Preface

All the content herein are from lectures given by Professor Ali Hajimiri. If you have the time, I would highly recommend watching the lectures as Professor Hajimiri is a great teacher who always goes back to first principles in his explanations where possible, and instills a great deal of clarity of the foundations.

Due to the lack to information on the web about the Heaviside Operator, I've typed up these notes from his lectures. All examples herein are from the lectures, and one can refer to them if the calculations don't make sense.

Please feel free to use these notes and the latex however you see fit. I hope you find this material as useful as I did.

Historical Aspect with Laplace Transform

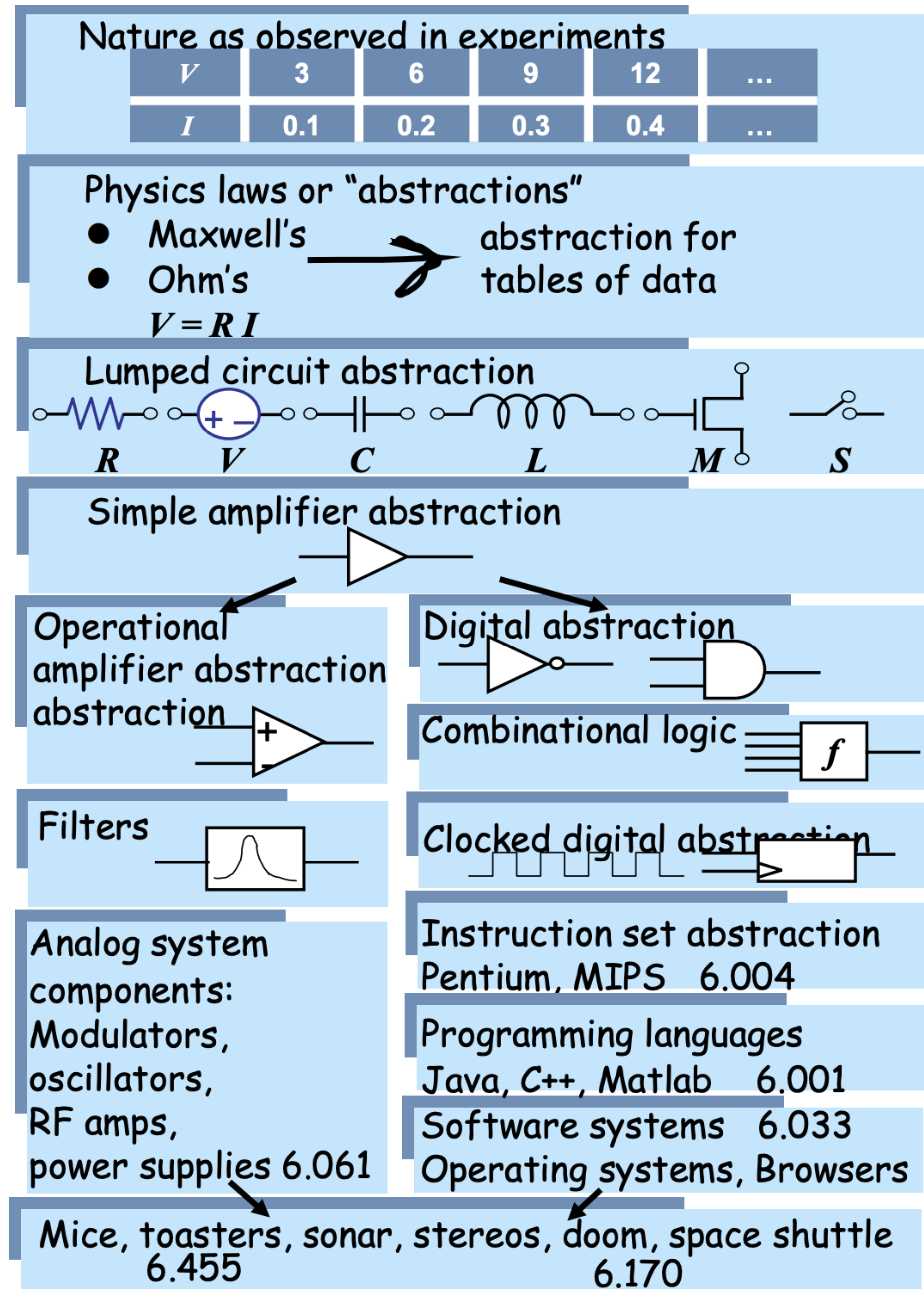
These notes **do not** assume you know Laplace Transform; however, the inquisitive reader who does know about the Laplace Transform will be wondering about how this method fits in with that. The notes do eventually go on to explain the Laplace Transform is a special case of the Operator Method, but till then, here is the interesting historical aspect from Professor Hajimiri to satisfy your curiosity:

"... Oliver Heaviside was a very interesting character. He was engineer really, but he developed all these mathematical techniques, and he was the one who applied this operator method to solving [differential] equations, and to solving circuit problems and so on. But he was not a classic mathematician in the sense that he wouldn't prove things, he wouldn't think that it was necessary to prove things. So what happened is that he got into these fights with people who were in the business of proving things, sometimes useless things, but nonetheless they were in that business. So he got into fights with these people and eventually some of these people said, "Okay, wait a second. It seems that this is working so they went and tried to prove it and the way they proved the background of these things was using complex contours, complex calculations, complex integrals and improper integrals, and eventually what it turned out that it is was some variation of something that Laplace had done at some point in their life and they called it Laplace Transform. Now, Laplace Transform is a lot less general than the Heaviside approach because it relies on improper integrals, and convergence of improper integrals. But because everyone hated Oliver Heaviside, everyone switched to Laplace Transform. They couldn't bring themselves to use his method, although it was a very good method. So the lesson in that is that if you come up with good methods, it is also helpful to be nice to people."

1. Overview

Where are we now? Where have we been?

Since these notes don't start form the beginning (i.e. from Observations of Nature and Maxwell's Equations), this section clarifies what we assume the reader to be familiar with.



We start by collection data about what we observe in nature through controlled experiments. These controlled experiments lead to tables of data from which we create abstractions that we call "Physical Laws", such as Maxwell's Equations. Maxwell's Equations give us a **mathematical model** [It's important to distinguish between mathematical and physical models. Two mathematically/electrically equivalent systems aren't physically equivalent. For example, if you take a circuit and put its Thevenin equivalent circuit in a black box and its Norton equivalent circuit in another black box, how would you know which is which? At their terminals, any external circuit would experience the same voltage and current from both boxes but the box with the Norton equivalent circuit would be warm, and would product a magnetic field as there is a current source present. This information is not captured by our mathematical model, and in this case we would need to update our model.] to understand how electric charges and electric current create electric and magnetic fields. These equations are however too unwieldy to use in general so we impose constrains on ourselves to play in a certain area of the playground of Electromagnetism.

More specifically, the constrains we discipline ourselves to adhere to are [Agarwal and Lang, Chapter 1.3]:

1. Dimensions of the circuit << The wavelength of light at the highest operating frequency of interest.
2. The boundaries of the discrete elements must be chosen so that $\dfrac{\partial \phi_B}{\partial t} = 0$ through any closed path outside the element for all time.
3. The elements must not include any net time-varying charge for all time. In other words, $\dfrac{\partial q}{\partial t} = 0$, where q is the total charge within the element.

To understand more about how these assumptions come about, and what their consequences are, Agarwal and Lang, Chapter 1.3 is a good resource.

