GATE CSE NOTES

by

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With best wishes from Joyoshish Saha

Set

* **Theorem:** Let U be the universe of discourse & A is a set. Then $A \subset U$.

* **Theorem:** Let A & B be sets. Then $A = B$ if & only if $A \subset B$ and $B \subset A$.

Cor. For any set A, $A \subset A$.

* **Theorem:** Let A, B & C be sets. If $A \subset B$ & $B \subset C$ then $A \subset C$.

* **Definitions:**
  - A set with no members is called an empty/null/void set.
  - A set with one member is called a singleton set.

* **Theorem:** Let ϕ be an empty set & A an arbitary set. Then, $\phi \subset A$.

* **Theorem:** Let ϕ & ϕ' be sets which are both empty. Then $\phi = \phi'$.

* Operations on Sets :

Let A & B be sets,

a) $A \cup B = \{x \mid x \in A \vee x \in B\}$

b) $A \cap B = \{x \mid x \in A \wedge x \in B\}$

c) $A - B = \{x \mid x \in A \wedge x \notin B\}$.

* If A & B are sets & $A \cap B = \phi$, then A & B are disjoint. If C is a collection of sets such that any two distinct elements of C are disjoint, then C is a collection of (pairwise) disjoint sets.

* Let * denote a binary operation, & let $x * y$ denote the resultant obtained by applying the operation * to the operands x & y. Then the operation * is commutative if $x * y = y * x$. The operation is associative if $(x * y) * z = x * (y * z)$.

* $A \cup B = B \cup A$
  $A \cap B = B \cap A$.

* $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$.

* Let Δ & $\square$ be binary operations, Then Δ distributes over $\square$ if

$$x \,\Delta\, (y \,\square\, z) = (x \,\Delta\, y) \,\square\, (x \,\Delta\, z).$$

$$(y \,\square\, z) \,\Delta\, x = (y \,\Delta\, x) \,\square\, (z \,\Delta\, x).$$

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

* $A \cup A = A$; $A \cap A = A$

$A \cup \phi = A$; $A \cap \phi = \phi$

$A - B \subset A$

* If $A \subset B$ & $C \subset D$, $(A \cup C) \subset (B \cup D)$

* If $A \subset B$ & $C \subset D$, $(A \cap C) \subset (B \cap D)$

* $A \subset A \cup B$

$A \cap B \subset A$

* $A \oplus B = B \oplus A$

$((A \oplus B) \oplus C) = (A \oplus (B \oplus C))$

* If $A \subset B$, $A \cup B = B$.

If $A \subset B$, $A \cap B = A$.

* $A - \phi = A$

* $A \cap (B - A) = \phi$

$A \cup (B - A) = A \cup B$

$(A \cup B) \cap C = A \cup (B \cup C)$ iff $A \subseteq C$

$(A \cap B) \cup C = A \cap (B \cup C)$ iff $C \subseteq A$

Modular laws.

$$\left\{\begin{array}{l} A - (B \cup C) = (A - B) \cap (A - C) \\ A - (B \cap C) = (A - B) \cup (A - C) \end{array}\right.$$

* $\bar{A} = U - A$. $\bar{\bar{A}} = A$

$A \cup \bar{A} = U$

$A \cap \bar{A} = \phi$.

* Uniqueness of complement.

Let A & B be subsets of a universe U. Then $B = \bar{A}$ iff

$A \cup B = U$

$A \cap B = \phi$.

* de Morgan's Law :

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

* Let $\mathcal{C}$ be a collection of subsets of some universe U.

(a) The union of the members of $\mathcal{C}$, denoted $\bigcup_{x \in \mathcal{C}} S$, is the set $\bigcup_{x \in \mathcal{C}} S =$

$$\{x \mid \exists S [S \in \mathcal{C} \wedge x \in S]\}$$

(b) If $\mathcal{C} \neq \phi$, the intersection of the members of $\mathcal{C}$, denoted $\bigcap_{x \in \mathcal{C}} S =$

$$\{x \mid \forall S [S \in \mathcal{C} \Rightarrow x \in S]\}$$

* Inductive definition of Set :

An inductive definition of a set always consists of ~~three~~ distinct components :

1. The basis, or basis clause, of the definition establishes that certain objects are in the set. This part of the definition has the dual function of establishing that the set being defined is not empty & of characterizing the "building blocks" which will be used to construct the remainder of the set.

2. The induction or inductive clause, of an inductive definition establishes the ways in which elements of the set can be combined to obtain new elements. The inductive clause always asserts that if objects $x, y, \ldots, z$ are elements of the set, then they can be combined in certain specified ways to create other objects which are also in the set. Thus, while the basis clause describes the building blocks of the set, the inductive clause describes the operations which can be performed on objects in order to construct new elements of the set.

3. The ~~external~~ extremal clause. asserts that unless an object can be shown to be a member of the set by applying the basis & inductive clauses a finite number of times, then the object is not a member of the set. The external clause of an inductive definition of a set S has a variety of forms, such as

(i) "No object is a member of S unless it's being so follows from a finite number of applications of the basis & inductive clauses."

(ii) "The set S is the smallest set which satisfies the basis & inductive clauses."

(iii) "The set S is the set such that S satisfies the basis & inductive clauses & no proper subset of S satisfies them. (i.e. if T is a subset of S such that T satisfies the basis & inductive clauses, then $T = S$)."

(iv) "The set S is the intersection of all sets which satisfy the properties specified by the basis & inductive clauses."

Eg. $N = \{0, 1, 2, 3, \ldots\}$

B : $0 \in N$ Basis

I : If $x \in N$ Inductive clause.
then $x + 1 \in N$

* Proof by Induction.

Eg. Let α be a well ~~formula~~ string of parantheses.

$L(\alpha)$ = no. of left paranthesess

$R(\alpha)$ = " " right "

If $x \in B$, then $L(x) = R(x)$

$\longrightarrow$ Basis.

$[\,]$ $\quad L(\alpha) = 1$ & $R(\alpha) = 1$

$\therefore L(\alpha) = R(x)$.

Induction

$x, y \in B$ | $L(x) = R(x)$

$[x] \in B$ | $L(y) = R(y)$

$L([x]) = L(x) + 1$.

$R([x]) = R(x) + 1$.

$L([x]) = R([x])$.

For $xy \in B$,

$L(xy) = L(x) + L(y)$

$R(xy) = R(x) + R(y)$

Since, $L(x) = R(x)$, $L(y) = R(y)$

$L(xy) = R(xy)$.

- Generally.

$$* \left\{ \begin{array}{c} P(0) \\ \forall n \, (P(n) \Rightarrow P(n+1)) \\ \hline \therefore \forall x \, P(x). \end{array} \right.$$

$$\begin{array}{c} P(k) \\ \forall n \, (P(n) \Rightarrow P(n+1)) \\ \hline \therefore \forall x \, [x \geq k \Rightarrow P(x)]. \end{array}$$

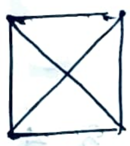
First Principle

Weak Induction.

Eg. Prove that the number of diagonal in an n-sided convex polygon is $\frac{n(n-3)}{2}$.

→ Basis.

$n = 3 \Rightarrow$ no. $= 0$



$n = 4 \Rightarrow$ no. $= \frac{4(4-3)}{2} = 2$.

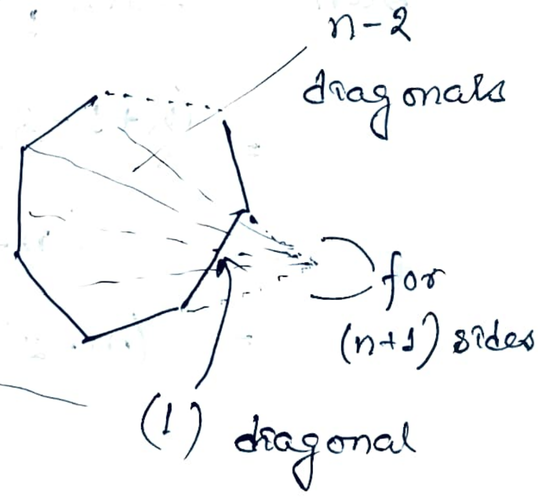
Inductive

Assume for n.

for n-sided no. $= \frac{n(n-3)}{2}$

for $(n+1)$ sided,

$\frac{n(n-3)}{2} + (n-2) + 1.$



$= \frac{n^2 - n - 2}{2}$

$= \frac{(n+1)(n+1-3)}{2}$

So, for $(n+1)$ sided polygon it's true.

So, proposition is true.

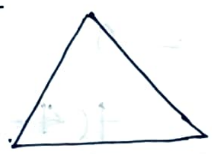
- **Strong Induction** / 2nd Principle.

$$\forall n \left[\forall k \left[k < n \Rightarrow P(k) \right] \Rightarrow P(n) \right]$$

$$\therefore \forall x \, P(x).$$

Eg. Prove that the sum of the interior angles of a n-sided convex polygon is $(n-2)\pi$.

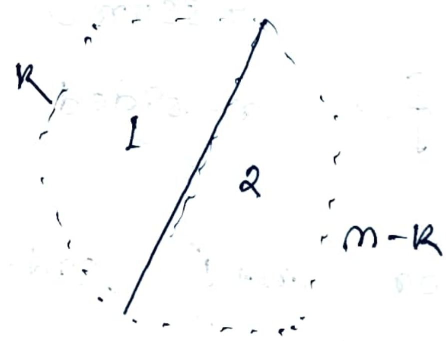
→ **Basis.** for $n = 3$, Sum $= \pi$.



$(3-2)\pi$.

Induction. Assume true for $k = 3, 4, \dots, n-1$

Prove for $k = n$.



$$\text{Sum} = (\text{Sum})_1 + (\text{Sum})_2$$

$$= (k+1-2)\pi +$$

$$(n-k+1-2)\pi$$

n sided polygon

$$= (n-2)\pi.$$

[Strong Induction]

* The Natural Numbers. ($\mathbb{N}$)

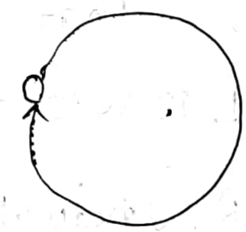
(View of Induction).

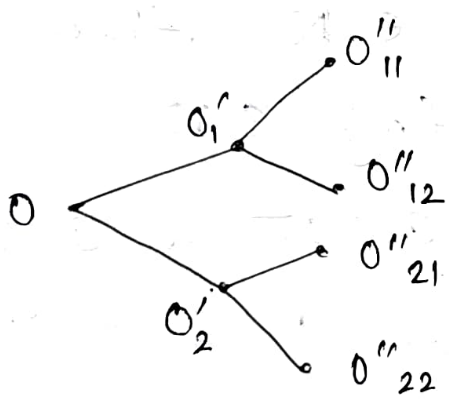
1. (Basis) $0 \in \mathbb{N}$
2. (Induction) If $n \in \mathbb{N}$, $n' \in \mathbb{N}$
3. (Extremal) If $S \subset \mathbb{N}$ & satisfies clauses 1 & 2 then $S = \mathbb{N}$

[n' is successor of n]

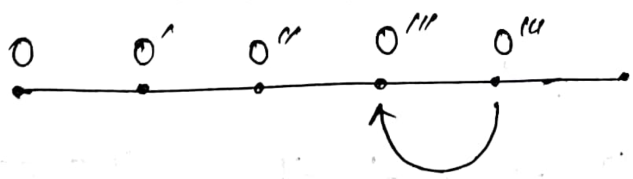
Problem 1. If 0 is successor of 0.

Problem 2 If 2 successors.





Problem 3



Predecessor is not unique.

Rectifying all these,

Each natural number will be set.

The first ~~se~~ natural number is defined to be ϕ, changing the basis step to,

1. (Basis) ϕ is a natural number.

For each natural number n, its successor n' is constructed as follows,

2. (Induction) If n is a natural number then $n \cup \{n\}$ is a natural number.

The extremal step remains unchanged.

Hence,

Definition: The set of natural numbers $\mathbb{N}$ is the set such that,

1. Basis – $\phi \in \mathbb{N}$
2. Induction – If $n \in \mathbb{N}$, then $n \cup \{n\} \in \mathbb{N}$
3. Extremal – If $S \subset \mathbb{N}$ & S satisfies clauses 1 & 2, then $S = \mathbb{N}$.

Theorems:

(i) 0 is not the successor of any natural number.

(ii) The successor to any natural number is unique.

(iii) If $n' = m'$, then $n = m$, i.e. if the successors of n & m are same, n & m will be same.

* Piano Postulates for the Natural Numbers

a) 0 is a natural number.

b) For each natural number n, there exists exactly one natural number n' which we call as the successor of n.

c) 0 is not the successor of any natural number.

d) If $n' = m'$, then $n = m$.

e) If $S \subset \mathbb{N}$, such that,

i) $0 \in S$

ii) If $n \in S$ the $n' \in S$,

then $S = \mathbb{N}$.

* Set Operations on Σ^*.

• Definition. Let Σ be an alphabet & x & y be elements of Σ^*.

If $x = a_1 a_2 \ldots a_m$ & $y = b_1 b_2 \ldots b_n$, $a_i, b_i \in \Sigma$; $m, n \in \mathbb{N}$, then the concatenation of x with y, denoted $x \cdot y$ or simply xy is the string $xy = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$.

If $x = \lambda$ [$\lambda \rightarrow$ empty string], then $xy = y$ for every y. Similarly if $y = \lambda$, then $xy = x$.

Eg. $x = abb$ $|x| = 3 = |y|$

$y = cba$

$xy = abbcba$

→ $|xy| = |x| + |y|$.

→ $xy \neq yx$.

→ $x(yz) = (xy)z$.

→ Definition. Let x be an element of Σ^*. For each $n \in N$ the string x^n is defined as follows:

1. $x^0 = \lambda$
2. $x^{n+1} = x^n . x$.

→ Definition. Let Σ be a finite alphabet. A language over Σ is a subset of Σ^*.

→ Definition. Let A & B be languages over Σ. The set of product of A with B, denoted by A.B, or simply AB, is the language

$$AB = (xy \mid x \in A \wedge y \in B).$$

The language AB consists of all strings which are formed by concatenating an element of A with an element of B.

- **Theorem.** Let A, B, C & D be arbitrary languages over Σ. The following relations hold.

a) $A\phi = \phi A = \phi$

b) $A\{\lambda\} = \{\lambda\}A = A$.

c) $(AB)C = A(BC)$

d) If $A \subset B$ & $C \subset D$, then $AC \subset BD$

e) $A(B \cup C) = AB \cup AC$

f) $(B \cup C)A = BA \cup CA$

g) $A(B \cap C) = AB \cap AC$

h) $(B \cap C)A = BA \cap CA$.

