Stationary stochastic processes

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Lecture 2

September 10 2024

Schedule for today

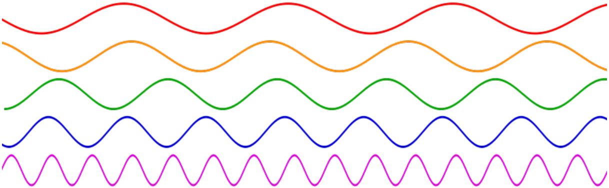
- Definition of spectral density and motivation
- Spectral density in continuous time
- Course information for computer exercises
- Spectral density in discrete time
- Sampling and aliasing
- Normalized frequency

A crash course on frequency

A cosine signal is defined as

$$x(t) = A\cos(2\pi f_0 t + \phi),$$

where f_0 represents the frequency. With increasing frequency the signal varies faster as the period length $T = 1/f_0$ becomes shorter.



Spectral density function

Theorem 4.1: If the covariance function $r(\tau), \tau \in \mathbb{R}$ of a stationary stochastic process $X(t), t \in \mathbb{R}$ is continuous, we have a corresponding **spectral density function** $R(f)$ such that

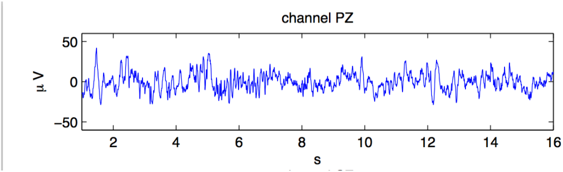
$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df,$$

and

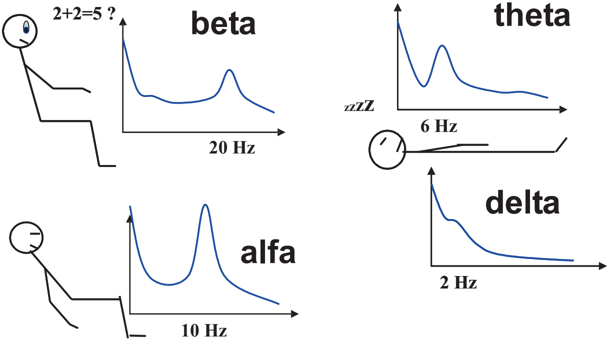
$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} d\tau.$$

The spectral density is always real-valued, positive, symmetric and integrable for real-valued processes.

Estimation of EEG

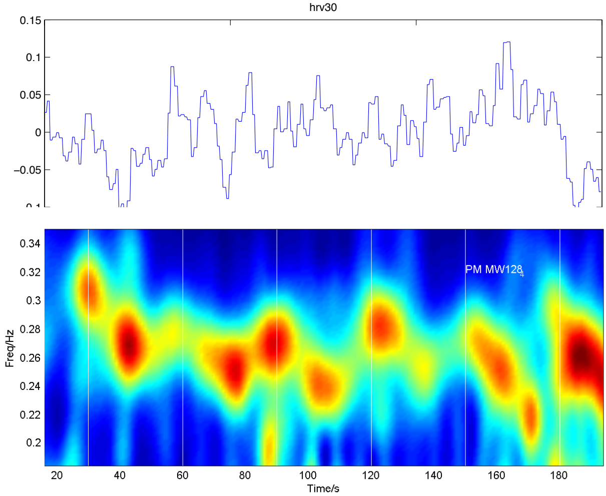


Different states in our brain can be classified with use of spectral information of the EEG-signal.



Estimation of HRV

The heart rate variability (HRV) is difficult to analyse in the time domain. A short-time spectral estimate (spectrogram) shows a variation in frequency pattern caused by fight-or-flight mode state.



Spectral density function

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The spectral density is always real-valued, positive, symmetric and integrable for real-valued processes.