As mentioned in the xxx first paragraph xxx, these constraints are assumptions we make about the Maxwell Equations that allow us to describe the voltage and currents across a set of elements in simple algebraic equations using Kirchhoff's Voltage Law, and Kirchhoff's Current Law. We then use these laws implicitly in Nodal Analysis, which can be used to solve any circuit. If we further discipline ourselves to operate such that all non-linear elements act linearly (constraining ourselves to operate in a narrow range around an operating point i.e. small signal/incremental method, for example), we can use Thevenin, Norton equivalent circuits, and Superposition to analyze circuits.

The next step on our journey is to understand the relationship of current and voltage captured by resistors, capacitors and inductors.

[Quick Recap Capacitors - At their basics, capacitors are elements whose charge depends on its voltage: $q(t) = C(t)xv(t)$ where C(t) is the capacitance of the capacitor at time t. We normally make the assumption that capacitance doesn't change with time, hence $q(t) = Cxv(t)$. Since current is the flow of charge, $i = \dfrac{dq}{dt}$, and so the current of a capacitor is $i = \dfrac{dq}{dt} = C\dfrac{dv}{dt}$.

Inductors - At their basics, an element whose magnetic flux depends on current: $\phi = L(t)i$ where L is the inductance of the inductor at time t. Again, we usually make assumption that inductance is time-invariant so $\phi = Li$. Now, voltage is the change in magnetic flux, $v(t) = \dfrac{d\phi}{dt} = L\dfrac{d\phi}{dt}$. Thus,

$$i(t) = \frac{1}{L}\int_{-\infty}^{\infty} v(t)dt$$

In summary, A resistor is an element whose current is proportion to its voltage. A capacitor is an element whose current is proportional to the derivative of its voltage. An inductor is an element whose current is proportional to the integral of its voltage.]

Now, given that we know the element laws of resistors, capacitors and inductors, and how to analyze circuits using Nodal Analysis, we are able to describe the relationship between a certain input into a circuit and output at some other point in a linear circuit. If we have non-linear elements, we can linearize them using the small-singal/incremental method (Agarwal and Lang, Chapter 8).
At this point, since we become increasingly interested in the relationship between input at a certain point and output at another point in a circuit, we start to refer to circuits as systems whose input and output is the voltage/current at two different points in the circuit.

Naturally, it becomes of interest to us to be able to describe the response of any arbitrary circuit with any number of different elements given some input. However, that is quite a complex task. By making decisions on the type of input signal (sinusoidal/non-sinusoidal), system (linear/non-linear, time variant/invariant) and response (transient/steady-state response) we'd like to specify and analyze, we're able to come up with assumptions and methods that allow us to simplify the way we specify and analyze systems.

Of great interest to us is the steady-state response of linear, time-invariant systems to sinusoidal inputs. What do these terms mean? Why this specific type of input, system and response?

- Sinusoidal inputs: Signals can be represented as a sum of scaled sinusoids. Thus, if we're able to understand the response of a system to a sinusoid with a parameterised angular frequency (i.e. $\sin(\omega_0 t)$ or $\cos(\omega_0 t)$), we can get the response to a sinusoid of any angular frequency.
- Linear System: From the above, we have the response of a single sinusoidal wave. But signals are represented as the sum of scaled sinusoids, not just one sinusoid. Hence, to analyze any arbirary signal, systems must be homogeneous (scaling the input signal scales the output signal by the same factor) and additive (the output of the sum of two inputs is the same as the sum of the outputs of each input). In other words, scaling/summing before or after the system is the same.
- Time-Invariant System: The response of the system at time t_1 is the same as the response at time $t_1 \pm t_2$ (i.e. the response of a system is not dependent on absolute time). We'll motivate the need for this requirement a bit later.
- Steady-State Response: The total response of a system consists of its transient response (the part of the response that decays to zero after a certain amount of time) and its steady state response (for a sinusoidal input, this will be a sinusoid with a possibly a different amplitude and phase shift).

Where are we going?

Assumptions: LTI System

At this point, EE courses/books tend to go on to analyze first and second-order circuits, and eventually generalize coming up with the transfer function by using the impedance model of elements and voltage divider relationships.

[What are first and second order circuits? Why are they imporant? Why don't we talk about/analyze third-order circuits?]

While that method is useful, most students have to hold tight through the grungy mathematics involved, and further, depending on how they're taught, the transfer function they end up with may not give them direct insight into the response of the circuit.

Here, we take a step to the side and forget about that path and only assume the reader only knows what's described in the "Where have we been?" section above.

What we're trying to achieve is to create a powerful framework that allows us to describe the transfer function of a circuit/system in simple terms and create simple ways to analyze those terms which in turn lead to a

To illustrate the point of where we're going, we'll work through an example using the heavyside operator which will highlight how powerful it is.

x. Motivating the Impulse Response

x. Convolution in Time-Domain

x. Heaviside Operator: Introduction, Basic Examples

Notationally, the Heaviside Operator is written as a lower-case p, p, and is defined as the time-domain derivative (or in simpler words; the derivative with respect to time)

$$p[f(x)] = \frac{df}{dt}$$

Now, it's inverse $\frac{1}{p}$ must satisfy the property that

$$p\left[\frac{1}{p}[f(x)]\right] = \frac{1}{p}\left[p[f(x)]\right] = f(x)$$

Well, we know that the inverse of a derivation is integration so $\frac{1}{p}$ must be

$$\frac{1}{p}[f(x)] = \int_{-\infty}^{t} f(\tau)\, d\tau$$

(τ is used to prevent ambiguity as t is one of our limits of integration)
But due to the limits of integration, we must have $f(-\infty) = 0$. This is a necessary condition whose consequences will be explore later. So if we have $f(-\infty) = 0$ then we have the inverse operator, $\frac{1}{p}$, defined as the integral from $-\infty$ to t.

Example 0.5.1. What is $\frac{1}{p}[\cos(5t)u(t)]$?

$$\begin{aligned}
\frac{1}{p}\Big[\cos(5t)u(t)\Big] &= \int_{-\infty}^{t} \cos(5\tau)u(\tau)\, d\tau \\
&= \int_{0}^{t} \cos(5\tau)\, d\tau && \text{since } u(t) = 0 \text{ for } t < 0
\end{aligned}$$

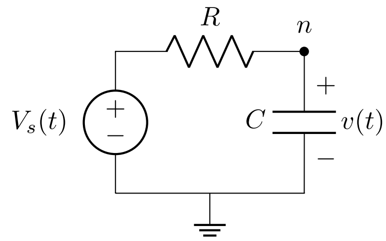
However, since $\cos(5t)u(t)$ is a one-sided function and $\int_{0}^{t} \cos(5\tau)\, d\tau$ could be a two sided function, we must multiply the integral by $u(t)$ so

$$\begin{aligned}
\frac{1}{p}\Big[\cos(5t)u(t)\Big] &= u(t)\int_{0}^{t} \cos(5\tau)\, d\tau \\
&= \frac{1}{5}\sin(5\tau)\Big|_{0}^{t} u(t) = \frac{1}{5}\sin(5t)u(t)
\end{aligned}$$

From hereon, although p is an operator, we'll start treating its notation more loosely in that we won't include the brackets on what it operates on anymore. The reason we do so is the same motivation behind using the operator; **the operator turns a differential equation into an algebraic problem, and so all the algebraic properties of association, commutation, distribution, identity, and inverse apply to this operator.** The aim of this section will be to illustrate how and why these properties apply to the operator.

The Low-Pass Operator
We'll define the low-pass operator and then the high-pass operator which will illustrate the algebraic

properties of the Heaviside operator. Although we haven't talked about frequencies, low-pass implies that a system passes the low frequencies to the output better than higher frequencies (this will discussed more in detail later on, but for now this is a simple understanding of the term 'low-pass'). Suppose we have a circuit as below.