- **Definition.** Let A be a language over Σ. The language A^n is defined inductively as follows:

1. $A^0 = \{\lambda\}$
2. $A^{n+1} = A^n . A$. for $n \in N$

The language A^n is the set product of A with itself n times. Therefore if $z \in A^n$ for $n \geq 1$ then $z = w_1 w_2 \cdots w_n$ where $w_i \in A$ for each i from 1 to n.

- **Definition.** Let A & B be subsets of Σ^* & m & n be arbitary elements of N. Then,

1. $A^m A^n = A^{m+n}$.
2. $(A^m)^n = A^{mn}$
3. $A \subset B \Rightarrow A^n \subset B^n$.

Definition. Let A be a subset of Σ^* & then the set A^* is defined to be

$$A^* = \bigcup A^n \quad , n \in \mathbb{N}.$$

$$= A^0 \cup A^1 \cup A^2 \cup \ldots$$

$$= \{\lambda\} \cup A \cup A^2 \cup \ldots$$

The set A^* is often called the star closure, Kleene closure or closure of A.

→ The set A^+ is defined to be

$$A^+ = \bigcup A^n \quad n \geq 1.$$

$$A^+ = A^1 \cup A^2 \cup A^3 \cup \ldots$$

It is called the positive closure of A.

Theorem Let A & B be languages over Σ & let $n \in \mathbb{N}$. Then,

→ $A^* = \{\lambda\} \cup A^*$

→ $A^n \subseteq A^*$ for $n \geq 0$.

→ $A^n \subset A^+$ for $n \geq 1$.

→ $A \subset AB^*$

→ $A \subset B^*A$

→ $(A \subset B) \Rightarrow A^* \subset B^*$

→ $(A \subset B) \Rightarrow A^+ \subset B^+$

→ $AA^* = A^*A = A^+$

→ $\lambda \in A \Leftrightarrow A^+ = A^*$

→ $(A^*B^*)^* = (A \cup B)^* = (A^* \cup B^*)^*$

→ $(A^+)^* = A^*A^* = A^*$

$\underline{o} \rightarrow \quad (A^*)^+ = (A^+)^* = A^*$

$\underline{o} \rightarrow \quad A^* A^+ = A^+ A^* = A^*$

- <u>Theorem.</u> Let A & B be arbitrary subsets of Σ^* such that $\lambda \notin A$. Then the equation $\underline{X = AX \cup B}$ has the unique solution $X = A^* B$.

$X = AX \cup B$

** <u>Proof</u>

$$
\begin{aligned}
X &= AX + B. \\
&= A(AX + B) + B \\
&= A^2 X + AB + B \\
&= A^2(AX + B) + AB + B \\
&= A^3 X + A^2 B + AB + B \\
&= A^{n+1} X + A^n B + A^{n-1} B + \dots + AB + B.
\end{aligned}
$$

1. If $w \in X$, then $w \in A^* B$.

   $X \subseteq A^* B$.

2. If $w \in A^* B$, then $w \in X$

   $A^* B \subseteq X$

To prove

1. $w \in X$. $|w| = n$.

   $w \in A^i B$ for some $i \leq n$.

   $\Rightarrow w \in A^* B$

2. $w \in A^* B$

   $\Rightarrow w \in A^j B$ for some j

Take $n > j \Rightarrow w \in X$.

$\therefore X = A^* B$.

→ If $\lambda \in A$; $X = A^* B$ is one of the solutions.

$\forall\ C \supset B$, $A^* C$ will also be a solution.

Relation.

* For $n > 0$, an ordered n-tuple with ith component a_i is a sequence of n objects denoted by $\langle a_1, a_2, \ldots, a_n \rangle$. Two ordered n-tuples are equal if & only if their ith components are equal for all i, $1 \le i \le n$.

* Let $\{A_1, A_2, \ldots, A_n\}$ be an indexed collection of sets with indices from 1 to n, where $n > 0$. The cartesian product or cross product of the sets A_1 through A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$ or $\times_{i=1}^{n} A_i$ is the set of n-tuples

$$\{\langle a_1, a_2, \ldots, a_n \rangle \mid a_i \in A_i\}.$$

When $A_i = A$ for all i, then $\times_{i=1}^{n} A_i$ will be denoted by A^n.

* Eg.

$A = \{1,2\}$ $\quad$ $A \times B = \{\langle 1,a \rangle, \langle 1,b \rangle, \langle 2,a \rangle, \langle 2,b \rangle\}$

$B = \{a,b\}$ $\quad$ $A \times C = \{\langle 1,\alpha \rangle, \langle 2,\alpha \rangle\}$

$C = \{\alpha\}$ $\quad$ $B \times D = \phi$

$D = \phi$

* $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

* Let $\{A_1, A_2, \ldots, A_n\}$ be sets. An n-ary relation R on $X_{i=1}^{n} A_i$ is a subset of $X_{i=1}^{n}$. If $R = \phi$, then R is called the empty or void relation.

If $R = X_{i=1}^{n} A_i$, then R is called the universal relation. If $A_i = A$ for all i, Then R is called ~~the universal relation.~~ an n-ary relation on A.

* Let R_1 be an n-ary relation on $X_{i=1}^{n} A_i$. Let R_2 be an m-ary relation is a subset of $X_{i=1}^{m} B_i$, then $R_1 = R_2$ iff $n = m$, & $A_i = B_i$ for all i, $1 \leq i \leq n$ & R_1, R_2 are equal sets of ordered n-tuples.

* Binary Relation : Let R be a binary relation over $A \times B$.

The set A is the domain of R ; B is the codomain. We denote $\langle a, b \rangle \in R$ by the infix notation aRb & $\langle a, b \rangle \notin R$ is denoted by $a \not{R} b$.

* Eg. $R = \{ \langle a,b \rangle \mid a = 2b \}$ $\quad x = \langle 6,3 \rangle$

Inductive definition.

Basis. $\langle 0,0 \rangle \in R$.

Induction. If $\langle x,y \rangle \in R$, then

$\langle x+2, y+1 \rangle \in R$.

Since $\langle 0,0 \rangle \in R$, $\langle 2,1 \rangle$, $\langle 4,2 \rangle$,

$\langle 6,3 \rangle \in R$.

* Eg. $R = \{ \langle a,b,c \rangle \mid a+b = c \}$

$x = \langle 1,1,2 \rangle$

Basis. $\langle 0,0,0 \rangle \in R$

Induction.

If $\langle x,y,z \rangle \in R$,

$\langle x+1, y, z+1 \rangle \in R$

$\langle x, y+1, z+1 \rangle \in R$.

$\langle 0,0,0 \rangle \in R$, $\langle 1,0,1 \rangle \in R$,

$\langle 1,1,2 \rangle \in R$.

* ~~Directed~~ Graph / ~~Digraph~~.

An ordered pair $D = \langle A,R \rangle$ where A is a set & R is a binary relation on A. The set A is the set of nodes (points, vertices) of D & the elements of R are the arcs (edges) of D. The relation R is called incidence relation of D.

* Source   Self loop

Destination

Indegree

Outdegree

* Let $D = \langle A, R \rangle$ be a digraph. If aRb, then the arc $\langle a, b \rangle$ originates at a & terminates at b.

An arc of the form $\langle a, a \rangle$ is called a loop. The number of arcs that originate at a node a is called the outdegree of node a; the number of arcs which terminate at a is called the indegree of node a.

* Let $D = \langle A, R \rangle$ be digraph with nodes a & b.

An undirected path p from a to b is a finite sequence of nodes $p = \langle c_0, c_1, \ldots, c_n \rangle$ such that

1. $c_0 = a$
2. $c_n = b$

For all c_i such that $0 \leq i \leq n$, either $c_i R c_{i+1}$ or $c_{i+1} R c_i$.

* Number of relations on A 2^{n^2}. | $\underset{m}{A} \to \underset{n}{B}$ 2^{mn}

* Properties of Relations:

1. A relation R on a set A is called reflexive if $(a,a) \in R$ $\forall a \in A$.

2. A relation R on a set is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ & $(b,a) \in R$ then $a=b$ is called antisymmetric.

Sym. $\forall a \forall b \, ((a,b) \in R \to (b,a) \in R)$.

Anti sym. $\forall a \forall b \, (((a,b) \in R \wedge (b,a) \in R) \to (a=b))$.

3. A relation R on a set A is called transitive if whenever $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$.

$\forall a \forall b \forall c \, (((a,b) \in R \wedge (b,c) \in R) \to (a,c) \in R)$.

- No. of reflexive relations. - $2^{n(n-1)}$.

No. of symmetric relations. - $2^{(n(n+1))/2}$.

No. of antisym. relations - $2^n \cdot 3^{(n^2-n)/2}$

Irreflexive - if $\forall a \in A$, $a \not R a$ or $(a,a) \notin R$.

- No. of sym. & antisym. both at a time $= 2^n$ (only diagonal elems are possible)

- No. of sym & asym. both at a time $= 1$. $[R = \phi]$.

- No. of ref & antisym. both at a time $= 3^{(n^2-n)/2}$.

- No. of asymmetric binary relations $= 3^{(n^2-n)/2}$.

* <u>Composition</u>. Let R be a relation from a set A to a set B & S a relation from B to a set C. The composite of R & S is the relation consisting of ordered pairs (a,c) where $(a) \in A$, $c \in C$ & for which there exists an element $b \in B$ such that $(a,b) \in R$ & $(b,c) \in S$. We denote the composite of R & S by <u>$S \circ R$</u>.

→ Let R be a relation of set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by $R^1 = R$ & $R^{n+1} = R^n \circ R$.

- → The relation R on a set A is transitive if & only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

* Representing Relations.

1. Using Matrices.

eg: $A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b$

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3),$
$(a_3, b_3), (a_3, b_5)\}$

$$M_R \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

- R is reflexive if all

main diagonal of M_R are

M_{R_1}, M_{R_2} matrices for R_1, R_2

$R_1 \cup R_2$

$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ (join)

$R_1 \cap R_2$

$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$ (meet)

Composite

$M_{S \circ R} = M_R \odot M_S$ — Boolean product

$R : A \to B$
$S : B \to C$

$M_{S \circ R} = [t_{ij}]$ | $t_{ij} = 1$ iff
$M_R = [r_{ij}]$ | $r_{ik} = s_{kj} = 1$
$M_S = [s_{ij}]$ | for some k

- R on the set A is symmetric iff

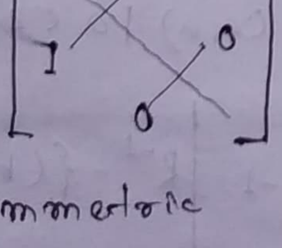
$m_{ji} = 1$ whenever $m_{ij} = 1$. This also means

$m_{ij} = 0$ whenever $m_{ji} = 0$.

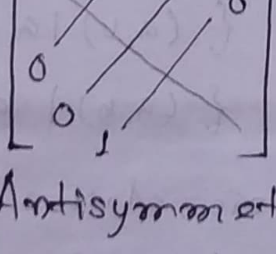
To be symmetric,

M_R = Transpose of M_R = $(M_R)^t$

$\Rightarrow M_R$ is symmetric.



Symmetric



Antisymmetric

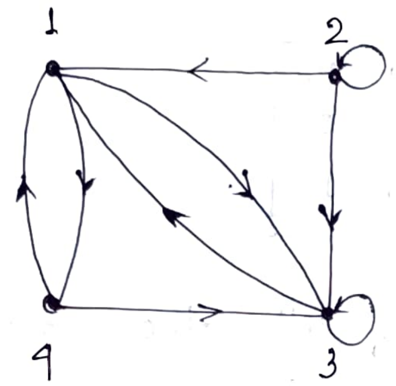
Matrix for an antisym. rel^n has the property that if $m_{ij} = 1$ with $i \neq j$ then $m_{ji} = 0$. Or, in other words, either

$m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

2. Using Digraph.

A digraph consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b) & the vertex b is called the terminal vertex of this edge.

eg.



Theorem R on A. There's a path of length n, from a to b, iff $(a, b) \in R^n$.

$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)\}$

* Closure of Relations.

If there is a relation S with property P containing R (relation on A set, may or may not have some property P) such that S is a subset of every relation with P containing R then S is called the closure of R wrt P.

- Reflexive closure : $R \cup \Delta$, Δ diagonal relⁿ

  eg. $R = \{(a,b) \mid a < b\}$

  $R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in \mathbb{Z}\}$

  $= \{(a,b) \mid a \leq b\}$

- Symmetric closure : $R \cup R^{-1}$ | $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

  eg. $R \cup R^{-1} = \{(a,b) \mid a \neq b\}$

- Transitive closure : Equals the connectivity relⁿ R^*

Theorem

R^* consists of the pairs (a, b) such that there is a path (in digraph) of length at least one from a to b in R.

$R^* = \bigcup_{n=1}^{\infty} R^n$

* Partial Ordering Relation.

A relation R on a set A is partial ordering ~~relating~~ relation if R is reflexive, antisymmetric, transitive.

* Partial Ordered Set. (POSET)

A set A with a partial order R, defined on A, is called partial ordered set & it is denoted by [A; R].

Eg. $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

L Partial order

POSET [A; R]

Eg. POSET $[\mathbb{R}; \leq]$. $\mathbb{R} \rightarrow$ real number.

* Equivalence Relation.

R is called equivalence relⁿ on a set A if it is ref, sym & transitive.

* Totally ordered Set. [linearly ordered set / chain]

A poset [A; R] is called a 'TOS' if every pair of elements in A are comparable, i.e. aRb or bRa $\forall a, b \in A$.

Functions.

* **Definition.** : Let A and B be two non-empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

• If f is a function from A to B, we say that A is the domain of f & B is the codomain of f. If $f(a) = b$, we say that b is the image of a & a is the preimage of b. The range of f is the set of all images of elements of A.

$$f: A \to B$$

f maps A to B.

• Let f_1 & f_2 be functions from A to R. Then $f_1 + f_2$ & $f_1 f_2$ are also functions from A to R defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

* **One-to-one function**: A function f is said to be one-to-one or injective iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

Choosing m elems from n elems in B to match with m elems of A. $(m \le n)$

$\underline{Q}$ ✓ • Number of one-one functions $A(n) \to B(n)$ is $n!$

• $A(m) \to B(n) \Rightarrow {}^{n}P_{m}$

• For one-one functions $|\text{Domain}| \le |\text{Co-domain}|$

- Taking contrapositive of the implication in the definition, f is one-to-one iff $f(a) \neq f(b)$ whenever $a \neq b$.

