Stationary Stochastic Processes
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Stochastic variables

- Distribution functions: $F_X(x_0) = P(X \leq x_0) = \begin{cases} \displaystyle\sum_{k \leq x_0} p_X(k) & (X \text{ discrete}) \\ \displaystyle\int_{-\infty}^{x_0} f_X(x)\,dx & (X \text{ continuous}) \end{cases}$

- Expected value: $E[X] = m_X = \begin{cases} \displaystyle\sum_{k} k\, p_X(k) & (X \text{ discrete}) \\ \displaystyle\int_{-\infty}^{\infty} x\, f_X(x)\,dx & (X \text{ continuous}) \end{cases}$

- Variance: $V[X] = E[X^2] - m_X^2 = \begin{cases} \displaystyle\sum_{k} (k - m_X)^2\, p_X(k) & (X \text{ discrete}) \\ \displaystyle\int_{-\infty}^{\infty} (x - m_X)^2\, f_X(x)\,dx & (X \text{ continuous}) \end{cases}$

- Rules for expected value and variance (a and b constants):
  * $E[aX + b] = aE[X] + b$
  * $V[aX] = a^2 V[X]$
  * $V[X + b] = V[X]$
  * $E[X + Y] = E[X] + E[Y]$
  * $V[X + Y] = V[X] + V[Y] + 2C[X, Y]$

- Covariance: $C[X, Y] = E[(X - m_X)(Y - m_Y)] = E[XY] - m_X m_Y$

- Correlation coefficient: $\rho[X, Y] = \dfrac{C[X, Y]}{\sqrt{V[X]\,V[Y]}}$

Stationary stochastic processes

- Estimation of expected value:

$$\hat{m}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$$

$$V[\hat{m}_n] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau)$$

$$V[\hat{m}_n] \approx \frac{1}{n} \sum_{\tau=-\infty}^{\infty} r_X(\tau) \qquad \text{for large } n$$

- If $\hat{m}_n \in N(m, V[\hat{m}_n])$, the confidence interval for m is

$$I_m : \quad \{\hat{m}_n - \lambda_{\alpha/2}\sqrt{V[\hat{m}_n]},\ \hat{m}_n + \lambda_{\alpha/2}\sqrt{V[\hat{m}_n]}\}$$

  with confidence level $1-\alpha$. For confidence level 0.95, $\alpha = 0.05$ and $\lambda_{\alpha/2} = \lambda_{0.025} = 1.96$.

- Estimation of covariance function:

$$\hat{r}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m_X)(X_{t+\tau} - m_X) \quad \text{for} \quad \tau \geq 0$$

  where m_X is replaced by $\hat{m}_n$ if m_X is unknown.

The Poisson process and the Wiener process

- A simply increasing process $\{X(t),\ t \geq 0\}$ is a homogeneous Poisson process, if $X(0) = 0$ and $X(t)$ has stationary, independent increments. If the intensity is λ,
  - $E[X(t)] = \lambda t$
  - $V[X(t)] = \lambda t$
  - $r_X(s,\ t) = \lambda \min(s,\ t)$

  The interarrival times are independent and exponentially distributed with mean value $1/\lambda$.

- A Gaussian process $\{X(t),\ t \geq 0\}$ is a Wiener process, if $X(0) = 0$, and $X(t)$ has independent increments, where $X(t) - X(t+h) \in N(0,\ \sigma^2 h)$,
  - $E[X(t)] = 0$
  - $V[X(t)] = \sigma^2 t$
  - $r_X(s,\ t) = \sigma^2 \min(s,\ t)$

Spectral representations

- Relations between covariance function $r_X(\tau)$ and spectral density $R_X(f)$:

Continuous time	Discrete time
$r_X(\tau) = \int_{-\infty}^{\infty} R_X(f)e^{i2\pi f\tau}\,df$	$r_X(\tau) = \int_{-1/2}^{1/2} R_X(f)e^{i2\pi f\tau}\,df$
$R_X(f) = \int_{-\infty}^{\infty} r_X(\tau)e^{-i2\pi f\tau}\,d\tau$	$R_X(f) = \sum_{\tau=-\infty}^{\infty} r_X(\tau)e^{-i2\pi f\tau}$

