GATE CSE NOTES

by

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Graph Theory

NPTEL - IISER Pune

- In a graph G, avg degree is $\frac{2e}{v}$. | $\sum d_v = 2e$

$$\delta(G) \le \frac{2e}{v} \le \Delta(G)$$

order – # vertices

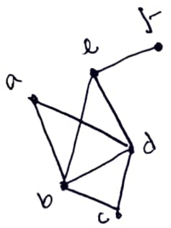
size – # edges

- # vertices with odd degree is even.

- Walk → edges distinct ⇒ trail (vertex can be repeated)

  Walk → vertices distinct ⇒ path (also automatically edges distinct too)

distinct vertices in a walk ⇒ distinct edges



abcdebd — trail

cbadef — path

→ If all vertices distinct except source & sink, then ⇒ cycle.

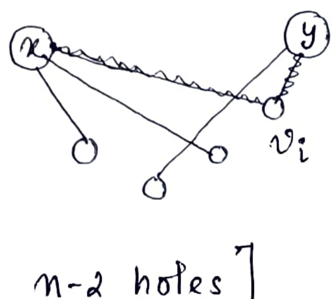
Theorem G has n vertices (order n). If $d(u) + d(v) \ge n-1$ for every 2 non-adj vertices u, v, then G is connected.

Proof Let $x, y \in V$

if $(x,y) \in E \Rightarrow x, y$ adjacent

Now, when $(x,y) \notin E$

$d(x) + d(y) \ge n-1$ implies there must be a v_i that adj. to both x, y. [By PHP, $n-1$ pigeons, $n-2$ holes]



$n = 6$

$d(x) + d(y) \ge 5$

3 2

Theorem If G is of order n with $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

Proof For every 2 non-adj vertices u, v

$$d(u) + d(v) \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$$

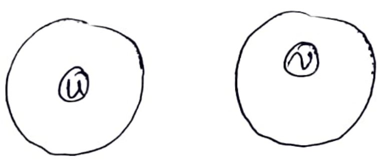
G is connected.

Theorem If G is disconnected, $\bar{G}$ is connected.

Proof If $u, v \in V(G)$, we find a path in $\bar{G}$ joining u, v.

Case 1

u, v in diff components



u, v not adj. in G

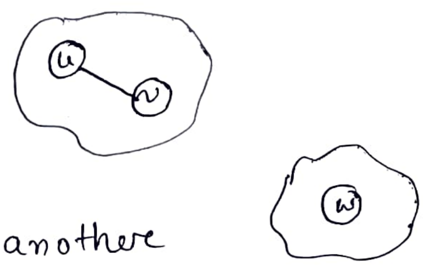
$\Downarrow$

u, v adj in $\bar{G}$

Case 2

u, v in same component

u, v may or may not be adj.



w in anothere component. In $\bar{G}$, w adj to u, v. Thus, there's path (u, w, v) in $\bar{G}$.

Thus, $\bar{G}$ is connected.

» **Trail** Walk - edges distinct.

Path Walk - vertices ⇒ edges distinct.

- **Eulerian trail** : Closed trail that includes all the edges of G.

☑ **Theorem** A connected graph or multigraph G is Eulerian iff every vertex is of even degree.

Proof ↗ In the trail, for each entry to a vertex there's an exit. This contributes 2 to the degree of the vertex.

if

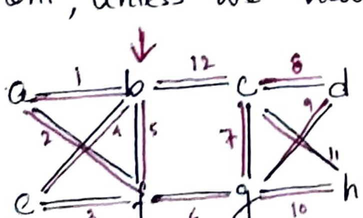
$d(v) = 2k \rightarrow$ even

↗ Given G with all $d(v) \rightarrow$ even

only if

Fleur's algo : At each step, we move across an edge whose deletion does not result in >1 component, unless we have no choices. At the end, there are no edges left.

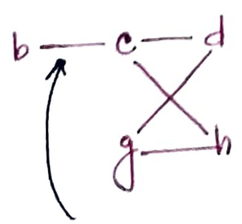
→ visiting an edge ⇒ delete it from G to consider further



Eulerian trail.

b a f e b f g c d g h c b.

7 → 8 Step



can't use as increases the # components

- Hamiltonian path

Path that includes all vertices of G.

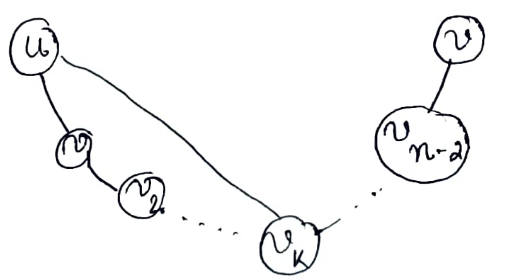
→ A graph is Hamiltonian if it contains a Hamiltonian cycle.

→ No known tests for Hamiltonian graph.

Theorem : If G is a simple graph with $n (\geq 3)$ vertices & if $d(v) \geq n/2$ then G is Hamiltonian.

Proof By contradiction,

u, v be 2 non-adj. vertices. If we join u, v there's Hamiltonian path joining u & v.



Let $d(u) = k \geq n/2$

If u is adj. to v_k then v_{k-1} cannot be adj to v.

If possible, let v be adj. to v_{k-1}. Then we have Hamiltonian cycle $v, v_{n-2} \ldots v_k, u, v_1, v_2, \ldots, v_{k-1}, v$

Since, we assumed it is not Hamiltonian graph, this is not possible.

Thus $d(v) \leq n-1-k \leq n-1-n/2 = n/2 - 1$

contradiction

So, G is Hamiltonian.

