Stationary stochastic processes

Maria Sandsten

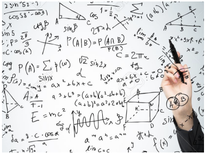
Lecture 1

September 3 2024

Short CV-Maria Sandsten

MSc in Electrical Engineering (1989). PhD in Signal Processing (1996). Professor in Statistical Signal Processing (2010).

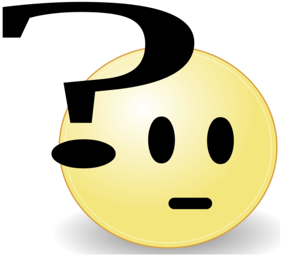
Research topic: Tailored spectral analysis methods for estimation, detection and classification of stationary and non-stationary processes.



Application areas over the years: machine learning of soundscapes for ecological monitoring, brain computer interfaces for sound processing, detection of acoustic vibrations from mining activities, bird song syllable clustering, characterization of dolphin echo location signals, speaker recognition, heart rate variability power estimation and classification of decay in living trees.

Stationary Stochastic Processes

- What is a **stochastic process**?
- When is a **stochastic process stationary**?
- Why do we want a **stochastic process** to be **stationary**?

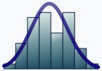


What is a stochastic process?

A stochastic process is a **statistical model of time series data**.

Statistics (from German: *Statistik*, orig. "description of a state, a country")[1][2] is the discipline that concerns the collection, organization, analysis, interpretation, and presentation of data.[3][4][5] In applying statistics to a scientific, industrial, or social problem, it is conventional to begin with a statistical population or a statistical model to be studied. Populations can be diverse groups of people or objects such as "all people living in a country" or "every atom composing a crystal". Statistics deals with every aspect of data, including the planning of data collection in terms of the design of surveys and experiments.[6]

Statistics



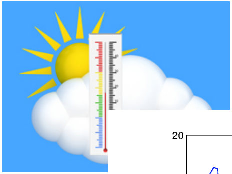
Outline · Statisticians · Glossary · Notation · Journals · Lists of topics · Articles · Category · $\sqrt{x}$ Mathematics portal

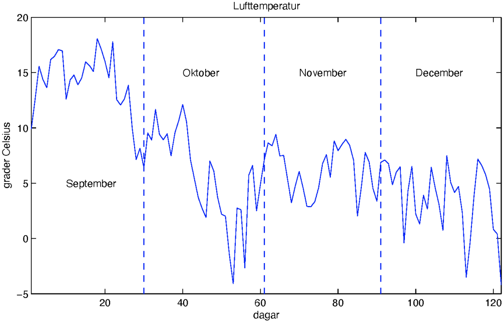
V · T · E

Part of a series on
Mathematics

What is a stochastic process?

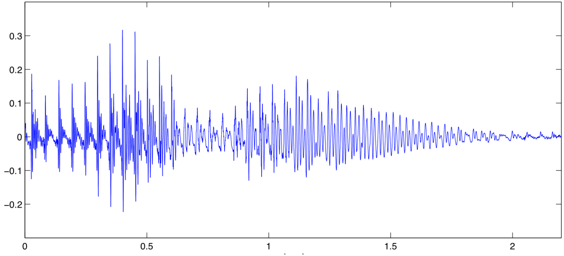
A stochastic process is a **statistical model of time series data**.





Daily temperature samples is an example of time series data. We can apply stochastic process models for reliable estimation of climate variations.

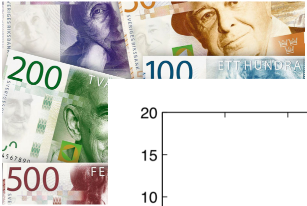
What is a stochastic process?

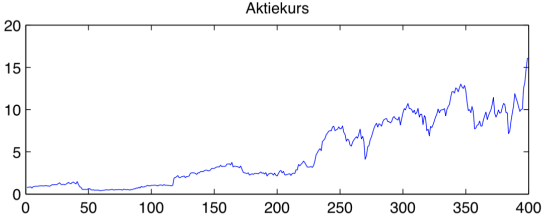


seconds

Speech data is often characterized using stochastic process models, e.g. in your mobile phone, where model parameters are transferred to reconstruct the speech in the connecting mobile phone.

