Mathematical Statistics, Centre for Mathematical Sciences
Lund University
Additional exercises 2023

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Additional exercises in Stationary Stochastic Processes

X1 Determine the covariance function $r_Y(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$, when

$$Y_t = \frac{X_t + X_{t-2}}{2},$$

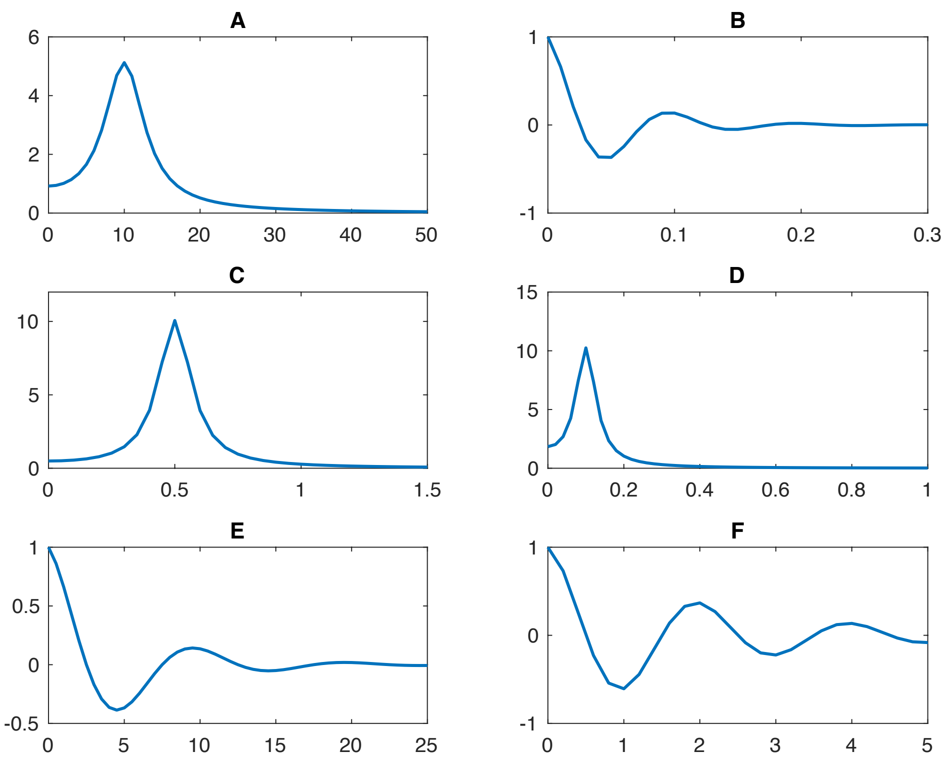
and the zero-mean weakly stationary process X_t, $t = 0, \pm 1, \pm 2, \ldots$, consists of independent variables and has variance $V[X_t] = 2$.

X2 Let X_t, $t = 0, \pm 1, \pm 2, \ldots$ be a stationary white noise process with $E[X_t] = 0$ and $V[X_t] = \sigma^2$. A new process is defined as

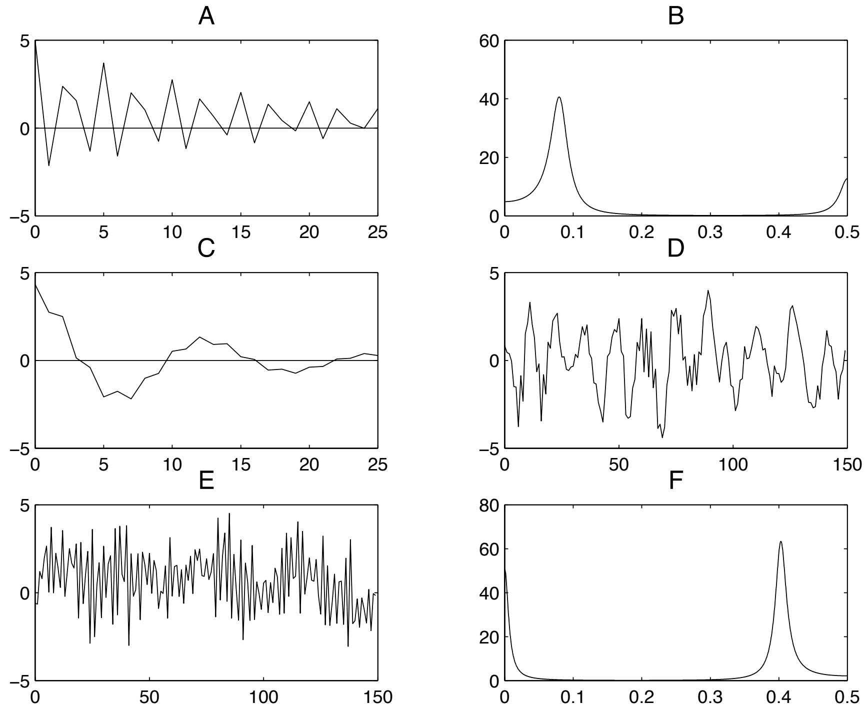
$$Y_t = X_t - 2X_{t-1} + X_{t-2} \quad t = 0, \pm 1, \pm 2 \ldots.$$

Calculate the covariance function $r_Y(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$.

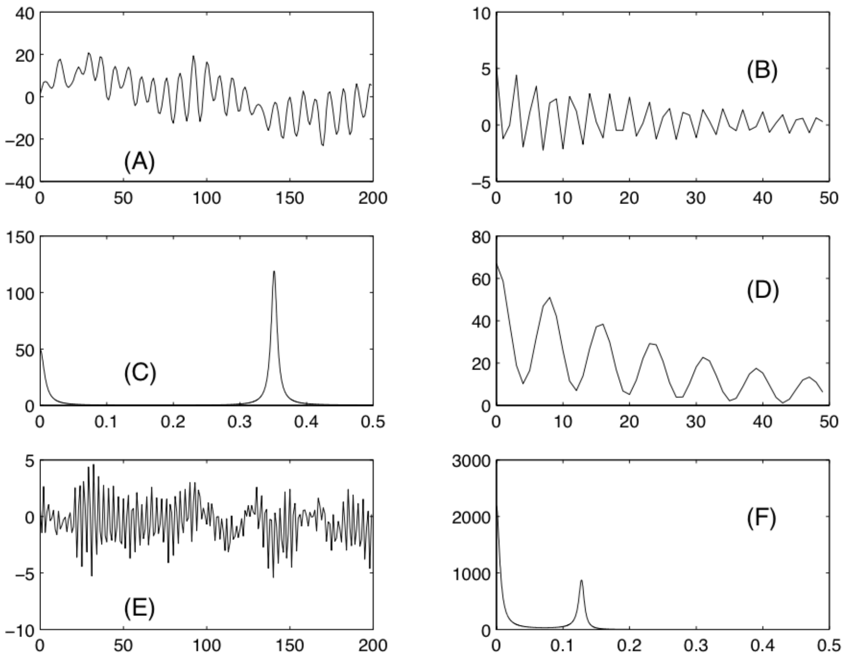
X3 Determine the three figures that are spectral densities and the three that are covariance functions. Combine the corresponding covariance function and spectral density.



X4 The figure below shows realizations of two different processes in discrete time with covariance functions and spectral densities. Determine which of the figures that belong to the same process.



X5 The figure below shows two realizations of two different processes in discrete time together with covariance functions and spectral densities. Combine the figures which belong to the same process.



X6 A real valued weakly stationary process $Y(t),\ t \in \mathbb{R}$ has the spectral density

$$R_Y(f) = \begin{cases} (1 - |f|/16) & |f| \leq 16 \\ 0 & |f| > 16 \end{cases}$$

a) Determine the covariance function of the process.

b) The process $Y(t)$ is sampled and a new process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$ is computed where $f_s = 1/d = 24$. Determine the spectral density for the sampled process Z_t.

X7 A continuous time zero-mean weakly stationary process $X(t)$, $t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \frac{|f| - 1}{2}, \qquad 1 \leq |f| \leq 3.$$

The process is sampled into a discrete time process, $Y_t = X(t)$, $t = 0, \pm d, \pm 2d, \ldots$. Determine if the following statements are correct.

a) The largest possible sampling distance to avoid aliasing is $d = 0.2$. Motivate your answer.

b) After sampling with sampling frequency $f_s = 4$, the resulting spectral density is

$$R_Y(f) = 1 - |f|, \quad |f| \leq 1.$$

Motivate your answer with drawings.

c) After sampling with sampling frequency $f_s = 2$, the resulting spectral density represents a discrete time stationary white noise sequence. Motivate your answer with drawings.

X8 A stationary process $Y(t),\ t \in \mathbb{R}$ has a spectral density that consists of two "boxes":

$$R_Y(f) = \begin{cases} 0 & \text{for } |f| < 0.75 \\ 1 & \text{for } 0.75 \leq |f| < 1 \\ 0 & \text{for } 1 \leq |f| < 2.5 \\ 1 & \text{for } 2.5 \leq |f| < 2.75 \\ 0 & \text{for } |f| \geq 2.75 \end{cases}$$

a) What smallest sampling frequency $f_s = 1/d$ is allowed if one should avoid aliasing?

b) Determine the spectral density for the sampled process $Z_t, t = 0, \pm d, \pm 2d, \ldots$ if $d = 0.5$.