The variance of the process is related to the spectral density as

$$r(0) = \int_{-\infty}^{\infty} R(f) df.$$

Frequency and spectral representation

We can divide the variance of the process into the variances at specific frequencies.

Exercise

For a low-frequency noise spectral density, $R(f) = 1/2$ for $-1 \leq f \leq 1$ and zero for all other values. Calculate the covariance function

Solution: The covariance function is given by

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f\tau} df = \frac{1}{2} \int_{-1}^{1} e^{i2\pi f\tau} df = \frac{\sin(2\pi\tau)}{2\pi\tau}.$$

Exercise

Exercise: The random harmonic function,

$$X(t) = A\cos(2\pi f_0 t + \phi), t \in \mathbb{R},$$

with $A > 0$ and $\phi \in U(0, 2\pi)$ has the spectral density

$$R(f) = \frac{E[A^2]}{4}(\delta(f - f_0) + \delta(f + f_0)),$$

where $\delta(f)$ is the Dirac delta function. What is the covariance function?

Solution: Use the definition giving

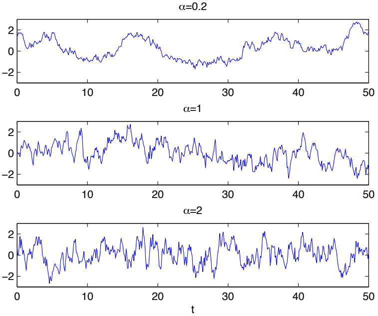
$$r(\tau) = \frac{E[A^2]}{4}(e^{i2\pi f_0 \tau} + e^{-i2\pi f_0 \tau}) = \frac{E[A^2]}{2}\cos(2\pi f_0 \tau), \quad \tau \in \mathbb{R}.$$

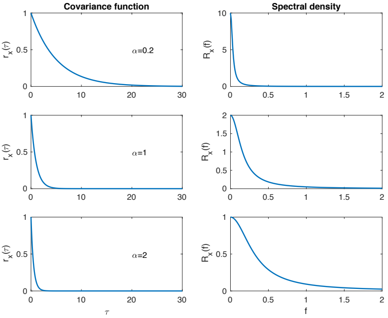
Hint: Useful for the solution of exercise 4.3 in the course book.

The Ornstein-Uhlenbeck process

For the Ornstein-Uhlenbeck process, often used for modelling in physics, evolutionary biology and financial mathematics, has the covariance function is $r(\tau) = e^{-\alpha|\tau|}$ and the spectral density $R(f) = \frac{2\alpha}{\alpha^2+(2\pi f)^2}$.

Examples:





The table of formulas

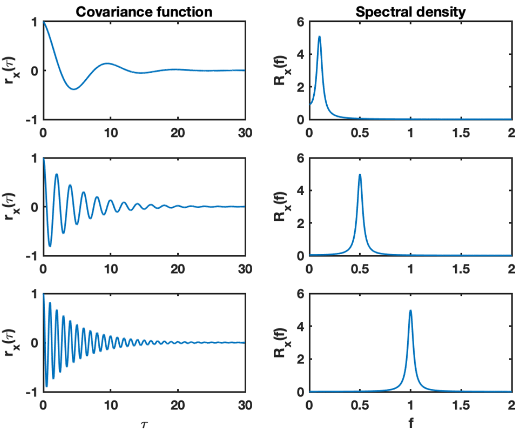
Someone else has computed what we need... Use the Fourier transform table to avoid heavy calculations for continuous time processes.