What is a stochastic process?

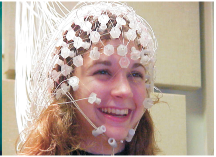


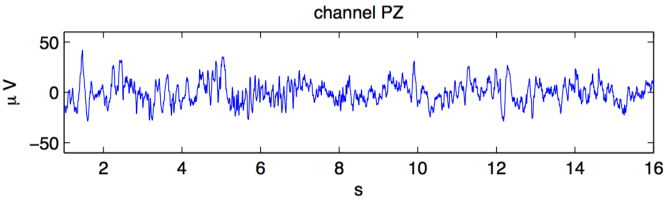


days

For financial data, e.g. stock prices, stochastic models are applied to predict the future value.

What is a stochastic process?





The frequency pattern of the the electrical signals from the brain (EEG) is estimated using stochastic process models to classify different states of the brain.

Definition: stochastic process

A stochastic process is a family of stochastic variables

$$X(t), \quad t \in T,$$

in continuous time. In discrete time, the stochastic process is denoted

$$X_t, \quad t = 0, \pm 1, \pm 2, \ldots.$$

All the stochastic variables in the stochastic process are defined on the same sample space Ω, and we have an n-dimensional random variable with an n-dimensional distribution function.

$$F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = P(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n).$$

Without restrictions the n-dimensional distribution function becomes difficult to handle.

Expected value and variance

For stochastic variables, the expected value and variance are defined as

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x) dx, \quad V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

when X is continuous with density function $f_X(x)$, $x \in R$.

Often we estimate the expected value and variance as the sample mean value and sample variance of available data.

The following rules are valid for stochastic variables,

$$\begin{aligned}
&E[aX + b] = aE[X] + b \\
&V[aX + b] = V[aX] = a^2 V[X] \\
&E[X + Y] = E[X] + E[Y] \\
&V[X + Y] = C[X + Y, X + Y] = V[X] + C[X, Y] + C[Y, X] + V[Y],
\end{aligned}$$

where a, b are constants and where the covariance $C[X, Y] = C[Y, X]$ for real valued stochastic variables.

Definition: stochastic process

For any stochastic process, we define a

- mean value function, $m(t) = E[X(t)]$,
- variance function, $v(t) = r(t,t) = V[X(t)] = C[X(t), X(t)]$,
- covariance function, $r(s,t) = C[X(s), X(t)]$,
- correlation function, $\rho(s,t) = \frac{C[X(s),X(t)]}{\sqrt{V[X(s)]V[X(t)]}}$.

What is the intuitive interpretation of these functions for our time series data examples? We start with the mean value and variance functions.

Example data - temperature

For any stochastic process, we define a

- mean value function, $m(t) = E[X(t)]$,
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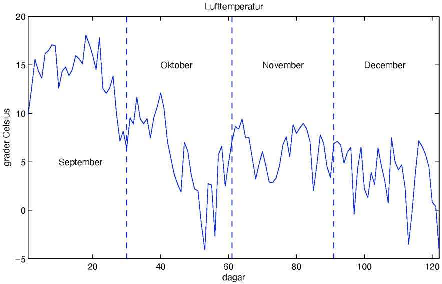

We want to use a stochastic process model for the time series. Can we calculate the estimates $\hat{m}(t)$ and $\hat{v}(t)$?

Example data - speech

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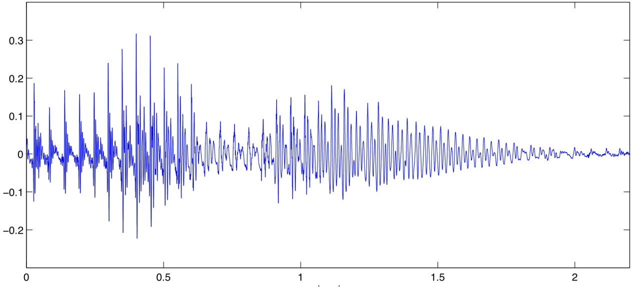

We want to use a stochastic process model for the time series. Can we calculate the estimates $\hat{m}(t)$ and $\hat{v}(t)$?