Bipartite Graph

V set divided into A, B.

Each $e \in E$ connects a vertex in A to a vertex in B.

eg 4-regular graph having 15 vertices can't be bipartite.

$|V| = 15$ $\quad V = A \cup B$ $\quad$ Let $|A| = x$

$$\sum_A d(v) = \sum_B d(v)$$

$$\Rightarrow 4x = (15 - x) 4$$

$\Rightarrow x =$ not an integer.

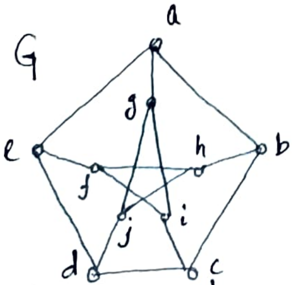
Theorem A graph is bipartite iff it does not have any odd cycles.

Diameter of G

$\text{diam}(G) = \max \{ d(u,v) \}$ $\quad u, v \in V$

$d(u,v) \rightarrow$ length of shortest path b/w u & v

G



Petersen graph

$d(u,v) = 1$ adj u,v

$= 2$ non-adj u,v

$\text{diam}(G) = 2$

☑ **Theorem** (G is a simple graph)

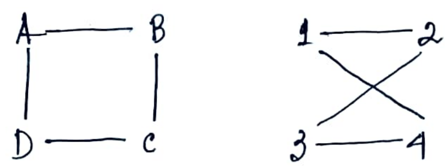
$$\boxed{\operatorname{diam}(G) \geq 3 \Rightarrow \operatorname{diam}(\bar{G}) \leq 3}$$

☑ **Theorem** If $\operatorname{diam}(G) \geq 4 \Rightarrow \operatorname{diam}(\bar{G}) \leq 2$

- **Isomorphism**

$G = (V, E)$, $G' = (V', E')$. $G \simeq G'$ if there exists a bijection $f: V \to V'$ with $(u, v) \in E \Leftrightarrow (f(u), f(v)) \in E'$ for all $u, v \in V$.

eg

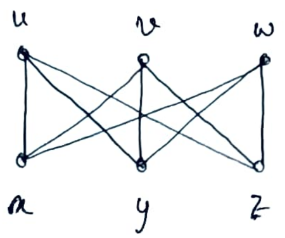


G G'

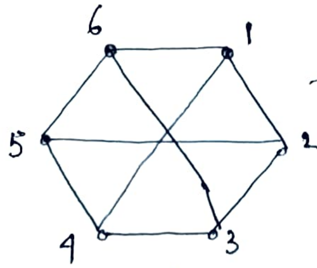
$f(A) = 1$ $f(C) = 3$
$f(B) = 2$ $f(D) = 4$.

$(A, B) \in E \Leftrightarrow (1, 2) \in E'$ & so on.

eg



G $K_{3,3}$



G'

→ (1, 3, 5)
(2, 4, 6)

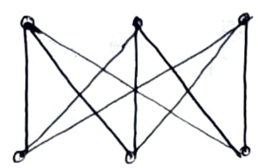
$f(u) = 1$	$f(x) = 2$
$f(v) = 3$	$f(y) = 4$
$f(w) = 5$	$f(z) = 6$

If 2 graphs are isomorphic, they have same number of vertices, edges, components, same degree sequence, diameter, length of longest path. If two graph differ in any of these, they are not isomorphic. However, having all these

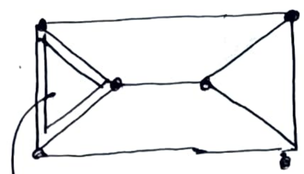
same does not imply that they are isomorphic.

✓ If two graphs are isomorphic & one of them contains a cycle of particular length, then the same must be true of the other graph.

eg.



$K_{3,3}$



G

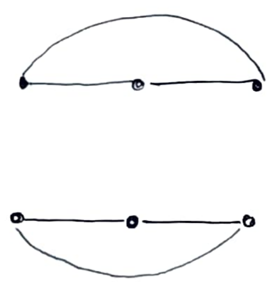
not isomorphic.

cycle of length 3

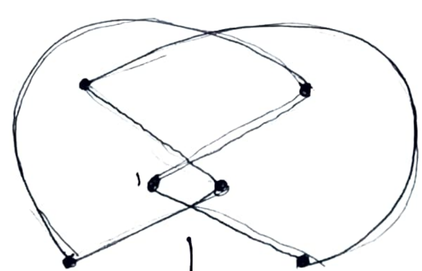
Can't be a cycle of odd length in $K_{3,3}$.

✓ G & G' are isomorphic iff $\bar{G} \simeq \bar{G'}$.

eg.



$\bar{K}_{3,3}$



$\bar{G}$

cycle of length 6

does not exist in $\bar{K}_{3,3}$

$K_{3,3}$ & G are not isomorphic.

Matching

- Set of disjoint edges.
- Every vertex of G is incident to at most one edge in the matching.
- Size of matching is the # edges in the matching M.

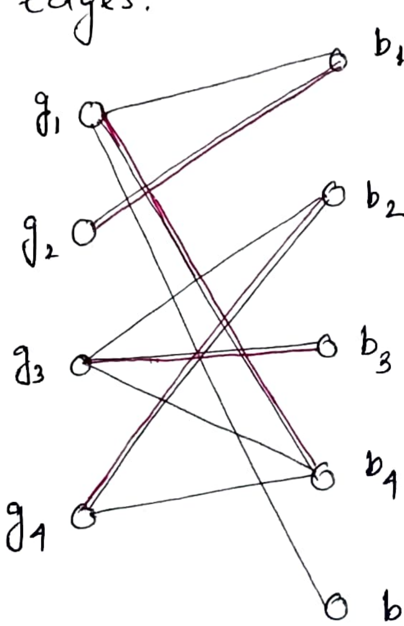
Matching is maximum when it has the largest possible size.