Fourier transforms

$g(\tau) \qquad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} g(\tau)\, d\tau$
$e^{-\alpha\|\tau\|}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}\, e^{-2\pi\alpha\|f\|}$
$\|\tau\| e^{-\alpha\|\tau\|}$	$2\, \frac{(\alpha^2-(2\pi f)^2)}{(\alpha^2+(2\pi f)^2)^2}$
$\|\tau\|^k e^{-\alpha\|\tau\|}$	$\frac{k!}{(\alpha^2+(2\pi f)^2)^{k+1}} \{(\alpha + i2\pi f)^{k+1} + (\alpha - i2\pi f)^{k+1}\}$
$e^{-\alpha\tau^2}$	$\sqrt{\pi/\alpha}\exp(-\frac{(2\pi f)^2}{4\alpha})$
$e^{-\alpha\|\tau\|}\cos(2\pi f_0 \tau)$	$\frac{\alpha}{\alpha^2+(2\pi f_0-2\pi f)^2} + \frac{\alpha}{\alpha^2+(2\pi f_0+2\pi f)^2}$

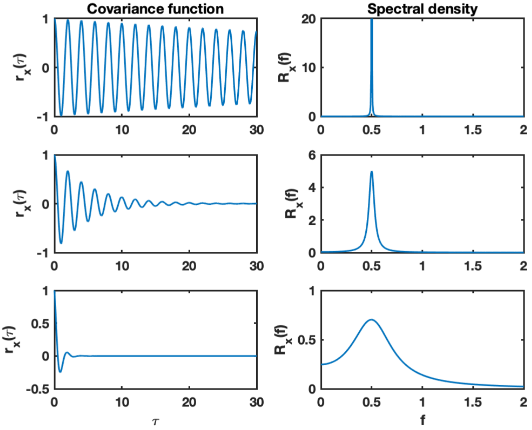
Examples

The main period of the covariance function (and data) is the strongest frequency in the spectral density.



More examples

A covariance function with high correlation at large values of τ is connected to a spectral density with strong peaks.



Exercise

A continuous time stationary process $X(t)$, $t \in \mathbb{R}$, has the spectral density

$$R(f) = \begin{cases} 1 - \frac{|f|}{4} & |f| \leq 4 \\ 0 & |f| > 4. \end{cases}$$

Determine the covariance function $r(\tau)$ of the process.

Solution: Use the Fourier table with switched variables, giving $r(0) = 4$, and

$$r(\tau) = \frac{1}{2(2\pi\tau)^2}(1 - \cos(8\pi\tau)), \quad \tau \neq 0.$$

$\begin{cases} \alpha & \text{if} \quad \tau = 0 \\ \frac{\sin(2\pi\alpha\tau)}{2\pi\tau} & \text{if} \quad \tau \neq 0 \end{cases}$	$\begin{cases} 1/2 & \text{if} \quad \lvert f\rvert \leq \alpha \\ 0 & \text{if} \quad \lvert f\rvert > \alpha \end{cases}$
$\begin{cases} 1 - \alpha\lvert\tau\rvert & \text{if} \quad \lvert\tau\rvert \leq \frac{1}{\alpha} \\ 0 & \text{if} \quad \lvert\tau\rvert > \frac{1}{\alpha} \end{cases}$	$\begin{cases} \frac{1}{\alpha} & \text{if} \quad f = 0 \\ \frac{2\alpha}{(2\pi f)^2}\left(1 - \cos\left(\frac{2\pi f}{\alpha}\right)\right) & \text{if} \quad f \neq 0 \end{cases}$
$g(\tau)h(\tau)$	$G(f) * H(f) = \int G(\nu)H(f-\nu)d\nu$
$g(\tau) * h(\tau) = \int g(t)h(\tau - t)dt$	$G(f)H(f)$

Computer exercise 1

2024 HT/Autumn

- Home
- Announcements
- **Assignments**
- Grades
- People
- Syllabus
- Modules
- Canvas Survey

The computer exercises are preferably performed in groups of two students. If you don't have a laptop, contact the coordinator, Oskar Keding, Canvas E-mail.

1) Schedule yourself for the 15-minutes presentation in the 'calendar' (to the left, available from September 9) and press the button 'find appointments' (to the right). Make sure that both of you are scheduled at the same occasion! (Usually more than one teaching assistant is available and therefore several pairs of students can sign up at one occasion at the same time.)