X9 A weakly stationary process $X(t),\ t \in \mathbb{R}$ has the spectral density

$$R_X(f) = \begin{cases} 1 & |f - 1| < 0.5, \\ 1 & |f + 1| < 0.5, \\ 0 & \text{otherwise.} \end{cases}$$

The process is sampled with $f_s = 1/d = 1$. Calculate the covariance function and the spectral density for the sampled process $Y_t = X(t)$, $t = 0, \pm 1, \pm 2, \ldots$. Also determine the variance for

$$\bar{Y} = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3}.$$

X10 Let $Y(t),\ t \in \mathbb{R}$ be a stationary stochastic process in continuous time with the covariance function

$$r_Y(\tau) \quad = \quad \frac{2}{1+(2\pi\tau)^2}.$$

The process is sampled at the time points $t = 0, \pm d, \pm 2d \ldots$. Find the spectral density of the sampled process $X_t = Y(t),\ t = 0, \pm d, \pm 2d \ldots$.

X11 The figure shows the daily stockprice of the 'Mathematical statistics special fund', april 2018-december 2018. The stock price at day t is modeled as

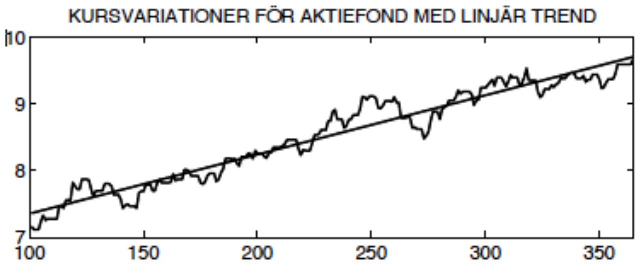
$$Y_t = b + a \cdot t + X_t, \qquad t = 100, 101, \ldots, 365,$$

where the linear trend increase $a = 0.00889$ and $b = 6.5$. The process

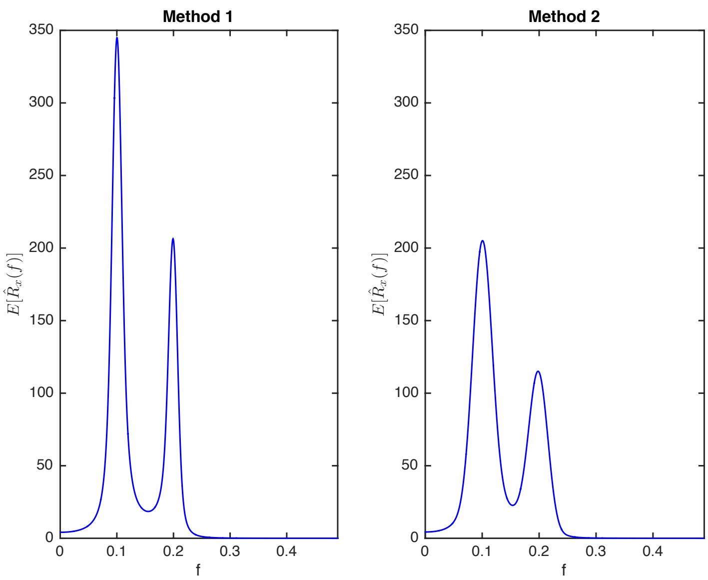
$$X_t = \frac{1}{16}\sum_{j=0}^{15} e_{t-j},$$

where the stationary Gaussian white noise process is defined with $e_t \in N(0, \sigma^2)$ and $\sigma^2 = 0.448$. Calculate the probability that the stock price increase from one day to another is larger than ten times the linear trend increase, i.e.,

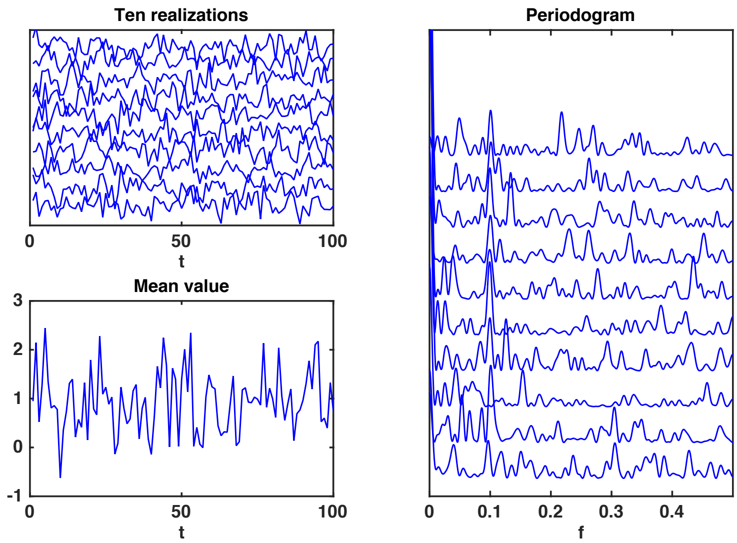
$$P(Y_{t+1} - Y_t > 10 \cdot a).$$



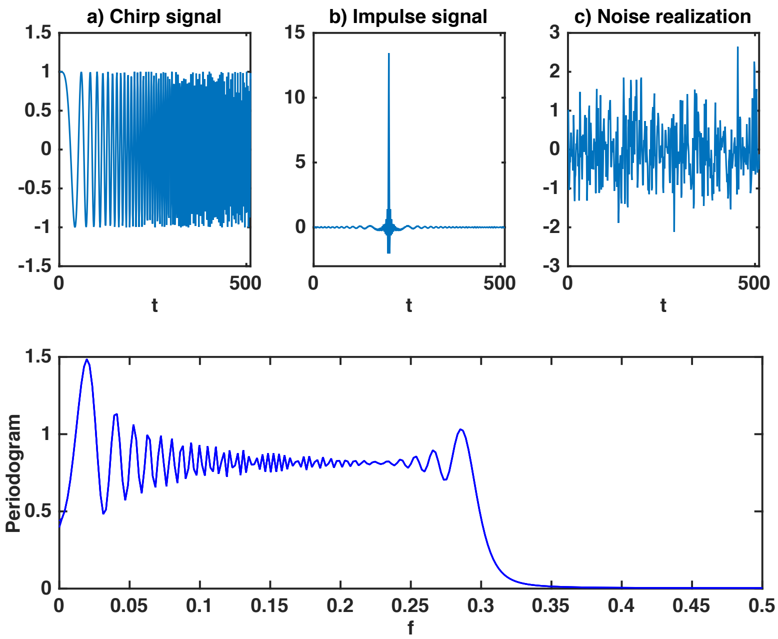
X12 The two pictures below show the expected values of the estimates of 100 samples of an AR(4)-process from two different methods, the modified periodogram and the Welch method with 4 windows and 50% overlap. Both methods use Hanning windows. Combine the figures with the correct method and motivate your answer.



X13 A colleague gives you a data set, where a stationary stochastic process, $X(t)$, is disturbed by noise that can be assumed to be zero-mean, white and stationary Gaussian. The data set should be analyzed and the stationary process $X(t)$ should possibly be modelled. The data set are depicted in the figure below, spread out on the y-axis to be viewable. From these ten realizations it is difficult to say anything about $X(t)$ and to get more information you compute the mean value of the ten realizations and also the individual periodograms (also spread out on the y-axis to be viewable). From these figures, could you judge what stationary stochastic process to use as a first model of the data set?



X14 The following data sequences are presented together with the periodogram in the lower figure. Can you judge from which signal a) b) or c), the periodogram is computed? Or could the periodogram belong to more than one of the signals? Consider that all the presented sequences actually involves many frequency components and that the periodogram does not specify when in time different frequencies appear.



X15 A real-valued weakly stationary process $X(t),\ t \in \mathbb{R}$ has the covariance function $r_x(\tau) = \frac{1}{1+\tau^2}$ and expected value $E[X(t)] = 2$. It is filtered through an ideal bandpass filter with frequency function

$$H(f) = 1, \quad 1 < |f| < 2,$$

and zero for all other values. Determine the expected value and the variance for the output signal from the filter.

X16 A real valued stationary Gaussian process $X(t),\ t \in \mathbb{R}$ has the expected value 0 and spectral density

$$R_X(f) = \pi e^{-2\pi|f|}.$$

a) Determine the covariance function.

b) Determine the spectral density after filtering through a linear filter with frequency function $H(f) = 1 + if$.

c) A new process $Y(t) = \int_2^3 X(u)du$ is defined. Is the process $Y(t)$ a Gaussian process?

X17 From the stationary sequence X_t, $t = 0, \pm1, \pm2, \ldots$, a new stationary sequence Y_t, $t = 0, \pm1, \pm2, \ldots$, is defined as

$$Y_t = X_t - \frac{X_{t-1} + X_t + X_{t+1}}{3}.$$

Determine the quote $R_Y(f)/R_X(f)$ between the spectral densities.

X18 A weakly stationary process $X(t),\ t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \begin{cases} 1000 - |f| & |f| < 1000, \\ 0 & \text{for all other values.} \end{cases}$$

a) Determine the covariance function for $X(t)$.

b) The process $X(t)$ is sampled with sample frequency $1/d = 1500$. Determine and draw the spectral density after the sampling ?

c) The process $X(t)$ is sampled with $1/d = 4000$ which gives the discrete time sequence Z_k, where $k = 0 \pm 1, \pm 2 \ldots$. A new process Y_k is created as $Y_k = (-1)^k Z_k$. Determine and draw the spectral density for the new process Y_k ? (Guidance: Investigate what happens with the covariance function for Y_k for different values of k.)

X19 A stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has the covariance function $r_X(\tau) = e^{-\tau^2/2}$.

a) Calculate the variances of $X(t)$, $X(t+2)$, and $X'(t)$.

b) Calculate the covariance and cross-covariances, $C[X(t), X(t+2)]$, $C[X(t), X'(t)]$ and $C[X'(t), X(t+2)]$.

c) Compute the probability, $P(X(t+2) > X(t) + 2X'(t))$.