- Folding (aliasing): Let $\{Z_t, t = 0, \pm d, \pm 2d, \dots\}$ be the continuous time process $Y(t)$ sampled with time interval d and sampling frequency $f_s = 1/d$:

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \quad \text{for} \quad -f_s/2 < f \le f_s/2$$

- Sum of harmonic components with random phase and amplitude:

$$X(t) \quad = \quad A_0 + \sum_{k=1}^{n} A_k \cos(2\pi f_k t + \varphi_k)$$

where $\varphi_k \in \text{Rect}(0,\ 2\pi)$, A_k, $k = 0,\ \dots,\ n$, are independent and $E[A_0] = 0$.

  * Covariance function:

$$r_X(\tau) \quad = \quad \sigma_0^2 + \sum_{k=1}^{n} \sigma_k^2 \cos 2\pi f_k \tau$$

  where $\sigma_0^2 = E\left[A_0^2\right]$ and $\sigma_k^2 = E\left[A_k^2\right]/2$.

  * Spectral density:

$$R_X(f) \quad = \quad \sum_{k=-n}^{n} b_k \delta_{f_k}(f),$$

  where $b_0 = \sigma_0^2 = E\left[A_0^2\right]$, and $b_k = \sigma_k^2/2 = E\left[A_k^2\right]/4$.

Linear filters - general theory

- Impulse response $h(u)$:

$$Y(t) = \begin{cases} \int_{-\infty}^{\infty} h(u)X(t-u)\,du & \text{(continuous time)} \\ \sum_{u=-\infty}^{\infty} h(u)X(t-u) & \text{(discrete time)} \end{cases}$$

- Relation between covariance functions:

$$r_Y(\tau) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\, r_X(\tau+u-v)\,du\,dv & \text{(continuous time)} \\ \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} h(u)h(v)\, r_X(\tau+u-v) & \text{(discrete time)} \end{cases}$$

- Frequency function $H(f)$ and impulse response $h(t)$:

Continuous time	Discrete time
$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\,df$	$h(t) = \int_{-1/2}^{1/2} H(f)e^{i2\pi ft}\,df$
$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,dt$	$H(f) = \sum_{t=-\infty}^{\infty} h(t)e^{-i2\pi ft}$

- Relation between spectral densities:

$$R_Y(f) = |H(f)|^2 R_X(f).$$

- Differentiation: $X'(t)$ exists (in quadratic mean) if $r''_X(t)$ exists. This is equivalent to $\int_{-\infty}^{\infty}(2\pi f)^2 R(f) df < \infty$. If $X'(t)$ exists, the following relations hold:

$$\begin{aligned}
r_{X'}(\tau) &= -r''_X(\tau) \\
R_{X'}(f) &= (2\pi f)^2 R_X(f) \\
V[X'(t)] &= \int_{-\infty}^{\infty}(2\pi f)^2 R_X(f)\, df \\
r_{X,X'}(\tau) &= C[X(t),\ X'(t+\tau)] = r'_X(\tau) \\
r_{X^{(j)},X^{(k)}}(\tau) &= (-1)^j r_X^{(j+k)}(\tau)
\end{aligned}$$

- Integration:

$$\begin{aligned}
E\left[\int g(s)X(s)\, ds\right] &= \int g(s)E[X(s)]\, ds \\
C\left[\int g(s)X(s)\, ds, \int h(t)Y(t)\, dt\right] &= \int\int g(s)h(t)\, C[X(s), Y(t)]\, ds\, dt
\end{aligned}$$