RC circuit that passes low frequencies

Let's give ourselves some intuition about the circuit. What do you think the voltage across the capacitor, $v(t)$, would be if the source voltage, $V_s(t)$, was constant? After a long period of time (till the capacitor charges up), there would be no current flowing through the capacitor, and so no current flowing through the resistor, and thus, no voltage drop across the resistor which mean $v(t) = V_s(t)$.

Now, if we vary $V_s(t)$ at quicker and quicker speeds, there will be some current across the capacitor as it charges and discharges, and there will be current flowing across the resistor to do that, which means there will be a voltage drop across the resistor. And now if we vary $V_s(t)$ at very high speeds, most of the voltage drop will be across the resistor, and very little will appear across the capacitor. Thus, if $v(t)$ is the output, at low frequencies, everything goes through, but lesser and lesser passes through at higher frequencies.

Nodal analysis at node n gives us

$$C\frac{dv}{dt} + \frac{v(t) - V_s(t)}{R} = 0$$
$$\frac{dv}{dt} + \frac{v(t)}{RC} = \frac{V_s(t)}{RC} \tag{1}$$

This is an example of a differential equation we get from a low-pass first-order system. We won't solve this equation, but will use this to create a generic form of a first-order differential equation (DE), replace the differentials with our Heaviside Operator and try to come up with some general result that we can apply to all first order DEs.

Looking at eq. (1), the general form of a first order DE is

$$y'(t) + ay(t) = x(t)$$

Removing (t) to make the notation simpler,

$$y' + ay = x$$

Using the Heaviside operator for the differentials,

$$py + ay = x$$
$$[p + a]y = x \tag{2}$$

where we define $[p + a]$ as an operator that given a function, takes the derivative of the function, adds it to the function multiplied by a constant a, and then returns the result. Now we want to calculate the inverse of the $[p + a]$ operator. We'll define the symbol of the inverse of the $[p + a]$ operator as $\frac{1}{p+a}$.

$$\therefore y = \frac{1}{p+a}x \tag{3}$$

Why is the $\frac{1}{p+a}$ operator useful? Because if we can find what it is, it solves the DE. So if we have a system and know its operator and if we know what that operator does, we can figure out the system's response $y(t)$ for any arbitrary input easily. Now, this might seem a little arbitrary but let's calculate what $p[e^{at}y(t)]$ is

$$\begin{aligned} p\left[e^{at}y\right] &= ae^{at}y + e^{at}py \\ &= e^{at}[py + ay] \\ &= e^{at}[p+a]y \end{aligned} \tag{4}$$

Now, if we multiply both sides of eq. (2) by e^{at}, we get

$$\begin{aligned} e^{at}[p+a]y &= e^{at}x \\ \therefore p\left[e^{at}y\right] &= e^{at}x && \text{(Substituting eq. (4))} \end{aligned}$$

Solving for y,

$$\begin{aligned} \frac{1}{\cancel{p}}\cancel{p}e^{at}y &= \frac{1}{p}\left[e^{at}x\right] \\ y &= e^{-at}\frac{1}{p}[e^{at}x] \end{aligned} \tag{5}$$

Equating eq. (3) and eq. (5), we see that

$$\boxed{\frac{1}{p+a}x(t) = e^{-at}\frac{1}{p}\left[e^{at}x(t)\right] = e^{-at}\int_{-\infty}^{t} e^{a\tau}x(\tau)\,d\tau} \tag{6}$$

Thus, this is the definition of the $\frac{1}{p+a}$ operator. This is the first-order low pass operator. Let's see what we can do with this; let's apply it and see how it works.

Example 0.5.2. Find the impulse response of a system with the differential equation of

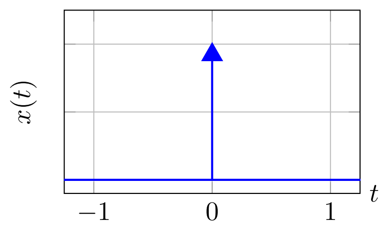
$$y' + ay = x$$

Answer. Since we're interested in the impulse response, our input is an impulse

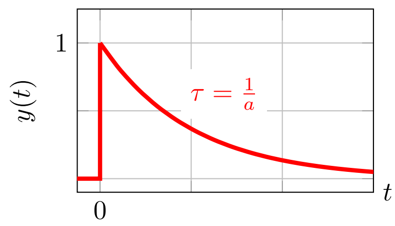
$$\begin{aligned} \therefore x(t) &= \delta(t) \\ y &= \frac{1}{p+a}x = \frac{1}{p+a}\delta(t) \\ &= e^{-at}\frac{1}{p}\left[e^{at}\delta(t)\right] \\ &= e^{-at}\int_{-\infty}^{t} e^{a\tau}\delta(\tau)\,d\tau \end{aligned}$$

Since, $\delta(\tau) = 0$ for all $\tau \neq 0$, the value of $e^{a\tau}$ only matters at $\tau = 0$, and so we can simplify $e^{a\tau}$ to $e^{a \cdot 0} = 1$. **This is a general result: whenever we have $\delta(t)$ in an integral, we'll leave $\delta(t)$ in the integral and evaluate everything else in the integral at $t = 0$.**

$$\begin{aligned} y &= e^{-at} \int_{-\infty}^{t} \cancelto{1}{e^{a\tau}} \delta(\tau)\, d\tau \\ &= e^{-at} \int_{-\infty}^{t} \delta(\tau)\, d\tau \\ &= e^{-at} u(t) \end{aligned}$$



$x(t) = \delta(t)$



$y(t) = e^{-at}u(t) = \frac{1}{p+a}\delta(t)$

Example 0.5.3. For the same system above, find the response for $x(t) = u(t)$
Answer.

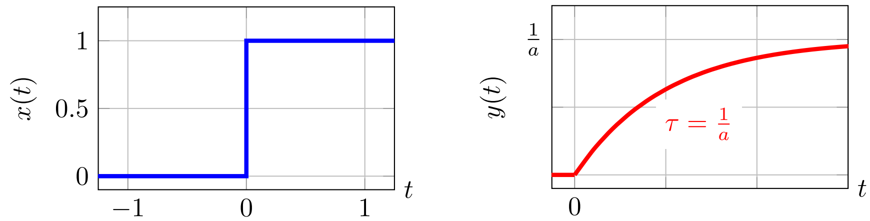
$$\begin{aligned} y' + ay &= u(t) \\ \therefore y &= \frac{1}{p+a} x = \frac{1}{p+a} u(t) \\ &= e^{-at} \int_{-\infty}^{t} e^{a\tau} u(\tau)\, d\tau \end{aligned}$$

Since, $u(\tau) = 0$ for all $\tau < 0$, the value of $e^{a\tau}$ only matters for $\tau \geq 0$, and so we can simplify the limits of the integral.

$$y = e^{-at} \int_{0}^{t} e^{a\tau}\, d\tau$$

This is wrong, however, as our input is a one-sided function and this equation results in a two-sided output; hence, we must multiply the equation by $u(t)$ to keep our output one-sided. **This is a general result: whenever we have $u(t)$ in an integral, we'll change the lower limit of the integral to 0 and multiply the integral by $u(t)$.**

$$\begin{aligned} \therefore y &= e^{-at} u(t) \int_{0}^{t} e^{a\tau}\, d\tau \\ &= e^{-at} u(t) \frac{1}{a} \left[e^{a\tau} \right]_{0}^{t} \\ &= \frac{1}{a} e^{-at} u(t) \left(e^{at} - 1 \right) \\ &= \frac{1}{a} \left(1 - e^{-at} \right) u(t) \end{aligned}$$

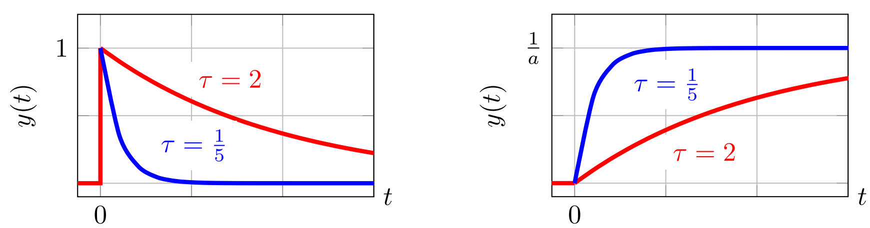


$x(t) = u(t)$ $\quad$ $y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t) = \frac{1}{p+a}u(t)$

Looking at the two responses of the low-pass filter for inputs of $\delta(t)$ and $u(t)$, what about the graphs tells us that we passed our input signal to a low-pass operator?