X20 A stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has the covariance function

$$r_X(\tau) = 2\cos(2\pi\tau) + \cos(4\pi\tau)$$

and expected value $m_X = 30$. Motivate that the process is differentiable and compute the probability,

$$P(X(t+2) > 30 + 0.5X(t) + 2X'(t)).$$

X21 A stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has the covariance function

$$r_X(\tau) = e^{-3|\tau|},$$

and unknown expected value $E[X(t)] = m$. You will be able to estimate m with

$$m^* = \alpha \frac{X(0) + X(10)}{2} + (1-\alpha)\frac{1}{10}\int_0^{10} X(t)dt.$$

Determine the constant α so that the variance of the estimate will become as small as possible.

X22 Let the input signal, $X(t)$, $t \in \mathbb{R}$, to a linear filter be a zero-mean weakly stationary signal with covariance function $r_X(\tau) = \frac{1}{1+\tau^2}$. The filter is defined by the impulse response:

$$h(t) = \begin{cases} 1 & 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the cross-covariance, $r_{X,Y}(\tau) = C[X(t),\ Y(t+\tau)]$, between the input and the output signal?

X23 You know that X_t, $t = 0, \pm 1, \pm 2 \ldots$ is an AR(2)-process and that $r_X(0) = 3$, $r_X(1) = 1$ and $r_X(2) = 0$. Calculate $r_X(3)$?

X24 Let X_t, $t = 0, \pm 1, \pm 2, \ldots$ be a sequence of independent stochastic variables with $E[X_t] = 0$ and $V[X_t] = 2$. A new process is defined as

$$Y_t = X_t - 2X_{t-1} + X_{t-2}.$$

Calculate the covariance function and spectral density of the process Y_t, $t = 0, \pm 1, \pm 2 \ldots$.

X25 From the white noise sequence, e_t, $t = 0, \pm 1, \pm 2, \ldots$, a new process is defined as

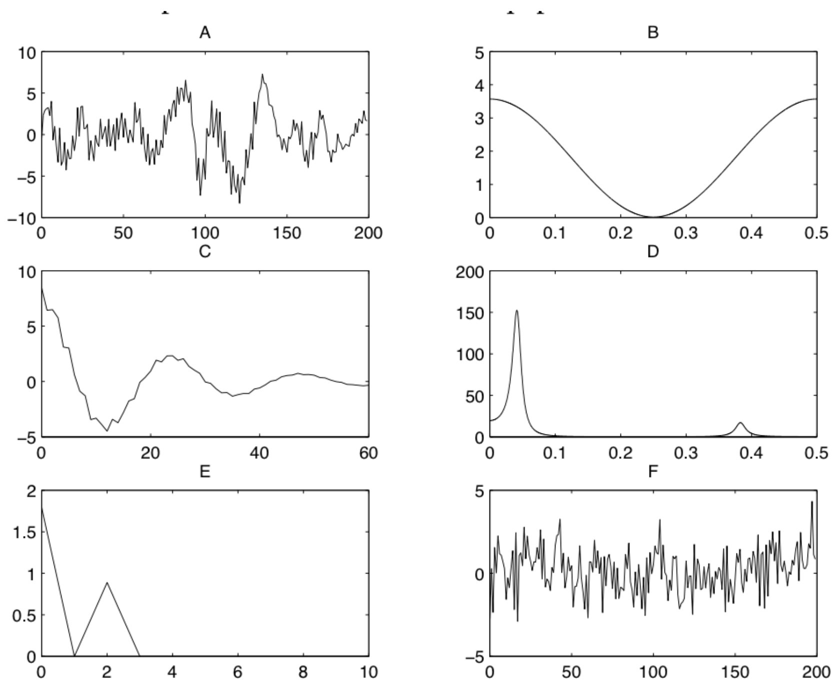
$$X_t = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \ldots + b_q e_{t-q},$$

which is known to have the covariance function

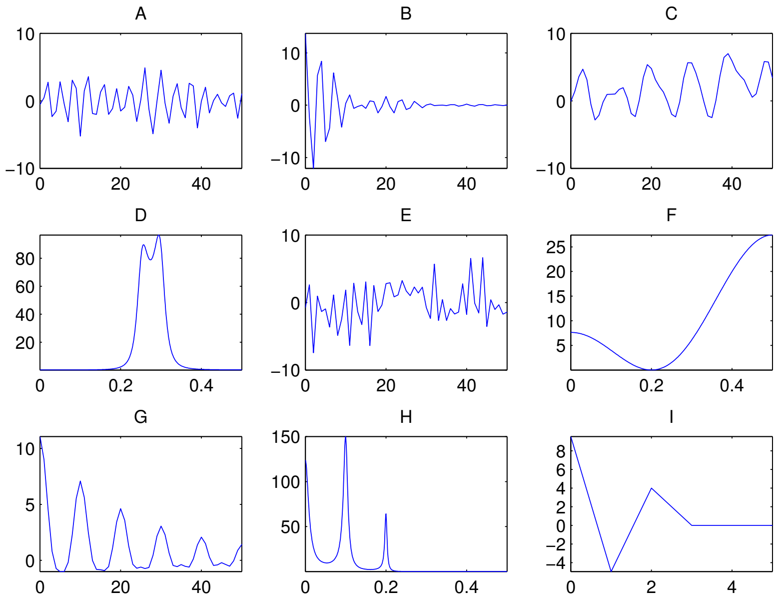
$$r_X(\tau) = \begin{cases} 2 & \tau = 0, \\ 0 & |\tau| = 1, \\ 0 & |\tau| = 2, \\ 1 & |\tau| = 3, \\ 0 & |\tau| > 3. \end{cases}$$

Determine the spectral density and the constants $b_1 \ldots b_q$ if $V[e_t] = 1$.

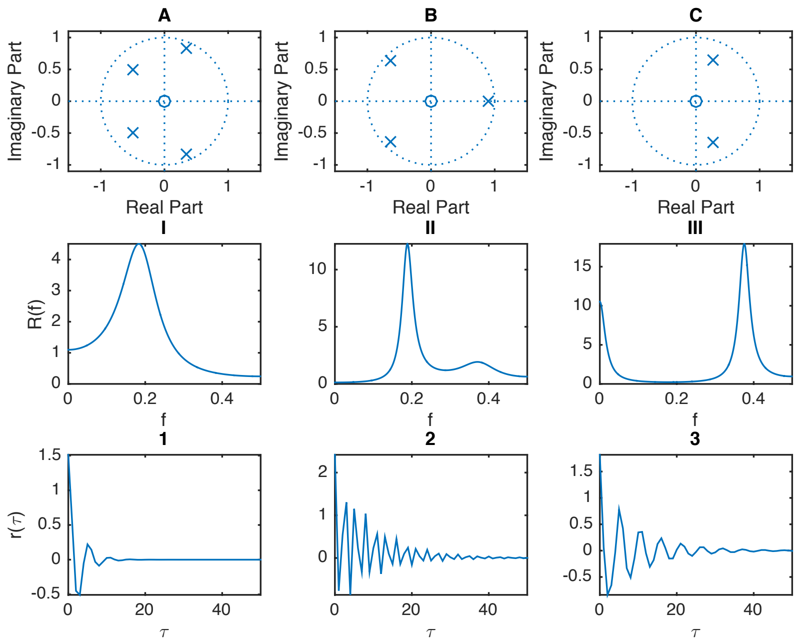
X26 The figure shows one AR(p)-process and one MA(q)-process. Combine realizations, covariance functions and spectral densities for the two process. Motivate.



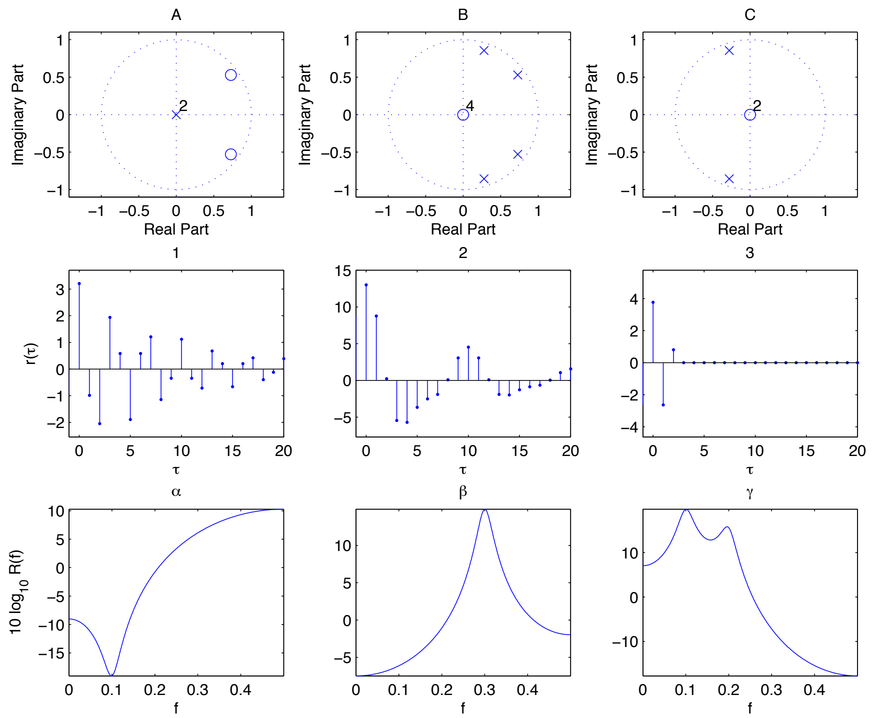
X27 The figures below belong to processes of the type AR or MA. State, with motivation, which of the figures that are realizations, covariance functions and spectral densities. Determine which realizations, covariance functions and spectral densities that belong to the same process. Also determine of which type the processes are and decide which order that can be seen in the figures. Hint: No process should be assumed to have an order larger than 5.