Definition: stationary stochastic process

A stochastic process is defined as **weakly stationary** when:

- the mean value function is constant, $m(t) = m$,
- the variance function is constant, $v(t) = r(0) = v$,
- the covariance function is only dependent in the distance between t and s, $r(s,t) = r(t-s) = r(\tau)$, for $\tau = t - s$.
- the correlation function is then $\rho(\tau) = \frac{r(\tau)}{\sqrt{r(0)r(0)}} = r(\tau)/r(0)$.

We usually mean weakly stationary processes when we talk about stationary processes. Again, we focus on the mean value and variance.

Example data - speech

For a stationary stochastic process,

- the mean value function is constant, $m(t) = m$,
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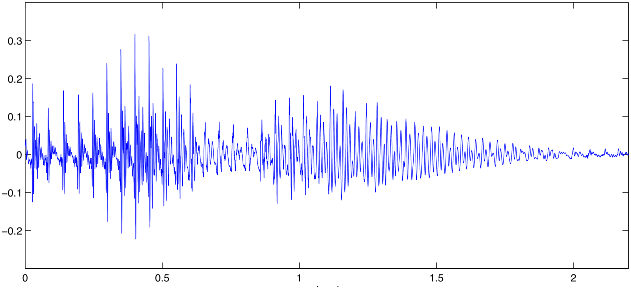

How can we find a mean value estimate $\hat{m}$ and a variance estimate $\hat{v}$?

Example data - temperature

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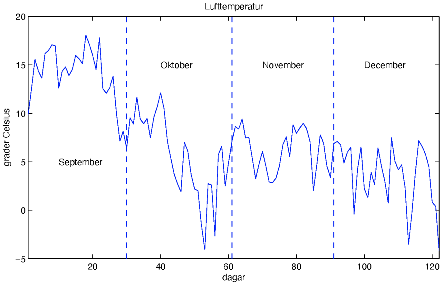

How can we find a mean value estimate $\hat{m}$ and a variance estimate $\hat{v}$?

Example data - stock price

For stationary stochastic process,

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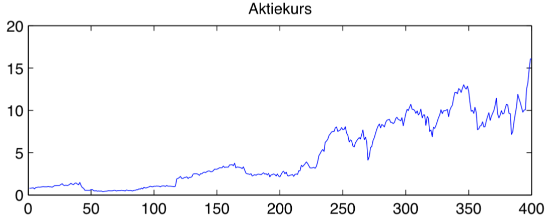
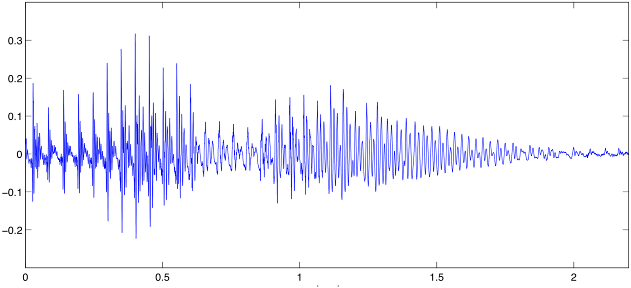


days

How can we find a mean value estimate $\hat{m}$ and a variance estimate $\hat{v}$?

Covariance and correlation functions

Many time series also have varying dependence between samples. This is especially visible for speech data.



Covariance and correlation functions

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For a stationary process, the correlation function is just a normalized covariance function.

Also note that the variance $v = r(0)$ is the covariance function at $\tau = 0$.

Review of covariance and correlation

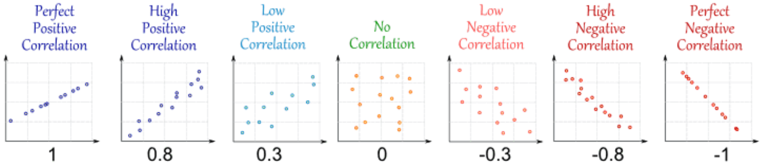
- The covariance between two stochastic variables is defined by

$$C[X, Y] = E[(X - m_X)(Y - m_Y)] = E[XY] - m_X m_Y$$

and measures linear dependence between X and Y.