2) A mandatory preparation for computer exercise 1 is found at https://quizms.maths.lth.se/ where you create a personal account using your Stil-login. The test opens September 9. When finished, document that you have passed the test, e.g. with a screenshot.

3) Work with computer exercise 1, found under 'Course material' above. Document your work with accessible scripts, figures and comments to be able to show and explain your results for the teaching assistant during your 15-minutes presentation slot.

Use the discussion forum and the question hours for assistance with the exercise, Thursdays 10-12 (MH:227), Thursdays 13-15 (MH:228) and Fridays 13-15 (MH:229).

4) The whole group should be prepared and appear at the scheduled session. Bring your own computer! Show the documentation of your passed Mozquizto-test.

Computer exercise 1

Time remaining

00:59:02

Questions

- 1. Three different processes 1p
- 2. Estimation of mean value 1p
- 3. Confidence interval 1p
- 4. White noise properties 1p
- 5. Sampling and aliasing 1p

Finish test

Question 4. White noise properties

White noise properties

Consider a discrete time stationary white noise process $X_n, n = 0, \pm 1, \pm 2 \ldots$, with zero mean and standard deviation 2. What is the spectral density?

- The spectral density is 4 for $f = 0$ and zero for all other values.
- The spectral density is 2 for $-0.5 < f \leq 0.5$.
- The spectral density is 4 for $-0.5 < f \leq 0.5$.
- The spectral density is 2 for $f = 0$ and zero for all other values.

Computer exercise 1

1 Introduction to estimation of expected value, covariance function and spectral density

In this section you will learn how to use the basic techniques for estimation of important functions of a stationary stochastic process.

1.1 Estimation of expected value

Load the file `data.mat` using the command `load data`. The file contains three realizations, each of 100 samples of Gaussian white noise. Plot the sequences,

```
>> plot([data1 data2 data3])
```

Q1. What conclusions can you make from this figure? Is it reasonable to assume that all realizations are from the same stationary stochastic process? Does this process have zero mean?

Computer exercise 1

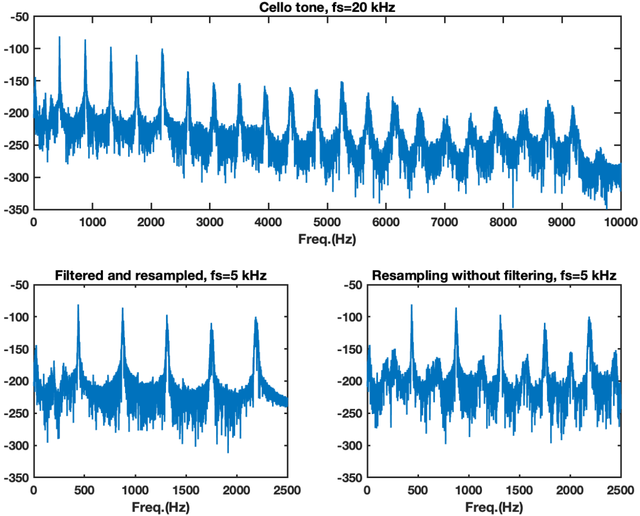
2 Instruments in a symphony orchestra

The sound from an acoustic instrument consists of a fundamental frequency, often termed a keynote, and some overtones or harmonics. The phases of the overtones typically depend on the instrument and are partly correlated with the oscillation of the keynote. This, together with the relation between the power of the overtones, produces the perceived sound of the instrument. If the keynote has frequency f_0, the frequencies of the overtones will depend on the type of instrument. For string instruments, the overtones can be well represented as $f_k = kf_0$, $k = 1, 2, \ldots$ The examples used in this exercise are recorded by the Philharmonia musicians, and are just a few examples of many, found at their webpage[1].