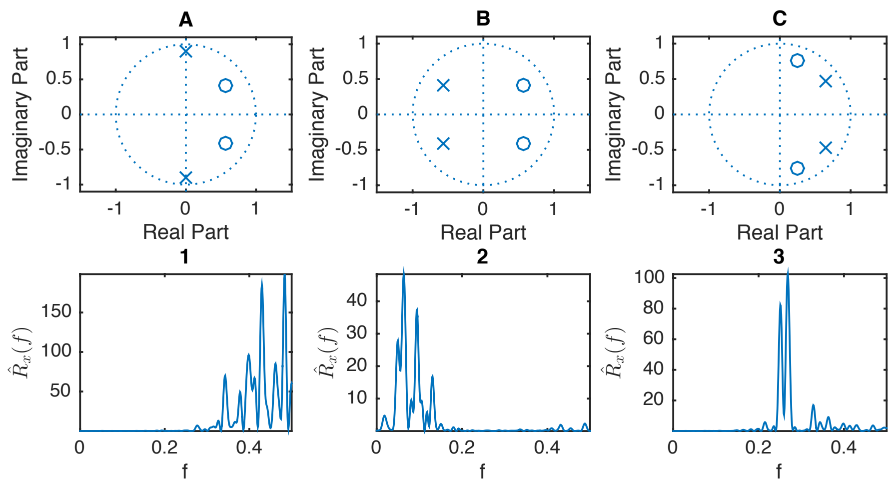
X28 The figures below belong to processes of type AR. Determine which pole-zero-plots, covariance functions and spectral densities that belong to the same process. Also, state which order each of the processes has. How does the order of the AR-process relate to the number of poles and to the number of peaks in the spectral density?



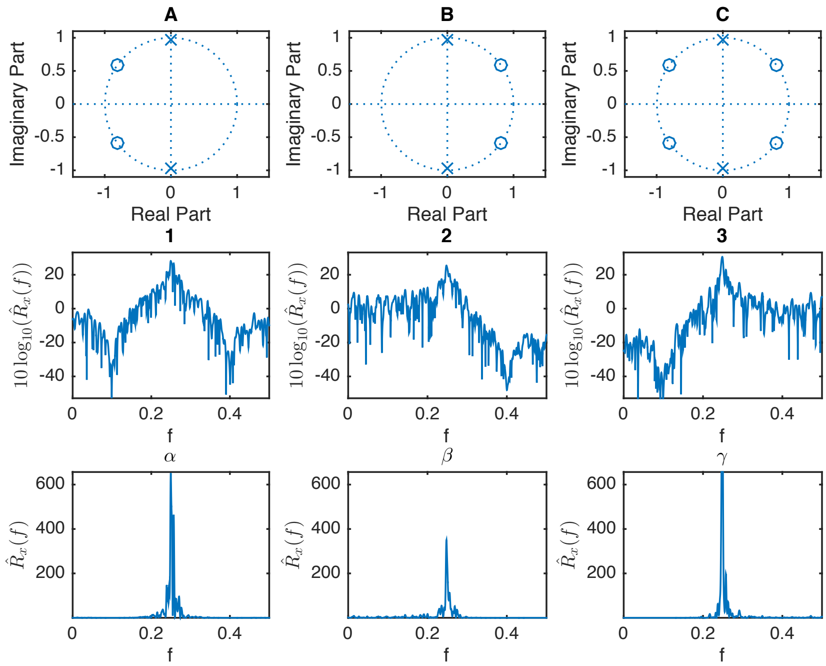
X29 The figures below belong to processes of type AR or MA. Determine which pole-zero-plots, covariance functions and spectral densities that belong to the same process and decide the type of each process. Also, state which order each of the processes has.



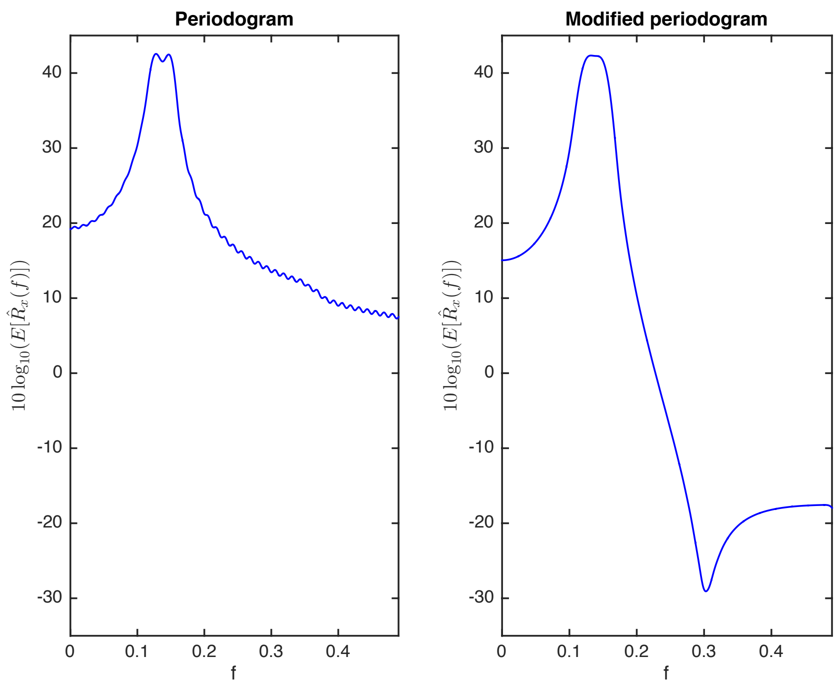
X30 The figure shows the poles and zeros of 3 different ARMA(2,2)-processes and the corresponding periodogram estimates from 3 realizations. Combine the pole-zero plot with the corresponding periodogram. Motivate your answer.



X31 The figure shows the poles and zeros of 3 different ARMA-processes and the corresponding modified periodogram estimates using a Hanning window from 3 realizations, both in dB-scale and linear scale. Combine the pole-zero plot with the corresponding logarithmic periodogram. Motivate your answer. How useful are the linear scaled periodograms?



X32 The two pictures below show the expected values in dB-scale of the estimates of two different methods, the periodogram and the modified periodogram using a Hanning window. The process is an ARMA-process of low order. Can you guess the order of the ARMA-process using the two expected values. Motivate your choice from the properties of the two methods.



X33 A stationary sequence S_t, $t = 0, \pm 1, \pm 2 \ldots$ is disturbed by noise N_t, $t = 0, \pm 1, \pm 2 \ldots$. One wants to improve the measured signal $Y_t = S_t + N_t$ by filtering Y_t in a filter with frequency function

$$H(f) = a_0 + a_1\, e^{-i2\pi f},$$

giving the output Z_t. Determine a_0 and a_1 so that the squared error, $E\left[(Z_t - S_t)^2\right]$ becomes as small as possible. The processes S_t and N_t are assumed to be independent with expected value 0 and covariance functions $r_S(\tau) = 0.5^{|\tau|}$ and $r_N(\tau) = (-0.5)^{|\tau|}$.

X34 A weakly stationary process $S_t,\ t = 0, \pm 1, \pm 2 \ldots$ has expected value zero and covariance function,

$$r_S(\tau) = \begin{cases} 2 & \tau = 0 \\ -1 & |\tau| = 1 \\ 0 & \text{for other values.} \end{cases}$$

The process is disturbed by colored noise $N_t,\ t = 0, \pm 1, \pm 2 \ldots$ with expected value zero and covariance function,

$$r_N(\tau) = \begin{cases} 2 & \tau = 0 \\ 1 & |\tau| = 1 \\ 0 & \text{for other values.} \end{cases}$$

The process and the disturbance are independent.

a) Determine the frequency function $H(f)$, $-1/2 < f \le 1/2$ and the impulse response $h(t)$, $t = 0, \pm 1, \pm 2 \ldots$, for the optimal Wiener filter that minimizes $E[(Y_t - S_t)^2]$ when Y_t is the output signal from the filter with $X_t = S_t + N_t$ as input signal.

b) In communication systems it is quite usual with echo, i.e.,you have a signal according to

$$X_t = S_t + 0.5 S_{t-1} + N_t,$$

where $0.5S_{t-1}$ is the damped and delayed original process realization. Determine the coefficients a, b for the non-causal impulse response

$$h(t) = \begin{cases} b & t = -1 \\ a & t = 0 \\ 0 & \text{for other values.} \end{cases}$$

for an optimal filter that minimizes

$$E[(Y_t - S_t)^2],$$

when Y_t is the output signal and X_t is the input signal to the optimal filter.

X35 A signal in discrete time is defined as

$$s_t = \begin{cases} 5 & t = 0, \\ 4 & t = 1, \\ 3 & t = 2, \\ 2 & t = 3, \\ 1 & t = 4. \end{cases}$$

The measurements are disturbed by Gaussian white noise, N_t, $t = 0, \pm 1, \pm 2, \ldots$ with expected value $E[N_t] = 0$ and variance $V[N_t] = 1$. The received signal is $Y_t = s_t + N_t$ or $Y_t = N_t$. Determine the decision time, the decision threshold and the impulse response of the causal matched filter so the error probabilities

$$\alpha = P(\text{detect signal if no signal sent})$$

$$\beta = P(\text{detect no signal if signal sent})$$

becomes equal and as small as possible.