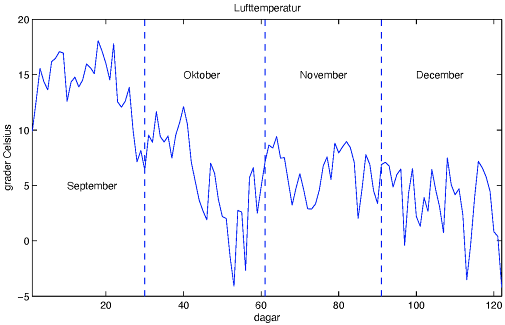
- The correlation coefficient is given from

$$\rho[X, Y] = \frac{C[X, Y]}{\sqrt{V[X]V[Y]}}.$$

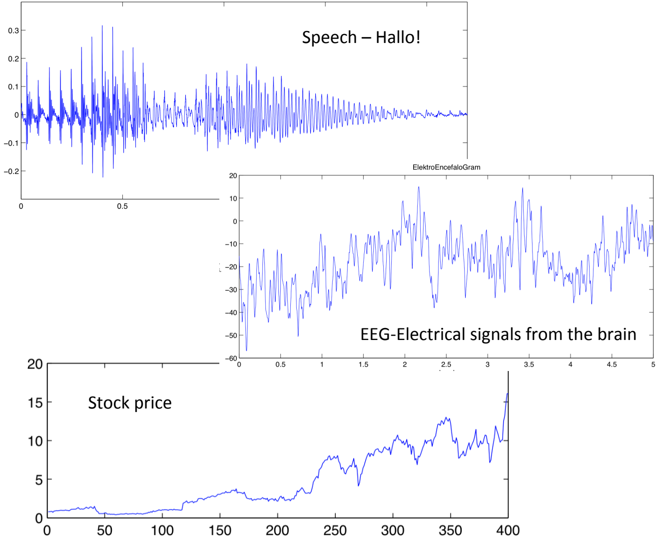


Temperature - correlation

We are often interested in the correlation between different time points (days) of the temperature data. Knowledge of this structure helps us predict the temperature the coming days.

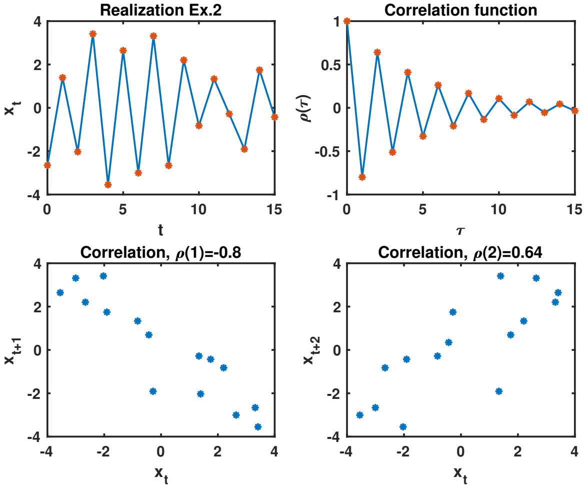


Example data - correlation



Covariance/correlation function

The discrete time stationary process X_t, $t = 0, \pm 1, \pm 2, \ldots$, has the covariance function $r(\tau) = (-0.8)^{|\tau|}$, $\tau = 0, \pm 1, \pm 2, \ldots$.