Sudden jumps are things that are high frequencies, and we can see from the graphs that low-pass filters don't allow, or at least attenuate high frequencies, and make the sudden jumps "smoother". By increasing a, thereby decreasing the time constant τ as $\tau = \frac{1}{a}$, the system reacts more abruptly, allowing higher frequencies to pass through, and the output resembles the input more.

[What is the time constant? exponentials go to completion in 5 time constants, a measure to quantize the rate of decay. Time constant has units of time. This phenomenon is obserever everytime the rate of something depends on its value - radio, active decay, Half-life.]

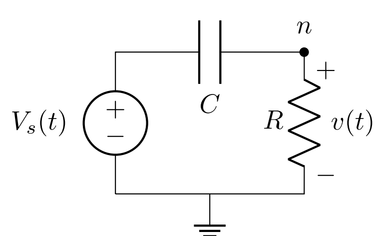


$y(t) = e^{-at}u(t) = \frac{1}{p+a}\delta(t)$ $\quad$ $y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t) = \frac{1}{p+a}u(t)$

Low-pass operator with time constants $\tau = \frac{1}{5}$ and $\tau = 2$

The High-Pass Operator

Referring to the RC circuit that passes low frequencies, can you think of how we could change components to pass high frequencies?



RC circuit that passes high frequencies

Nodal analysis at node n give us

$$C\frac{dv(t) - V_s(t)}{dt} + \frac{v(t)}{R} = 0$$
$$\frac{dv}{dt} + \frac{v(t)}{\tau} = \frac{dV_s(t)}{dt}$$

The general form of the differential equation is

$$
\begin{aligned}
y' + ay &= x' \\
\big[p + a\big] y &= px \\
y &= \frac{1}{p+a}\big[px\big]
\end{aligned}
$$

From the notation above, it's clear that we can think of this as applying the Heaviside operator p to the input $x(t)$, and then applying the low-pass operator to the result. Another way to think of $\frac{1}{p+a}p$ is that it is a new operator with the symbol $\frac{p}{p+a}$ operating on our input. So let's try to figure out what this new operator, the high-pass operator, does to our input. We'll use the low-pass operator to figure this out.
Restating the low-pass operator for ease of reference,

$$
\boxed{\frac{1}{p+a}x(t) = e^{-at}\frac{1}{p}\Big[e^{at}x(t)\Big] = e^{-at}\int_{-\infty}^{t} e^{a\tau}x(\tau)\,d\tau}
$$

$$
\begin{aligned}
\therefore y &= \frac{p}{p+a}\big[x(t)\big] = \frac{1}{p+a}\big[p\,x(t)\big] \\
&= e^{-at}\int_{-\infty}^{t} e^{a\tau}px(\tau)\,d\tau \\
&= e^{-at}\int_{-\infty}^{t} e^{a\tau}x'(\tau)\,d\tau
\end{aligned}
$$

Integrating by parts with $u = e^{a\tau}$ and $v' = x'(\tau)$,

$$
\begin{aligned}
&= e^{-at}\left[e^{at}x(t) - a\int_{-\infty}^{t} e^{a\tau}x(\tau)\,d\tau\right] \\
&= x(t) - a\left[e^{-at}\int_{-\infty}^{t} e^{a\tau}x(\tau)\,d\tau\right]
\end{aligned}
$$

The equation in the brackets is exactly our low-pass operator, so

$$
\begin{aligned}
y &= x(t) - a\frac{1}{p+a}x(t) \\
&= \left[1 - a\frac{1}{p+a}\right]x(t)
\end{aligned}
$$

where 1 is defined as the unity operator. Thus,

$$
\frac{p}{p+a}\big[x(t)\big] = \left[1 - a\frac{1}{p+a}\right]x(t)
$$

If we treat p as a rational function algebraically, the right hand side is

$$
\begin{aligned}
&= \left[1 - \frac{a}{p+a}\right]x(t) \\
&= \frac{p+a-a}{p+a}\big[x(t)\big] \\
&= \frac{p}{p+a}\big[x(t)\big]
\end{aligned}
$$

We can see from the result above that the Heaviside operator behaves as a rational function algebraically, and so the algebraic properties of association, commutation, and distribution apply to the Heaviside operator, which is motivation behind using the operator to solve differential equations.

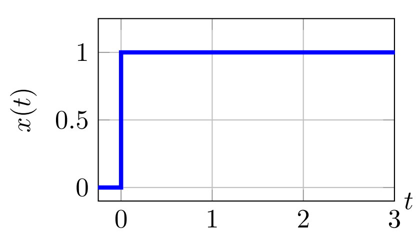
The power of the operator is that it allows us deal with differential equations as rational functions. This examples illustrates that the Heaviside operator behaves as a rational function algebraically and from now on, we'll treat the operator as a rational function, and no formal proof will be given. (For those who scoff at the lack of rigour, Oliver Heaviside would say to you,"Shall I refuse my dinner because I do not fully understand the process of digestion?")

Now, in general, if we have the operator of the form $\dfrac{p+a}{p+b}$, then

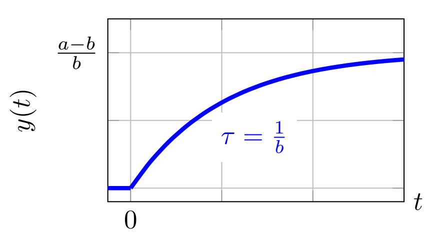
$$\begin{aligned}\frac{p+a}{p+b} &= \frac{p+b-b+a}{p+b} \\ &= \frac{p+b}{p+b} + \frac{a-b}{p+b} \\ &= 1 + \frac{a-b}{p+b}\end{aligned}$$

So if we want to know the step response of this system,

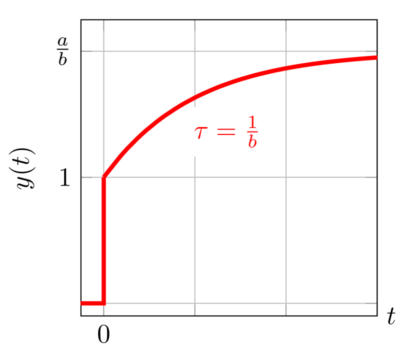
$$\begin{aligned}y(t) &= \left[1 + \frac{a-b}{p+b}\right] u(t) \\ &= u(t) + \frac{a-b}{p+b} u(t)\end{aligned}$$



$x(t) = u(t)$



$y(t) = \frac{a-b}{b}\left(1 - e^{-bt}\right) u(t) = \frac{a-b}{p+b} u(t)$



$y(t) = \frac{a-b}{b}\left(1 - e^{-bt}\right) u(t) + u(t) = \left[\frac{a-b}{p+b} + 1\right] u(t)$

Thus, the high-pass operator, and any first order system in general, can be broken down into a unity and low-pass operator.