X36 a) A matched filter for detection of the signal

$$s_t = \begin{cases} 1 & t = 0, \\ 2 & t = 1, \\ 4 & t = 2, \end{cases}$$

should be designed. The signal is disturbed by Gaussian zero mean white noise N_t, $t = 0, \pm 1, \pm 2, \ldots$ with variance $V[N_t] = 4$. The received signal is $Y_t = s_t + N_t$ or $Y_t = N_t$. Determine the coefficients of the impulse response h_t, $t = 0, 1, 2$, the decision threshold and the decision time so the error probabilities

$$\alpha = P(\text{detect signal if no signal sent})$$

$$\beta = P(\text{detect no signal if signal sent})$$

becomes equal and as small as possible.

b) Now assume that the noise has the covariance function,

$$r_N(\tau) = \begin{cases} 4 & \tau = 0, \\ 1 & \tau = 1, \\ 0 & \tau \geq 2. \end{cases}$$

Determine the coefficients h_t, $t = 0, 1, 2$ for optimal detection of the signal s_t. Also, determine the equal error probabilities for this optimal filter and compare with the error probabilities you would have received if you used the non-optimal filter you designed in a).

Solutions

X1 The variance is

$$r_Y(0) = V[Y_t] = C[Y_t, Y_t] = \frac{1}{4}C[X_t + X_{t-2}, X_t + X_{t-2}] = \frac{1}{4}(V[X_t] + V[X_{t-2}]) = 1.$$

The covariances are

$$r_Y(1) = C[Y_{t-1}, Y_t] = \frac{1}{4}C[X_{t-1} + X_{t-3}, X_t + X_{t-2}] = 0,$$

and

$$r_Y(2) = C[Y_{t-2}, Y_t] = \frac{1}{4}C[X_{t-2} + X_{t-4}, X_t + X_{t-2}] = \frac{1}{4}V[X_{t-2}] = \frac{1}{2},$$

and zero for larger values of τ, The symmetry property gives $r_Y(-2) = r_Y(2) = \frac{1}{2}$ and

$$r_Y(\tau) = \begin{cases} 1 & \tau = 0, \\ 0 & \tau = \pm 1, \\ \frac{1}{2} & \tau = \pm 2, \\ 0 & \text{for all other values.} \end{cases}$$

X2 The expression for the variance is given from

$$r_Y(0) = C[Y_t, Y_t] = C[X_t - 2X_{t-1} + X_{t-2}, X_t - 2X_{t-1} + X_{t-2}],$$
$$= V[X_t] + 4V[X_{t-1}] + V[X_{t-2}] = 6\sigma^2.$$

The covariances are

$$r_Y(1) = C[Y_{t-1}, Y_t] = C[X_{t-1} - 2X_{t-2} + X_{t-3}, X_t - 2X_{t-1} + X_{t-2}],$$
$$= -2V[X_{t-1}] - 2V[X_{t-2}] = -4\sigma^2,$$
$$r_Y(2) = C[Y_{t-2}, Y_t] = C[X_{t-2} - 2X_{t-3} + X_{t-4}, X_t - 2X_{t-1} + X_{t-2}],$$
$$= V[X_{t-2}] = \sigma^2.$$

and zero for larger values of τ, The symmetry property $r_Y(-\tau) = r_Y(\tau)$ gives the covariances for negative τ and the final answer is

$$r_Y(\tau) = \begin{cases} 6\sigma^2 & \tau = 0, \\ -4\sigma^2 & \tau = \pm 1, \\ \sigma^2 & \tau = \pm 2, \\ 0 & \text{for all other values.} \end{cases}$$

X3 The covariance functions are B, E and F as all these have their largest value for $\tau = 0$. A, C and D are all positive which is necessary for spectral densities.The period time of B is $T = 0.1$, which matches a spectral density frequency of $f_0 = 10$ as in A. The period time of E is $T = 10$, which corresponds to a frequency peak of $f_0 = 0.1$ in D. Finally the period time of F is $T = 2$ which matches the spectral density in C with a peak at $f_0 = 0.5$.

X4 Figures D and E are realizations: they can not be covariance functions as the largest value are not at zero and they can not be spectral densities as they are negative. Figures A and C are covariance functions as they are negative and the maximum is at zero, and accordingly B and F must be spectral densities (also seen from the scale of the x-axis, 0 to 0.5).

B has a strong peak for a low frequency, ≈ 0.08 (period ≈ 12) and a smaller one at 0.5 (period 2). The corresponding covariance function should have a period of 12 and a smaller fast variation (jittering) as well, which fits with C and the realization D, (note that the scales of the x-axes are different). The spectral density F corresponds to a fast variation as there is a strong peak at 0.4 (period 2.5), which fits with A and E.

X5 Figures C and F are spectral densities (non-negative and scale 0 to 0.5) and B and D are covariance functions (maximum at zero). Then A and E are realizations. C has a strong peak at 0.35, where F has one at a lower value ≈ 0.12. B has a higher frequency than D. This is also the case for E compared to A. Therefore, A-D-F belong to one process and E-B-C to the other process. Note the tricky time-scaling of the figures of the realizations compared to the figures of the covariance functions, which might lead to the conclusion that A and B belong together, which is wrong.

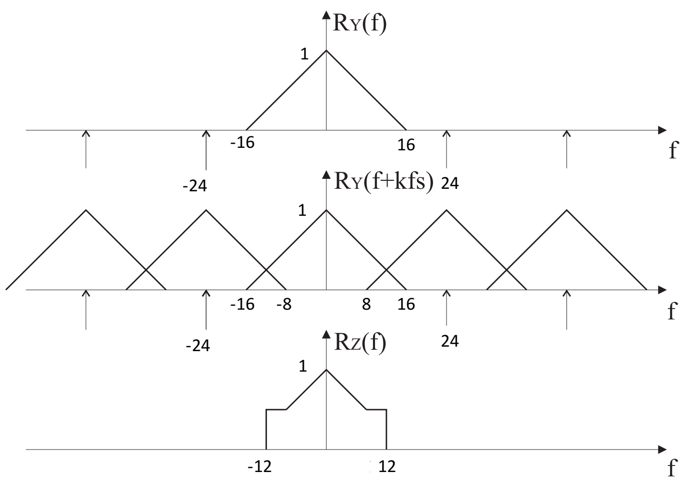
X6 a) The covariance function is found from the table of formulas if τ and f are switched. We find

$$r_Y(\tau) = \frac{2}{16(2\pi\tau)^2}(1 - \cos(2\pi\tau \cdot 16)) = \frac{1}{32(\pi\tau)^2}(1 - \cos(32\pi\tau)) \quad \tau \neq 0,$$

and $r_Y(0) = 16$.

b) We find the solution according to $R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + 24k)$, $-12 < f \leq 12$, where the contributing spectral densities in the important frequency range is,

$$R_Z(f) = R_Y(f + 24) + R_Y(f) + R_Y(f - 24), \quad -12 < f \leq 12.$$



The spectral density for the sampled process is

$$R_Z(f) = \begin{cases} (1 - |f|/16) & |f| \leq 8 \\ 1/2 & 8 \leq |f| \leq 12. \end{cases}$$

X7 a) Wrong. $f_s \geq 6$ i.e. $d \leq 1/6$.

b) Wrong. The resulting spectral density is

$$R_Y(f) = 1, \quad 1 \leq |f| \leq 2.$$

c) Correct.

X8 a) The maximum frequency is $f_0 = 2.75$. To avoid aliasing the sampling interval d should fulfill $\frac{1}{2d} = \frac{f_s}{2} \geq f_0$, dvs $f_s \geq 11/2$, or $d \leq 2/11$.

b) With $f_s = 2$, we get

$$\begin{aligned} R_Z(f) &= \sum_{k=-\infty}^{\infty} R_Y(f+2k) \\ &= R_Y(f) + R_Y(f-2) + R_Y(f+2), \quad -1 < f \leq 1, \end{aligned}$$

giving

$$R_Z(f) = \begin{cases} 0 & 0 \leq |f| < 0.5, \\ 1 & 0.5 \leq |f| \leq 1, \\ 0 & 1 < |f|. \end{cases}$$

X9 The spectral density is

$$R_Y(f) = \sum_{k=-\infty}^{\infty} R_X(f + kf_s) = 1 + 1 = 2, \quad -1/2 < f < 1/2.$$

The spectral density of the sampled process is constant. Accordingly it is a white noise process of independent stochastic variables. The variance of the process is

$$r_Y(0) = \int_{-1/2}^{1/2} R_Y(f)df = \int_{-1/2}^{1/2} 2df = 2.$$

The average $\bar{Y}$ consists of 3 independent stochastic variables and the variance is

$$V[\bar{Y}] = V[\frac{1}{3}\sum_{k=t-1}^{t+1} Y_k] = C[\frac{1}{3}\sum_{k=t-1}^{t+1} Y_k, \frac{1}{3}\sum_{l=t-1}^{t+1} Y_l] = \frac{1}{3^2}\sum_{k=t-1}^{t+1} V[Y_k] = \frac{1}{3^2}3V[Y_t] = \frac{V[Y_t]}{3} = 2/3.$$

X10 We have $r_Y(\tau) = \frac{2}{1+(2\pi\tau)^2}$ and $R_Y(f) = \frac{1}{2\pi} \cdot 2\pi \cdot e^{-2\pi \cdot |f|/2\pi} = e^{-|f|}$. The spectral density of the sampled process is

$$\begin{aligned} R_X(f) &\overset{|f| \leq \frac{f_s}{2}}{=} \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) = \sum_{k=-\infty}^{\infty} e^{-|f+f_s k|} \\ &= \left(e^{-|f|} + \sum_{k=1}^{\infty} e^{-f_s k - f} + \sum_{k=-\infty}^{-1} e^{f_s k + f}\right) \\ &= \cdot\left(e^{-|f|} + e^{-f} \cdot \frac{e^{-f_s}}{1-e^{-f_s}} + e^{f} \cdot \frac{e^{-f_s}}{1-e^{-f_s}}\right) \\ &= \left(e^{-|f|} + \frac{1}{e^{f_s}-1}\left(e^{f} + e^{-f}\right)\right), \quad -\frac{f_s}{2} < f \leq \frac{f_s}{2} \end{aligned}$$

X11 The increase from day t to day $t+1$ is

$$Y_{t+1} - Y_t = b + a \cdot (t+1) + X_{t+1} - b - a \cdot t - X_t = a + X_{t+1} - X_t.$$

We simplify

$$\begin{aligned}
X_{t+1} - X_t &= \frac{1}{16}\left(\sum_{j=0}^{15} e_{t+1-j} - \sum_{j=0}^{15} e_{t-j}\right) \\
&= \frac{1}{16}(e_{t+1} + e_t + \ldots + e_{t+1-15} - e_t - \ldots - e_{t-14} - e_{t-15}) \\
&= \frac{1}{16}(e_{t+1} - e_{t-15}),
\end{aligned}$$

where most of the terms are cancelled out.