- Cross-covariance and cross-spectrum:

$$\begin{aligned}
r_{X,Y}(\tau) &= C[X(t),\ Y(t+\tau)] = \int e^{i2\pi f\tau} R_{X,Y}(f)\, df \\
R_{X,Y}(f) &= H(f)R_X(f) = A_{X,Y}(f)e^{i\Phi_{X,Y}(f)}
\end{aligned}$$

  where $A_{X,Y}(f)$ is the amplitude spectrum and $\Phi_{X,Y}(f)$ the phase spectrum. The squared coherence spectrum is

$$\kappa^2_{X,Y}(f) = \frac{A^2_{X,Y}(f)}{R_X(f)R_Y(f)}$$

AR- MA- and ARMA-models

- White noise in discrete time: $\{e_t,\ t = 0, \pm 1, \ldots\}$, $E[e_t] = 0$ and $V[e_t] = \sigma^2$:

$$R_e(f) = \sigma^2 \quad \text{for } -1/2 \le f \le 1/2$$

- AR(p)-process: $(a_0 = 1)$

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t$$

  ⋆ Yule-Walker equations for covariance function:

$$r_X(k) + a_1 r_X(k-1) + \ldots + a_p r_X(k-p) = \begin{cases} \sigma^2 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, \ldots \end{cases}$$

  ⋆ Spectral density:

$$R_X(f) = \frac{1}{|\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2} \sigma^2$$

- MA(q)-process: $(b_0 = 1)$

$$X_t = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \ldots + b_q e_{t-q}$$

  ⋆ Covariance function:

$$r_X(\tau) = \begin{cases} \sigma^2 \sum_{j-k=\tau} b_j b_k & \text{for } |\tau| \le q \\ 0 & \text{for } |\tau| > q \end{cases}$$

  ⋆ Spectral density:

$$R_X(f) = |\sum_{k=0}^{q} b_k e^{-i2\pi fk}|^2 \sigma^2$$

Matched filter and Wiener filter

- Matched filter:

  ⋆ with white noise:

$$\begin{aligned} h(t) &= s(T-t), \quad 0 \le t \le T \\ SNR_{opt} &= \frac{\int_0^T s^2(T-u)\,du}{\sigma_N^2} \end{aligned}$$

  ⋆ with colored noise:

$$\begin{aligned} s(T-t) &= \int_0^T h(u) r_N(t-u)\,du \\ SNR_{opt} &= \int_0^T \int_0^T h(u)h(v) r_N(u-v)\,du\,dv \end{aligned}$$

- Wiener filter:

$$\begin{aligned} H(f) &= \frac{R_S(f)}{R_S(f) + R_N(f)} \\ SNR &= \frac{\int R_S(f)\,df}{\int \frac{R_S(f) R_N(f)}{R_S(f) + R_N(f)}\,df} \end{aligned}$$

Spectral estimation

- Periodogram of the sequence $\{x(t), t = 0, 1, 2, \dots n-1\}$,

$$\hat{R}_x(f) = \frac{1}{n}|\mathcal{X}(f)|^2$$

where $\mathcal{X}(f) = \sum_{t=0}^{n-1} x(t)e^{-i2\pi ft}$.

$$\begin{aligned} E\left[\hat{R}_x(f)\right] &= \sum_{\tau=-\infty}^{\infty} k_n(\tau) r_X(\tau) e^{-i2\pi f\tau} \\ &= \int_{-1/2}^{1/2} K_n(f-u) R_X(u) du \end{aligned}$$

where $k_n(\tau) = 1 - \frac{|\tau|}{n}$ for $-n+1 \le \tau \le n-1$ and $K_n(f) = \sum_{\tau=-n+1}^{n-1} k_n(\tau)e^{-i2\pi f\tau}$.

$$V\left[\hat{R}_x(f)\right] \approx \begin{cases} R_X^2(f) & \text{for } 0 < |f| < 1/2 \\ 2R_X^2(f) & \text{for } f = 0, \pm 1/2 \end{cases}$$

The distribution of the periodogram estimate is

$$\frac{\hat{R}_x(f)}{R_X(f)} \approx \frac{\chi^2(2)}{2} \quad \text{for} \quad 0 < f < 1/2$$

- Modified periodogram

$$\begin{aligned} \hat{R}_w(f) &= \frac{1}{n}\left|\sum_{t=0}^{n-1} x(t)w(t)e^{-i2\pi ft}\right|^2 \\ &= \frac{1}{n}\left|\int_{-1/2}^{1/2} \mathcal{X}(\nu)W(f-\nu)d\nu\right|^2 \end{aligned}$$

