Inequality Proofs — Problem Set

Warm-up

1. Assuming all pronumerals are real numbers, prove the following:

   a) $a^2 + b^2 > 2ab$
   b) $x^2 + 4y^2 > 4xy$
   c) $\frac{a+b}{2} \geq \sqrt{ab}$

2. The triangle inequality states that $|a + b| \leq |a| + |b|$.

   a) In your own words, what does this mean?
   b) How is this related to triangles? Draw a diagram representing the triangle inequality.

3. Assuming all pronumerals are real numbers, prove the following using the triangle inequality:

   a) $|a + 1| \leq |a| + 1$
   b) $|2x + 3y| \leq 2|x| + 3|y|$
   c) $|3m - 4n| \leq 3|m| + 4|n|$

Skill-building

1. Assuming all pronumerals are real numbers, prove the following:

   a) $a + \frac{1}{a} \geq 2$
   b) $1 \geq 2\sin(x)\cos(x)$ (Hint: $1 = \sin^2 x + \cos^2 x$)
   c) $-1 \leq 2\sin(x)\cos(x)$ (Hint: $1 = \sin^2 x + \cos^2 x$)
   d) $e^{2x} + 1 \geq 2e^x$

2. Prove the triangle inequality, that that $|a + b| \leq |a| + |b|$.

3. Assuming all pronumerals are real numbers, prove the following:

   a) $|x - y| \leq ||x| - |y||$
   b) $|a + b + c| \leq |a| + |b| + |c|$
   c) $|\ln(x\sin(x))| \leq |\ln(x)| + |\ln(\sin(x))|$

4. Use the squeeze theorem to prove the following

   a) Given that $-1 \leq \sin(x) \leq 1$, prove that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$
   b) $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
   c) $\lim_{x\to\infty} e^{-x}\sin(x) = 0$

Easier Exam Questions

1. (Sydney Tech 2020 Q5) Assume that a and b are negative real numbers with $a > b$. Which of the following might be false?

   (A) $\frac{1}{a-b} < 0$

   (B) $\frac{a}{b} - \frac{b}{a} < 0$

   (C) $a + b > 2b$

   (D) $2a > 3b$

2. (Sydney Tech 2020 Q14b) If a, b and $x \geq 0$,

   i. Show that $a^2 + b^2 \geq 2ab$ **1**

   ii. Hence, show that $\frac{x}{x^2+4} \leq \frac{1}{4}$ **2**

3. (Blacktown Boys 2023 Q15c)If a, b, c and d are positive real numbers, prove that:

   (i) $\frac{a+b}{2} \geq \sqrt{ab}$ **1**

   (ii) $(a+b+c+d)^2 \geq 4(ac+bc+bd+ad)$ **2**

   (iii) $(a+b+c+d)^2 \geq \frac{8}{3}(ab+ad+bc+cd+bd+ac)$ **2**

4. (St George Girls 2021 Q14a) It can be shown that $\frac{a+b}{2} \geq \sqrt{ab}$. DO NOT PROVE THIS.

   (i) Prove that $(a+b)(b+c)(c+a) \geq 8abc$ for all positive real numbers a, b, and c. **1**

   (ii) Suppose that x, y, and z are the lengths of the sides of a triangle. Using the result from part (i), deduce that

   $$xyz \geq (y+z-x)(z+x-y)(x+y-z)$$

   2

Harder Exam Questions

1. (Cheltenham Girls 2024 Q14d)

   (i) Using the AM-GM inequality, or otherwise, prove that, if $x > 1$, then $\frac{x}{\sqrt{x-1}} \geq 2$. **2**

   (ii) Hence, or otherwise, prove that, for $a > 1, b > 1$ the following inequality holds

   $$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8$$

   2

2. (Normanhurst Boys 2025 Q15a)Let a, b and c be positive real numbers.

   i. Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$. **2**

   ii. Hence or otherwise, prove that $a^3 + b^3 + c^3 \geq 3abc$. **1**

   iii. Hence prove that $ac(a+c) + ab(a+b) + bc(b+c) \geq 6abc$. **2**

3. (Killara 2020 Q14a)

   (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. **1**

   (ii) Hence, show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. **2**

(iii) Hence, or otherwise prove that

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc,$$

Where a, b and c are distinct positive real numbers. **2**

4. (Killara 2023 Q14b) (i) For $a, b > 0$, prove that

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

2

(ii) Let $a_1, a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_1a_2a_3 \ldots a_n = 1$.
Prove that,

$$(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n) \geq 2^n$$

2

(iii) Prove that for $a, b, c, d > 0$,

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a+b+c+d$$

2

5. (Killara 2024 Q12a) (i) If $a, b, c > 0$, prove that:

$$a^2 + b^2 + c^2 \geq bc + ca + ab$$

2

(ii) Hence, or otherwise, prove that:

$$2(a^3 + b^3 + c^3) \geq bc(b+c) + ca(c+a) + ab(a+b)$$

2