To compute the probability of a Gaussian process, we only need the expected value and the variance. We find the expected value

$$E[Y_{t+1} - Y_t] = a + E[X_{t+1} - X_t] = a,$$

as $E[X_{t+1} - X_t] = 0$. The variance is

$$V[Y_{t+1} - Y_t] = V[X_{t+1} - X_t] = \frac{1}{16^2} V[e_{t+1} - e_{t-15}] = \frac{2\sigma^2}{16^2}.$$

The probability is

$$\begin{aligned}
P(Y_{t+1} - Y_t > 10a) &= 1 - P\left(Y_{t+1} - Y_t \leq 10a\right) = \\
= 1 - \Phi\left(\frac{16 \cdot (10a - a)}{\sqrt{2}\sigma}\right) &= 1 - \Phi(1.35242) = 0.088,
\end{aligned}$$

with $a = 0.00889$ and $\sigma = \sqrt{0.448}$. The probability to sell at ten times the linear increase is 0.088. Change ten times to 5 times or 2 times and see the how much the probability increases if you lower your claims.

X12 Method 1 is the modified periodogram and method 2 is the Welch method. With $n = 100$, the Hanning window mainlobe in the modified periodogram is $\approx 4/n = 0.04$. The Hanning window mainlobe of the Welch method will be much wider as the Welch method with 4 windows and 50% overlap will use Hanning windows of length $n = 40$ with corresponding mainline width of $\approx 4/n = 0.1$. Therefore the more peaked spectrum must belong to the modified periodogram and the more smoothed one to the Welch method.

X13 In the ten periodograms, peaks are viewable in almost all cases at the frequency of $f_0 = 0.1$. This indicates that a sinusoid, or a periodic signal, is present in the realization. The periodic signal is difficult to see in the different realizations, but also in the mean value. We can then suspect that the phase is different in the realizations. The mean value average level seems to differ from zero, rather it is close to 0.5. This indicates that there is at least some of the realizations that have a mean value differing from zero. This is confirmed from some of the periodograms which show strong peaks at $f = 0$. It is then reasonable to include a stochastic level in the model, alternatively make sure that each realization of the data set has a mean value of zero.

An appropriate model could be,

$$X(t) = A0 + A\cos(2\pi 0.1t + \phi), \quad t = 0\ldots 99,$$

where $A0$ and A are probably independent stochastic variables and ϕ is uniform in the interval $[0, 2\pi]$.

X14 The figure shows the periodogram corresponding to all three of the proposed signals! When computing the Fourier transforms of the sequences, the differences will show up in the argument, the phase. These differences are then hidden in the squared absolute value, the periodogram. Please verify using the following MATLAB code, where `x` is the chirp signal, `y` is the impulse signal and `z` is a noise realization.

```
>> n=512;
>> x=cos(2*pi*0.0003*([0:n-1]').^2);
>> y=real(ifft(abs(fft(x)).*exp(j*2*pi*[[0:-1:-n/2]';[n/2-1:-1:1]']/n*200)));
>> b=rand(n/2,1)-0.5;
>> z=real(ifft(abs(fft(x)).*exp(j*2*pi*[b;0;-b(n/2:-1:2)])));
```

The following expression with `x` replaced for `y` and `z` produces the periodogram.

```
>> plot([0:n-1]/n,1/n*abs(fft(x,n)).^2)
```

X15 The spectral density is

$$R_X(f) = \pi e^{-2\pi|f|}.$$

The variance is

$$V[Y(t)] = 2\int_1^2 \pi e^{-2\pi|f|}df = e^{-2\pi} - e^{-4\pi},$$

and the expected value is

$$m_Y = H(0)m_X = 0 \cdot 2 = 0.$$

X16

a) The transform table with $\alpha = 1$ gives $r_X(\tau) = 1/(1+\tau^2)$.

b) $R_y(f) = |1 + if|^2\pi e^{-2\pi|f|} = (1 + f^2)\pi e^{-2\pi|f|}$.

c) A linear combination of Gaussian processes becomes a Gaussian process.

X17 The quote is given from $R_Y(f)/R_X(f) = |H(f)|^2$ as

$$R_Y(f) = |H(f)|^2 R_X(f).$$

The sequence can be simplified into

$$Y_t = -\frac{1}{3}X_{t+1} + \frac{2}{3}X_t - \frac{1}{3}X_{t-1},$$

and from definition the output from a linear filter is given as

$$Y_t = \sum_{u=-\infty}^{\infty} h(u)X_{t-u} = \sum_{u=-1}^{1} h(u)X_{t-u},$$

where the impulse response is identified as $h(-1) = -1/3$, $h(0) = 2/3$, and $h(1) = -1/3$. The frequency function is

$$H(f) = \sum_u h(u)e^{-i2\pi fu} = \frac{1}{3}e^{i2\pi f} + \frac{2}{3} - \frac{1}{3}e^{-i2\pi f} = \frac{2}{3}(1 - \cos(2\pi f)).$$

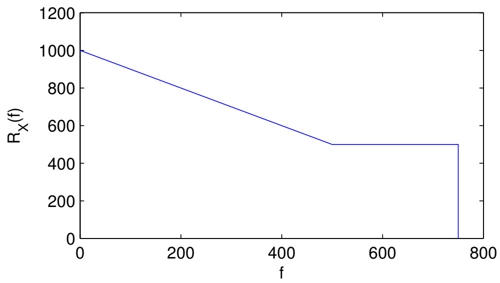
resulting in

$$\frac{R_Y(f)}{R_X(f)} = |H(f)|^2 = \frac{4}{9}(1 - \cos(2\pi f))^2.$$

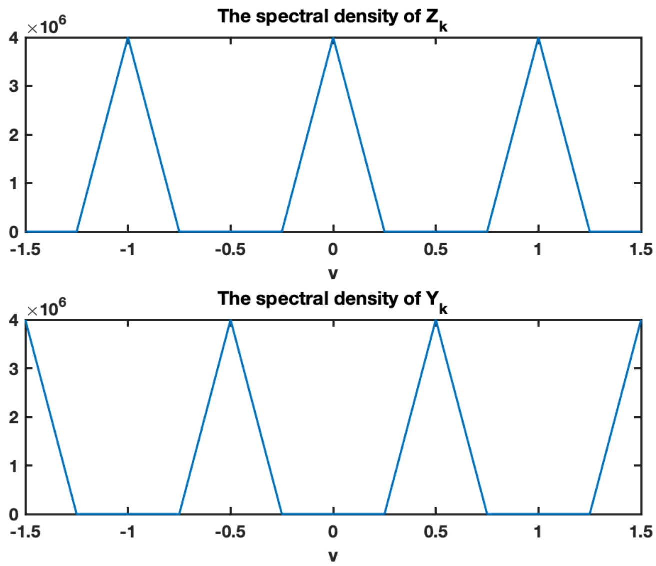
X18 a) The covariance function is given by

$$r_X(\tau) = \begin{cases} 10^6 & \tau = 0 \\ \frac{2}{(2\pi\tau)^2}(1 - \cos(2\pi 1000\tau)) & \tau \neq 0 \end{cases}$$

b) The sample frequency causes aliasing,



c) The sampled process Z_k, $k = 0, \pm 1, \pm 2 \ldots$ will have a periodic spectral density as shown in the first figure below expressed in the normalized frequency scale $\nu = f/f_s = fd$. The corresponding covariance function is $r_Z(k) = r_X(\tau)$, where $k = \tau/d$. We find the covariance function for the process Y_k to be $r_Y(k) = (-1)^k r_Z(k) = e^{i\pi k} r_Z(k) = e^{i2\pi 0.5k} r_Z(k)$. The transform table gives that $g(\tau)e^{i2\pi f_0\tau}$ has the Fourier transform $G(f - f_0)$. With $f_0 = 0.5$ we find the spectral density $R_Y(f) = R_X(f - 0.5)$, according to the second figure below.



X19 The derivatives of $r_X(\tau) = e^{-\tau^2/2}$ are $r'_X(\tau) = -\tau e^{-\tau^2/2}$ and $r''_X(\tau) = (\tau^2-1)e^{-\tau^2/2}$.

a) The variances are $V[X(t)] = V[X(t+2)] = r_X(0) = 1$, $V[X'(t)] = -r''_X(0) = 1$.

b) The covariance and cross-covariance functions are

$$C[X(t), X(t+2)] = r_X(2) = e^{-2},$$
$$C[X(t), X'(t)] = r'_X(0) = 0, \text{ Always the case, see remark 6.3 p. 149}$$
$$C[X'(t), X(t+2)] = -r'_X(2) = 2e^{-2}.$$

c) Rewrite $P(X(t) - X(t+2) + 2X'(t) < 0)$. For the new Gaussian process $Y(t) = X(t) - X(t+2) + 2X'(t)$ we get

$$P(Y(t) < 0) = \Phi(\frac{m_Y}{\sigma_Y}).$$

Usually, to find σ_Y the variance $V[Y(t)]$ is needed including variances covariance and cross-covariances as calculated above. However, as the expected value is $E[Y(t)] = E[X(t) - X(t+2) + 2X'(t)] = m_X - m_X + 2\cdot 0 = 0$, then

$$P(Y(t) < 0) = \Phi(\frac{0}{\sigma_Y}) = 0.5,$$

without calculating $V[Y(t)]$!