2.1 Keynotes and overtones

Download the file `cellodiffA`, where you find three variables, `celloA2`, `celloA3` and `celloA4`, each representing a recording of the note A. The sampling frequency is `fs=44100` Hz. Listen to the tones by using the command `sound`. Note: if the sampling frequency is not given as an

Computer exercise 1



Discrete time and spectral density

Theorem 4.4: For every covariance function $r(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$, of a stationary stochastic process, X_t, $t = 0, \pm 1, \pm 2, \ldots$, there exists a spectral density function $R(f)$ such that

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i 2\pi f \tau} df, \quad \tau = 0, \pm 1, \pm 2, \ldots,$$

and

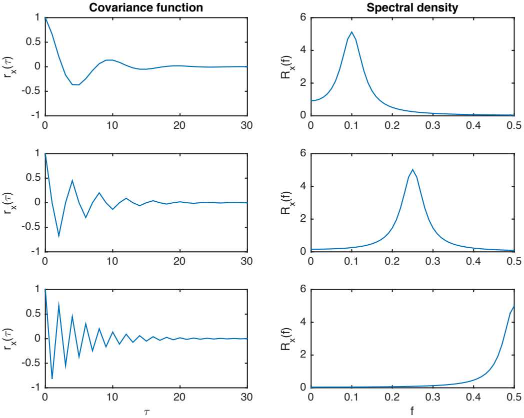
$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i 2\pi f \tau}, \quad -1/2 < f \leq 1/2.$$

The spectral density is always real-valued, positive, symmetric and integrable for real-valued processes.

The variance of the process is related to the spectral density as

$$r(0) = \int_{-1/2}^{1/2} R(f) df.$$

Note the restricted frequency scale



The white noise process

For a sequence of independent stochastic variables e_t, $t = 0, \pm 1, \pm 2 \ldots$ (white noise), the covariance function is

$$r(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots \end{cases}$$

We find the spectral density as

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} = r(0) = \sigma^2, \;\; -1/2 < f \leq 1/2.$$

Exercise

Last week we calculated the covariance function of the stationary stochastic process,

$$X_t = 2e_t - e_{t-1}, \quad t = 0, \pm 1, \pm 2, \ldots,$$

to be $r(0) = 10$, $r(\pm 1) = -4$ and zero for all other values.
The process e_t is a stationary white noise process with $E[e_t] = 0$ and $V[e_t] = 2$, $t = 0, \pm 1, \pm 2, \ldots$.

Now, calculate the spectral density of the discrete time process X_t.

Solution: Use the definition,

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} = 10 - 8\cos(2\pi f), \;\; -1/2 < f \leq 1/2.$$

Sampling - connection in time

The process $X(t)$, $t \in \mathbb{R}$, is stationary with covariance function $r_X(\tau)$, $\tau \in \mathbb{R}$.

Then the sampled sequence $Z_t = X(t)$, $t = 0, \pm d, \pm 2d, \ldots$, has the same covariance function as $X(t)$ at the corresponding values of $\tau = t - s = 0, \pm d, \pm 2d, \ldots$, i.e.

$$r_Z(\tau) = C[Z_s, Z_t] = C[X(s), X(t)] = r_X(\tau).$$

The Nyquist-Shannon sampling theorem

If $f_s \geq 2f_{max}$, where $f_s = 1/d$, the spectral density for the sampled process Z_t is given by

$$R_Z(f) = R_X(f), \quad -f_s/2 < f \leq f_s/2,$$

and the original continuous time signal can be perfectly recovered from the corresponding discrete time samples.

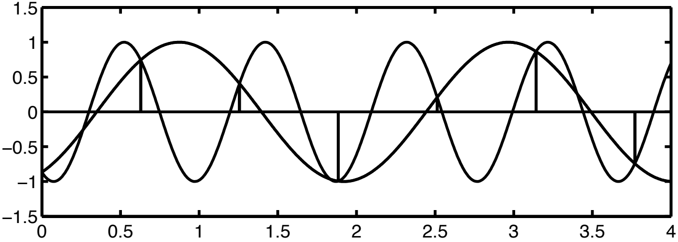
If the sampling frequency

$$f_s < 2f_{max},$$

the resulting spectral density of Z_t will affected by **aliasing**.

