^1(\tau)\mathrm{d}\tau = 1.$$

For $k = 2$, we consider the nodes $x_0 = 0$ and $x_1 = h$ and obtain the basis functions

$$\varphi_1^2(\tau) = \frac{\tau - x_1}{x_0 - x_1} = \frac{\tau - h}{-h}, \quad \varphi_2^2(\tau) = \frac{\tau - x_0}{x_1 - x_0} = \frac{\tau - 0}{h}.$$

Thus, the weights are given by

$$b_1^2 = \frac{1}{h}\int_h^{2h} \frac{\tau - h}{-h}\mathrm{d}\tau = \frac{\frac{4}{2}h^2 - 2h^2 - \frac{1}{2}h^2 + h^2}{-h^2} = -\frac{1}{2},$$
$$b_2^2 = \frac{1}{h}\int_h^{2h} \frac{\tau}{h}\mathrm{d}\tau = \frac{\frac{4}{2}h^2 - \frac{1}{2}h^2}{h^2} = \frac{3}{2}.$$

For $k = 3$, we consider the nodes $x_0 = 0$, $x_1 = h$ and $x_2 = 2h$ and obtain the basis functions

$$\begin{aligned}
\varphi_1^3(\tau) &= \frac{(\tau - x_1)(\tau - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(\tau - h)(\tau - 2h)}{(-h)(-2h)}, \\
\varphi_2^3(\tau) &= \frac{(\tau - x_0)(\tau - x_1)}{(x_1 - x_0)(x_1 - x_2)} = \frac{\tau(\tau - 2h)}{h(-h)}, \\
\varphi_3^3(\tau) &= \frac{(\tau - x_0)(\tau - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\tau(\tau - h)}{2h \cdot h}.
\end{aligned}$$

Thus, the weights are given by

$$\begin{aligned}
b_1^3 &= \frac{1}{h}\int_{2h}^{3h} \frac{(\tau - h)(\tau - 2h)}{(-h)(-2h)} \mathrm{d}\tau = \frac{1}{h}\int_{2h}^{3h} \frac{\tau^2 - 3h\tau + 2h^2}{2h^2} \mathrm{d}\tau \\
&= \frac{1}{2h^3}\Big(\frac{27}{3}h^3 - \frac{27}{2}h^3 + 6h^3 - \frac{8h^3}{3} + \frac{12}{2}h^3 - 4h^3\Big) = \frac{5}{12}, \\
b_2^3 &= \frac{1}{h}\int_{2h}^{3h} \frac{\tau(\tau - h)}{(\frac{h}{2})(-\frac{h}{2})} \mathrm{d}\tau = \frac{1}{h}\int_{2h}^{3h} \frac{\tau^2 - 2h\tau}{-h^2} \mathrm{d}\tau \\
&= \frac{-1}{h^3}\Big(\frac{27}{3}h^3 - 9h^3 - \frac{8}{3}h^3 + 4h^3\Big) = \frac{-1}{h^3} \cdot \Big(\frac{19}{3} - \frac{15}{3}\Big)h^3 = -\frac{4}{3}, \\
b_1^3 &= \frac{1}{h}\int_{2h}^{3h} \frac{\tau(\tau - h)}{2h \cdot h} \mathrm{d}\tau = \frac{1}{h}\int_{2h}^{3h} \frac{\tau^2 - h\tau}{2h^2} \mathrm{d}\tau \\
&= \frac{1}{2h^3}\Big(\frac{27}{3}h^3 - \frac{9}{2}h^3 - \frac{8}{3}h^3 + \frac{4}{2}h^3\Big) = \frac{23}{12}.
\end{aligned}$$

12. Find the order of the method

$$y_{n+2} = y_n + h\left(\frac{1}{3}f_{n+2} + \frac{4}{3}f_{n+1} + \frac{1}{3}f_n\right)$$

Solution: We check the method by inserting the exact polynomial solutions $p(t) = t^q$, $q = 0, 1, \ldots$. Then the solution at the grid points becomes $y(t_{n+k}) = t_{n+k}^q$ and $f(y(t_{n+k})) = \dot{p} = qt_{n+k}^{q-1}$.