X20 The covariance function is (indefinite many times) differentiable and therefore the process is differentiable in quadratic mean. We have

$$C[X(t+2), X(t)] = r_X(2) = 3,$$
$$C[X(t+2), X'(t)] = r'_X(-2) = 0,$$
$$C[X(t), X'(t)] = 0,$$
$$V[X'(t)] = -r''_X(0) = 24\pi^2,$$
$$E[X(t+2) - 0.5X(t) - 2X'(t)] = 30 - 15 - 2\cdot 0 = 15,$$
$$V[X(t{+}2){-}0.5X(t){-}2X'(t)] = V[X(t{+}2)]{+}0.25V[X(t)]{+}4V[X'(t)]{-}C[X(t{+}2), X(t)]{+}0 = 96\pi^2{+}0.75$$
$$P(X(t+2) > 30 + X(t) + 2X'(t)) = 1 - P(X(t+2) - X(t) - 2X'(t) \le 30) =$$
$$= 1 - \Phi(\frac{30-15}{\sqrt{96\pi^2+0.75}}) = 1 - \Phi(0.49) = 1 - 0.688 = 0.312.$$

X21 The variance is given as

$$\begin{aligned}
V[m^*] &= V[\alpha\frac{X(0)+X(10)}{2}] + V[\frac{1-\alpha}{10}\int_0^{10} X(t)dt] + 2C[\alpha\frac{X(0)+X(10)}{2}, \frac{1-\alpha}{10}\int_0^{10} X(t)dt], \\
&= \frac{\alpha^2}{4}(V[X(0)] + V[X(10)] + 2C[X(0), X(10)]) + \frac{(1-\alpha)^2}{100}\int_0^{10}\int_0^{10} C[X(t), X(s)]dtds, \\
&+ \frac{\alpha(1-\alpha)}{10}(\int_0^{10} C[X(0), X(t)]dt + \int_0^{10} C[X(10), X(t)]dt), \\
&= \frac{\alpha^2}{4}(2+2e^{-30}) + \frac{(1-\alpha)^2}{100}2\int_0^{10}\int_0^{t} e^{-3(t-s)}dsdt + \frac{\alpha(1-\alpha)}{5}\int_0^{10} e^{-3t}dt, \\
&= \frac{\alpha^2}{4}(2+2e^{-30}) + \frac{2(1-\alpha)^2}{100}(\frac{10}{3} - \frac{1-e^{-30}}{9}) + \alpha(1-\alpha)\frac{1-e^{-30}}{15}, \\
&\approx \frac{\alpha^2}{2} + (1-\alpha)^2\frac{29}{450} + \frac{\alpha(1-\alpha)}{15}.
\end{aligned}$$

Derivation for the optimal α,

$$\frac{\partial V}{\partial \alpha} = \alpha - (1-\alpha)\frac{29}{225} + \frac{1}{15} - \frac{2\alpha}{15} = 0$$

gives

$$\alpha = \frac{1}{16}.$$

X22 We start with the definition of the output signal of a linear filter:

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} h(u)X(t-u)du, \\ &= \int_0^1 X(t-u)du, \end{aligned}$$

where $X(t)$ is the input signal. We would like to compute the cross-covariance between the input and the output (in that order), which is, by its definition:

$$\begin{aligned} r_{X,Y}(\tau) &= C[X(t),\ Y(t+\tau)] = C\left[X(t),\ \int_0^1 X(t+\tau-u)du\right], \\ &= \int_0^1 C\left[X(t),\ X(t+\tau-u)\right] du = \int_0^1 r_X(\tau-u)du, \\ &= \int_0^1 \frac{1}{1+(\tau-u)^2}du = \Big[-\arctan(\tau-u)\Big]_0^1, \\ &= \arctan(\tau) - \arctan(\tau-1). \end{aligned}$$

X23 The Yule-Walker equations give

$$\begin{cases} 3 + a_1 &= \sigma^2 \\ 1 + 3a_1 + a_2 &= 0 \\ a_1 + 3a_2 &= 0 \end{cases}$$

with solution $a_2 = 1/8$ and $a_1 = -3/8$. From the Yule-Walker equations

$$r_X(3) = -a_1 r_X(2) - a_2 r_X(1) = 3/8 \cdot 0 - 1/8 \cdot 1 = -1/8.$$

X24 The covariance function is given from

$$r_Y(\tau) = C[Y_t, Y_{t+\tau}] = C[X_t - 2X_{t-1} + X_{t-2}, X_{t+\tau} - 2X_{t+\tau-1} + X_{t+\tau-2}],$$

where the variance $r_X(0) = 2$ and $r_X(\tau) = 0$ for all other values of τ. We get

$$r_Y(\tau) = \begin{cases} 12, & \tau = 0, \\ -8, & \tau = \pm 1, \\ 2, & \tau = \pm 2, \\ 0, & \text{for all other values.} \end{cases}$$

The spectral density is given from the Fourier transform of the covariance function,

$$
\begin{aligned}
R_Y(f) &= \sum_{\tau=-\infty}^{\infty} r_Y(\tau)e^{-i2\pi f\tau}, \\
&= 2e^{i2\pi f2} - 8e^{i2\pi f} + 12 - 8e^{-i2\pi f} + 2e^{-i2\pi f2}, \\
&= 12 - 16\cos(2\pi f) + 4\cos(4\pi f).
\end{aligned}
$$

X25 The spectral density is given from the known covariance function as

$$R_X(f) = \sum_{\tau} r_X(\tau)e^{-i2\pi f\tau} = 2 + e^{-i2\pi f3} + e^{i2\pi f3}.$$

The spectral density output of the filter with impulse response coefficients, $b_0, b_1, \dots, b_q$ for the white noise spectral density input $R_e(f) = \sigma^2 = 1$ is given from

$$R_X(f) = |H(f)|^2 R_e(f) = |\sum_{u=0}^{q} b_u e^{-i2\pi fu}|^2.$$

Reformulating into

$$R_X(f) = \sum_{u=0}^{q}\sum_{v=0}^{q} b_u b_v e^{-i2\pi f(u-v)}$$

and comparing with $R_X(f) = 2 + e^{-i2\pi f3} + e^{i2\pi f3}$ yields

$$R_X(f) = b_0^2 + b_3^2 + b_0 b_3 e^{i2\pi f3} + b_0 b_3 e^{-i2\pi f3},$$

where only the possible non-zero coefficients are retained. The solution becomes $b_0 = b_3 = 1$ and all other $b_k = 0$.

X26 Only B and D can be spectral densities, they are positive and the scale range between 0 and 0.5. Then only C and E can be covariance functions and A and F are realizations. D has one low and one high strong frequency, which corresponds with the covariance function in C and the realization in A, and is most possibly an AR(4)-process. The covariance function in E is probably an MA(2)-process, with corresponding spectral density B and realization F.

X27 The spectral density is positive which coincides with D,F,H. The covariance function is largest at zero, i.e., B,G,I. Accordingly, A,C,E are realizations. The covariance function I belongs to a MA(2)-process as it is zero when $\tau > 2$ where B and G belong to AR-processes. The highest frequencies are found in E connected to F. Somewhat lower frequencies are found in A and D where C must be connected with H with the lowest frequency content. The covariance function (the MA(2)-process) fits with the spectral density F where the other spectral densities have peaks (AR-processes). Realization A fits with the covariance function B, where the realization C has the same period as G. D is probably an AR(4)-process as we see two frequency peaks and H is possibly an AR(5)-process as we see two frequency peaks and a peak at the frequency zero. In summary: MA(2)-process E,F,I, AR(4)-process D,A,B, AR(5)-process H,C,G.

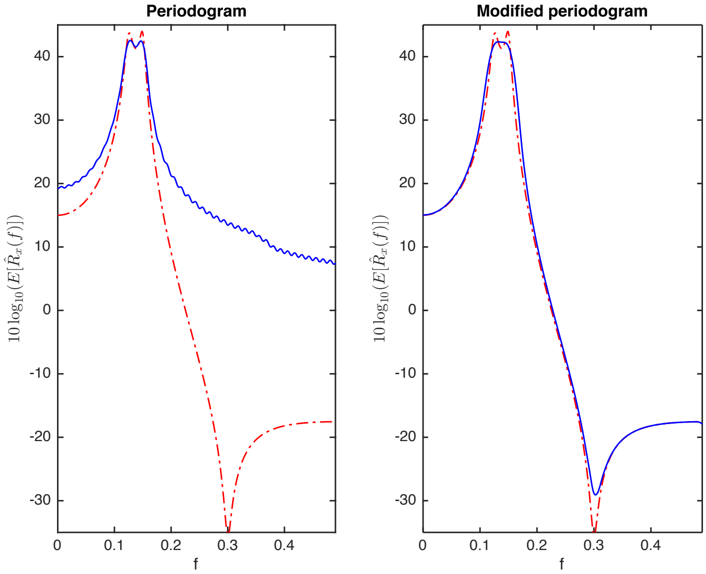
X28 The pole-zero plots and the spectral densities can be combined as A-II, B-III and C-I. The strongest peak in III has a high frequency which connects to the covariance function in 2. The spectral density in I is more damped than the one in II although they have the same frequency, which corresponds to the more damped covariance function in 1 compared to the one in 3. Therefore I and 1 belong together and II and 3. The number of poles found in the upper half, with angle 0 to π including the real axis, correspond to the number of peaks in the spectral density, i.e., A-II-3 is AR(4), B-III-2 is AR(3), C-I-1 is AR(2).