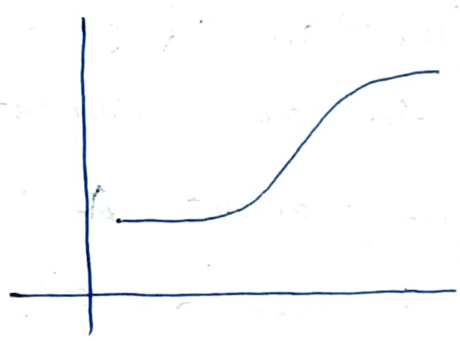
- Using quantifiers, for f to be injective,

$$\forall a \forall b \, (f(a) = f(b) \rightarrow a = b).$$

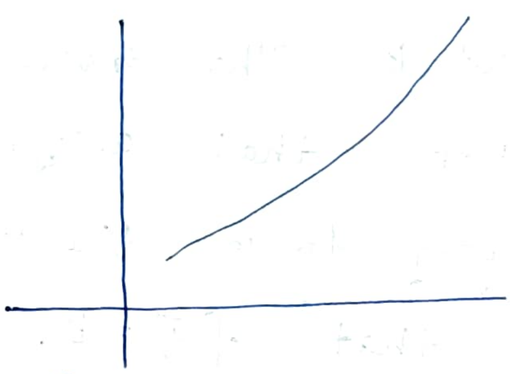
or $\forall a \forall b \, (a \neq b \rightarrow f(a) \neq f(b))$.

Universe of discourse is the domain of the function.

- A function f whose domain & codomain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$ & strictly increasing if $f(x) < f(y)$, whenever $x < y$ & x & y are in the domain of f. Similarly, decreasing & strictly decreasing.



Increasing



Strictly increasing.

* Every function is a relation, but not the reverse.

* Onto function: A function f from A to B is called onto or surjective iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

For f to be onto,

$$\forall y \exists x \, (f(x) = y).$$

- For codomain & range of onto functions, we can say

codomain = range

- Number of onto functions from a set of m elements to a set of n elements is

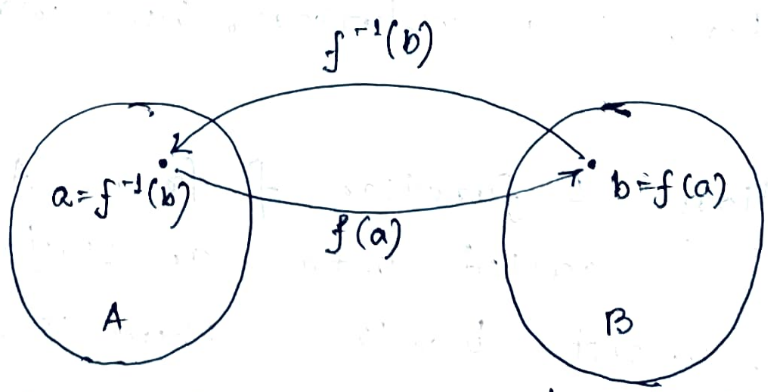
$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m \quad \Big| \quad \text{or} \quad n! \left\{ {m \atop n} \right\} \to S_2(m,n)$$

* <u>Bijection</u>: A function f is one-to-one correspondence or a bijection if it is both one-to-one & onto.

- For bijection $f: D \to C$

$|C| = |D|$

* <u>Inverse function</u>: Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A, such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

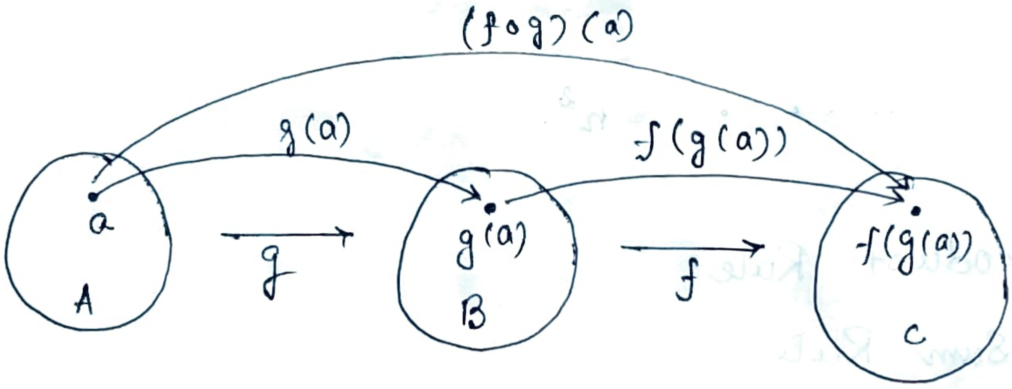


- A one-to-one correspondence is called invertible because we can define inverse of this function.

* Composition of the functions:

Let g be a function from set A to B & let f be a function from set B to C. The composition of the functions f and g, denoted by $f \circ g$ is defined by

$$(f \circ g)(a) = f(g(a)).$$



- $f \circ g \neq g \circ f$.
- $(f \circ f^{-1})(a) = a. = I(a)$.

Functions

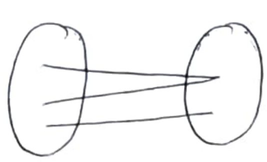
* Relation having no 2 ordered pairs with the same first component. $R: A \to B$

Every elem of A is mapped to only one elem of B.

$>$ ✓ $<$ ×

* f on $(x, y) = \{(x, y) \mid y = x^2\}$

$\equiv f(x) = x^2$



Domain — Codomain

Preimage — Image

Range

* Checking if a f^n or not:

i) $\forall x \in A$, $f(x)$ defined & $\in B$

ii) $f(x)$ is unique, single valued

* One-One / Injection $\quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ✓

* Onto / Surjection $\quad f(A) = B \quad$ Co-Domain = Range

↓ Checking: Every elem of B has a preimage in A.

↳ Bijection = One-one + Onto

↳ Also, one-one correspondence

⇓

$|A| = |B|$

* Composite f^n

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd} \longrightarrow g\{f(x)\}$$

$f: R \to R \quad f(x) = x + 2$

$g: R \to R \quad g(x) = x^2$

$(g \circ f)(x) = g(f(x)) = (x+2)^2$

$(f \circ g)(x) = f(g(x)) = x^2 + 2$

* $g \circ f \neq f \circ g$

$g \circ (f \circ h) = (g \circ f) \circ h$

* If f & g are one-one, $g \circ f$ is one-one

* If f & g are onto, $g \circ f$ is onto

More properties on (4).

* If f & g are bijections, $g \circ f$ is a bijection.

* Identity mapping $f(x) = x$

* Inverse mapping Exists if bijection.

* No. of functions

A = {a, b} (m), B = {1, 2, 3} (n)

Total # fns $= n^m$

one-one f^ns $= \begin{cases} {}^nP_m & m \le n \\ 0 & m > n \end{cases}$

onto f^ns $= \begin{cases} 0 & m < n \\ n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \dots \\ \dots (-1)^{n-1} \binom{n}{n-1} \end{cases}$, $m \geq n$

When $n = m$,

onto f^ns $= n! = m!$

* # onto f^ns $= \sum_{K=0}^{n} (-1)^K \binom{n}{K} (n-K)^m$.

or $\boxed{n! \left\{ {m \atop n} \right\}}$

Stirling 2nd kind

$\left\{ {m \atop n} \right\}$ # ways to partition set of m elems into n subsets

$n!$ # ways of assigning the subsets to the n elements (Which elem of B is to be mapped to a subset of A)

eg {1, 2, 3} {a, b}

$\left\{ {m \atop n} \right\} = \left\{ {3 \atop 2} \right\}$

1 | 2,3

1,2 | 3

1,3 | 2

= 3

$n! = 2!$

$\Rightarrow$ ~~12~~ 6

⇒ **Stirling's no. of 2nd kind :** $\left\{ {m \atop n} \right\}$

$$S(m,n) = S(m-1,n-1) + n\,S(m-1,n)$$

→ # ways to partition set of m elements into n subsets.

$S(3,2)$
↓ set $\{0,1,2\}$
$\{0,1\}\ \{2\}$
$\{0\}\ \{1,2\}$
$\{1\}\ \{0,2\}$
$\Rightarrow S(3,2) = 3$

$$S(m,n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

Table

m \ n	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

$$S(4,2) = S(3,1) + 2\,S(3,2) = 1 + 2 \cdot 3 = 7$$

eg # onto f^ns from A $(|A| = 6)$ to B $(|B| = 3)$.

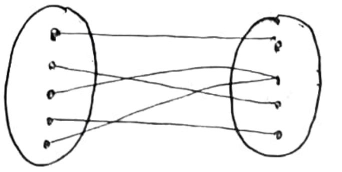
→ $3! \left\{ {6 \atop 3} \right\} = 6 \cdot 90 = 540$

* eg How many ways are there to assign 5 different jobs to 4 employees if each emp. is assigned to at least one job (also each job has to be taken by at least one emp.) ?

→ # onto f^ns

$4! \left\{ {5 \atop 4} \right\} = 24 \times 10 = 240$

jobs emp.



5 4

[2 employees should not work at same job]

* Symmetric f^n. If $f = f^{-1}$

- If $|X| > |Y|$ & $|Y| = 2$ then
  ✓ # onto f^ns $X \to Y = 2^{n(X)} - 2$

* If $g \circ f$ is one - one, then
  ✓ f is one - one but g may not be.

* If $g \circ f$ is onto, then
  ✓ g is onto, but f may not be.

* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

* $f : X \to Y$ be a f^n & A, B be arbitrary non-empty subsets of X.
  1. If $A \subseteq B$, $f(A) \subseteq f(B)$
  2. ~~If~~ $f(A \cup B) = f(A) \cup f(B)$
  3. $f(A \cap B) \subseteq f(A) \cap f(B)$.
     Equality holds when f is one - one

* Domain of fun^n

Trigonometric	Inverse trig
• $\sin x$, $\cos x$ defined for all real values.	• $\sin^{-1}x$, $\cos^{-1}x$ for $-1 \le x \le 1$
• $\tan x$, $\sec x$ " real except $x = (2n+1)\pi/2$	• $\tan^{-1}x$, $\cot^{-1}x$ for $x \in R$
• $\cot x$, $\operatorname{cosec} x$ " except $x = n\pi$.	• $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ for $x \le -1$, $x \ge 1$

Relations.

Q. If $A = \{1, 2, \ldots, n\}$ then the number of reflexive relations possible on A.

→ Assume the relation is R s.t. $R \subseteq A \times A$.

In matrix representation, R to be reflexive the elements on the diagonal must be 1 & rest of the elements can be 0 or 1.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Number of elements except the 1's of the diagonal $= n^2 - n$

∴ Number of reflexive relations possible is 2^{n^2-n}.

∴ Number of irreflexive relations on A is $\cancel{2^{n^2} - 2^{n^2-n+1}}$ 2^{n^2-n}.

NB. → The relation '$\leq$' is reflexive on any set of real numbers.

→ The relation 'is a divisior of' is reflexive on any set of non-zero real numbers.

→ The relation 'is a subset of' denoted by '$\subseteq$' is reflexive on any collection of sets.

→ The relation 'is parallel to' is reflexive on a set of all straight lines.

→ $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y \text{ is even integer}\}$ is reflexive.

→ '$\equiv \bmod 5$' on $\mathbb{Z}$ is reflexive.

8. Which of the following is false?

a) If R_1 is reflexive, then every superset of R_1 is reflexive. True.

b) If R_1 is reflexive, then every subset of R_1 is reflexive. False

c) If R_1, R_2 are reflexive then $R_1 \cap R_2$ is reflexive. True [R_1, R_2 defined on A]

d) If R_1, R_2 are reflexive, then $R_1 \cup R_2$ is reflexive. True [$R_1 \cup R_2$ is a superset of R_1 or R_2]

NB.
- There is no relation that is reflexive & irreflexive at the same time.
- There can be relations that are not reflexive & not irreflexive.
- Smallest irreflexive relation = { }
- Cardinality of largest irreflexive relation = $n^2 - n$.

Cardinality of largest reflexive relation = n^2

Cardinality of smallest irreflexive relation = 0

Cardinality of smallest reflexive relation = n

Q. Number of irreflexive relations on A, st $|A| = n$.

$\rightarrow$ $\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$ Diagonal elements must be zero & the rest elements can be either 0 or 1.

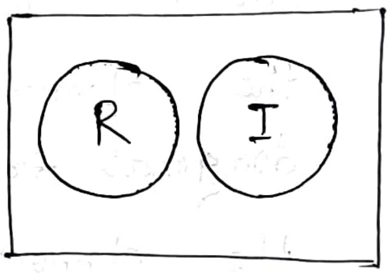
So, the number of irreflexive relations is 2^{n^2-n}.

Q. No. of relations that are either ref. or irreflexive.

* $\rightarrow$ $2 \times 2^{n^2-n} = 2^{n^2-n+1}$ [both ref & irr. is not an option]

Q. No. of relations that are ~~neit~~ neither ref. nor irreflexive.

* $\rightarrow$ $2^{n^2} - (2^{n^2-n+1})$



NB. Any relation that is reflexive can never be irreflexive.
i.e. the R & I are ~~no~~ disjoint sets.

NB $\rightarrow$ The relation '<' on set of all real numbers is irreflexive.
$\rightarrow$ The relation '$\subset$' on set of all sets is irreflexive.
$\rightarrow$ The relation '$\perp$' on set of all straight lines is irreflexive.

Q. Which of the following is false?

a) Every subset of irreflexive relation is irreflexive. True.

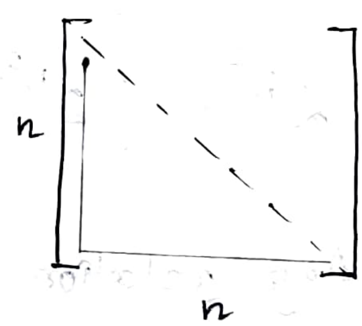
b) Every superset of irreflexive relation is irreflexive. False

c) If R_1 is irreflexive, R_2 is irreflexive then $R_1 \cap R_2$ is irreflexive. True.

d) If R_1, R_2 is irreflexive, then $R_1 \cup R_2$ is irreflexive. True.

Q. Number of symmetric relations on A.
$|A| = n$.

→ For a symmetric relation the elements on the lower diagonal triangle determine the elements of upper triangle. And the elements on the diagonal can be anything (0 or 1)



So, the number of elements on the lower triangle & the diagonal =

$$\frac{n(n+1)}{2}$$

1 ×
2 × ×
3 × × ×
⋮
n × × × . . . . . ×

So, the number of symmetric relations is

$$2^{\frac{n(n+1)}{2}}.$$

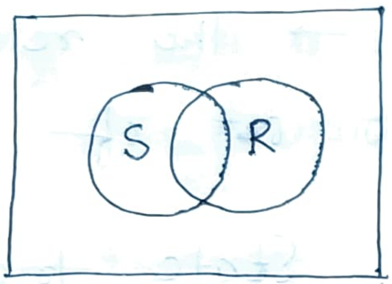
or,

$$2^n \times 2^{(n^2-n)/2}$$

$$= 2^{\frac{n^2+n}{2}}$$

Q. Number of relations that are both symmetric & reflexive. (Called Compatibility rel^n)

→ $A \times A = \{ \underbrace{\overbrace{(1,1), (2,2), \ldots\ldots}^{n}}_{\text{diagonal}}, \underbrace{\overbrace{(1,2), (1,3), \ldots}^{\text{non-diagonal}}}_{n^2-n} \}$



To be reflexive all diagonal elements must be present.