A perfect matching is that matches every vertex.

- Alternating path is a path that alternates b/w matched & unmatched edges.

eg. $b_5\, g_1\, b_4\, g_3\, b_3$

- Augmenting path is an alternating path that starts & ends on unmatched vertices.



Th: A matching is maximum iff there's no augmenting path wrt M.

$M = \{(g_1, b_4), (g_2, b_1), (g_3, b_3), (g_4, b_2)\}$

- Every graph with $\delta(G) \geq 2$ has a cycle of length at least $\delta(G) + 1$.

- Each vertex in matching M has degree one.
- Maximum matching for a G is not unique.

⇒ <u>Matching number</u> (of a graph): Size of a maximum matching of the graph.

Matching in Bipartite Graphs

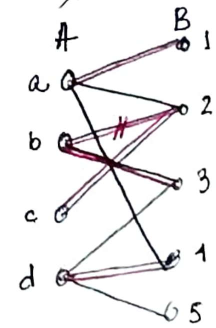
$G = ((A,B),E)$ be a bipartite graph. If $|A| \leq |B|$, size of maximum matching is at most $|A|$. We want to decide whether there exists a matching saturating A.

If there is such a matching, M, then for any subset S of A, the edges of M link the vertices of S to as many vertices of B.

Matching condition

<u>Hall's condⁿ</u> (existence of matching saturating A)

$$|N(S)| \geq |S| \quad \forall \; S \subseteq A.$$



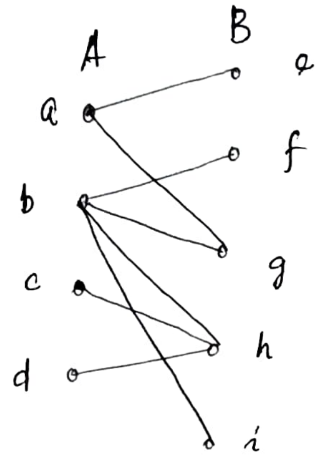
N(S) is set of vertices of $G \setminus S$ adjacent to at least one vertex of S. N(S) is the neighbourhood of S

$S = \{a,b,c\}$ G
$G \setminus S = \{d,1,2,3,4\}$
$N(S) = \{1,2,3,4\}$
$|N(S)| \geq |S|$

Hall's matching theorem

$G = ((A,B),E)$ has a matching saturating A iff

$|N(S)| \geq |S|$ for all $S \subseteq A$.



eg for $S = \{a,b,c,d\}$

$G \setminus S = \{e,f,g,h,i\}$

$N(S) = \{e,f,g,h,i\}$ $|N(S)| \geq |S|$

for $S = \{c,d\}$

$G \setminus S = \{a,b,e,f,g,h,i\}$

$N(S) = \{h\}$ $|N(S)| \ngeq |S|$

So, no matching exists saturating A.

- Stable Marriage Problem

N men, N women.

Rogue couple. A man & a woman such that they prefer each other to the person they are currently paired with in current matching.

Stable matching no rogue couples.

$\Rightarrow$ There's always an stable matching.

⇒ Gale - Shapely Algorithm for stable matching

⇒ Algo terminates in at most $N(N-1)+1$ rounds.

i) In the beginning of each round, every man not engaged proposes to his favorite girl not yet crossed off his preference list.

ii) Then each woman looks at her new proposals, & also the man she is provisionally engaged to (if has one) & then picks favorite from them. They become engaged (as of now) while all other men get permanently rejected from her.

iii) All rejected men cross off their lists the woman who rejected them.

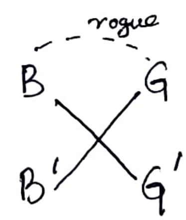
Cycle repeats until no man is rejected in a round.

⇒ This algo guarantees that everyone gets married.

⇒ Algo is male optimal, female pessimal.

⇒ It produces a stable matching (no rogue couples).

Proof Suppose there's rogue couple B, G.

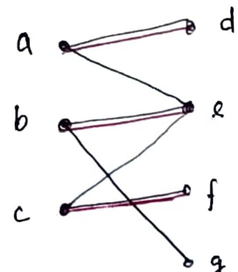


B must have proposed to G & G rejected B. Only way to reject B for G is that she got someone better. But woman can only trade up. So, she must like B' better than B. Contradiction! So stable.

Vertex cover

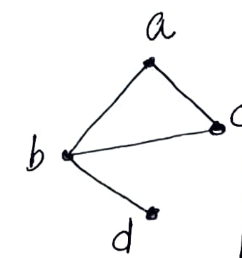
$G = (V, E)$. A set $K \subset V$ is a vertex cover of E if any edge of G is incident to a vertex in K.

The vertex cover number of G, $\tau(G)$, is the min. size of a vertex cover of G.

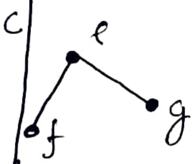


Matching
$M = \{ad, be, cf\}$

Vertex cover
$K = \{a, d, e, f, g\}$
or
$\{a, e, f, g\}$
or
$\{a, b, c\}$.



$\{a, b\}$ or $\{b, c\}$



$\{e\}$

$\Downarrow$

For any matching M, K contains at least one end vertex of each edge of M.