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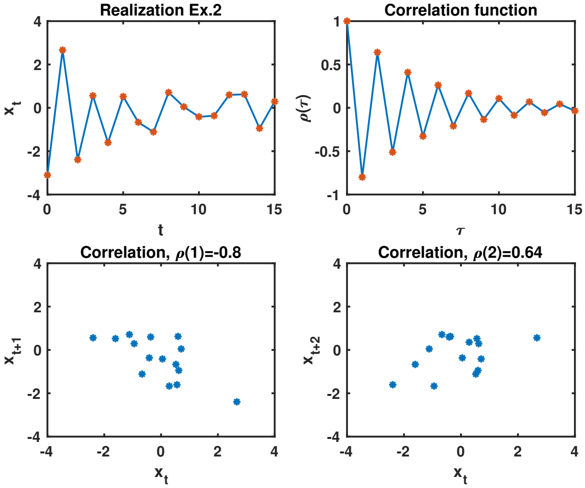
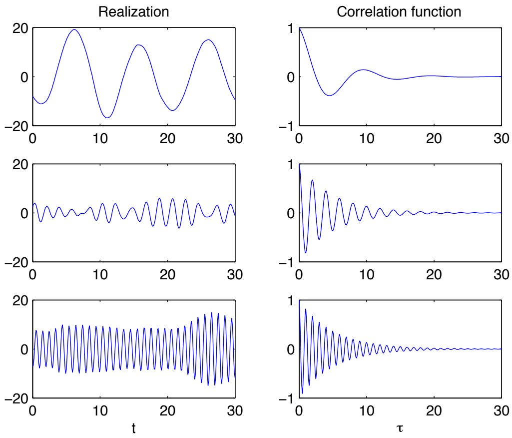


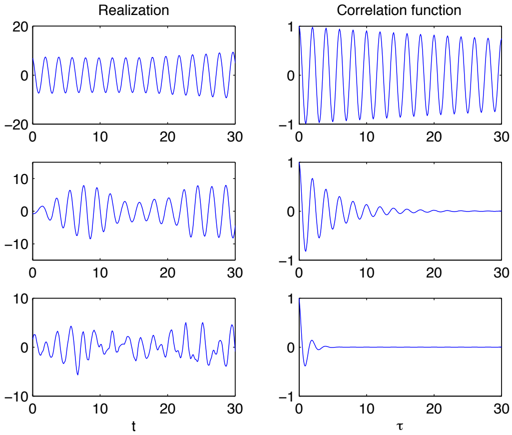
Data and covariance function

The main period of data is reflected in the period of the correlation/covariance function.



Data and covariance function

High correlation/covariance for large values of τ is connected to a more periodic data sequence.



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We also have the following general properties:

- non-negative variance, $r(0) \geq 0$,
- variance is the maximum of the covariance function, $r(0) \geq |r(\tau)|$,
- symmetrical covariance function, $r(-\tau) = r(\tau)$, (real-valued processes).

Schedule for the course

- Week 1: Covariance properties & Mean value estimation.
- Week 2: Spectral density, Sampling & Aliasing.
- Week 3: Harmonic functions, Gaussian processes & Filtering.
- Week 4: Parametric modelling with AR- MA- and ARMA-models.
- Week 5: Differentiation, Integration & Optimal filtering.
- Week 6: Spectral estimation & Cross-spectrum.
- Week 7: Summary.

Canvas course information

2024 HT/Autumn

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- Grades
- People
- Syllabus
- Modules
- Canvas Survey

▾ GENERAL INFORMATION

- Course contents
- Staff
- Course material
- Old exams
- Examination Friday November 1 8.00-13.00
- Re-examination preliminary date

Master thesis proposals !!!

▾ W1 Covariance properties & Mean value estimation

- W1 Overview lecture 1
- W1 Seminar 1
- W1 Exercises and short lectures

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W1 Exercises and short lectures

Covariance properties & Mean value estimation

Topics of week 1 are covered in the course book by the following chapters:

1, 2.1, 2.2, 2.3.2, 2.3.3, 2.5.1, 2.5.2, (2.5.3), 2.6

and the exercises:

2.1(-1), 2.2(-1), 2.9(0), X1(0), X2(-1), 2.3(0), 2.6(+1), 2.8(0), 2.14(0), 2.18(+1), 2.10(+1), 2.16(0), 2.15(0), 2.12(+1), 2.13(+1)

An mp4-video (10-20 minutes) with slides presents the topic, where the corresponding slides also are included in a separate pdf-file. Exercises that correspond to the video-material are listed.