To be symmetric, i.e. for $(x,y) \in R$ implying $(y,x) \in R$, there are $\frac{n^2-n}{2}$ pairs.

∴ Number of relations that are both symmetric & reflexive $= 2^{\frac{n^2-n}{2}} = 2^{\frac{n(n-1)}{2}}$.

Note.

Sym, not ref $\quad |S - R| = n(S) - n(S \cap R)$

$= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

Ref, not sym $\quad |R - S| = n(R) - n(S \cap R)$

$= 2^{n^2-n} - 2^{\frac{n(n-1)}{2}}$

not sym, not ref. $\quad |\overline{S \cup R}| = 2^{n^2} - \left(n(S) + n(R) - n(S \cap R)\right)$

NB. → The relation 'x is brother of y' is symmetric on set of all men.

→ The relation 'is parallel to' is symmetric on set of all straight lines.

→ The relation 'is perpendicular to' is symmetric on set of all straight lines.

→ The relation '$\leq$' is not symmetric on set of all real numbers.

→ The relation '$\subseteq$' is not symmetric on set of all sets.

Q. State true or false

a) Every subset of a symmetric relation is symmetric. (False)

b) Every superset of a symmetric relation is symmetric. (False)

c) If R_1 & R_2 are symmetric, $R_1 \cap R_2$ is sym. (True)

d) $R_1 \cup R_2$ are sym. (True)

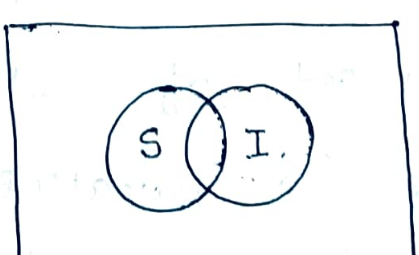
e) $R_1 - R_2$ is sym. (True).

Similar Logic.

R_1	R_2	$R_1 \cap R_2$
(x,y)	(x,y)	(x,y)
$\downarrow$	$\downarrow$	
(y,x)	(y,x)	$\longrightarrow (y,x)$

* Symmetric & Irreflexive Relations –

* $S \cap I = \underbrace{\{(1,2),(2,1),\ldots\}}_{\text{non-diagonal}}$



$n(S) = 2^{\frac{n(n+1)}{2}}$

$n(I) = 2^{n^2-n}$

$n(S \cap I) = 2^{\frac{n^2-n}{2}}$

$n(S \cup I) = n(S) + n(I) - n(S \cap I)$

$n(S - I) = n(S) - n(S \cap I)$

$n(I - S) = n(I) - n(S \cap I)$

$n(S \Delta I) = n(S-I) + n(I-S)$

$n(\overline{S \cup I}) = 2^{n^2} - n(S \cup I)$

NB. Cardinality of smallest antisymmetric relation $= 0$.

* & for the largest $= n + \frac{n^2-n}{2}$

$= \frac{n^2+n}{2}$

Q. Number of antisymmetric relations on A.

→ $A \times A = \{\underbrace{(1,1),(2,2),\ldots,(n,n)}_{n},\ \underbrace{(1,2),(2,1),(2,3),(3,2),\ldots}_{n^2-n}\}$

**

Number of antisymmetric relations $=$

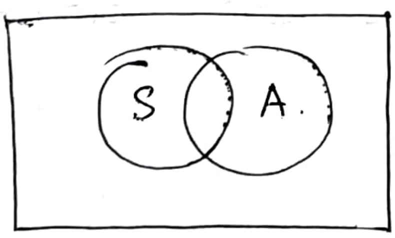
$$\left(2^n \times 3^{\frac{n^2-n}{2}}\right)$$

because,

for diagonal elements there are 2 possibilities — to be present or not.

for rest of the elements the half of the n^2-n elements can be present, not present.
eg. for a pair $\underline{(1,2)}, \underline{(2,1)}$, $(1,2)$ or $(2,1)$ may be present or none of them.
(3 possibilities

* Symmetric & Antisymmetric Relation.



$n(S) = 2^{\frac{n(n+1)}{2}}$

$n(A) = 2^n\, 3^{\frac{n(n-1)}{2}}$

$S \cap A \subseteq \{(1,1),(2,2),\ldots,(n,n)\}$

* $n(S \cap A) = 2^n$

$n(S \cup A) = n(S) + n(A) - n(S \cap A)$

$n(S - A) = n(S) - n(S \cap A)$

$n(A - S) = n(A) - n(S \cap A)$

$n(\overline{S \cup A}) = n(U) - n(S \cup A)$

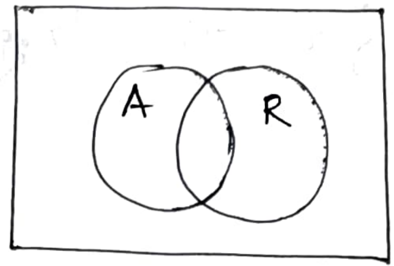
* Reflexive & Antisymmetric Relations :

$n(A) = 2^n 3^{n(n-1)/2}$

$n(R) = 2^{n(n-1)}$

* $n(U) = 2^{n^2}$

$n(A \cap R) = 3^{n(n-1)/2}$



↳ $A \cup R$, $A - R$, $R - A$, $\overline{R \cup A}$

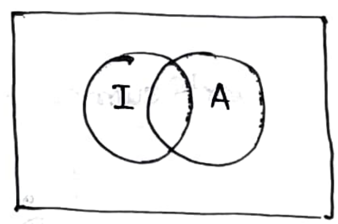
* Irreflexive & Antisymmetric Relations.

$n(I) = 2^{n(n-1)}$

* $n(A) = 2^n 3^{n(n-1)/2}$

$n(U) = 2^{n^2}$

$n(A \cap I) = 3^{n(n-1)/2}$



↳ $A \cup I$, $A - I$, $I - A$, $\overline{A \cup I}$

Q. State true or false.

a) Every subset of an antisymmetric relation is antisymmetric. (True)

b) Every superset of an antisymmetric relation is antisymmetric. (False)

c) Antisymmetric relations are closed under set union. (F)

d) 〃 〃 〃 〃 under set intersection. (T)

e) 〃 〃 〃 〃 under set difference. (T)

f) 〃 〃 〃 〃 under set complementation. (F)

NB. • The relation '$\leq$' is antisymmetric on any set of real numbers.

• The relation '$<$' is antisymmetric on any set of real numbers.

• The relation 'is a divisor of' is an antisymmetric relation on any set of positive real numbers.

• The relation 'is subset of' is antisym. relation on any set of sets.

* Asymmetric Relation.

A relation R on a set A is said to be asymmetric if (xRy) then $(y \not R x)\ \forall\ x, y \in A$.

eg. $R = \{(1,2), (2,2)\}$
- Not asymmetric
- Antisymmetric

** Diagonal elements can be present in antisymmetric but not in asymmetric.

$R' = \{\}$ — Sym, Asym, Antisym.

→ Cardinality of smallest asym. $rel^n = 0$

Cardinality of largest asym. $rel^n = \frac{n^2-n}{2}$

Q. Number of asym. rel^ns.

$3^{(n^2-n)/2}$

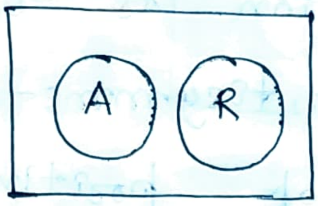
$A \times A = \{(1,1), (2,2), \ldots, \underbrace{(1,2), (2,1)}, \ldots\}$

- (1,2)
- (2,1)
- None

Possibilities.

* Reflexive & Asymmetric Relations.

→ If a relⁿ is reflexive, then it cannot be asymmetric.



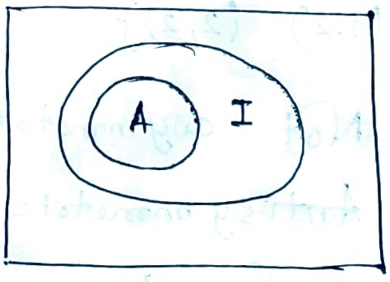
$n(A) = 3^{n(n-1)/2}$

$n(R) = 2^{n(n-1)}$

* Irreflexive & Asymmetric Relation.

→ ** Every asymmetric relation, is irreflexive & ~~xxxxxxx~~ the reverse is not true always.

eg. $R = \{(1,2), (2,1)\}$
- Irreflexive
- Not asymmetric.



$n(I) = 2^{n(n-1)}$

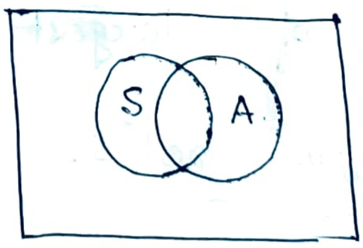
$n(A) = 3^{n(n-1)/2}$

* Symmetric & Asymmetric Relation.

$R = \{\}$ ← sym, asym.

$n(S) = 2^{n(n+1)/2}$

$n(A) = 3^{n(n-1)/2}$



$n(S \cap A) = 1$

$n(S \cup A)$ $\quad$ $n(\overline{S \cup A})$ $\quad$ $n(S - A)$ $\quad$ $n(A - S)$

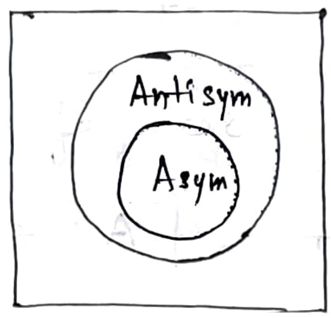
---

transitive relⁿs = 2, n = 1
13, n = 2
171, n = 3
3994, n = 4

* Antisymmetric & Asymmetric Relations.

→ Every asymmetric relation is antisymmetric. But, the reverse need not be true.

*



* Q. Asymmetric relation is closed under

a) Subset operation. True

b) Superset operation False

c) Union operation False

d) Intersection operation True

e) Set difference operation True

f) Complementation False

Q. Which of the following is not an equivalence rel^n

a) $R_1 = \{(a,b) \mid a-b$ is an integer$\}$

b) $R_2 = \{(a,b) \mid a-b$ is divisible by $5\}$

c) $R_3 = \{(a,b) \mid a-b$ is an odd number$\}$ transitivity fails

d) $R_4 = \{(a,b) \mid a-b$ is an even number$\}$

* NB. Cardinality of smallest partial order = n.

* Find whether following are Totally ordered sets or not -

1. If A is any set of real no.'s then the poset $[A; \leq]$ is TOS.

2. If $A = \{1, 2, \ldots, 10\}$ then the poset $[A; \leq]$ is a TOS.

3. If $A = \{1, 2, 6, 30, 60, 300\}$, then $[A; /]$ is TOS.

✓ 4. If $S = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, then $[S, \subseteq]$ is NOT TOS.

✓ 5. If $S = \{\phi, \{a\}, \{b, a\}, \{a, b, c\}\}$, then $[S; \subseteq]$ is TOS.

G'Q. Let $A = \{a, b, c\}$. Which is true?

✓a) $R_1 = \{(a, a), (c, c)\}$ is symmetric, antisym, & transitive on A.

b) $R_2 = \{(a, b), (b, a), (a, c)\}$ is sym, antisym.

✓c) $R_3 = \{(a, b), (b, a), (c, c)\}$ is sym but not antisym.

✓d) $R_4 = \{(a, b), (b, c), (c, c)\}$ is antisym but not sym.

G' Q. Let $A = \{a, b, c, d\}$ & a relation on set A is defined as $R = \{(a,a), (b,a), (b,b), (b,c), (b,d), (c,a), (c,b), (c,c), (c,d)\}$.

Which is true?

a) R is an equivalence relation.

b) R is irreflexive or antisymmetric.

c) R is symmetric or asymmetric.

✓d) R is transitive.

G' Q. Let A = Set of all real numbers.

$R = \{(a,b) \mid b = a^k$ for some integer $k\}$

* i.e. $aRb \iff b = a^k$. Then,

a) R is an equivalence relation.

✓b) R is a partial order.

c) R is reflexive & symmetric but not transitive.

d) R is a TOS.

Soln. a) $(2,4) \in R \not\Rightarrow (4,2) \in R$.

Not symmetric.

b) Antisymmetric.

$(2,4) \in R$ & $(4,16) \in R$.

$\Rightarrow (2,16) \in R$. hence transitive.

| $b = a^k \quad c = b^l$
| $c = (a^k)^l = a^{kl}$
| $= a^m$

c) Not sym.

d) $(2,3) \notin R$, $(3,2) \notin R$.

Not a total order.

G'8. Which is not true?

~~a)~~ If a relation R on A is symmetric & transitive then R is reflexive.

* b) If a relation R on A is irreflexive & transitive then R is antisym.

~~c)~~ If R & S are antisym. rel^{n}s on a set A then $R \cup S$ & $R \cap S$ are antisym.

d) If R & S are trans., $R \cap S$ is always transitive but $R \cup S$ need not be transitive.

Sol^n. a) $A = \{a, b, c\}$

$R = \{(a,b), (b,a), (a,a)\}$ Not ref

b) If $(a,b), (b,a)$ is present in R, (a,a) can't be prese as irreflexive

assuming R is not antisym. it contradicts.

c) $R = \{(a,b)\}$ $R \cup S$ not antisym.

$S = \{(b,a)\}$

d) $R = \{(a,b), (b,c), (a,c)\}$

$S = \{(a,c), (c,d), (a,d)\}$

$R \cap S = \{(a,c)\}$ ✗ trans.

G'96 Let A & B be sets & let A^c & B^c denote their complements. The set $(A-B) \cup (B-A) \cup (A \cap B)$ is

~~a)~~ $A \cup B$

b) $A^c \cup B^c$

c) $A \cap B$

d) $A^c \cap B^c$

G'96. Let R be a non-empty relation on a collection of sets defined by ARB iff $A \cap B = \phi$. Then.

a) R is ref. & trans.

✓ b) R is sym. & non-trans.

c) R is equivalence relⁿ.

d) R is non-reflexive & not sym.

Solⁿ. a)

Not transitive. $(A,B) \wedge (B,C) \not\Rightarrow (A,C)$

Not reflexive

b)

Symmetric.

G'97 The number of equivalence relation on the set $\{1,2,3,4\}$ is

Bell Number
Set Partition

✓ a) 15   b) 16   c) 24   d) 4.

Remember ***

$n = 3$
no. of eqv. relⁿ $= 5$

$n = 4$
no. of eqv. relⁿ $= 15$.

1
1 2
2 3 5
:

G'98 The no. of elements on the smallest & largest eqr. relⁿ on A (n elements) is -

smallest - n

largest - n^2.