So, $|M| \leq |K|$. So, the max size of a matching is at most the min. size of a vertex cover for any graph.

König's theorem Let G be a bipartite graph. Then the size of a max. matching equals the size of a min. vertex cover.

i.e. $\mu(G) = \tau(G)$.

$\mu(G)$ → matching no.

$\tau(G)$ → vertex cover no.

- Edge cover.

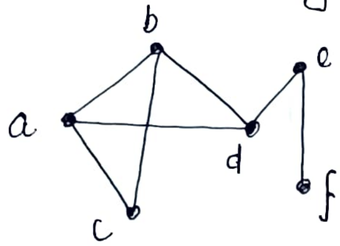
Set of edges $F \subset E(G)$ such that every vertex $v \in V(G)$ is incident to an edge $e \in F$. [Edge cover can exist if there's no isolated vertices.]

$\rho(G)$, edge cover number Min size of edge cover.

⇒ Stable set, C or Independent set Subset of V such that for each edge e of G, $e \not\subset C$.

↓

no two vertices are adjacent



$\{a, e\}$ or $\{b, f\}$ or $\{c, d, f\}$

↑ max stable set.

Stable set number $\alpha(G)$. / Independence no:

$$\alpha(G) = \max\{|C| : C \text{ is a stable set}\}$$

- Gallai's theorem

For G w/o isolated vertices,

$$\boxed{\alpha(G) + \tau(G) = |V| = \mu(G) + \rho(G)}$$

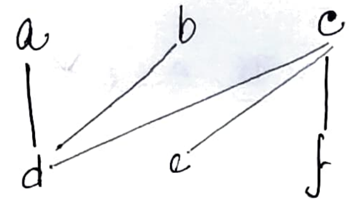
- for bipartite graph with no isolated vertices, $\boxed{\begin{matrix}\alpha(G) = \rho(G) \\ \mu(G) = \tau(G).\end{matrix}}$

$\tau(G)$ ⇝ min size of vertex cover

$\mu(G)$ ⇝ max. matching size.

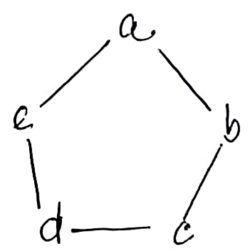
$\rho(G)$ ⇝ min. size of edge cover

eg Independent set.



$S = \{a, b, e, f\}$

$\bar{S} = \{c, d\} \rightsquigarrow$ min vertex cover



$S = \{a, d\}$

$\bar{S} = \{b, c, e\} \rightsquigarrow$ min vertex cover.

Theorem In a graph G, $S \subseteq V$ is an independent set iff $\bar{S}$ is a vertex cover of hence

$$\alpha(G) + \tau(G) = |V|$$

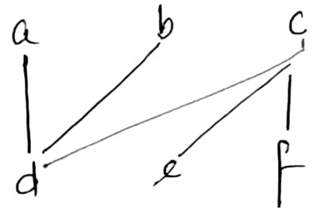
Proof: $\Rightarrow$ If S is a stable set, every edge is incident to at most one vertex in S. So, every edge is incident to at least one vertex of $\bar{S}$. So, $\bar{S}$ covers all edges.

$\Leftarrow$ $\bar{S}$ being vertex cover, there are no edges joining vertices of S. So, S is a stable set.

Hence, every max. independent set is complement of min. vertex cover.

$$\alpha(G) + \tau(G) = |V|.$$

eg Edge cover, L

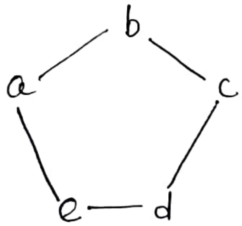


L, edge cover =

{ad, bd, ec, fc}

min edge cover

Max. matching = {ad, cf}



L = {ae, bc, cd}

M = {ab, de}

MAX
1. Independent vertex set $\alpha(G)$
2. Independent edge set (or Matching) $\mu(G)$

MIN
3. Vertex Cover $\tau(G)$
4. Edge Cover $\rho(G)$

S → stable set
⇓
$\bar{S}$ → vertex cover.

$\boxed{\alpha(G) + \tau(G) = |V|}$

$\boxed{\mu(G) + \rho(G) = |V|}$

stable set ~ vertex cover

matching ~ edge c

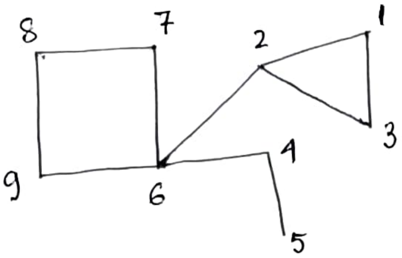
Theorem For graph G, with no isolated vertices,

$$\boxed{\mu(G) + \rho(G) = |V|.}$$

Tutte's theorem

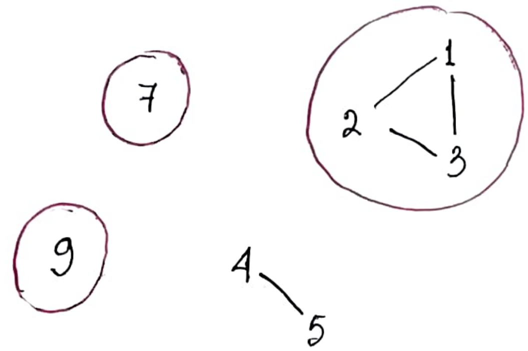
G = (V, E) & S be subset of V, $S \subseteq V$. Let odd($G \setminus S$) denote the number of odd components that get generated if S is removed from G.