$p(t)$	$p'(t)$	$p(t_n + 2h) - p(t_n) + h\left(\frac{1}{3}p'(t_n + 2h) + \frac{4}{3}p'(t_n + h) + \frac{1}{3}p'(t_n)\right)$	
1	0	$1 - 1 - h\left(\frac{1}{3}0 + \frac{4}{3}0 + \frac{1}{3}0\right) = 0$	okay
t	1	$t_n + 2h - t_n - h\left(\frac{1}{3}1 + \frac{4}{3}1 + \frac{1}{3}1\right) = 0$	okay
t^2	$2t$	$(t_n + 2h)^2 - t_n^2 - h\left(\frac{2}{3}t_n + \frac{4}{3}h + \frac{8}{3}(t_n + h) + \frac{2}{3}t_n\right)$	
		$= 4ht_n + 4h^2 - 4ht_n - 4h = 0$	okay
t^3	$3t^2$	$(t_n + 2h)^3 - t_n^3 - h\left((t_n + 2h)^2 + 4(t_n + h)^2 + t_n^2\right) =$	
		$= 6ht_n^2 + 12h^2t_n + 8h^3 - h\left(6t_n^2 + 12ht_n + 8h^2\right) = 0$	okay
t^4	$4t^3$	$(t_n + 2h)^4 - t_n^4 - h\left(\frac{4}{3}(t_n + 2h)^3 + \frac{16}{3}(t_n + h)^3 + \frac{4}{3}t_n^3\right)$	
		$= 8ht_n^3 + 24h^2t_n^2 + 32h^3t_n + 16h^4$	
		$-h\left(8t_n^3 + 24t_n^2h + 32t_n^2h + 16h^3\right) = 0$	okay
t^5	$5t^4$	$(t_n + 2h)^5 - t_n^5 - h\left(\frac{5}{3}(t_n + 2h)^4 + \frac{20}{3}(t_n + h)^4 + \frac{5}{3}t_n^4\right)$	
		$= \cdots + 32h^5 - h(\cdots + \frac{100}{3}h^4) \neq 0$	

We observe that up to $q = 4$ the method solves the equation exact but not any longer for $q = 5$. Thus, the methods os consistent of order $q = 4$.

13. The k-step BDF method, BDFk, is of order k. It is given by the formula

$$\sum_{j=0}^{k} a_j y_{n+j} = hf(t_{n+k}, y_{n+k}).$$

Find the coefficients a_j for BDF3. Is it zero-stable? Is it convergent? Justify.

Solution: The generating polynomials are given by

$$\rho(w) = \sum_{j=0}^{k} a_j w^j \quad \text{and} \quad \sigma(w) = w^k.$$

In order to obtain a third order method we need the following conditions to be fulfilled:

$$\sum_{j=0}^{k} j^m a_j = m \sum_{j=0}^{k} j^{m-1} b_j, \quad m \in \{0, 1, 2, 3\}.$$

Thus, for the method to be of third order, we obtain the order conditions

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 0 \\ a_1 + 2a_2 + 3a_3 &= 1 \\ a_1 + 4a_2 + 9a_3 &= 6 \\ a_1 + 8a_2 + 27a_3 &= 27. \end{aligned}$$

Solving this linear system, the coefficients are given by

$$[a_0, a_1, a_2, a_3] = [-1/3, 3/2, -3, 11/6].$$

We check the root condition for the ρ to investigate if the method is zero-stable. The polynomial

$$\begin{aligned} \rho(w) &= \frac{11}{6}w^3 - 3w^2 + \frac{3}{2}w - \frac{1}{3} \\ &= \frac{11}{6}\Big(w^3 - \frac{18}{11}w^2 + \frac{9}{11}w - \frac{2}{11}\Big) \\ &= \frac{11}{6}(w-1)\Big(w^2 - \frac{7}{11}w + \frac{2}{11}\Big) \\ &= \frac{11}{6}(w-1)\Big(w^2 - \frac{7}{11}w + \frac{2}{11}\Big)\Big(w^2 - \frac{7}{11}w + \frac{2}{11}\Big) \\ &= \frac{11}{6}(w-1)\Big(w - \frac{7 + i\sqrt{39}}{22}\Big)\Big(w - \frac{7 - i\sqrt{39}}{22}\Big) \end{aligned}$$

The root 1 is on the unit circle but simple. The other two root are in the unite circle as

$$\left|\frac{7 \pm i\sqrt{39}}{22}\right|^2 = \frac{49+39}{22^2} < 1.$$

Thus, the method is zero-stable. As the method is zero stable and consistent of order 3, it is also convergent of order 3.