G'98 If R_1 & R_2 are eqr. relⁿ. then state T/F

* a) $R_1 \cup R_2$ is eqr. relⁿ (F) True for ref, sym; false for transitive

b) $R_1 \cap R_2$ is eqr relⁿ. (T)

G'01. Which one is equivalentce?

i) $R_1(a,b)$ iff $(a+b)$ is even over the set of integers.

ii) $R_2(a,b)$ iff $(a+b)$ is odd over the set of integers.

iii) $R_3(a,b)$ iff $(ab>0)$ over set of non-zero ~~relational~~ rational numbers.

iv) $R_4(a,b)$ iff $|a-b| \le 2$ over the set of natural numbers.

$(1,3) \in R_4$

$(3,5) \in R_4$

$\not\Rightarrow (1,5) \in R_4$

Not transitive.

$a \cdot b > 0 \quad b \cdot c > 0$

$ab^2c > 0$

$\Rightarrow ac > 0 \; [b^2 > 0]$

Transitive.

G'06 R is defined on ordered pairs of integers as follows. $(x,y) \; R \; (u,v)$ if $x<u$ & $y>v$.
Then R is

a) Neither a partial order nor an eqv.

b) A partial order but not total order

c) Total order

d) An eqv. reln.

Soln. $R = \{<(1,3),(2,0)>, \ldots\}$

a) $(x,y),(y,x)$ not reflexive.

G'06 E, F & G are finite sets, let

$X = (E \cap F) - (F \cap G)$ &

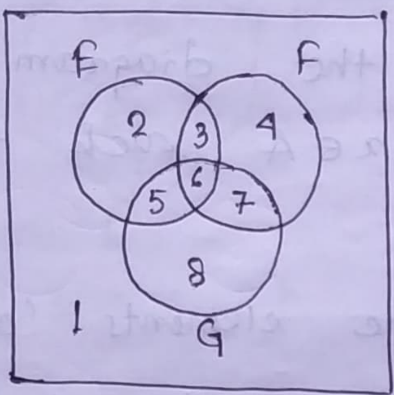
$Y = (E - (E \cap G)) - (E - F)$

Which is true?

a) $X \subset Y$ c) $X = Y$

b) $X \supset Y$ d) $X - Y \neq \phi$, $Y - X \neq \phi$

Sol^n.



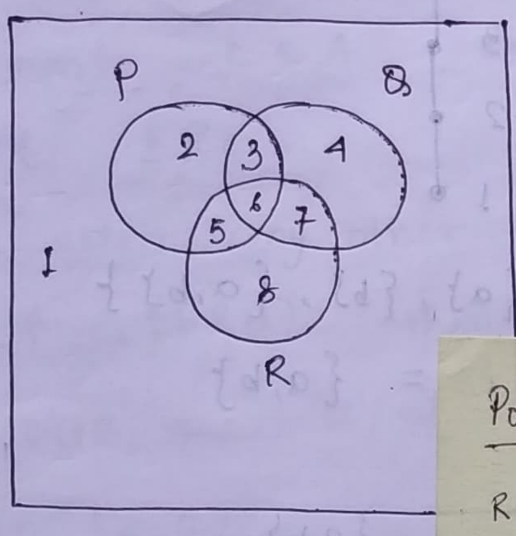
$X = (3, 6) - (6, 7)$
$= 3$

$Y = (2, 3) - (2, 5)$
$= 3$

G'08. If P, Q, R are subsets of universal set U then $(P \cap Q \cap R) \cup (P' \cap Q \cap R) \cup Q^c \cup R^c$ is

a) $Q^c \cup R^c$ b) $P \cup Q^c \cup R^c$ c) $P^c \cup Q^c \cup R^c$ d) U

Sol^n.



$6 \cup ((1, 8, 7, 4) \cap$
$(3, 4, 6, 7) \cap (5, 6, 7, 8))$
$\cup (1, 2, 5, 8) \cup (1, 2, 3, 4$

- Composite of relations. SoR for R ...

R : A → B 2nd elem of ordered pair ...

S : B → C 1st elem of " " ...

eg. R : {1, 2, 3} → {1, 2, 3, 4} {(1, 1), (1, 4), ...

S : {1, 2, 3, 4} → {0, 1, 2} {(1, 0), (2, 0), ...

SoR = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}

Powers of R

$R^1 = R$

$R^{n+1} = R^n \circ R$

R on A is transitive iff

$R^n \subseteq R$ for $n = 1, 2, \ldots$

Partial Orders and Lattices.

* Hasse Diagram (POSET Diagram).

Let $[A; R]$ be a POSET. The poset diagram is as follows ~

i) There is a vertex corresponding to each element of 'A'.

ii) An edge between the elements 'a' & 'b' is not present in the diagram if there exists an element $x \in A$ such that (aRx) and (xRb).

iii) An edge b/w the elements 'a' & 'b' is present iff aRb & there is no element $'x' \in A$ such that (aRx) & (xRb).

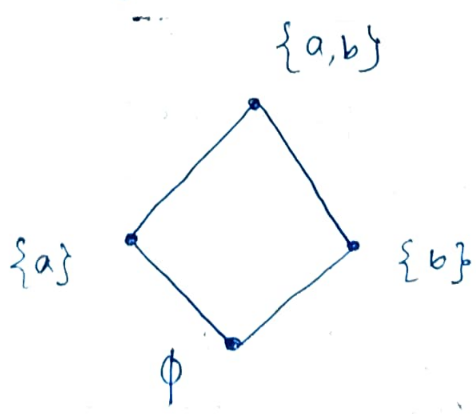
eg. $A = \{1, 2, 3, 4, 5\}$ $[A; \leq]$ POSET (reflexive, antisym., trans.)

Total order relⁿ (Chain)

eg $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$S = P(A)$ $A = \{a, b\}$ $[S, \subseteq]$ POSET.

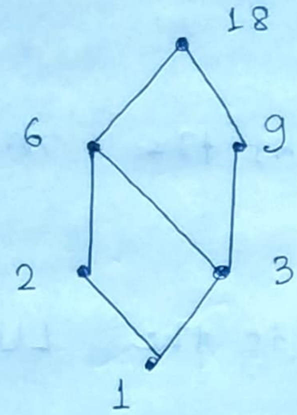
'Subset' relⁿ



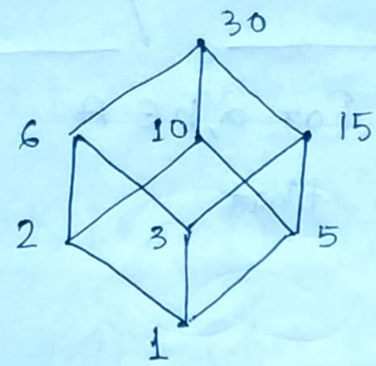
Not Total order

eg. $A = \{1, 2, 3, 9, 6, 18\}$ $[A, \div]$

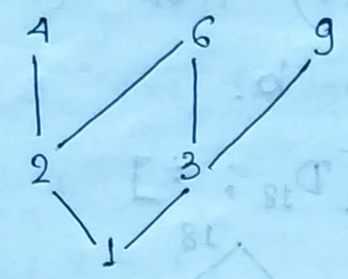
divides rel^n



eg. $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ $[A, \div]$



eg. $A = \{1, 2, 3, 4, 6, 9\}$ $[A, \div]$



* <u>Least upper bound.</u> (LUB / Join / supremum)

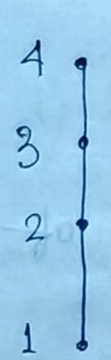
Let $[A; R]$ be a poset. For $a, b \in A$, if there exists an element $c \in A$ such that

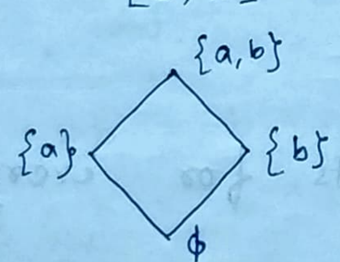
i) aRc & bRc

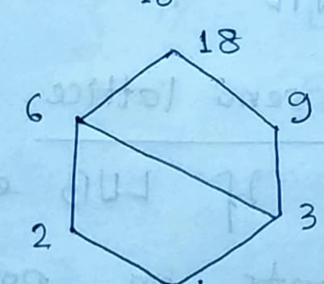
ii) if there exists any other element d such that (aRd) & (bRd) then (cRd), then

c is called LUB of 'a' & 'b'.

$[A; \leq]$ $[S, \subseteq]$ $[D_{18}; \div]$







LUB 4

$\{a, b\}$ LUB

concerned with

$\{\phi, \{a\}, \{b\}, \{a,b\}\}$

LUB 18.

- In set inclusion relation ($\subseteq$), union of sets is the LUB.

- In $\le$ rel^n, LUB is the maximum element.

- In $\div$ rel^n, LCM is the LUB.

* GLB (Greatest Lower Bound / meet).

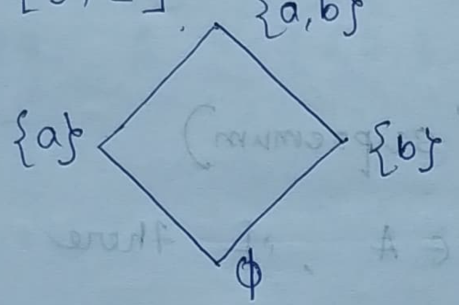
Let [A; R] be a poset. For $a, b \in A$, if there exists an element 'c' such that

i) (cRa) & (cRb)

&

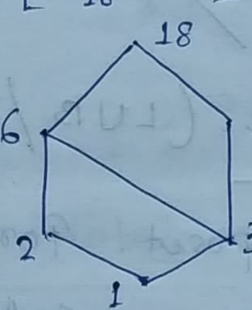
ii) If there exists any other element 'd' such that (dRa) & (dRb) then (dRc), then 'c' is called GLB of a & b.

[S; $\subseteq$]



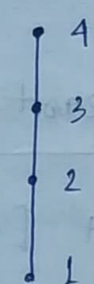
GLB ϕ.

[D_{18} ; $\div$]



For {6, 9}, GLB 3

For {2, 3}, GLB 1



GLB 1

- In set inclusion, GLB is the intersection.

- In $\le$ rel^n, least element is the GLB.

- In $\div$ rel^n, GCD is the GLB.

* Join semi lattice.

If LUB exists for every pair of elements in poset.

Meet semi lattice

If GLB exists for every pair of elements in poset.

Lattice

If both LUB & GLB exist for every pair of elements in poset.

eg $A = \{1, 2, 3, \dots, 10\}$ $[A; \div]$ is meet semi lattice.

$S = \{\{a\}, \{b\}, \{a, b\}\}$ $[S; \subseteq]$ is join semi lattice.

$A = \{1, 2, 3, 4\}$ $[A; \leq]$ is a lattice.

→ If A is set of all +ve integers, then poset $[A; \div]$ is a lattice.

→ If n is a +ve integer then D_n = set of all +ve divisors of n.

$D_6 = \{1, 2, 3, 6\}$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Every $[D_n; \div]$ is a lattice.

→ If $P(A)$ denotes powerset of 'A' then $[P(A); \subseteq]$ is a lattice.

- **Properties of Lattice.**

In a lattice for any 3 elements $a, b, c \in A$,

$\vee$ – LUB / Join
$\wedge$ – GLB / Meet

i) ~~Cumutative~~ Commutative law.

$a \vee b = b \vee a$

$a \wedge b = b \wedge a$

ii) Associative law.

$(a \vee b) \vee c = a \vee (b \vee c)$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

iii) Idempotent Law.

$$a \wedge a = a$$
$$a \vee a = a$$

iv) Absorption.

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

- In a lattice $(a \vee b) = b$ iff

$$(a \wedge b) = a, \ \forall\, a, b \in L$$

- Distributive law might not ~~g~~ hold good for a lattice.

- Distributive lattices :
1. Every Boolean algebra
2. Every TOSET / Chain
3. $(\mathbb{N}, \leq) \to (\wedge \text{ gcd}, \vee \text{ lcm})$
4. Set of +ve divisors, D_n

→ Boolean algebra when square free. (prime factor^n has only one factor for each prime)

• Distributive Lattice :

A lattice is said to b the following distributive

i) $a \vee (b \wedge c) = (a \vee b) \wedge$

ii) $a \wedge (b \vee c) = (a \wedge b)$

★ Sublattice. Let L be a lattice $[L, \vee, \wedge]$. A subset M of L is called a sublattice of L if

i) M is a lattice i.e. $[M, \vee, \wedge]$

ii) For any pair of elements $a, b \in M$, the LUB & GLB are same in M & L.

★ • Bounded lattice. Let L be a lattice wrt R, if there exists an element $I \in L$ such that $(aRI)\ \forall\, a \in L$, then I is called upper bound of the lattice L.

Similarly, if there exists an element $0 \in L$, such that $(0Ra)\ \forall\, a \in L$, then 0 is called lower bound of the lattice L.

In a lattice if upper bound & Lower bound exists then it is called a bounded lattice.

$[\{1,2,3,4\} ; \leq]$

Bounded lattice.

→ In a bounded lattice,

1. LUB of a & I i.e. $a \vee I = I$
2. GLB of a & I i.e. $a \wedge I = a$
3. LUB of a & 0 i.e. $a \vee 0 = a$
4. GLB of a & 0 i.e. $a \wedge 0 = 0$.

I - upper bound

0 - Lower bound.

- Complement of an element.

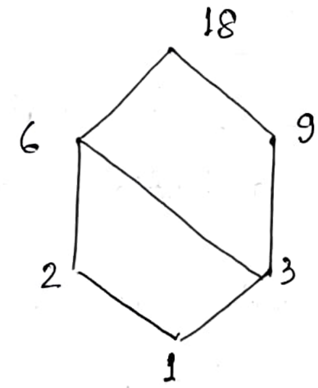
Let L be ~~an~~ bounded lattice, for any element $a \in L$, if there exists an element $b \in L$ such that $(a \vee b) = I$ and $(a \wedge b) = 0$, the 'b' is called 'complement of a' written as $\bar{a}$. and 'a' & 'b' are complements of each other.

eg $[D_{18} ; \div]$

$(2 \vee 9) = 18$

$(2 \wedge 9) = 1$

$\bar{2} = 9$ | $\bar{9} = 2$



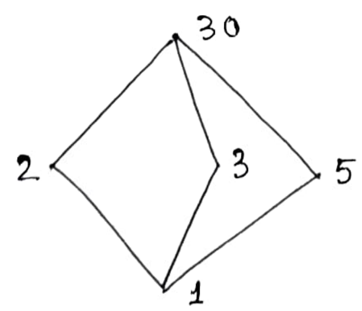
$\vee$ Join

$\wedge$ Meet

I UB

0 LB

eg $[\{1,2,3,5,30\} ; \div]$



$(2 \vee 3) = 30$

$(2 \wedge 3) = 1$

$\bar{2} = 3$

$\bar{3} = 2$.

* Complemented Lattice.