G.



$S = \{6, 8\}$

$G \setminus S$



odd components = 3

Tutte's Theorem : G has a perfect matching iff for every subset $S \subseteq V$,

$$odd(G \setminus S) \leq |S|$$

- **Maximal Matching** If we cannot add any edge to the existing matching.

→ If M is maximum ⇒ it is maximal.

→ Max. matching isn't unique.

- If G is a k-regular bipartite graph with $k > 0$, G has a perfect matching.

Social Networks & Stable Marriage.

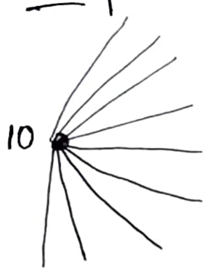
- $\sum d(v) = 2e$.

- <u>Ramsey's Theorem</u> (3,4 version)

Any group of 10 people must contain 3 mutual friends (K_3) or 4 mutual strangers. (K_4).

$\Rightarrow$ If every edge of K_{10} is colored red or blue, then it must contain a red K_3 or a blue K_4.

<u>Proof</u> Each vertex has 9 ~~10~~ edges adj. to ~~leaving~~ it.



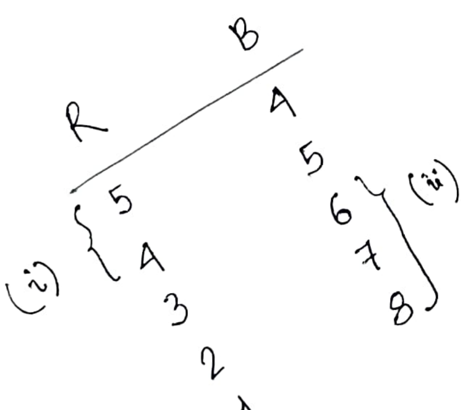
9 edges have to be colored with blue or red.

$\Rightarrow$ It must have at least 5 red edges or 5 blue edges. (PHP)

2 holes
9 pigeons

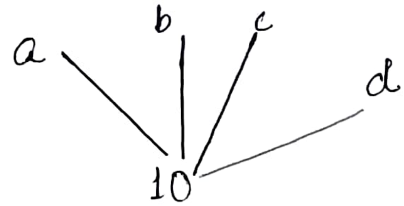
Also, vertex 10 must have either at least $\underbrace{\text{4 red edges}}_{(i)}$ <u>or</u> at least $\underbrace{\text{6 blue edges}}_{(ii)}$. (If not $\rightarrow$ 3R+5B $= 8$ ✗)

if none is correct then 3R + 5B



B, R interchangeable.

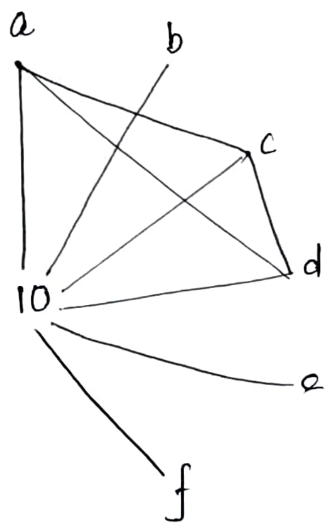
case a) 10 has at least 4 red edges.



If one of a-b, b-c, a-c, a-d, b-d, c-d is red, that gives a red K_3.

If none of a-b, b-c, c-d, a-c, a-d, b-d is red, then it gives a blue K_4.

case b) 10 has at least 6 blue edges.



Among a, b, c, d, e, f, we must have a red K_3 or a blue K_3. (Ramsey (3,3)).

If a red K_3 is present, then done.

If a blue K_3 is there, we get a blue K_4 in K_{10}.

- Stable Marriage

$K_{N,N}$ - N vertices for men, N vertices for women, edge connecting each man to each woman.

Complete bipartite.

Perfect matching

Selection of N edges such that every vertex is included on exactly one edge.

There are N! perfect matchings. (for $K_{n,n}$).

⇒ Stable matching

Problem: Each person has an ordered list of some members of the opposite sex as his/her preference list.

If the resulting set of marriages (matching) contains no pairs of the form $\{m_i, w_j\}$, $\{m_k, w_l\}$ such that m_i prefers w_l to w_j and w_l prefers m_i to m_k, the marriage is stable.

rogue pair

(Stable matching can always be achieved)

eg

Men	Women
1	~~A~~ B C
2	~~B~~ A C
3	~~A~~ ~~B~~ C

Women	Men
A	2 1 3
B	1 2 3
C	1 2 3

i)		ii)		iii)	
A	1, 3	A	1	A	1
B	2	B	2, 3	B	2
C	–	C	–	C	3

Stable matching

- Tree : Acyclic, connected graph.

⇒ # ~~un~~labeled tree graphs with n nodes $= n^{n-2}$
[Cayley's theorem]

binary trees with n nodes $= \dfrac{(2n)!}{(n+1)!\,n!}$
(unlabeled)

BSTs with n nodes $= C_n$

⇒ Every tree with n vertices has $n-1$ edges.

⇒ Any 2 vertices are connected by a unique ~~edge~~ path in a tree.

- Spanning tree. Any tree inside a graph that uses all vertices of G is a spanning tree.

⇒ # spanning trees are calculated using Kirchoff's matrix tree theorem.

$$L_{ij} = \begin{cases} d(i) & i = j \\ -1 & i \neq j \;;\; i,j \text{ are adj.} \\ 0 & \text{otherwise.} \end{cases}$$

delete one row & one column

$L^* \rightarrow$ find det (L^*)

↓

span. trees

- Planar graph

$$\boxed{v+f-e=2}$$

$\Downarrow$

Proof: Induct on e.