Solving a Second-Order System

All this is well and good for a first-order system, but how about a second-order system? The general equation of a second order system is of the form

$$\begin{aligned}\frac{d^2y}{dt^2} + A\frac{dy}{dt} + By &= x \quad \text{or,}\\ y'' + Ay' + By &= x\end{aligned}$$

y'' is p operating on y twice — $p\Big[p\big[y\big]\Big]$ — so we can write it as p^2y

$$\begin{aligned}p^2y + Apy + By &= x\\ \left[p^2 + Ap + B\right] y &= x\end{aligned}$$

This is a polynomial to the second degree, so we can factor it. Letting $A = a + b$, and $B = ab$,

$$\begin{aligned}\left[p^2 + (a+b)p + ab\right] y &= x\\ \left[p^2 + ap + bp + ab\right] y &= x\\ \left[p(p+a) + b(p+a)\right] y &= x\\ \left[p+a\right]\left[p+b\right] y &= x\end{aligned}$$

Note that we can even change the order of $[p+a]$ and $[p+b]$, and we'll get the same result.

$$\therefore y = \frac{1}{p+a}\frac{1}{p+b}x = \frac{1}{p+b}\frac{1}{p+a}x$$

Example 0.5.4. Solve

$$\begin{aligned}y'' + 3y' + 2y &= x \quad \text{where,}\\ x(t) &= \delta(t)\end{aligned}$$

Answer. Using the Heaviside operator,

$$\begin{aligned}\left[p^2 + 3p + 2\right] y &= \delta(t)\\ \left[p+1\right]\left[p+2\right] y &= \delta(t)\\ y &= \frac{1}{p+2}\frac{1}{p+1}\delta(t)\\ &= \frac{1}{p+2}\left[e^{-t}\int_{-\infty}^{t} e^{\tau}\delta(\tau)\,d\tau\right]\\ &= \frac{1}{p+2}\left[e^{-t}\int_{-\infty}^{t} \cancelto{1}{e^{\tau}}\delta(\tau)\,d\tau\right]\\ &= \frac{1}{p+2}\left[e^{-t}u(t)\right]\\ &= e^{-2t}\int_{-\infty}^{t} e^{2\tau}e^{-\tau}u(\tau)\,d\tau\\ &= e^{-2t}\int_{-\infty}^{t} e^{\tau}u(\tau)\,d\tau\\ &= e^{-2t}u(t)\int_{0}^{t} e^{\tau}\,d\tau\\ &= e^{\tau}\Big|_{0}^{t} e^{-2t}u(t)\\ &= (e^{t}-1)e^{-2t}u(t)\\ &= (e^{-t}-e^{-2t})u(t)\end{aligned}$$

We mentioned previously that the impulse was a gigantic shock to the system that brings about all its characteristic behaviors (i.e. the system's natural frequencies), and we can see that the time constants — represented by reciprocal of the roots of the characteristic polynomial — 1 and $\frac{1}{2}$, are present in the exponentials in the response. We will not discuss this more now as we will treat it more formally later.

Actually, there is an easier way to arrive at the result above. From previous calculations, we know that

$$\frac{1}{p+a}\delta(t) = e^{-at}u(t)$$

So if we're able to simplify $\frac{1}{p+2}\frac{1}{p+1}$ to something of the form $\frac{K_1}{p+2} + \frac{K_2}{p+1}$, we can can get the result of $\frac{K_1}{p+2}\delta(t)$ and $\frac{K_2}{p+1}\delta(t)$ individually, and use superposition to get the final result. We want K_1 and K_2 such that

$$\frac{1}{p+2}\frac{1}{p+1} = \frac{K_1}{p+2} + \frac{K_2}{p+1} \tag{7}$$

Let's try to figure out K_1. Multiplying both sides by $(p+2)$,

$$\frac{1}{p+1} = K_1 + \frac{K_2(p+2)}{p+1}$$

Setting p to -2,

$$\begin{aligned} \frac{1}{-2+1} &= K_1 + \frac{K_2(-2+2)}{p+1} \\ \therefore K_1 &= -1 \end{aligned}$$

Similarly, we can get K_2 by multiplying eq. (7) by $(p+1)$ and setting p to -1,

$$k_2 = \frac{1}{p+2} = \frac{1}{-1+2} = 1$$

This process is called Partial Fraction Expansion. Thus, applying an impulse to our system, we get

$$\begin{aligned} y(t) &= \left[\frac{-1}{p+2} + \frac{1}{p+1}\right]\delta(t) \\ &= -e^{-2t}u(t) + e^{-t}u(t) \end{aligned}$$

which is the same result as we got before. Let's do another example with the same system, but this time let's hit the system at one of its natural frequencies.

Example 0.5.5. Solve

$$\begin{aligned} y'' + 3y' + 2y = x \quad &\text{where,} \\ x(t) = e^{-t}u(t) \end{aligned}$$

Answer.

$$y = \frac{1}{p+2}\frac{1}{p+1}e^{-t}u(t)$$

Now, we know that $e^{-t}u(t) = \frac{1}{p+1}\delta(t)$, so substituting that in we get

$$y = \frac{1}{p+2}\frac{1}{(p+1)^2}\delta(t)$$

Performing partial fraction expansion will give us an equation of the form

$$\frac{1}{p+2}\frac{1}{(p+1)^2} = \frac{K_1}{p+2} + \frac{K_2}{(p+1)^2} + \frac{K_3}{p+1}$$

Unfortunately, we don't know the general result of $\frac{1}{(p+a)^2}\delta(t)$, so going through the partial fraction expansion and superposition route won't be useful. Nevertheless, we will calculate the result using $\frac{1}{p+a}x(t) = e^{-at}\int_{-\infty}^{t} e^{a\tau}x(\tau)\,d\tau$.

$$\begin{aligned}
y &= \frac{1}{p+2}\frac{1}{(p+1)^2}\delta(t) && (8)\\
\therefore y &= \frac{1}{p+2}\left[e^{-t}\int_{-\infty}^{t} e^{\tau}e^{-\tau}u(\tau)\,d\tau\right]\\
&= \frac{1}{p+2}\left[e^{-t}\int_{-\infty}^{t} u(\tau)\,d\tau\right]\\
&= \frac{1}{p+2}\left[e^{-t}u(t)\int_{0}^{t} d\tau\right]\\
&= \frac{1}{p+2}\left[t\,e^{-t}u(t)\right] && (9)
\end{aligned}$$

When we hit the system at it's natural frequency, the system gets more excited, loosely speaking, than if we hit it at some other frequency and the result is the natural frequency multiplied by t which grows over time, as opposed to the frequency we hit it at being multiplied by a constant. Continuing,

$$\begin{aligned}
&= e^{-2t}\int_{-\infty}^{t} e^{2\tau}\,\tau\,e^{-\tau}u(\tau)\,d\tau\\
&= e^{-2t}\int_{-\infty}^{t} e^{\tau}\,\tau\,u(\tau)\,d\tau\\
&= e^{-2t}u(t)\int_{0}^{t} e^{\tau}\tau\,d\tau
\end{aligned}$$

Integrating by parts with $u = \tau$ and $v' = e^{\tau}$,

$$\begin{aligned}
&= e^{-2t}u(t)\left[\tau e^{\tau} - \int_{0}^{t} e^{\tau}\,d\tau\right]\\
&= e^{-2t}u(t)\left[te^{t} - e^{\tau}\Big|_{0}^{t}\right]\\
&= e^{-2t}u(t)\left[te^{t} - (e^{t}-1)\right]\\
&= e^{-2t}u(t)\left[e^{t}(t-1)+1\right]\\
&= \left[e^{-t}(t-1) + e^{-2t}\right]u(t)
\end{aligned}$$

Thus, even the natural frequency we didn't hit the system at came out, but just not as strongly as the natural frequency that we did hit it at.

