Harder Induction — Problem Set

Warm-up

1. Prove the following by induction for positive integers n:

   a) $\sum_{k=1}^{n} \frac{1}{2}k(k+1) = \frac{1}{6}n(n+1)(n+2)$
   b) $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
   c) $\sum_{k=1}^{n} \frac{1}{2}k(k+1) = \frac{1}{6}n(n+1)(n+2)$
   d) $2^n > n$
   e) $n^2 > 2n+1$ for $n \geq 3$
   f) If $T_1 = 3$ and $T_n = T_{n-1} + 4$ then $T_n = 4n - 1$
   g) If $T_1 = 5$ and $T_n = 2T_{n-1} + 1$ for $n \geq 2$, that $T_n = 6 \times 2^{n-1} - 1$ for $n \geq 1$
   h) $\frac{d}{dx}x^n = nx^{n-1}$ (hint: use the product rule)

Skill-building

1. Prove the following by induction for positive integers n:

   a) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
   b) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$
   c) $n! > n^2$ for $n \geq 4$
   d) $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for $n \geq 2$
   e) If $T_1 = a$ and $T_n = T_{n-1} + d$ for $n \geq 2$, that $T_n = a + (n-1)d$ for $n \geq 1$
   f) If $T_1 = a$ and $T_n = rT_{n-1}$ for $n \geq 2$, that $T_n = ar^{n-1}$ for $n \geq 1$
   g) Given $\frac{d}{dx}e^x = e^x$, prove by induction that $\frac{d}{dx}e^{nx} = ne^{nx}$for all $n \geq 1$.

Easier Exam Questions

1. (Cheltenham Girls 2023 Q10) The following induction proof to show that $5^n + 2 \times 11^n$ is a multiple of 3 for all positive integers n, contains an algebraic error.

   In which section does the error occur? (note: the options flow from one to another as one whole inductive proof)

   (A) Show true for $n = 1$. $5^1 + 2 \times 11^1 = 27$, which is a multiple of 3. $\therefore$ True for $n = 1$.

   (B) Assume true for $n = k$, where $k \in \mathbb{N}$. i.e. $5^k + 2 \times 11^k = 3M$ for $M \in \mathbb{N}$.

   (C) Prove true for $n = k+1$. $5^{k+1} + 2 \times 11^{k+1} = 5 \times 5^k + 2 \times 11^k \times 11 = 5(5^k + 2 \times 11^k) + 6 \times 2 \times 11^k$

   (D) $= 5(3M) + 3(6 \times 11^k)$ from $n = k$ assumption $= 3(5 + 6 \times 11^k) = 3N, \quad N \in \mathbb{N}$ Hence, divisible by 3.

2. (Barker 2025 Q13a) Given that $u_1 = 1$ and $u_{n+1} = 2u_n + (-1)^{n+1}$,
   prove that $u_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$, where $n \in \mathbb{Z}^+$, by mathematical induction. **3**

3. (Killara 2020 Q13a) Prove by mathematical induction that $\forall n \in \mathbb{Z}^+$

$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}$$

3

4. (Killara 2021 Q13a) Using principle of mathematical induction, prove that $\forall n > 5, n \in \mathcal{N}$,

$$n! < \left(\frac{n}{2}\right)^n$$

3

5. (St George Girls 2020 Q14a) Prove by induction that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for integers $n \geq 1$. **3**

Harder Exam Questions

1. (Barker 2025 Q16b) Given that the product of n consecutive numbers is divisible by $n!$, use mathematical induction to prove that $n^5 - n$ is divisible by 2, 3 and 5 for integers $n \geq 2$. **4**

2. (Manly 2024 Q13a) Prove by mathematical induction that $7^n + 13^n + 19^n$ is divisible by 13 for all positive odd integers n. **4**

3. (St George Girls 2024 Q16a) A sequence is defined by the following formula for $n \in Z^+$:

$$T_0 = 0$$

$$T_n = \sqrt{T_{n-1} + 2}$$

   Prove by mathematical induction that $T_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$ for integer $n \geq 0$. **4**

4. (St George Girls 2020 Q15) A sequence is given by the recurrence relation $u_1 = 7$, $u_{n+1} = 2u_n + 3$ for $n \geq 1$.

   Prove by induction that the general formula for the sequence is $u_n = 5(2^n) - 3$. **4**

5. (St George Girls 2020 Q16) Use mathematical induction to prove that $4^{n+1} + 6^n$ is divisible by 10 when n is even. **3**