#

$\rightarrow$ $e=0$, $v=1$, $f=1$

$$v+f-e=2$$

$\rightarrow$ IHOP True for $e=k$

$\rightarrow$ $e=k+1$

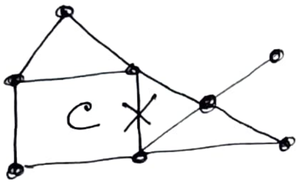
Case I: G is a tree.

$$e=v-1 \qquad f=1$$

$$v+f-e = v+1-v+1=2.$$

Case II: G not a tree.

G must contain a cycle.



Remove any edge from cycle C to obtain a smaller graph G'.

	v	f	e
G	v	f	$k+1$
G'	v	$f-1$	k

$*$

$*$ G' has k edges. So, using IHOP,

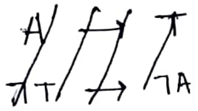
$$v+(f-1)-k=2$$

$$v-(k+1)+f=2$$

proved for $k+1$ edges

- For a cube

$$8+6-12=2$$



Converse not true

⇒ If G is planar with $n \geq 3$ vertices, e edges then,

$3 \cdot 2 \leq 2 \cdot 5$

$\boxed{e \leq 3n - 6}$

If $e > 3n - 6$ for a G with $n \geq 3$ then G is non-planar.

⇓

K_5 non-planar

$\boxed{3f \leq 2e}$

$\boxed{v + f - e = 2}$

$3(e - v + 2) \leq 2e$

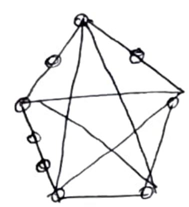
$\Rightarrow e \leq 3v - 6$

Every non-planar G contains either K_5 or $K_{3,3}$ or a sub-division of K_5 or $K_{3,3}$.

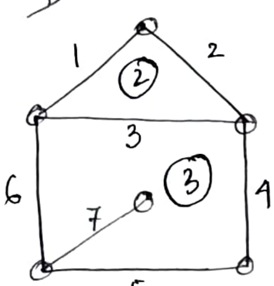
↓

K_5

Kuratowski's theorem



Proof



Edge-face matrix

if edge i borders face j, then $M_{ij} = 1$

e \ f	1	2	3
1	1	1	0
2	1	1	0
3	0	1	1
4	1	0	1
5	1	0	1
6	1	0	1
7	0	0	1

Counting col & row wise.

$x = \#1\text{'s in } M$

row wise addition - how many faces does an edge border.

$\Rightarrow x \leq 2e.$

as an edge can border 1 or 2 faces.

col wise - how many edges border a face.

$\Rightarrow x \geq 3f$ as min 3 edges required to form a face.

So, $3f \leq x \leq 2e$

Connectivity

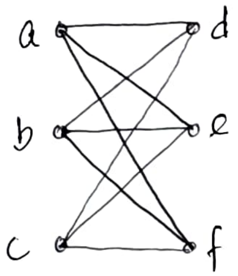
K vertices removed ⇒ graph disconnected
$K(G) = K$

- A graph is K-connected if there does not exist a set of K-1 vertices whose removal disconnects the graph. (how many vertices we can remove such that G stays connected)

- Vertex connectivity $K(G)$ of G is the largest K such that G is K-connected. [Size of minimal vertex cut]

$\boxed{K(G) \le \delta(G)}$ min degree

eg $K_{3,3}$



1-connected
2- "
3- "
not 4-connected.

$K(K_{3,3}) = 3$

(removal of a,b,c disconnects the graph).

$\{a,b,c\}$

- Vertex set (of a connected graph) or Vertex cut ~~Cutset~~
Set $S \subseteq V$ whose removal disconnects the graph.

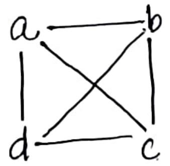
- Edge set / cut Set
Minimal Set of edges whose removal disconnects the graph.

⇒ Edge connectivity $K'(G)$: size of smallest edge set/cutset.

Min # edges to be removed to disconnect G.

- **Edge cut** : Set of edges of the form $[S, \bar{S}]$ where S is a non-empty proper subset of V & $\bar{S} = V - S$.

eg.



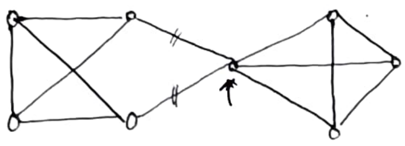
K_4

$S = \{a\}$

$\bar{S} = \{b, c, d\}$

$[S, \bar{S}] = \{(a,b), (a,c), (a,d)\}$

eg



$\kappa(G) = 1$ $\kappa'(G) = 2$ $\delta(G) = 3$ → min degree of vertices

☑ **Theorem** : If G is a simple graph,

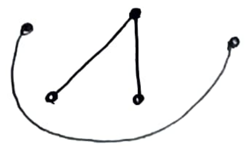
$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

Expansion lemma

Let G be a k-connected graph. If G' is obtained from G by adding a new vertex x adj. to at least k vertices in G, then G' is k-connected.

☑ **Theorem** Every graph with v vertices & e edges has at least $v - e$ connected components.

(fewer edges than vertices)



$|V| = 5$

$|E| = 3$

- **Articulation point / Cut Vertex**

Removing it & edges incident on it disconnects G.
(or produce more connected components)

- **Bridge / Cut edge**

Removal of this edge disconnects G.
(or produce more conn. components)

- Strongly connected components
  - in case of digraphs.