What we've done

In this section, we defined the Heaviside operator, p, as the differential with respect to time. We motivated the operator by explaining that all the algebraic properties apply to it, and we saw a glimpse of this when deriving the High-Pass Operator. We further explained that this fact turned

solving differential equations into solving algebraic problems.
We then saw the consequence of this in solving a second-order system; we treated the operator on $y(t)$ as a function and we were hence able to factorize it, and apply the inverse of the factors to the input. Thus, we fulfilled what we set out to examine in this section which was to WHATTTTTTTTTTT.

Where we're going
We also saw a glimpse of where we're going in Example 0.5.4; by performing partial fraction expansion on the operator with factors of p in the denominator (i.e. $\frac{1}{(p+2)(p+1)}$), we were able to simplify the operator into a sum of simpler operator with irreducible factors of p in the denominator (i.e. $\frac{-1}{p+2}+\frac{1}{p+1}$). Thus, if we knew the general result of the simpler operators on our specific input (i.e. $\frac{1}{p+a}x(t)$), we could get the result of the simpler operators individually and then by superposition, we could sum the individual responses to get to the final response.

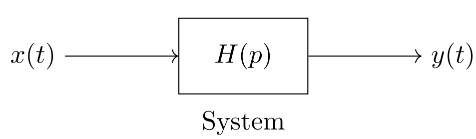
But this is not that useful; the effect an operator has on one type of input is not generalize-able to another type of input (for example, $\frac{1}{p+a}\delta(t)$ is different from $\frac{1}{p+a}u(t)$). Hence, for this to be useful we would have to figure out the effect an operator has on every type of input and create a lookup table of some sort. That would be tedious, but in Example 0.5.5 we saw the solution to this problem; we can also convert the input into some operator operating on an impulse $\delta(t)$ (i.e. $x(t) = e^{-t}u(t) = \frac{1}{p+1}\delta(t)$). This meant that after multiplying to obtain the system operator and factorizing it (i.e. $\frac{1}{p+2}\frac{1}{(p+1)^2}\delta(t)$), we could use partial fraction expansion to simplify the operator into a sum of simpler operators and then from the general result of each operator acting on $\delta(t)$, we could figure out the specific result! Hence, we simplified the only input we need to consider to the impulse function, $\delta(t)$.
But we quickly saw the limitation of this method; we generated an operator of the form $\frac{K_2}{(p+a)^2}$ in the partial fraction expansion, and since we didn't know the general result of that operator, the partial fraction expansion and superposition method wasn't so useful. But the good news is that if we did know the result, the response would have been very easy to calculate.

But surely we can't do this for every single system; imagine trying to come up with operators to solve 3rd, 4th, 5th order system. In fact, we'll explore this now, but let's define a few terms first.

Definitions
The general form of a system is

$$\begin{aligned}
y^{(m)}(t) + a_{m-1}y^{(m-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) &= x^{(n)}(t) + b_{n-1}x^{(n-1)}(t) + \cdots + b_1 x'(t) + b_0 x(t) \\
\left[p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0\right] y(t) &= \left[p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0\right] x(t) \\
y(t) &= \frac{p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0}{p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0} x(t)
\end{aligned}$$

where $y(t)$ is the output, or system's response, and $x(t)$ is the input. Note that the derivatives of the input present in the above equation is due to the system (as in the case of the circuit used to derived the High-Pass operator), and is not part of what we feed into the system.
The above equation can also be represented by the following diagram.



The entire operator that operates on our input, $x(t)$, to produce the system's response, $y(t)$, is called the **System Operator**. It is denoted by $H(p)$.

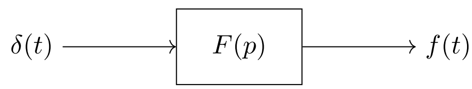
$$y(t) = H(p)\,x(t) \tag{10}$$

Now, the impulse response of a system is the response of the system to an impulse.

$$\begin{aligned} x(t) &= \delta(t) \\ h(t) = y(t) &= H(p)\,x(t) \\ \therefore h(t) &= H(p)\,\delta(t) \end{aligned} \tag{11}$$

We'll generalize the notation in Equation (11) to all functions.

$$f(t) = F(p)\,\delta(t)$$



Function Operator

Thus, $F(p)$ is the operator that generates $f(t)$ from an impulse $\delta(t)$. Similarly,

$$x(t) = X(p)\,\delta(t) \tag{12}$$

$$y(t) = Y(p)\,\delta(t) \tag{13}$$

where $X(p)$ is known as the **input operator** and $Y(p)$ is known as the **output operator**.

Substituting Equation (12) and Equation (13) into Equation (10),

$$\begin{aligned} Y(p)\,\delta(t) &= H(p)\,X(p)\,\delta(t) \\ \iff Y(p) &= H(p)\,X(p) \end{aligned}$$

To find a system's response, our aim will be to find $H(p)$ (by using the Heaviside operator, p, for differentials with respect to time in the equation of a system) and $X(p)$ (by converting our input into the operator that generates our input from an impulse using a catalog), then simplify $Y(p)$ using long division, factorization and partial fraction expansion, and lastly, converting the result of each of the simplified operators acting on $\delta(t)$ back to time-domain functions using a catalog, and summing them up.

The point of all this is that it makes our systems tractable and understandable. We can see what parts of the system or input contributes what to the response, and can therefore make informed design decisions. But won't our catalog be big? What's the point of this exercise if we have to memorise a big look-up table? Let's go back to exploring this equation.

The importance of first and second order systems

The general form of a system is

$$\begin{aligned} y^{(m)}(t) + a_{m-1}y^{(m-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) &= x^{(n)}(t) + b_{n-1}x^{(n-1)}(t) + \cdots + b_1 x'(t) + b_0 x(t) \\ \left[p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0\right] y(t) &= \left[p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0\right] x(t) \\ y(t) &= \frac{p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0}{p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0} x(t) \end{aligned}$$

where n is the order of the numerator and m is the order of the denominator.
Case 1: If $n >= m$, we can perform long division and we'll end up

$$y(t) = \left[c_{n-m}p^{n-m} + \cdots + c_0 p^0 + \frac{b_{m-1}p^{m-1} + \cdots + b_1 p + b_0}{p^m + \cdots + a_1 p + a_0}\right] x(t)$$

where c is some natural number. We don't have to worry about the operators of the form $c_i p^i$ as we can apply them directly by differentiating $x(t)$ i times and multiplying by c_i. We'll treat the proper fraction the same as the next case.

Case 2: If $n < m$, we have a proper fraction.
According to the Fundamental Theorem of Algebra, any m^{th} order polynomial with real coefficients can be factored into first and second order polynomials. The second order polynomials can be further factored into two first order polynomials with complex roots that are conjugates of each other. Thus,

$$\begin{aligned} y(t) &= \frac{p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0}{p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0} x\left(t\right) \\ &= \frac{p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0}{(p - r_m)(p - r_{m-1}) \cdots (p - r_1)} x(t) \end{aligned}$$

where $r_i \in \mathbb{C}$.
We can then perform partial fraction expansion, and simplify the system operator into a sum of simpler operators with the operator p to the first or second order in the denominator. **This means that the response of any m^{th} order system can be broken down into the sum of responses of first and second order system!** This is why the response of first and second order systems are studied in detail; the dynamics of any order system simplify to the dynamics present in first and second order systems.