X29 The following combinations belong together: A-3-α which is an MA(2)-process, as there are two zeros and the covariance function is zero for $\tau > 2$ and the spectral density shows a clear zero (low value) at the frequency f corresponding to the angle $2\pi f$ i A. B-2-γ is an AR(4)-process as there is 4 poles and C-1-β is an AR(2)-process. The covariance function in 2 has a lower frequency than the one in 1 which correspond to the notes in B and C. Similarly, the spectral density in γ has the two peaks at a lower frequency than the peaks in the spectral density in β.

X30 Periodogram 1 belongs to a high frequency process as it has the main power at $f > 0.3$, which could be B, with the pole at a high angle. Periodogram 2 is the most low-frequency process, which then must be C, which has the pole at a low angle. Periodogram 3 has most power at $f = 0.25$ which corresponds to A.

X31 The logarithmic scaled spectrum estimates indicate where the zeros are located, which are where the differences are found. We then combine periodogram 1 with C, 2 with A, 3 with B, according to the location of the zeros. The linear called periodograms do not give any information on the zeros location and as the power from the poles are found around $f = 0.25$ in all cases, the figures do not give enough information. It is always advisable to visualize in both scales!

X32 The periodogram shows two peaks (poles) close to $f = 0.1$ where the modified periodogram shows a broader peak but also one zero at $f = 0.3$. As we know that the periodogram has a more narrow mainlobe than the modified periodogram and therefore can resolve peaks that are close, the assumption of two peaks is reasonable. The modified periodogram, however, has lower sidelobes, which makes it possible to also resolve zeros, and the zero seen at $f = 0.3$ is probably one zero. We find the mirrored picture on the negative side of the frequency axis, and therefore in total 4 poles and 2 zeros, i.e., an ARMA(4,2)-model.



X33 The impulse response coefficients, $h(0) = a_0$ and $h(1) = a_1$ are identified from the given frequency function. The output from the filter is

$$Z_t = a_0 Y_t + a_1 Y_{t-1},$$

and the squared error becomes

$$E\left[(a_0 Y_t + a_1 Y_{t-1} - S_t)^2\right] =$$

$$E\left[(a_0 S_t + a_1 S_{k-1} - S_t + a_0 N_t + a_1 N_{t-1})^2\right].$$

As all processes are zero-mean we rewrite and expand the expression as

$$a_0^2 V[S_t] + a_1^2 V[S_{t-1}] + a_0^2 V[N_t] + a_1^2 V[N_{t-1}] + V[S_t] +$$

$$2a_0 a_1 C[S_t, S_{t-1}] + 2a_0 a_1 C[N_t, N_{t-1}] - 2a_0 v V[S_t] - 2a_1 C[S_t, S_{t-1}] =$$

$$2a_0^2 + 2a_1^2 + 1 - 2a_0 - a_1 = \epsilon.$$

Differentiation yields

$$\begin{aligned} \frac{\partial \epsilon}{\partial a_0} &= 4a_0 - 2 = 0 \quad \rightarrow a_0 = \frac{1}{2} \\ \frac{\partial \epsilon}{\partial a_1} &= 4a_1 - 1 = 0 \quad \rightarrow a_1 = \frac{1}{4} \end{aligned}$$

X34 a) The Wiener filter is given by

$$H(f) = \frac{R_s(f)}{R_S(f) + R_N(f)},$$

where

$$R_S(f) = \sum_\tau r_S(\tau) e^{-i2\pi f\tau} = 2 - 2\cos(2\pi f),$$

and

$$R_N(f) = \sum_\tau r_N(\tau) e^{-i2\pi f\tau} = 2 + 2\cos(2\pi f).$$

We get

$$H(f) = 0.5 - 0.5\cos(2\pi f) = 0.5 - 0.25(e^{-i2\pi f} + e^{i2\pi f}) \quad -0.5 \le f < 0.5,$$

and the impulse response $h(0) = 0.5$, $h(\pm 1) = -0.25$ and zero for all other t.

b) For the given impulse response

$$\min_{a,b} E[(aX_t + bX_{t+1} - S_t)^2],$$

$$\min_{a,b} E[(aS_t + 0.5aS_{t-1} + aN_t + bS_{t+1} + 0.5bS_t + bN_{t+1} - S_t)^2],$$

$$\min_{a,b}((a-1+0.5b)^2 + 0.25a^2 + b^2)r_S(0) + (a^2+b^2)r_N(0) +$$
$$+2(0.5a(a-1+0.5b) + b(a-1+0.5b))r_S(1) + 2abr_N(1)).$$

We get

$$\min f(a,b) = \frac{7}{2}a^2 + \frac{7}{2}b^2 + \frac{3}{2}ab - 3a + 2.$$

Differentiation with respect to a, b yield

$$\begin{aligned} \frac{\partial f}{\partial a} &= 7a + \frac{3b}{2} - 3 = 0, \\ \frac{\partial f}{\partial b} &= 7b + \frac{3}{2}a = 0, \end{aligned}$$

resulting in $b = -18/187 \approx -0.0963$ and $a = 84/187 \approx 0.449$. The impulse response is $h(0) = 0.449$, $h(-1) = -0.0963$ and zero for all other t.

X35 The causal impulse response is given as $h(0) = c$, $h(1) = 2c$, $h(2) = 3c$, $h(3) = 4c$, $h(4) = 5c$, where the constant is assumed to be $c = 1$ and the decision time is $T = 4$. The expected value for received signal when the signal is sent becomes

$$E[Y_T \mid \text{signal sent}] = E[\sum_{u=0}^{4} h(u)(s_{T-u} + N_{T-u})] = \left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right) = 55.$$

With equal error probabilities the threshold is $k = E[Y_T \mid \text{signal sent}]/2 = 27.5$. The variance for of the received signal is

$$V[Y_T] = V[N_4 + 2N_3 + 3N_2 + 4N_1 + 5N_0] = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The equal error probabilities become

$$\alpha = P(Y_T > 27.5 \mid \text{no signal sent}) = 1 - \Phi(27.5/\sqrt{55}) = 1 - \Phi(3.708) = 1.044 \cdot 10^{-4}$$
$$\beta = P(Y_T < 27.5 \mid \text{signal sent}) = \Phi((27.5 - 55)/\sqrt{55}) = 1 - \Phi((55 - 27.5)/\sqrt{55}) = 1.044 \cdot 10^{-4}$$

X36 a) The coefficients $h(t)$ are determined by $h(t) = s_{T-t}$. If $T = 2$ we get the causal filter

$$h(t) = s_{2-t} = \begin{cases} 4 & t = 0, \\ 2 & t = 1, \\ 1 & t = 2. \end{cases}$$

For the case when signal is sent we get

$$\begin{aligned}
E[Y_2] &= E[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = \sum_{u=0}^{2} h(u)^2 = 16 + 4 + 1 = 21, \\
V[Y_2] &= V[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = V[\sum_{u=0}^{2} h(u)N_{2-u}] = \\
&= \sum_{u=0}^{2} h(u)^2 V[N_{2-u}] = (16 + 4 + 1)4 = 84.
\end{aligned}$$

We get $Y_2 \in N(0,\, 84)$ if no signal is sent and $Y_2 \in N(21,\, 84)$ if signal is sent. The error probabilities become equal with the decision threshold $k = \frac{21}{2} = 10.5$ and

$$\beta = \alpha = P(Y_2 > 10.5 \mid \text{no signal sent}) = 1 - \Phi(\frac{10.5 - 0}{2\sqrt{21}}) \approx 0.125.$$

b) The optimal coefficients are received from the solution of $s_{2-t} = \sum_{u=0}^{2} h(u) r_N(t-u)$ for $t = 0,\, 1,\, 2,\, (c = 1)$.

$$\begin{aligned}
4 &= 4h(0) + h(1), \\
2 &= h(0) + 4h(1) + h(2), \\
1 &= h(1) + 4h(2),
\end{aligned}$$

with solution $h(0) = 0.95$, $h(1) = 0.21$ and $h(2) = 0.20$.

$$\begin{aligned}
E[Y_2] &= E[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = h(0)s_2 + h(1)s_1 + h(2)s_0 = 4.41, \\
V[Y_2] &= V[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = V[\sum_{u=0}^{2} h(u)N_{2-u}] = \sum_{u=0}^{2}\sum_{v=0}^{2} h(u)h(v)r_N(u-v) = \\
&= 4(h(0)^2 + h(1)^2 + h(2)^2) + 2h(0)h(1) + 2h(1)h(2) = 4.41.
\end{aligned}$$

The error probabilities with the threshold $k = \frac{E(Y_2)}{2} = 2.20$ become

$$\beta = \alpha = P(Y_2 > 2.20 \mid \text{no signal sent}) = 1 - \Phi(\frac{2.20}{\sqrt{4.41}}) \approx 0.147.$$

If we instead had used the filter from a) (which not is optimal any longer as the noise is not white) we receive for signal sent

$$\begin{aligned}
E[Y_2] &= E[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = h(0)s_2 + h(1)s_1 + h(2)s_0 = 21, \\
V[Y_2] &= V[\sum_{u=0}^{2} h(u)(s_{2-u} + N_{2-u})] = V[\sum_{u=0}^{2} h(u)N_{2-u}] = \sum_{u=0}^{2}\sum_{v=0}^{2} h(u)h(v)r_N(u-v) = \\
&= 4(h(0)^2 + h(1)^2 + h(2)^2) + 2h(0)h(1) + 2h(1)h(2) = 104.
\end{aligned}$$

The error probabilities with the threshold $k = \frac{E(Y_2)}{2} = 10.5$ become

$$\beta = \alpha = P(Y_2 > 10.5 \mid \text{no signal sent}) = 1 - \Phi(\frac{10.5}{\sqrt{104}}) \approx 0.152,$$

which is not much, but still larger than the optimal value 0.147.