☑ **Theorem** In any G, if $\exists$ exactly 2 vertices of odd degree, namely x, y, then $\exists\ x \rightsquigarrow y$.

☑ **Theorem** In a simple graph, e edges, n vertices, k components (note: 1 component $\Rightarrow$ G is connected)

then

$$\boxed{n-k \le e \le \frac{(n-k)(n-k+1)}{2}}$$

$\Rightarrow$ When $k=1$,

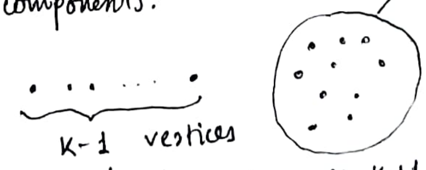
$$n-1 \le e \le \frac{n(n-1)}{2} \quad ({}^{n}C_2)$$

$n-1$: min. $n-1$ edges for n vertices to be connected

$\frac{n(n-1)}{2}$: for complete graph

k components.

(≤)



→ complete K_{n-k+1} (to maximise # edges)

$\Rightarrow e = |E(K_{n-k+1})|$

$$= \frac{(n-k)(n-k+1)}{2}$$

(≥) min # edges $= (n-k+1)-1 = n-k$
(at least has to be a tree; hence, $\#e = \#v - 1$)

Q. $|V| = 10,\ k = 3$

min # edges $= 10 - 3 = 7$

max # edges $= \dfrac{7 \cdot 8}{2} = 28$

Q $n = 10,\ e = 6,\ k = ?$

*

$10 - k \leq 6$

$\Rightarrow k \geq 4$

$6 \leq \dfrac{(10-k)(10-k+1)}{2}$

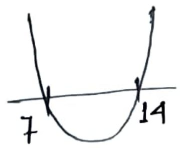
$\Rightarrow (10-k)(11-k) \geq 12$

$\Rightarrow k^2 - 21k + 110 \geq 12$

$\Rightarrow k^2 - 21k + 98 \geq 0$

$\Rightarrow (k-7)(k-14) \geq 0$

$k \leq 7$ or $k \geq 14$ as a parabola.



Q Max #edges in a disconnected graph G with n vertices.

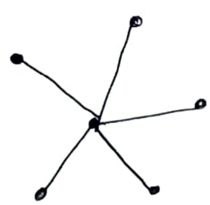
*

$\rightarrow k \geq 2$

$n - k \leq e \leq \dfrac{(n-k)(n-k+1)}{2}$

$e \leq \dfrac{(n-2)(n-1)}{2}$

• Star graph. $K_{1,n}$



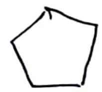
$K_{1,5}$

cut edges $= 5$ or n

cut vertices $= 1$

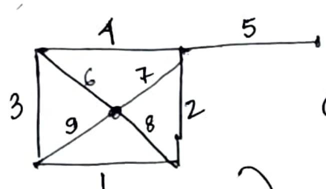
Produces max # connected components by removing cut vertex.

$\text{diam}(K_{1,n}) = 2$ if $n+1 = |V| \geq 3$

- For cycle graph , wheel graph , complete graph, there's no cut vertices.  3

- Cut Set : Minimal set of ~~vertices~~ edges that if removed disconnects the graph.

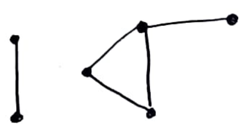
eg



cut-set $= \{e1, e9, e6, e4\}$

cut

Minimal cut-set $= \{e5\}$



- Cut : Partition of vertices into 2 disjoint subsets. Any cut determines a cut-set, set of edges that have one end in each subset of partition.

- Connected component : Maximal connected subgraph of und. graph.

$\Sigma d = 2e$

- Vertex connectivity $\kappa(G)$

$$\kappa(G) \leq \delta(G) \leq \underbrace{\frac{2e}{n}}_{\text{avg deg.}} \leq \Delta(G)$$

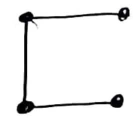
- Edge connectivity $\kappa'(G)$.

$\kappa'(G) \le \delta(G)$

(with each vertex we remove, we remove at least one edge)

$\underline{\kappa(G) \le \kappa'(G)} \le \delta \le \frac{2e}{n} \le \Delta$

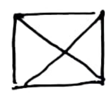
- G is separable iff $\kappa(G) = 1$.
  - ↳ 1 connected
  - ↳ if has a cut vertex.

eg 

- k-line connected if $\kappa'(G) = k$

For K_n, $\kappa(K_n) = n-1$

$\kappa'(K_n) =$ ~~not defined~~ $n-1$



For C_n (cycle), $\kappa(C_n) = \kappa'(C_n) = 2$

$\kappa(W_{n+1}) = 3$

$\kappa'(W_{n+1}) = 3$

$K_{n,n}$

$\kappa(K_{n,n}) = n$

$\kappa'(K_{n,n}) = n.$

$\kappa(K_{m,n}) = \min(m,n)$

$\kappa'(K_{m,n}) = \min(m,n)$

- For directed graphs,

connectivity
- → unilateral → $\forall x, y$ $x \rightsquigarrow y$ or $y \rightsquigarrow x$
- → strong
- → Weak → undir. graph is connected

Unilateral

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ d & c \arrow[l] \end{tikzcd}$$

for a, c $a \rightsquigarrow c$

but $c \not\rightsquigarrow a$

☑ Theorem: If G has at least 3 vertices, the following are equivalent :

1. G is 2-connected.
2. G is connected & has no cut vertex.
3. For all distinct vertices u, v, w in G, there's a path from u to v that does not contain w.