The numbered exercises (2.1, 2.2 ...) are found in the course book and the solutions in the solutions manual pdf-file. The additional exercises (X1, X2 ...) with solutions are found in the additional exercises pdf-file. Exercises are marked with the level of difficulty: (-1) a start-up or rehearsal, (0) to be calculated and (+1) if you like challenges!

Use the discussion forum and the question hours for assistance, Thursdays 10-12 (MH:227), Thursdays 13-15 (MH:228) and Fridays 13-15 (MH:229).

Review of stochastic variables. Ex. 2.1(-1), 2.2(-1)

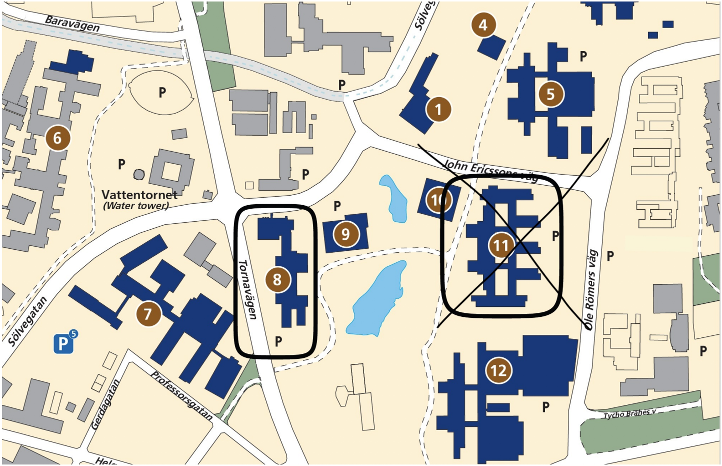
Review_stoch_var.mp4 Review_stoch_var.pdf

Why stochastic processes: Ex. 2.9(0)

Intro_stoch_proc.mp4 Intro_stoch_proc.pdf

Where to go?

E:A is only for todays lecture (located in 11). MH:R is where we meet tomorrow and the following weeks (located in 8). One occasion, Sept 18, we meet in MA:1 (located in 9). The question hours for calculation and computer exercises are in MH:227,228,229 (second floor in 8).



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▾ **W2 Spectral density, Sampling & Aliasing**

- **W2 Overview lecture 2**
- **W2 Exercises and short lectures**
- **W2 Computer exercise 1**
  9 Sep 0 pts

▾ **W3 Harmonic functions, Gaussian processes & Filtering**

- **W3 Overview lecture 3**
- **W3 Seminar 2**
- **W3 Exercises and short lectures**
- **W3 Presentation of computer exercise 1**
  16 Sep 0 pts

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The computer exercises are preferably performed in groups of two students. If you don't have a laptop, contact the coordinator, Oskar Keding, Canvas E-mail.

1) Schedule yourself for the 15-minutes presentation in the 'calendar' (to the left, available from September 9) and press the button 'find appointments' (to the right). Make sure that both of you are scheduled at the same occasion! (Usually more than one teaching assistant is available and therefore several pairs of students can sign up at one occasion at the same time.)

2) A mandatory preparation for computer exercise 1 is found at https://quizms.maths.lth.se/ where you create a personal account using your Stil-login. The test opens September 9. When finished, document that you have passed the test, e.g. with a screenshot.

3) Work with computer exercise 1, found under 'Course material' above. Document your work with accessible scripts, figures and comments to be able to show and explain your results for the teaching assistant during your 15-minutes presentation slot.

Use the discussion forum and the question hours for assistance with the exercise, Thursdays 10-12 (MH:227), Thursdays 13-15 (MH:228) and Fridays 13-15 (MH:229).

4) The whole group should be prepared and appear at the scheduled session. Bring your own computer! Show the documentation of your passed Mozquizto-test.

Examination

- Presentation computer exercise 1, W3 resulting in 0.5 credits.
- Presentations computer exercise 2 and 3, W5 and W7 resulting in 1 credit.
- Written examination Friday November 1, 8.00-13.00 in Vic:3A-3D and Vic:2D, Victoriastadion resulting in 6 credits.