An example of aliasing

A high frequency signal will be interpreted as a low frequency signal if the sampling frequency is too small. The sampled signal will be aliased.



Sampling - connection in frequency

Theorem 4.5: The covariance function, $r_Z(\tau)$, $\tau = 0, \pm d, \pm 2d, \dots$ is given from

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

where the spectral density for the sampled process Z_t is

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f + kf_s) \quad -f_s/2 < f \leq f_s/2.$$

A nice proof is found at p. 97-98 in the course book!

Exercise

A stationary continuous time process $X(t),\ t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \begin{cases} 1 - \frac{|f|}{4} & |f| \leq 4 \\ 0 & |f| > 4. \end{cases}$$

The process $X(t)$ is sampled with $f_s = 6$ resulting in the discrete time process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$.

a) Is the sampling frequency large enough to avoid aliasing?

b) Which is the smallest possible sampling frequency to avoid aliasing?

c) Determine the spectral density for the sampled process Z_t when $f_s = 6$.

We use

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f + kf_s) \quad -f_s/2 < f \leq f_s/2.$$

Example

A cosine signal with $f_0 = 60$ Hz,

$$X(t) = \cos(2\pi 60 t), \quad t \in \mathbb{R},$$

is sampled with $f_s = 90$ Hz. We sample at time points $t = nd = n/f_s$ and find a discrete time signal

$$Y_n = \cos(2\pi \frac{60}{90} n) = \cos(2\pi \frac{2}{3} n) = \cos(2\pi \frac{1}{3} n) = \cos(2\pi \frac{30}{90} n).$$

The sampling causes aliasing to a discrete time signal with normalized frequency $\nu_0 = 1/3$ which corresponds to a continuous time signal of $f_0 = 30$ Hz.

Remember that the spectral density $R(f)$ of a discrete time process is defined only for $|f| \leq 1/2$.

More examples

The cosine signals are all sampled with different sample frequencies, i.e. $t = nd = n/f_s$, with $f_s = 180, 120, 60$. We study the resulting discrete time signals

$$
\begin{aligned}
X(t) = \cos(2\pi 60 t) &\quad \rightarrow \quad Y_n = \cos(2\pi \frac{60}{180} n) = \cos(2\pi \frac{1}{3} n) \\
X(t) = \cos(2\pi 60 t) &\quad \rightarrow \quad Y_n = \cos(2\pi \frac{60}{120} n) = \cos(2\pi \frac{1}{2} n) = (-1)^n \\
X(t) = \cos(2\pi 60 t) &\quad \rightarrow \quad Y_n = \cos(2\pi \frac{60}{60} n) = \cos(2\pi n) = 1
\end{aligned}
$$

Remember that the spectral density $R(f)$ of a discrete time process is defined only for $|f| \leq 1/2$.

Normalized frequency

The sampled covariance function, $r_Z(\tau)$, $\tau = 0 \pm d, \pm 2d, \ldots$,

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

is converted to the covariance function of a discrete time covariance function $r_Y(n)$, $n = 0, \pm 1, \pm 2, \ldots$ using $\tau = nd$,

$$r_Y(n) = r_Z(nd) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f n d} df = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi n f / f_s} df.$$

With the change of variables $f = \nu f_s$, we find

$$r_Y(n) = f_s \int_{-1/2}^{1/2} R_Z(\nu f_s) e^{i2\pi \nu n} d\nu = \int_{-1/2}^{1/2} R_Y(\nu) e^{i2\pi \nu n} d\nu,$$

where $R_Y(\nu) = f_s R_Z(\nu f_s)$. The normalized frequency $\nu = fd = f/f_s$ represents the frequency of the discrete time process.

Canvas course information