Proof: $1 \Rightarrow 3$ Since G is 2-connected, G with w removed is a connected graph G'. Thus, in G', $u \rightsquigarrow v$ exists, which in G is path $u \rightsquigarrow v$ avoiding w.

$3 \Rightarrow 2$ 3 implies G is connected as $u \rightsquigarrow v$ always exists if 3 holds true. Suppose, w is a cut vertex, so that $G' = G - w$ is disconnected. Let, u, v be vertices in 2 different components in G', so that $u \rightsquigarrow v$ does not exist. Then every path joining u to v must use w. Contradiction!

$2 \Rightarrow 1$ Since, G has $\geqslant 3$ vertices & has no cut vertex its connectivity is at least 2.

☑ Th: If G has at least 3 vertices, then G is 2-connected iff every 2 vertices u & v are contained in a cycle.

Cor. If G has ≥ 3 vertices, then G is 2-connected iff between every 2 vertices u, v there are 2 internally disjoint paths, i.e, paths that share only the vertices u, v.

☒ Menger's Theorem If G has at least $k+1$ vertices then G is k-connected iff b/w every 2 vertices u, v there are k pairwise internally disjoint path.

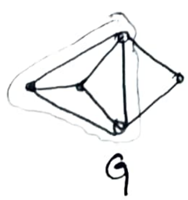
Coloring

- If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Chromatic no.

- $\chi(G) = \max \{ \chi(C) \,;\, C \text{ is conn. component of } G \}$
- $\chi(K_n) = n$
- Clique : Complete subgraph.

Clique number $\omega(G)$: size of max. clique



$\omega(G) = 4$.

- Lower bounds for χ

$\Rightarrow \boxed{\chi(G) \geq \omega(G)}$ ⇝ size of max clique (clique number)

$\boxed{\chi(G) \leq \Delta(G) + 1}$

$\Rightarrow \chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

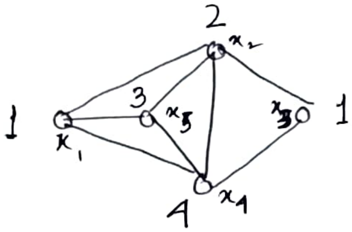
$\alpha(G)$ ⇝ size of max indep. set

↓ Proof

Let G be colored with $1, 2, \ldots, \chi(G)$

S_i = Set of vertices colored with i

$|S_i| \leq \alpha(G)$ Which vertices we color with the same colour? → Ones that are not adj. to each other



$$|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G)$$

$$= \chi(G)\,\alpha(G)$$

$S_1 = \{x_1, x_3\}$

$S_2 = \{x_2\}$ $S_3 = \{x_5\}$

$S_4 = \{x_4\}$

$$\Rightarrow \boxed{\chi(G) \geq \frac{|V(G)|}{\alpha(G)}}.$$

$\boxed{\chi \geq n/\alpha}$ $\boxed{\chi \geq \omega}$

- Greedy coloring can't force to use more than $\Delta(G) + 1$ colors.

$\chi(K_n) = n = \Delta(K_n) + 1$

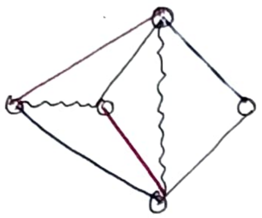
Ordering of vertices determines $\chi(G)$ # colors.

- Brooks theorem

If G is not K_n or C_{2n+1} then

$$\chi(G) \le \Delta(G).$$

- Edge Coloring

eg 

4-edge colorable.

$\chi'(G)$ chromatic index for edge coloring

⇒ Since edges sharing an end vertex need diff. colors.

Vertex having Δ degree, has Δ edges incident on it. All Δ edges have to be colored with diff. colors.

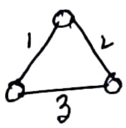
$$\boxed{\chi'(G) \geq \Delta(G)}$$

Th If G is a simple graph, $\chi'(G) \le \Delta(G) + 1$

So, $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$

Class 1 Class 2

- Regular graphs of odd order are class 2 graphs.



$\Delta(G) = 2$

$\chi'(G) = 3.$

- Every bipartite graph is of class 1.

$\chi' = \max(m,n)$

bipartite graph

If G is bipartite, $\chi'(G) = \Delta(G)$.

- If a graph is properly colored, the vertices that are assigned a particular color form an independent set. (Each color class is " ")

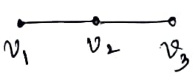
- Girth : Smallest cycle in the graph.

If G is 2-colored, it is bipartite. (2 set of indep. vertices)

- Ordering in Greedy algorithm

Pick a vertex v_n & list the vertices of G as $v_1, v_2, \dots, v_n$ so that if $i < j$, $d(v_i, v_n) \geq d(v_j, v_n)$, that is, we list the vertices from v_n first.

(v_1, v_2, v_3)



If G is not regular (all vertices having same degree Δ), $\chi \leq \Delta$.

If G is not an odd cycle (C_{2n+1}) or a complete graph (K_n), $\chi \leq \Delta$.

Every k-chromatic graph has at least $\binom{k}{2}$ edges.

Proof For each pair of colors i, j $(1 \le i < j \le k)$ there must be at least 1 edge joining a vertex of color i to a vertex of color j (otherwise we could collapse those 2 color classes to one).

Since, at least one edge for each pair of color classes, G has at least $\binom{k}{2}$ edges.