Definition: stationary stochastic process

A stochastic process is a family of stochastic variables

$$X(t), \quad t \in T, \qquad X_t, \quad t = 0, \pm 1, \pm 2, \ldots,$$

in continuous and discrete time. For any stochastic process, we define a

- mean value function, $m(t)$,
- variance function, $v(t) = r(t,t)$,
- covariance function, $r(s,t)$,
- correlation function, $\rho(s,t)$.

A stochastic process is defined as weakly stationary when:

- the mean value function is constant, $m(t) = m$,
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- the covariance function is $r(s,t) = r(t-s) = r(\tau)$, for $\tau = t - s$.
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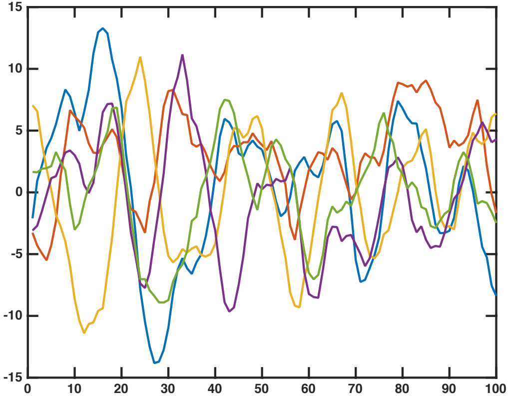
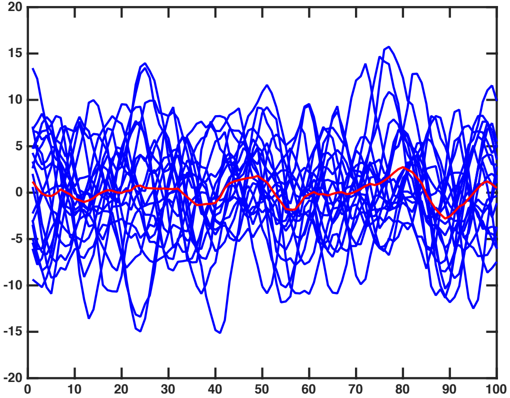
Estimation of mean value

How do we estimate m from one or a few realizations?



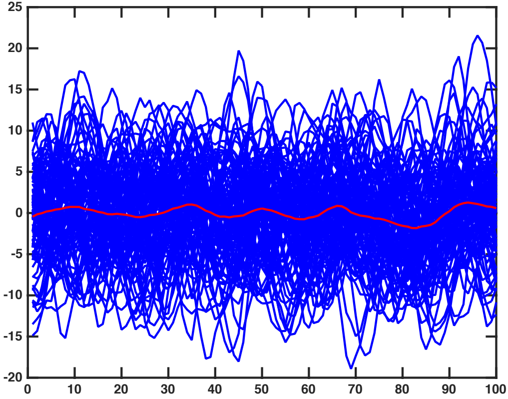
Ensemble mean

For a general time series, modeled as a stochastic process, we average over realizations at each value t giving $\hat{m}(t)$.



Ensemble mean

If the properties for a stationary process model is fulfilled, the mean value estimates for all t approach m.



Estimation of mean value

For a stationary process, we can use the mean value over time for one realization,

$$\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X_t,$$

as an unbiased estimator of m as

$$E[\hat{m}_n] = \frac{1}{n}\sum_{t=1}^{n} E[X_t] = \frac{1}{n}\underbrace{(m + m + \ldots + m)}_{n} = m.$$

A stationary process is called **linearly ergodic**, or **ergodic of first order**, as the mean value can be estimated using the mean value over time.

Variance of $\hat{m}_n$

The variance is calculated as

$$V[\hat{m}_n] = C[\frac{1}{n}\sum_{t=1}^{n} X_t, \frac{1}{n}\sum_{s=1}^{n} X_s] = \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n} r(s-t).$$

With $s - t = u$ we get

$$V[\hat{m}_n] = \frac{1}{n^2}\sum_{u=-n+1}^{n-1} (n - |u|)r(u).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{u} r(u).$$

If $V[\hat{m}_n] \to 0$ when $n \to \infty$, $\hat{m}_n$ is consistent.