We'll show that there are, in fact, only four different kinds of simplified operators we can get after partial fraction expansion. We're building to this look-up table (or catalog) and we'll do so through examples in the following section.

x. A big a-side: Solving differential equations with the p operator

x. The Catalog - convert between time and operator domain

$x(t) = X(p)\delta(t)$	$X(p)$
$\delta(t)$	1
$u(t)$	$\frac{1}{p}$
$r(t) = t \cdot u(t)$	$\frac{1}{p^2}$
$\frac{t^m}{m!}\, u(t)$	$\frac{1}{p^{m+1}}$
$e^{-rt}\, u(t)$	$\frac{1}{p+r}$
$\frac{t^m}{m!} e^{-rt}\, u(t)$	$\frac{1}{(p+r)^{m+1}}$

Our Current Catalog

Now that we've listed waveforms and their corresponding heavyside operator that we're familiar with, we will derive the rest of the possible second order rational functions we could get. Instead of listing the operator and figuring out the response to $\delta(t)$, we will start with the waveform and see if we can derive the operator (i.e. the rational function) that could produce it using what we know from the current catalog.

What about $\cos(\omega_0 t)u(t)$?

Can we use any of the previous operators to figure this out? I'd suggest you try before continuing.
From ....., we know that $\cos(\omega_0 t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$. Thus,

$$\begin{aligned}
\cos(\omega_0 t)u(t) &= \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] u(t) \\
&= \frac{1}{2}\left[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)\right] \\
&= \frac{1}{2}\left[\frac{1}{p - j\omega_0} + \frac{1}{p + j\omega_0}\right] && \text{(from the catalog)} \\
&= \frac{1}{2}\left[\frac{p + j\omega_0 + p - j\omega_0}{p^2 - (j\omega_0)^2}\right] \\
&= \frac{1}{2}\left[\frac{2p}{p^2 + \omega_0^2}\right] \\
&= \frac{p}{p^2 + \omega_0^2}
\end{aligned}$$

What about $\sin(\omega_0 t)u(t)$?

If you had to guess, what do you think it could be?
We know that $\sin(\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$. Thus,

$$\begin{aligned}
\sin(\omega_0 t)u(t) &= \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] u(t) \\
&= \frac{1}{2j}\left[e^{j\omega_0 t}u(t) - e^{-j\omega_0 t}u(t)\right] \\
&= \frac{1}{2j}\left[\frac{1}{p - j\omega_0} - \frac{1}{p + j\omega_0}\right] && \text{(from the catalog)} \\
&= \frac{1}{2j}\left[\frac{p + j\omega_0 - p + j\omega_0}{p^2 - (j\omega_0)^2}\right] \\
&= \frac{1}{2j}\left[\frac{2j\omega_0}{p^2 + \omega_0^2}\right] \\
&= \frac{\omega_0}{p^2 + \omega_0^2}
\end{aligned}$$

What about $e^{-\sigma t}\cos(\omega_0 t)u(t)$?

If you had to guess, what do you think it could be? (your guess should reduce to the operator for $\cos(\omega_0 t)u(t)$ for $\sigma = 0$, and to $e^{-\sigma t}$ for $\omega_0 = 0$)

$$\begin{aligned}
e^{-\sigma t}\cos(\omega_0 t)u(t) &= e^{-\sigma t}\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]u(t) \\
&= \frac{1}{2}\left[e^{(j\omega_0-\sigma)t}+e^{(-j\omega_0-\sigma)t}\right]u(t) \\
&= \frac{1}{2}\left[\frac{1}{p+(\sigma-j\omega_0)}+\frac{1}{p+(\sigma+j\omega_0)}\right]u(t) \qquad \text{(from the catalog)} \\
&= \frac{1}{2}\left[\frac{2p+2\sigma}{(p+\sigma+j\omega_0)(p+\sigma-j\omega_0)}\right]u(t) \\
&= \frac{p+\sigma}{(p+\sigma)^2+\omega_0^2}
\end{aligned}$$

What about $e^{-\sigma t}\sin(\omega_0 t)u(t)$?

Extrapolating from $\cos(\omega_0 t)u(t)$ and $\sin(\omega_0 t)u(t)$, a good guess could be $\frac{\omega_0}{(p+\sigma)^2+\omega_0^2}$. Let's see what it really is.

$$\begin{aligned}
e^{-\sigma t}\sin(\omega_0 t)u(t) &= e^{-\sigma t}\left[\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right]u(t) \\
&= \frac{1}{2j}\left[e^{(j\omega_0-\sigma)t}-e^{(-j\omega_0-\sigma)t}\right]u(t) \\
&= \frac{1}{2j}\left[\frac{1}{p+(\sigma-j\omega_0)}-\frac{1}{p+(\sigma+j\omega_0)}\right]u(t) \qquad \text{(from the catalog)} \\
&= \frac{1}{2j}\left[\frac{2j\omega_0}{(p+\sigma+j\omega_0)(p+\sigma-j\omega_0)}\right]u(t) \\
&= \frac{\omega_0}{(p+\sigma)^2+\omega_0^2}
\end{aligned}$$

The Final Catalog

What is special about this table? According to the fundamnetal theorem of algebra, we can factor any n^{th} order polynomial into first and second order polynomials with real coefficients. This table contains all the first and second order polynomials you can get from the partial fraction expansion of rational functions!
Thus, these waveforms are the only sort of waveforms that we can get out of a system that can be described with rational functions, which is all circuits with lumped circuit elements. There are other systems that are not lumped whose responses do not look like this, but if it is lumped and if we're dealing with circuit elements of a linear kind, these are the responses.

Example to motivate the usefulness of the catalog:
Suppose we would like to solve the differential equation

$$y'' + 3y' + 2y = x(t)$$

where the input is

$$x(t) = 4\cos(t)u(t)$$

$x(t) = X(p)\delta(t)$	$X(p)$
$\delta(t)$	1
$u(t)$	$\frac{1}{p}$
$r(t) = t \cdot u(t)$	$\frac{1}{p^2}$
$\frac{t^m}{m!}\, u(t)$	$\frac{1}{p^{m+1}}$
$e^{-rt}\, u(t)$	$\frac{1}{p+r}$
$\frac{t^m}{m!} e^{-rt}\, u(t)$	$\frac{1}{(p+r)^{m+1}}$
$\cos(\omega_0 t)\, u(t)$	$\frac{p}{p^2+\omega_0^2}$
$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{p^2+\omega_0^2}$
$e^{-\sigma t}\cos(\omega_0 t)\, u(t)$	$\frac{p+\sigma}{(p+\sigma)^2+\omega_0^2}$
$e^{-\sigma t}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(p+\sigma)^2+\omega_0^2}$

Final Catalog

$x(t) = X(p)\delta(t)$	$X(p)$
$\delta(t)$	1
$\frac{t^m}{m!}\, u(t)$	$\frac{1}{p^{m+1}}$
$\frac{t^m}{m!} e^{-rt}\, u(t)$	$\frac{1}{(p+r)^{m+1}}$
$\cos(\omega_0 t)\, u(t)$	$\frac{p}{p^2+\omega_0^2}$
$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{p^2+\omega_0^2}$
$e^{-\sigma t}\cos(\omega_0 t)\, u(t)$	$\frac{p+\sigma}{(p+\sigma)^2+\omega_0^2}$
$e^{-\sigma t}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(p+\sigma)^2+\omega_0^2}$