- Lag-windowing

$$\begin{aligned} \hat{R}_{lw}(f) &= \sum_{\tau=-\infty}^{\infty} k_{L_n}(\tau)\hat{r}_x(\tau)e^{-i2\pi f\tau} \\ &= \int_{-1/2}^{1/2} K_{L_n}(f-\nu)\hat{R}_x(\nu)d\nu \end{aligned}$$

- Averaging of spectrum

$$\hat{R}_{av}(f) = \frac{1}{K}\sum_{j=1}^{K} \hat{R}_{x,j}(f)$$

where K different spectrum estimates, $\hat{R}_{x,j}(f)$, $j = 1 \dots K$, are used. The distribution is

$$\frac{\hat{R}_{av}(f)}{R_X(f)} \approx \frac{\chi^2(2K)}{2K} \quad \text{for} \quad 0 < f < 1/2$$

Fourier transforms

$g(\tau) \qquad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} g(\tau)\,d\tau$
$e^{-\alpha\|\tau\|}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}\, e^{-2\pi\alpha\|f\|}$
$\|\tau\| e^{-\alpha\|\tau\|}$	$2\,\frac{(\alpha^2-(2\pi f)^2)}{(\alpha^2+(2\pi f)^2)^2}$
$\|\tau\|^k e^{-\alpha\|\tau\|}$	$\frac{k!}{(\alpha^2+(2\pi f)^2)^{k+1}}\{(\alpha+i2\pi f)^{k+1}+(\alpha-i2\pi f)^{k+1}\}$
$e^{-\alpha\tau^2}$	$\sqrt{\pi/\alpha}\exp(-\frac{(2\pi f)^2}{4\alpha})$
$e^{-\alpha\|\tau\|}\cos(2\pi f_0\tau)$	$\frac{\alpha}{\alpha^2+(2\pi f_0-2\pi f)^2}+\frac{\alpha}{\alpha^2+(2\pi f_0+2\pi f)^2}$
$e^{-\alpha\|\tau\|}\sin(2\pi f_0\tau)$	$\frac{2\pi f_0-2\pi f}{\alpha^2+(2\pi f_0-2\pi f)^2}+\frac{2\pi f_0+2\pi f}{\alpha^2+(2\pi f_0+2\pi f)^2}$
$\begin{cases}\alpha & \text{if} \quad \tau=0\\ \frac{\sin(2\pi\alpha\tau)}{2\pi\tau} & \text{if} \quad \tau\neq 0\end{cases}$	$\begin{cases}1/2 & \text{if} \quad \|f\|\le\alpha\\ 0 & \text{if} \quad \|f\|>\alpha\end{cases}$
$\begin{cases}1-\alpha\|\tau\| & \text{if} \quad \|\tau\|\le\frac{1}{\alpha}\\ 0 & \text{if} \quad \|\tau\|>\frac{1}{\alpha}\end{cases}$	$\begin{cases}\frac{1}{\alpha} & \text{if} \quad f=0\\ \frac{2\alpha}{(2\pi f)^2}\left(1-\cos\left(\frac{2\pi f}{\alpha}\right)\right) & \text{if} \quad f\neq 0\end{cases}$
$g(\tau)h(\tau)$	$G(f)*H(f)=\int G(\nu)H(f-\nu)d\nu$
$g(\tau)*h(\tau)=\int g(t)h(\tau-t)dt$	$G(f)H(f)$
$g'(\tau)$	$i2\pi f\, G(f)$
$g(\alpha\tau)$	$\frac{1}{\alpha}G(\frac{f}{\alpha})$
$\frac{1}{\alpha}g(\frac{\tau}{\alpha})$	$G(\alpha f)$
$g(\tau-\tau_0)$	$G(f)e^{-i2\pi f\tau_0}$
$g(\tau)e^{i2\pi f_0\tau}$	$G(f-f_0)$

Parseval's theorem:

$$\sum_{t=0}^{n-1}|x(t)|^2=\int_{-1/2}^{1/2}|X(f)|^2 df,$$

where $X(f)=\sum_{t=0}^{n-1}x(t)e^{-i2\pi ft}$.

Gaussian distribution table

$$F(x) = \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998