Choice of mean value estimator

Exercise: The mean value, m, of a stationary stochastic process, defined by $Y_t = m + X_t$, $t = 0, \pm 1, \pm 2, \ldots$, should be estimated. A model of the process X_t is

$$X_t = e_t - 2e_{t-1} + e_{t-2},$$

where e_t, $t = 0, \pm 1, \pm 2 \ldots$, is independent stochastic variables (white noise) with expected value zero and variance one. One can choose between two estimates for m,

$$\hat{m}_1 = \frac{Y_t + Y_{t-1}}{2}$$

or

$$\hat{m}_2 = \frac{Y_t + Y_{t-2}}{2}.$$

Which is the most optimal estimator, $\hat{m}_1$ or $\hat{m}_2$, i.e. which estimator has the lowest variance?

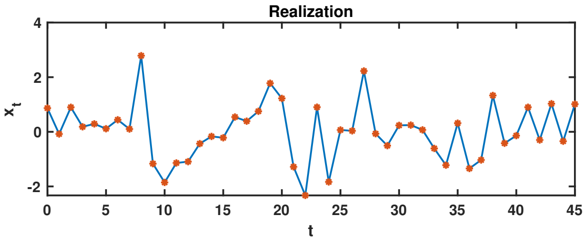
The white noise process

The white noise process e_t, $t = 0, \pm 1, \pm 2 \ldots$ is a time series of independent stochastic variables defined by a variance

$$V[e_t] = r(0) = \sigma^2,$$

and where all covariances

$$r(\tau) = C[e_s, e_t] = 0, \quad \tau = t - s = \pm 1, \pm 2, \ldots.$$



We sometimes also see the notation $r(\tau) = \sigma^2 \delta_\tau$ using the Kronecker delta function.

Variance of mean value estimates for different processes

Exercise: Suppose that, $X_t, t = 0, \pm 1, \pm 2...$, is a stationary stochastic process with unknown mean m, covariance function

$$r_X(\tau) = \alpha^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \ldots .$$

a) We would like to express the variance of the estimator $\hat{m}$ by averaging n consecutive samples of the process. Suppose that n is large and approximate the variance of the estimator.

b) Compare and discuss the resulting variances for some values of α.

List of exercises for tomorrow

- Stochastic variables
- Stationary stochastic processes
- Valid covariance functions
- Data and covariance function
- Choice of mean value estimator
- Variance of mean value estimates for different processes

Canvas course information

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W1 Exercises and short lectures

Covariance properties & Mean value estimation

Topics of week 1 are covered in the course book by the following chapters:

1, 2.1, 2.2, 2.3.2, 2.3.3, 2.5.1, 2.5.2, (2.5.3), 2.6

and the exercises:

2.1(-1), 2.2(-1), 2.9(0), X1(0), X2(-1), 2.3(0), 2.6(+1), 2.8(0), 2.14(0), 2.18(+1), 2.10(+1), 2.16(0), 2.15(0), 2.12(+1), 2.13(+1)

An mp4-video (10-20 minutes) with slides presents the topic, where the corresponding slides also are included in a separate pdf-file. Exercises that correspond to the video-material are listed.

The numbered exercises (2.1, 2.2 ...) are found in the course book and the solutions in the solutions manual pdf-file. The additional exercises (X1, X2 ...) with solutions are found in the additional exercises pdf-file. Exercises are marked with the level of difficulty: (-1) a start-up or rehearsal, (0) to be calculated and (+1) if you like challenges!

Use the discussion forum and the question hours for assistance, Thursdays 10-12 (MH:227), Thursdays 13-15 (MH:228) and Fridays 13-15 (MH:229).

Review of stochastic variables. Ex. 2.1(-1), 2.2(-1)

Review_stoch_var.mp4 Review_stoch_var.pdf

Why stochastic processes: Ex. 2.9(0)

Intro_stoch_proc.mp4 Intro_stoch_proc.pdf