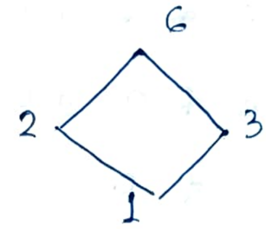
Let L be a bounded lattice, if each element of L has a complement in L then L is called a complemented lattice. In a complemented lattice, each element has at least one complement.

— In a distributive lattice, complement of an element of exists is unique i.e. each element has at most one complement.

eg. $[\{1,2,3,6\} ; \div]$.

$\bar{1} = 6$

$\bar{2} = 3$



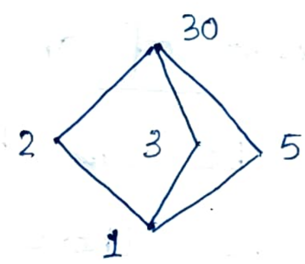
— $(P(A), \subseteq)$ is complemented

eg. $[\{1,3,2,5,30\} ; \div]$

$\bar{1} = 30$

$\bar{2} = 3 \quad \bar{2} = 5$

$\bar{5} = 3 \quad \bar{5} = 2$



Not distributive.

eg.

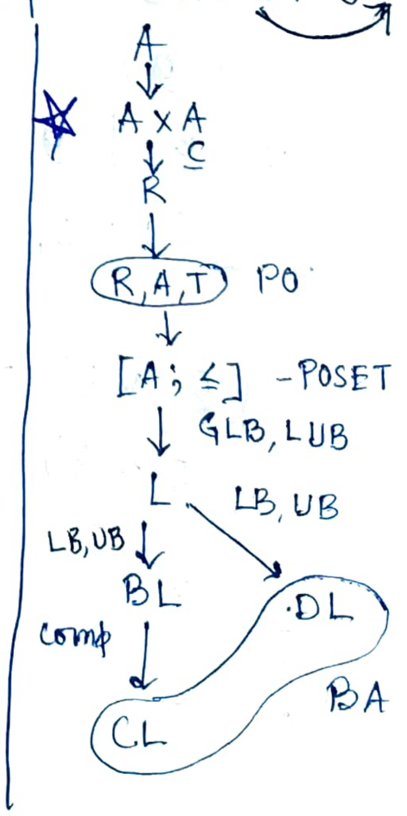
$[A ; \leq]$

bounded but not complemented.

* Boolean Algebra.

A lattice L is called Boolean algebra if it is distributive & complemented. (& bounded)

Every element has unique complement in BA.



* Maximal element.

If in a poset, an element is not related to any other element, then it is called maximal element.

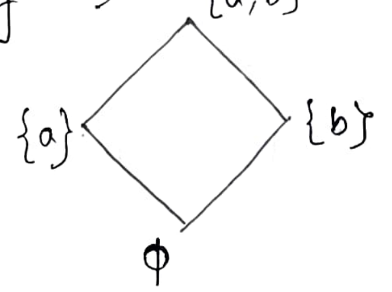
e, x

Minimal element.

If in a poset, if no element is related to an element, then it is called minimal element.

x, e

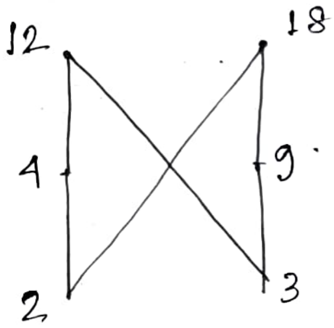
eg ↩ {a,b} [P(A) ; ⊆]



Maximal {a,b}

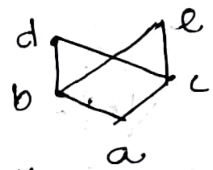
Minimal ϕ.

Q. The poset [{2,3,4, 9, 12, 18} ; ÷] is neither join or meet semi lattice.



* { 2 Maximal ⇒ no LUB
  2 minimal ⇒ no GLB

Q. poset diagram.



P. Which's true?

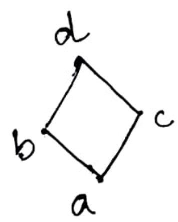
No. LUB. for d, e

a) P is not a lattice. ✓

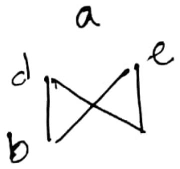
b) {a,b,c,d} is a lattice ✓

c) {b,c,d,e} " "

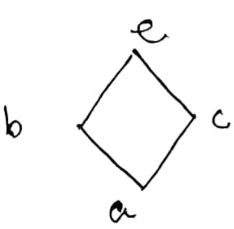
d) {a,b,c,e} " " ✓



It's a lattice



It's not a lattice



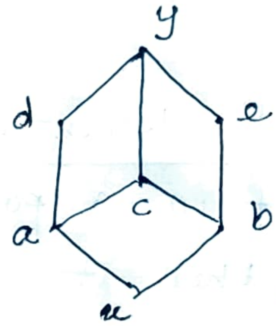
It's a lattice.

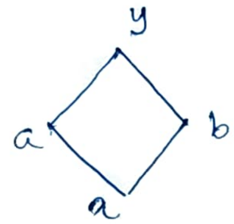
- $(S, \leq)$ is a <u>well-ordered set</u> if it is a poset such that $\leq$ is a total ordering & every nonempty subset of S has a least element.

-

Q. Hasse diagram of $L = \{x, a, b, c, d, e, y\}$ lattice..
Which of the following subsets of L are sublattice of L?

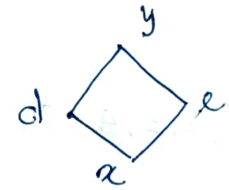
a) $\{x, a, b, y\}$
b) $\{x, a, c, y\}$
c) $\{x, a, e, y\}$
d) $\{x, d, e, y\}$
e) $\{x, c, d, y\}$



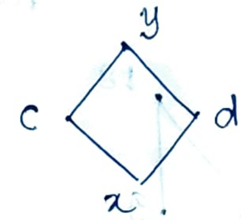
→ 

LUB of a, b is y
LUB of a, b in L is c.
Not a sublattice (but a lattice)

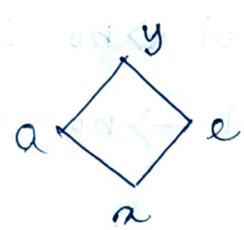
Sublattice



Sublattice

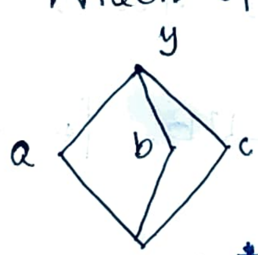


$GLB(c, d) = x$
$GLB(c, d)_L = a$
Not a sublattice

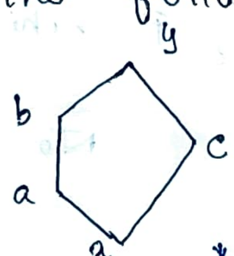


Sublattice

Q. Which of the following lattices is not distributive?



L_1^*



L_2^*

If in some lattice, we find L_1^* or L_2^* as sublattice, then that lattice is not distributive.

→ L_1^* : $a \vee (b \wedge c) \overset{?}{=} (a \vee b) \wedge (a \vee c)$

$a \vee x \qquad y \wedge y$

$a \neq y$

Not distributive.

L_2^* : $a \vee (b \wedge c) \overset{?}{=} (a \vee b) \wedge (a \vee c)$

$a \vee x \qquad b \wedge y$

$a \neq b$

Not distributive.

Q. Which are not True? $\boxed{***}$

T a) A lattice with 4 or fewer elements is distributive.

T b) Every totally ordered set is a distributive lattice.

T c) Every sublattice of a distributive lattice is also distributive.

✓ d) Every distributive lattice is a bounded lattice.

→ If L_1^* / L_2^* is contained in a lattice, it's not distributive.

If not contained, that is distributive.

* Totally ordered set is chain shaped.

It's not isomorphic to L_1^* / L_2^*.

c) Each distributive lattice does not contain L_1^* / L_2^*. So, sublattice of distributive lattice is distributive.

d) $[N ; \leq]$ POSET Lattice.

– Not bounded, but distributive.

Q. Which is not a distributed lattice?

a) $[P(A) ; \subseteq]$ where $A = \{a, b, c, d\}$

b) $[D_{81} ; \div]$

c) $[R ; \leq]$ R is set of real numbers.

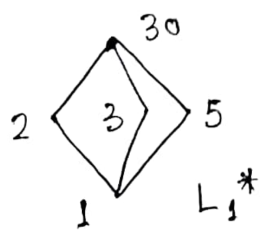
✓ d) $[\{1, 2, 3, 5, 30\} ; \div]$

→ a) For any set of set union & intersection are distributive.

b) $[\{1, 3, 9, 27, 81\} ; \div]$ Total order. ⇒ Distributive.

* c) Total order ⇒ Distributive.

d)



⇒ Not distributive

Q. $[D_{18} ; \div]$. Which is not true?

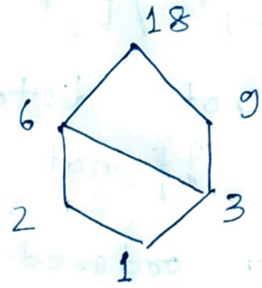
a) $\bar{1} = 18$ ✓c) $\bar{3} = 6$

b) $\bar{2} = 9$ d) $\bar{6}$ doesn't exist.

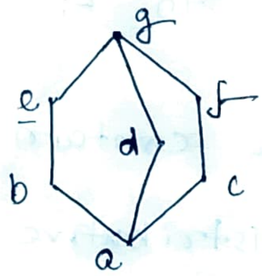
→ $D_{18} = \{1, 2, 3, 6, 9, 18\}$ | LB = 1, UB = 18

LUB (a,b) = LCM $(a,b) = 18$

GLB (a,b) = GCD $(a,b) = 1$.



Q. How many complements does the element e have?



→ $\bar{e} = d$ | $\bar{e} = f$ | $\bar{e} = c$

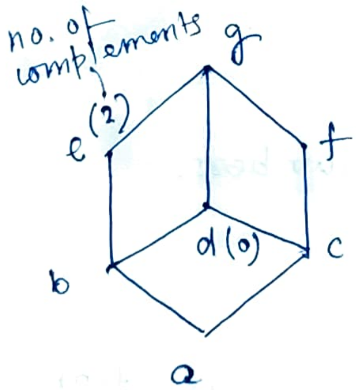
⇒ Not distributive.

→ $\bar{b} = d, f, c$. ⇒ Complemented lattice.

$\bar{a} = g$.

★ eg. D_{18} is distributive, not complemented.

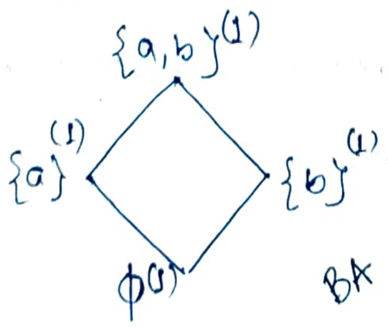
eg.



not distributive

not complemented

eg $[P(A) ; \subseteq]$.

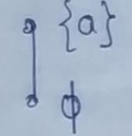


distributive

complemented

eg $A = \{a\}$ POSET

$[P(A) ; \subseteq]$ LATTICE $\rightarrow$ (Distributive & complemented) $\rightarrow$ Boolean algebra

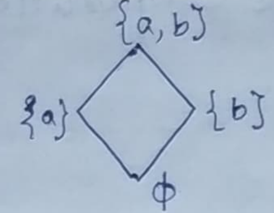


✓ - any total order with 2 elements is BA.

- Can't find complement for every element in chain.

eg. $A = \{a,b\}$

$[P(A) ; \subseteq]$

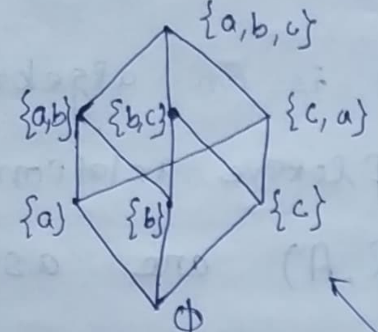


D & C
⇓
BA.

✓ - If we have any 4 elements set, if it has to be a BA, it must be isomorphic to ◇.

* eg. $A = \{a,b,c\}$

✓ $[P(A) ; \subseteq]$



If we have any 8 elements, & it needs to be BA, it must look like this.

✓ → If any lattice has to be a BA, it has to have 2^n elements.

✓ No. of edges in BA, $= n2^{n-1}$ $[n > 2]$

n - no. of elements.

* Th. The poset $[D_n ; \div]$ is a BA iff n is a square free number. * (prime fact^n - unique primes)

✓ If the poset $[D_n ; \div]$ is a BA then complement of $x = \frac{n}{x}$ $\forall x \in D_n$.

Q. Which is not BA ? *

a) $[D_{110} ; /]$ ✓c) $[D_{45} ; /]$

b) $[D_{91} ; /]$ ✓d) $[D_{64} ; /]$

$D_{110} = \{1,2,5,10,11,22,55,110\}$ 2^3 elements $\bar{5} = \frac{110}{5} = 22$

↓ 110 square free number.

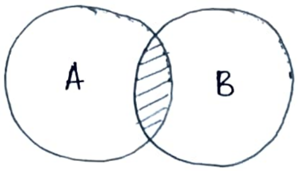
no square no.

$D_{91} = \{1,7,13,91\}$

$D_{45} = \{1,3,5,9,15,45\}$ $6 \neq 2^n$

$D_{64} = \{1,2,4,8,16,32,64\}$ $7 \neq 2^n$

- Absorption Law



$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

$$A \oplus B$$
$$= (A \cup B) - (A \cap B)$$
$$= (A - B) \cup (B - A)$$
$$= (A' \cap B) \cup (A \cap B')$$

$$A \oplus A' = U$$

$$A \oplus \phi = A$$

$$A \oplus U = A'$$

from $(A \cup B) - (A \cap B)$

- Cartesian Product

If $A \times B = B \times A$, the one of the following must be true:

1. $A = B$
2. $A = \phi$
3. $B = \phi$.

- Multiset

$$A = \{m_1 a_1, m_2 a_2, \ldots, m_k a_k\}$$
$$B = \{n_1 a_1, n_2 a_2, \ldots, n_k a_k\}.$$

$m, n \rightsquigarrow$ # times a_i occures in the multiset

$$A \cup B = \{\max(m_i, n_i) \cdot a_i\} \quad \forall i \in [1, k]$$

$$A \cap B = \{\min(m_i, n_i) \cdot a_i\} \quad \forall i \in [1, k]$$

$$A - B = \{(m_i - n_i) a_i, \text{ if } m_i > n_i; \quad 0 \cdot a_i, \text{ otherwise}\}$$

$$A + B = \{(m_i + n_i) a_i, \quad \forall i \in [1, k]\}$$

• Relation

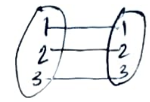
$A_R B \subseteq A \times B$

eg. $R_{\leq}(A,A) = \{ (x,y) \mid x \leq y \ ; \ x \in R, y \in R \}$

Possible $rel^n s = 2^{mn}$ $|A| = m, \ |B| = n$

Complement of R : R' or $\bar{R} = (A \times B) - R$.