Simplified Final Catalog

Now, combining the above two equations and converting to operator domain using the catalog,

$$\left[p^2 + 3p + 2\right] y = \frac{4p}{p^2+1}\delta(t)$$

Factoring and rearranging for y(t),

$$y(t) = \frac{4p}{(p+1)(p+2)(p^2+1)}\delta(t)$$

$$\begin{aligned}
Y(p) &= \frac{4p}{(p+1)(p+2)(p^2+1)} \\
&= \frac{-2}{p+1} + \frac{8}{5}\left(\frac{1}{p+2}\right) + \frac{1}{5}\left(\frac{1+3j}{p+j}\right) + \frac{1}{5}\left(\frac{1-3j}{p-j}\right) \\
&= \frac{-2}{p+1} + \frac{8}{5}\left(\frac{1}{p+2}\right) + \frac{2}{5}\left(\frac{p+3}{p^2+1}\right) \\
&= \frac{-2}{p+1} + \frac{8}{5}\left(\frac{1}{p+2}\right) + \frac{2}{5}\left(\frac{p}{p^2+1}\right) + \frac{2}{5}\left(\frac{3}{p^2+1}\right)
\end{aligned}$$

(the partial fraction expansion working out can be seen here)

Using the catalog to convert to time domain,

$$y(t) = \left(-2e^{-t} + \frac{8}{5}e^{-2t} + \frac{2}{5}\cos(t) + \frac{6}{5}\sin(t)\right) u(t)$$

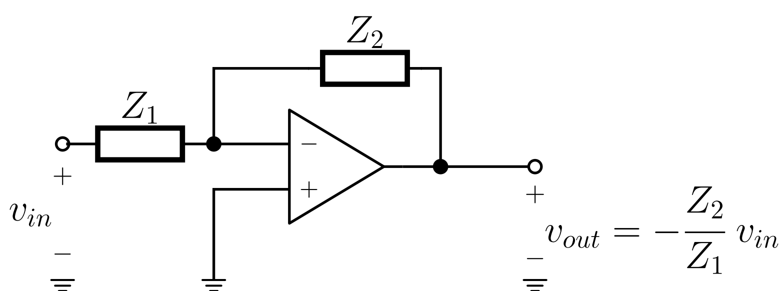
Putting it all together
The catalog is extremely useful because it allows us to get the response of a system to any input using partial fraction expansion, multiplication, and the catalog. There's no need for any integration or convolution!

Summary of the method:

1. Get the differential equation of the system in terms of the p operator.
2. Convert input from time domain to operator domain using catalog (i.e. Find $X(p)$ where $x(t) = X(p)\delta(t)$).
3. Rearrange to get $y(t)$ on one side such that $y(t) = Y(p)\,\delta(t)$, and simplify $Y(p)$ using partial fraction expansion.
4. Convert $Y(p)\,\delta(t)$ to time domain using the catalog.

Concluding Notes
We have now developed a powerful method that not only allows you to do the calculations in certain ways that avoid integration or convolution, which make it easier to do by hand, but also this method gives us additional insight when it comes to design which helps us choosing elements to get the output we desire.

Example to motivate how the operator domain helps with design



Suppose we have an op-amp in this configuration. The square boxes represent the impedance of elements that we have to choose. We know that since this is an op-amp with negative feedback,

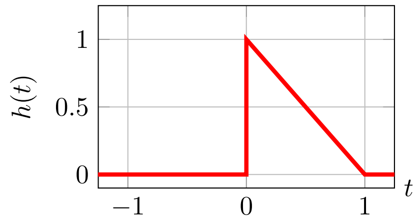
$$V_{out} = -\frac{Z_2}{Z_1} V_{in} \tag{14}$$

Now, suppose we want to build an integrator. What components should we choose for elements 1 and 2? Well, we know that the $\frac{1}{p}$ operator is integration. So we have to choose Z_2 and Z_1 such that we have p in the denominator of eq. (14). If Z_2 is a capacitor, this fulfills our criteria (making Z_1 an inductor also fulfills our criteria, but inductors are generally avoided where possible due to their interference from their magnetic fields, large series resistance and stray capacitance). Since we don't want any other p operators, Z_1 must be a resistor. This is in fact how integrators are built.

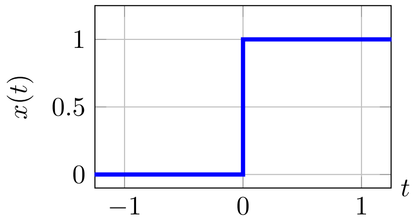
Similarly, what if we wanted to differentiate the input voltage with the same circuit setup as above? We know that the p operator is differentiation so we must have p in the numerator of eq. (14). Avoiding inductors, Z_1 must be a capacitor and since we don't want any other operators, Z_2 must be a resistor.

x. Delays and Time Shifts - The last piece of the puzzle

Suppose we have a system whose impulse response and input are respectively



$h(t) = u(t) - t \cdot u(t) + (t-1) \cdot u(t-1)$



$x(t) = u(t)$

Now, converting the input to the operator domain, we get

$$X(p) = \frac{1}{p}$$

and converting it's impulse response to the operator domain, we get

$$H(p) = \frac{1}{p} - \frac{1}{p^2} + \mathbf{?}$$

How do we represent the delay in the operator domain? The operator $\dfrac{1}{p^2}$ is obviously incorrect, so what could it be? Let's try to figure this out by looking at the definition of the impulse response

$$h(t) = H(p)\delta(t)$$

From this definition, can we figure out what is $h(t-\tau)$?
Let's try to use the Taylor Series Expansion to figure this out.

$$\begin{aligned} h(t-\tau) &= \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} h^k(t) \\ &= h(t) - \frac{\tau}{1!}h'(t) + \frac{\tau^2}{2!}h''(t) - \frac{\tau^3}{3!}h'''(t) + \cdots \end{aligned}$$

Since the p-operator is the derivative, $h'(t) = p[h(t)] = p\,H(p)\delta(t)$. In general, $h^k(t) = p^k\,H(p)\delta(t)$. Thus,

$$\begin{aligned} h(t-\tau) &= H(p)\delta(t) - \tau p H(p)\delta(t) + \frac{(\tau p)^2}{2!}H(p)\delta(t) - \frac{(\tau p)^3}{3!}H(p)\delta(t) + \cdots \\ &= \left[1 - \tau p + \frac{(\tau p)^2}{2!} - \frac{(\tau p)^3}{3!} + \cdots\right] H(p)\delta(t) \\ &= e^{-\tau p} H(p)\delta(t) \end{aligned}$$

In general, we have,

$$f(t-\tau) = e^{-\tau p} F(p)\delta(t)$$

But we generally work with $f(t-\tau)u(t-\tau)$

Going back to our example above,

$$\begin{aligned} h(t) &= u(t) - t \cdot u(t) + (t-1) \cdot u(t-1) \\ \therefore H(p) &= \frac{1}{p} - \frac{1}{p^2} + e^{-p}\frac{1}{p^2} \end{aligned}$$

Now,

$$y(t) = H(p)X(p) = \frac{1}{p^2} - \frac{1}{p^3} + e^{-p}\frac{1}{p^3}$$

Using the catalog to convert back to waveforms,

$$y(t) = t \cdot u(t) - \frac{t^2}{2}u(t) + \frac{(t-1)^2}{2}u(t-1)$$

Thus, the complete catalog for systems with lumped elements, and hence rational system operators is