Numerical Methods for Differential Equations, FMNN10 / NUMN32
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Review questions and study problems, week 2

1. True or false (justify your answer): *All explicit Runge-Kutta methods of order 3 are convergent.*

2. Construct the Butcher tableau for the 3-stage Heun method,

$$\begin{aligned} Y_1' &= f(t_n, y_n) \\ Y_2' &= f(t_n + h/3, y_n + hY_1'/3) \\ Y_3' &= f(t_n + 2h/3, y_n + 2hY_2'/3) \\ y_{n+1} &= y_n + h\,(Y_1' + 3Y_3')/4 \end{aligned}$$

3. Write the equations for the Runge-Kutta method (RK4) with the Butcher tableau

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

   Is this an explicit or implicit method?

4. Suppose you apply the RK4 method to the linear test equation $y' = \lambda y$. You then get $y_{n+1} = P(h\lambda)y_n$, where the polynomial $P(h\lambda)$ is called the *stability function* of the method. Derive $P(h\lambda)$ for the RK4 method by hand. If you look at the polynomial, you probably recognize it. What does the polynomial approximate? Can you explain this?

5. Find the stability function of the Runge-Kutta method given by

1/3	1/3	0
2/3	1/3	1/3
	1/2	1/2

   Is the method A-stable?

6. What is an *embedded Runge–Kutta method?*

7. Can you give an example of an A-stable explicit Runge-Kutta method? Can you give an example of an A-stable multistep method of order 3? Motivate the answers.

8. What is the difference between Runge–Kutta and multistep methods?

9. Determine which of the following methods are 0-stable (zero-stable).

   - $y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f_{n+2} + \frac{8}{12}f_{n+1} - \frac{1}{12}f_n\right)$
   - $y_{n+2} = y_n + 2hf_{n+2}$
   - $y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}hf_{n+2}$
   - $y_{n+2} = 3y_{n+1} - 2y_n + hf_{n+2}$

10. The following is a method of order 4. To what family does it belong?

$$y_{n+3} = y_{n+2} + h\left(\frac{9}{24}f_{n+3} + \frac{19}{24}f_{n+2} - \frac{5}{24}f_{n+1} + \frac{1}{24}f_n\right).$$

    Is it explicit or implicit?

11. Find the coefficients of the Adams-Bashforth methods with $k = 1$, 2 and 3 steps by computing

$$b_j^k = \frac{1}{h}\int_{t_{n+k-1}}^{t_{n+k}} \varphi_j^k(\tau)\mathrm{d}\tau,$$

    where φ_j^k is the j:th Lagrange basis polynomial when interpolating on a grid with k nodes.

12. Find the order of the method

$$y_{n+2} = y_n + h\left(\frac{1}{3}f_{n+2} + \frac{4}{3}f_{n+1} + \frac{1}{3}f_n\right)$$

13. The k-step BDF method, BDFk, is of order k. It is given by the formula

$$\sum_{j=0}^{k} a_j y_{n+j} = hf(t_{n+k}, y_{n+k}).$$

    Find the coefficients a_j for BDF3. Is it zero-stable? Is it convergent? Justify.