Diagonal Relation $(\Delta_A) : A \to A$

$\Delta_A = \{ (x,x) \mid x \in A \}$. eg. $R = (A,A)$ 

→ Identity Matrix $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $|\Delta_A| = n, \ |A| = n$

Reflexive $\forall x \in A, \ (x,x) \in R$

- → If R is reflexive, R^{-1} too.
- → Superset of R is reflexive.
- → $R \cup S$, $R \cap S$ are reflexive, when R, S are reflexive.
- → Smallest reflexive rel^n on A $= \Delta_A$ | $|\Delta_A| = n$
- → Largest " " " $= A \times A$ | $|A \times A| = n^2$
- → # reflexive $rel^n s$ possible $= 2^{n^2 - n}$.

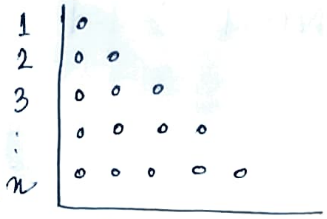
Irreflexive $\forall x \in A, \ (x,x) \notin R$. $R : A \to A$

does not mean 'non-reflexive'

- → If R is reflexive, $\bar{R}$ is irreflexive.
- → Smallest irreflexive $rel^n = \phi$
- → Largest " " $= A \times A - \Delta_A$ | size $= n^2 - n$
- → # irreflexive $rel^n s$ possible $= 2^{n^2 - n}$

* Symmetric $(x,y) \in R \Rightarrow (y,x) \in R \quad \forall x,y \in A$

- If R is symmetric, R^{-1} too.
- $R \cup S$, $R \cap S$ are symmetric, if R,S symmetric.
- Smallest sym. relⁿ $= \phi$
- Largest sym. relⁿ $= A \times A$
- # sym. relⁿs on $A = 2^{n(n+1)/2}$

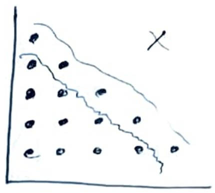


$(a,b) \to (b,a)$

* Antisymmetric $(x,y) \in R$ and $(y,x) \in R \Rightarrow x = y$.

- Smallest $= \phi$.
- Largest $\rightarrow$ #elems $= n(n+1)/2$

Largest



- If R, S are antisymmetric, $R \cup S$ need not be antisymmetric, $R \cap S$ is always antisym.

$(a,b) \in R$
$(b,a) \in S$

- Every subset of an antisym. relⁿ is also antisym.
- R being antisym, $R \cap R^{-1} \subseteq \Delta_A = \{(x,x) \mid x \in A\}$
- # possible antisym. relⁿs $= 2^n \cdot 3^{(n^2-n)/2}$

$$\{\underbrace{(1,1),(2,2),\ldots,(n,n)}_{n \text{ elems}},\ \underbrace{(1,2),(2,1),(1,3),(3,1)\ldots}_{n^2-n \text{ elems}}\}$$

n elems: present or not $\Rightarrow 2^n$

n^2-n elems: (a,b) — $(n^2-n)/2$; (b,a) — $(n^2-n)/2$

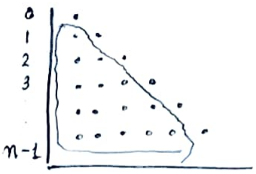
for pair $(a,b),(b,a)$, [such #pairs $= \frac{n^2-n}{2}$]

1) (a,b) present
2) (b,a) present
3) none present

$\Rightarrow 3^{(n^2-n)/2}$

Asymmetric $\quad (x,y) \in R \Rightarrow (y,x) \notin R$

- Both antisymmetric & irreflexive.
- Every asymm. rel^n is antisym., not vice versa.
- Smallest $= \phi$
- Largest size $= \frac{n(n-1)}{2} = \frac{n^2-n}{2}$



- # asymmetric rel^{n}s on $A = 3^{(n^2-n)/2}$

  $(a,b), (b,a)$ or neither

Transitive $\quad xRy$ and $yRz \Rightarrow xRz \quad \forall x,y,z \in A$

- Smallest $= \phi$
- Largest $= A \times A \quad$ size n^2

R on A is transitive if & only if $R^n \subseteq R$.

- # transitive rel^{n}s $\rightsquigarrow$ don't have closed form formula.

$n=0$	$\# = 1$	$n=2$	$\# = 13$
$n=1$	$\# = 2$	$n=3$	$\# = 171$

(a,a) can be in an antisym. rel^n but not in an asymmetric rel^n

- If R is transitive, R^{-1} too.

  R^c need not be.
- $R \cap S$ is transitive, if R,S are.

  $R \cup S$ need not be.

2^4 16

A transitive rel^n is asymmetric iff it is irreflexive.

- tra. $\quad aRb,\ bRc \Rightarrow aRc \quad \forall a,b,c \in A$

asy $\quad aRb \Rightarrow b \not R a \quad \forall a,b \in A$

irre. $\quad a \not R a \quad \forall a \in A$

(a,b) (b,a) (a,a) $\quad 2 \cdot 3$

$2 \cdot 5$ 10	$3 \cdot 4$ 12	$4 \cdot 5$ 20
	$3 \cdot 5$ 15	

Intransitive $\quad \neg(\forall a,b,c : aRb \wedge bRc \Rightarrow aRc)$

or $\exists a,b,c : aRb \wedge bRc \wedge \neg(aRc)$

for some a,b,c: $aRb,\ bRc \Rightarrow a \not R c$

Antitransitive $\quad \forall a,b,c : aRb \wedge bRc \Rightarrow \neg(aRc)$

$\forall a,b,c$: $aRb,\ bRc \Rightarrow a \not R c$

Equivalance rel^n : reflexive $\wedge$ symmetric $\wedge$ transitive

- Smallest = Δ_A. size $= n$
- Largest = $A \times A$ size $= n^2$
- # equivalence rel^{n}s $\rightarrow$ Bell number $B(n)$

eg. for $n=4$, $B(4) =$ ~~200~~ 15

4th row in Bell triangle last elem.

1
1 2
2 3 5
5 7 10 15
15 20 27 37 52
52 67 87 114 151 203

partitions of a set | Bell number

eg. $n=2$ $\{1,2\}$

(1,2) (1)(2)

2 ways

eg $n=3$ $\{1,2,3\}$

(1,2,3) (1,2)(3) (1,3)(2) (2,3)(1) (1)(2)(3)

5 ways.

Total rel^n

aRb, or bRa
$\forall a,b \in A$

Total order / Linear order

Rel^n that is ref, antisym, transitive & total.

$$\text{Bell}(n) = \sum_{k=1}^{n} S(n,k)$$

total # partitions of n elems into k subsets

↳ Called Stirling's no. of 2nd kind

eg.

$$\text{Bell}(2) = S(2,1) + S(2,2) = 1 + 1 = 2$$

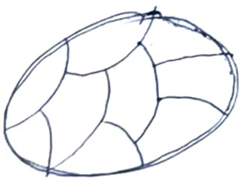
Equivalence classes of an eqv. rel^n partition a set into disjoint, non-empty subsets.

$\forall a,b \in A$,

aRb

$\Leftrightarrow [a] = [b]$

$\Leftrightarrow [a] \cap [b] \neq \phi$



Eqv. classes of 2 elems are either identical or disjoint.

- Equivalence classes

R be an equivalence relⁿ on A. Set of all elems that are related to an element a of A is called equivalence class of a. Denoted as $[a]_R = \{ s \mid (a,s) \in R \}$

⇝ All the equivalence classes of relⁿ R on A are either equal or disjoint & $\bigcup [a]_R = A$.

⇝ Equivalence classes are also called partitions since they are disjoint & their union gives the set on which the relⁿ is defined.

eg $R = \{ (a,b) \mid a \equiv b \pmod m \}$

1. $\forall a \in A$, $a - a = 0$ divisible by m. $a \equiv a \pmod m$
2. $a, b \in A$, $a \equiv b \pmod m \Rightarrow m \mid (a-b) \Rightarrow m \mid (b-a) \Rightarrow b \equiv a \pmod m$
3. $a, b, c \in A$, $(a-b) \bmod m = 0$, $(b-c) \bmod m = 0$

$\Rightarrow (a-c) \bmod m = 0 \Rightarrow a \equiv c \pmod m$

So, R is equivalence relⁿ.

Equivalence classes: Possible values for remainder when divided by m : $0, 1, 2, \ldots, m-1$

Classes : $[0]_m, [1]_m, [2]_m, \ldots, [m-1]_m$

$a \equiv b \bmod m$

$[0]_m = \{ \ldots, -2m, -m, 0, m, 2m, \ldots \}$

$[1]_m = \{ \ldots, -2m+1, -m+1, 1, m+1, 2m+1, \ldots \}$

$\vdots$

$[m-1]_m = \{ \ldots, -2m-1, -m-1, -1, m-1, 2m-1, \ldots \}$

Congruence class of an integer i mod m

$[i]_m = \{ \ldots, i-2m, i-m, i, i+m, i+2m, \ldots \}$

* **Partial order relⁿ** Reflexive $\wedge$ Antisym. $\wedge$ Transitive.

eg $\subseteq$ on a set of sets

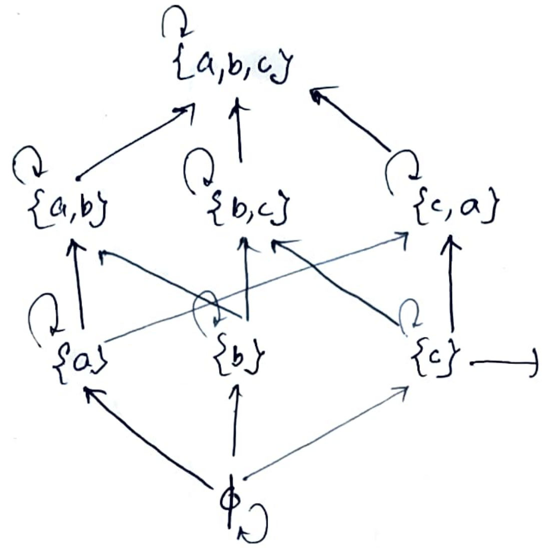
$A \subseteq A$ | $A \subseteq B,\ B \subseteq A \Rightarrow A = B$ | $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$

↳ $A = \{a, b, c\}$ Power set of A for $\subseteq$ is PO relⁿ.

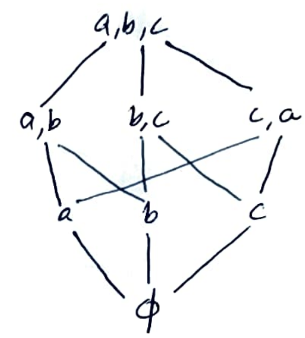
$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

Diagrammatically,

omit loops.
all arrow upwards
skip all that can be inferred from transitivity.



Partial order



Hasse Diagram

Functions

- $\forall x \in X, \exists (x, y) \in R$, $y \in Y$
- $(x, y) \in R \wedge (x, z) \in R \Rightarrow y = z$

X, Y related
Domain
Codomain
Range.

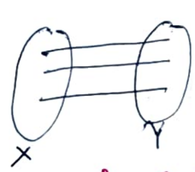
① **Partial f^n**: Subset of elems in X has mapping to elems in Y.

eg. $f(x) = \sqrt{x}$ $f: R \to [0, \infty]$

It's not a f^n.

subset of ordered pairs from $A \times B$ s.t. no 2 ordered pairs have same first component.
$(1,2)\ (1,9)$ ✗
each elem of A is mapped to only one of B.

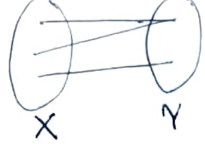
② **One-one** (injective)



$x_1, x_2 \in X$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ { each $a \in Y$ has only one preimage

③ Many-one

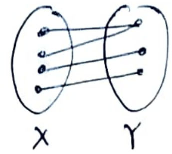


eg. $f(x) = |x|$

$f : R \to R$

④ Onto (Surjective)

codomain = range



Every elem. of Y, codomain has a preimage.

⑤ Into

range $\subset$ codomain

⑥ Bijective : Injective $\wedge$ Surjective

- If f is bijective $|X| = |Y|$
- Permutation f^n : Bijection from set A to itself.

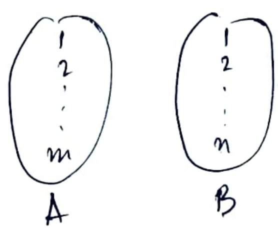
functions

$f : A \to B$

$|A| = m$

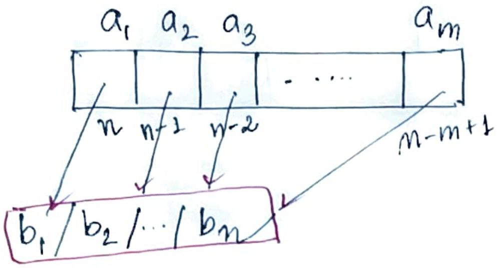
$|B| = n$

① # possible f^ns $= n^m$ ← (Each elem in A has n options to map to)



② # partial f^ns $= (n+1)^m$ → for every elem in A, we can choose to map to one of the n elems or choose not to connect at all
⇒ so $n+1$ options

③ # one-one $f^n s$



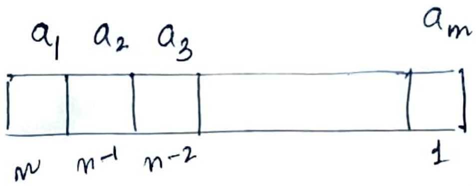
$$n(n-1)(n-2)\cdots(n-m+1)$$

$$= \frac{n!}{(n-m)!} = {}^nP_m$$

many-one $f^n s$ = $n^m - {}^nP_m$

④ # Bijections

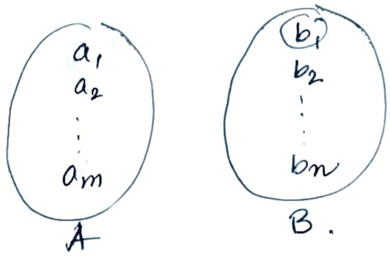
$n = m$



$$n(n-1)(n-2)\cdots 1 = n!$$

Permutation $f^n s$ = $n!$

⑤ # Into $f^n s$



$f^n s$ ~~whose~~ where b_1 has no preimage.

$|S_1| = (n-1)^m$

$f^n s$ where b_2 …. $|S_2| = (n-1)^m$

$m = n$

$|S_n| = (n-1)^m$ | S_n where b_n has no preimage

$S = S_1 \cup S_2 \cup \cdots \cup S_n$ ⇝ at least one elem in B has no preimage (Set of into $f^n s$)

$$|S| = (|S_1| + \cdots + |S_n|) - (|S_1 \cap S_2| + |S_1 \cap S_3| + \cdots) + (|S_1 \cap S_2 \cap S_3| + \cdots) - \cdots$$

$$= n(n-1)^m - {}^nC_2 (n-2)^m + {}^nC_3 (n-3)^m - \cdots \pm {}^nC_{n-1} 1^m$$

$$= \sum_{i=1}^{n-1} {}^nC_i (-1)^{i+1} (n-i)^m$$

or $n^m - n!\, S\{m,n\}$

onto $f^n s$ = $n^m - |S|$.

$n!\, S\{m,n\}$

- Inverse of f^n

$f^{-1}: Y \to X \quad \{(y, x) \mid (x, y) \in f\}$

✓ ⇝ f is invertible iff f is bijective.

✓ ⇝ f is bijective iff f^{-1} is also bijective.

⇝ $f^{-1}(f(x)) = x. \quad | \quad f(f^{-1}(x)) = x$

eg $f(x,y) = (x+y, x-y)$

✓
$x+y = a \qquad 2x = a+b \qquad y = \frac{a-b}{2}$

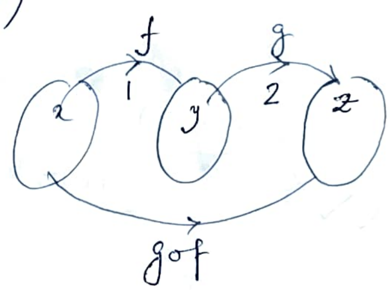
$x-y = b. \qquad x = \frac{a+b}{2}$

*

$f^{-1}(a,b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$

$f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

- Composition of f^n $\quad f \circ g(x) = f(g(x))$



- $f \circ g \neq g \circ f$
- $f \circ (g \circ h) = (f \circ g) \circ h$
- $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$
- If f & g are one-one, $g \circ f$ is also one-one. ✓
- If f & g are onto, $g \circ f$ is also onto. ✓
  ↳ if bijective, $g \circ f$ " bijective.
- If $g \circ f$ is also onto, g, f need not be onto.

- Algebra of f^ns

- Special f^ns
  - Identity f^n $\quad I(x) = x \quad I : R \to R$
  - Constant f^n $\quad f(x) = c \quad \forall x$
  - Sign f^n $\quad f(x) = 1 \quad x \geq 0$; $\quad -1 \quad x < 0$
  - GIF $[x] = n \quad n \leq x \leq n+1$.

- POSET

(set, partial order relation)

eg. $(\mathbb{R}, \leq)$ | $(Z^+, /)$ → div^y set

$A = \{1,2,3\}$

$(P(A), \subseteq)$

- TOSET / chain / Linearly ordered set

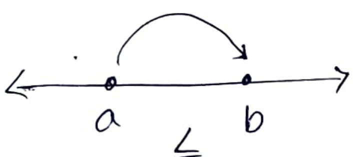
★ (S, R) is TOSET iff

1. (S, R) is POSET.
2. $\forall\, x, y \in S$, xRy or yRx (or total rel^n)

eg. $(P(A), \subseteq)$ not TOSET
↳ $\{a,b\} \nsubseteq \{c,d\}$
$\{c,d\} \nsubseteq \{a,b\}$

TOSET'S Hasse diagram is a chain.

eg. $(R, \leq)$ is TOSET.



→ Every pair is related.

eg. Divisibility set of a prime number D_{prime}

- $(D_{prime}, /) \longrightarrow$ TOSET

eg $\quad D_7 = \{1, 7\}$

7
|
1

chain

- Hasse Diagram

eg. $(P(A), \subseteq)$. ★

$|A| = n$

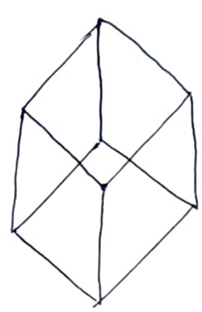
$\#$ vertices $= 2^n$

$\#$ edges $= n \cdot 2^{n-1}$

$A = \{1,2,3\}$

$\# v = 2^3$

$\# e = 3 \cdot 2^2$



TOSET

$\# v = n$

$\# e = n-1$

Maximal element

(S, R)

$a \in S$ is maximal iff

$\nexists\ b \in S$ s.t. $a \leq b$

or aRb

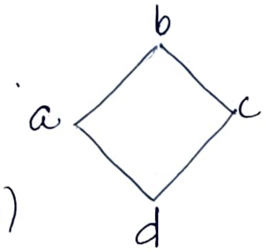
Minimal element

$a \in S$ iff $\nexists\ b \in S$ st. $b \leq a$

or bRa

eg.

$(S, \leq)$

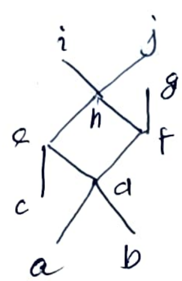


maximal $\{b\}$

minimal $\{d\}$

eg

$(S, \leq)$



maximal $\{i, j, g\}$

minimal $\{a, b, c\}$

Greatest element $\boxed{1}$ $a \in S$ iff $b \leq a$ $\forall\ b \in S$

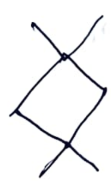
bRa

Least element $\boxed{0}$ $a \in S$ iff $a \leq b$ $\forall\ b \in S$

aRb

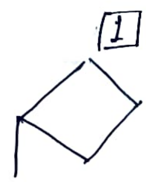
eg.

eg.



no $\boxed{1}$ or $\boxed{0}$

eg.



no $\boxed{0}$

eg.

no $\boxed{1}$

- Poset that has 1 & 0 is said to <u>be bounded</u>

Upper bound (A)

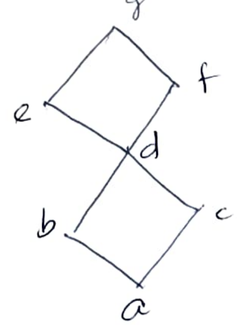
$A \subseteq S$

$(S, \leq)$

$a \in UB(A)$ iff $b \leq a$ $\forall\ b \in A$

bRa

eg



$A = \{b, c\}$ $\quad B = \{b, d\}$

$UB(A) = \{d, e, f, g\}$

$UB(B) = \{d, e, f, g\}$

$LUB(A) = d \mid LUB(B) = d$

$LB(A) = \{a\}$

$LB(B) = \{b, a\}$

$GLB(A) = a$

$GLB(B) = b$

Lower bound (A) $\quad A \subseteq S$

$(S, \leq)$ $\quad a \in LB(A)$ iff $a \leq b \ \forall b \in A$

aRb

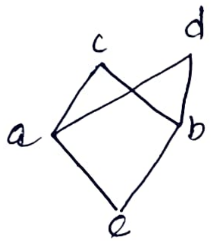
Lowest upper bound (A) LUB $\quad \vee$ Join

$A \subseteq S \quad a \in A$

$a = LUB(A)$ iff $a \in UB(A)$ and $a \leq b \ \forall \cancel{b \in A}$ $b \in UB(A)$

Greatest lower bound (A) GLB $\quad \wedge$ Meet

$a = GLB(A)$ iff $a \in LB(A)$ and $b \leq a \ \forall \cancel{b \in A}$ $b \in LB(A)$



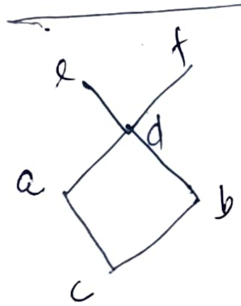
$A = \{a, b\}$

$LB(A) = \{e\}$ $\quad GLB(A) = e$

$UB(A) = \{c, d\}$ $\quad LUB(A) = X$

- If $LUB(S)$ exists for $(S, \leq)$ then $LUB(S) = 1$ greatest elem

- If $GLB(S)$ exists then it is 0.



$A = S = \{a, b, c, d, e, f\}$

$LB(A) = \{c\}$ $\quad GLB(A) = c$

$UB(A) = \phi$ $\quad LUB = X$

Lattice

POSET is a lattice iff $\forall a, b \in S$

$\underbrace{LUB(\{a,b\})}_{\text{Join}(a,b)}$ and $\underbrace{GLB(\{a,b\})}_{\text{Meet}(a,b)}$ exist.

$a \vee b, \; a \wedge b$

Join (a,b) — Meet (a,b)

$a \vee b$ — $a \wedge b$

Supremum(a,b) — Infimum (a,b)

eg $(R, \leq)$ POSET as well as TOSET. → Lattice

✓ ⇝ Every TOSET is a lattice. ★

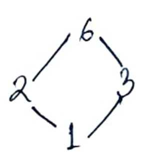
eg $(P(A), \subseteq)$ is lattice.

eg $(S, \subseteq)$ lattice ; S → set of all sets

eg $(Z^+, /)$

$LUB(\{a,b\}) = a \vee b = LCM(a,b)$ ★ Lattice ✓

$GLB(\{a,b\}) = a \wedge b = GCD(a,b)$



- Semi Lattice.

Poset $(S, \leq)$ is semi lattice iff

$\forall a, b \in S$ $a \vee b$ or $a \wedge b$ exists.

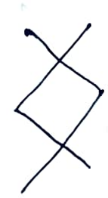
$a \vee b$ → Join semi lattice

$a \wedge b$ → Meet semi lattice

eg.

Join semi lattice

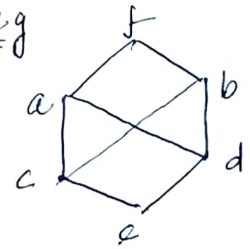
Meet semi lattice.



not semi lattice

not lattice

eg



$\{a,b\}$ $LB = \{c,d,e\}$ Closure

$GLB = \times$ indeterminate

$\{c,d\}$ $UB = \{a,b,f\}$

$LUB = \times$

not semi lattice

not lattice

Properties of lattices $(S, \leq)$ ✡

① **Closure** $\forall a,b \in S$, $a \vee b$, $a \wedge b$ exist ✓

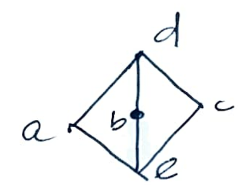
② **Commutative** $\forall a,b \in S$, $a \vee b = b \vee a$ $a \wedge b = b \wedge a$

③ **Associative** $\forall a,b,c \in S$ $a \vee (b \vee c) = (a \vee b) \vee c$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

④ **Distributive** $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

not! $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$



Where this property holds, → distributive lattice. ✓

⑤ Lattice need not be bounded. ↪ both 1, 0 exist.

eg. $(Z, \leq)$ no 1, 0 Unbounded lattice

↳ Finite lattice is always bounded.

⑥ **Idempotent** $\forall a \in S$, $a \vee a = a$ $a \wedge a = a$

⑦ **Absorption** $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

⑧ **Consistency** $\forall a,b \in S$

$a \leq a \vee b$	$a \wedge b \leq a$
$b \leq a \vee b$	$a \wedge b \leq b$

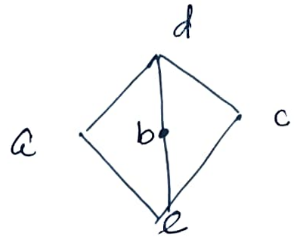
eg $a \le b,\ b \le c$

$a \wedge c = a$

$a \vee c = c$

$a \vee b = b \Rightarrow a \le b$

⑨ Distributive inequality



* ✓

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) \le (a \wedge b) \vee (a \wedge c)$$

⑩ Complement

✓ *

$$\left.\begin{array}{l}\forall a \in S\ \exists\ a' \in S \\ \text{st} \quad a \vee a' = 1 \\ \qquad a \wedge a' = 0\end{array}\right\} \longrightarrow \text{Complemented lattice}$$

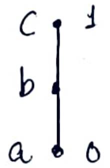
Every lattice is not a complemented lattice.

$$\begin{cases} 1 \vee 0 = 1 \\ 1 \wedge 0 = 0 \end{cases}$$

Have maximum, $\boxed{1}$ minimum elem. $\boxed{0}$

✓ * A complemented lattice is always bounded.

eg



$b \vee c = c = 1$

$b \wedge c = b \ne 0$

c not a complement of b.

$b \vee a = b \ne 1$

$c \ne b'$

$a \ne b'$

$b \ne b'$

$b \vee b = b$

$b \wedge b = b$

So, b has no complement

not a comp. lattice.

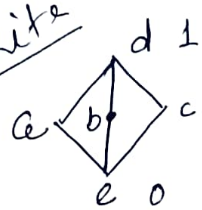
- A TOSET (Chain) can't be a complemented lattice if

$|S| \geq 3$.

TOSET is comp. lattice iff $|S| = 2$.

$\leq$

eg kite



$a' = b, c$

$b' = a, c$

$c' = a, b$.

Complemented lattice.

$d \vee e = d = 1$

$d \wedge e = e = 0$

$d' \neq a, b, c$

$d \vee a = d = 1$

$d \wedge a = a$

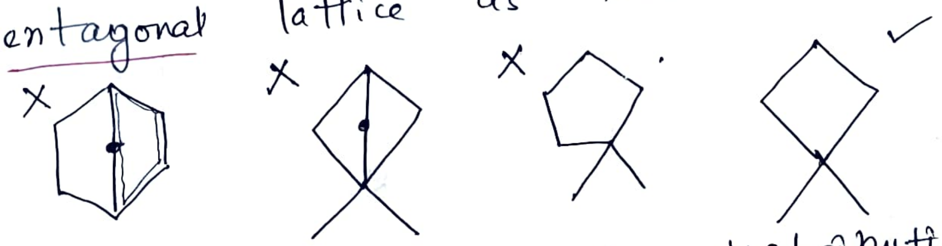
$e' \neq a, b, c$

$e \vee a = a$

$d' = e$

$e' = d$

- Distributive lattice
  - → A lattice is dist. iff it doesn't contain kite or pentagonal lattice as sub-lattice.



  - → If a complemented lattice is distributive then complement of each element is unique.

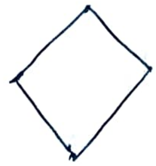
- A lattice is a Boolean algebra iff distributive, bounded & complemented.

Rel^n → POSET → Lattice → Boolean Algebra

In BA, every elem has a unique complement.

eg $(\mathcal{P}(A), \subseteq)$ example of Boolean algebra

$A = \{1, 2\}$



eg $(D_n, /)$ also BA.

Lattice, whose Hasse diagram is isomorphic to Hasse diagram of $(\mathcal{P}(A), \subseteq)$ then it is a <u>BA</u>.

In BA, if

$$\left.\begin{aligned} a \vee b &= a \vee c \\ \underline{\text{and}} \quad a \wedge b &= a \wedge c \end{aligned}\right\} \Rightarrow \text{then } b = c.$$