

The language and notation of proof — Problem Set

Warm-up

- Write the following statements in words if c represents the statement “I will eat ice cream” and d represents “I will be happy”.
 - $d \Leftrightarrow c$
 - $c \Rightarrow d$
 - $c \vee d$
- State the contrapositive of the following statements
 - If I am happy, I will dance.
 - If my computer is powered, it will turn on.
 - If the sun is outside, I will wake up.
- Write the following as sentences
 - $\forall m \in \mathbb{R} \exists n \in \mathbb{R}$ such that $n = \cos(m)$
 - $\forall m \in \mathbb{Z}^+ \exists n \in \mathbb{Z}$ such that $m^3 = n$
 - $\forall m \in \mathbb{Z}^+ \exists n \in \mathbb{Z}$ such that $n = -m$
- Prove the following by proving their contrapositives
 - If a^2 is odd then a is odd
 - For integers m and n , if $m - n$ is odd, then either m is odd or n is odd.
 - For integers m and n , if mn is even, then either m is even or n is even.
- Prove the following by contradiction (in words)
 - If it is raining, the ground is wet.
 - If you turn off the lights (with no other light sources), the room is dark.
 - If you drop ice into a warm drink, it gets colder.

Skill-building

- Write the following statements in words if c represents the statement “I will run today” and d represents “I will be fast”.
 - $d \wedge \neg c$
 - $\neg d \Rightarrow c$
 - $c \not\Rightarrow \neg d$
- State the contrapositive of the following statements
 - If I am happy, I will dance.

- b) If my computer is powered, it will turn on.
 - c) If the sun is outside, I will wake up.
3. Write the following as sentences
- a) $\forall m \in \mathbb{R} \exists n \in \mathbb{R}$ such that $n = \cos(m)$
 - b) $\forall m \in \mathbb{Z}^+ \exists n \in \mathbb{Z}$ such that $m^3 = n$
 - c) $\forall m \in \mathbb{Z}^+ \exists n \in \mathbb{Z}$ such that $n = -m$
4. Prove the following by proving their contrapositives
- a) If $a^2 + 2a + 6$ is even then a is even
 - b) For integers m and n , if $m - n$ is odd, then either m is odd or n is odd.
 - c) For integers m and n , if mn is even, then either m is even or n is even.
5. Prove the following by contradiction
- a) $\sqrt{2}$ is irrational
 - b) The sum of a rational and irrational number is irrational
 - c) $\log_3 5$ is irrational
6. State if the following are true or false, and if false, provide a counterexample
- a) For any 2 numbers s and t , $|s + t| \leq |s| + |t|$
 - b) For any number x , $x^2 > x$
 - c) For any 2 numbers a and b , If $a < b$ then $|a| < |b|$

Easier Exam Questions

1. (Blacktown Boys 2020 Q5) The following slogan is used at a particular swimming centre.
- “If you are a strong swimmer, then you are unlikely to drown.”
- Which of the following statements is logically equivalent to the above slogan?
- A. If you are unlikely to drown, then you are a strong swimmer.
 - B. If you are a strong swimmer, then you are likely to drown.
 - C. If you are not a strong swimmer, then you are likely to drown.
 - D. If you are likely to drown, then you are not a strong swimmer.
2. (Blacktown Boys 2022 Q4) Let $x \in \mathbb{Z}$. The contrapositive of the statement “If x is even, then $5x - 11$ is odd.” is
- A. If x is odd, then $5x - 11$ is even.
 - B. If $5x - 11$ is even, then x is odd.
 - C. If x is odd, then $5x - 11$ is odd.
 - D. If $5x - 11$ is odd, then x is even.

3. (Blacktown Boys 2023 Q3) The negation of the statement “I listen to music and I play games.” is:
- A. I don’t listen to music or I don’t play games.
 - B. I don’t listen to music and I don’t play games.
 - C. I don’t listen to music or I play games.
 - D. I don’t listen to music and I play games.
4. (Girraween 2021 Q8) The negation of the statement “If n is a positive integer, $3^n - 1$ is divisible by 4” is
- (A) “ n is a positive integer so $3^n - 1$ is divisible by 4”
 - (B) “ n is NOT a positive integer BUT $3^n - 1$ is divisible by 4”
 - (C) “ n is a positive integer BUT $3^n - 1$ is NOT divisible by 4”
 - (D) “ n is NOT a positive integer and $3^n - 1$ is NOT divisible by 4”
5. (Blacktown Boys 2020 Q4) Which of the following statement is true?
- A. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 10$
 - B. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y = x^2$
 - C. $\exists x \in \mathbb{R}$, such that $\forall y \in \mathbb{R}, xy = 10$
 - D. $\exists x \in \mathbb{R}$, such that $\forall y \in \mathbb{R}, y = x^2$

Harder Exam Questions

1. (Blacktown Boys 2023 Q9) Given that $x, y \in \mathbb{Z}$, where $x, y \geq 0$, which of the following is a FALSE statement?
- A. $\exists x(\forall y : y = 1 - 3x)$
 - B. $\exists x(\exists y : y = 5 + 2x)$
 - C. $\forall x(\exists y : y = x^2)$
 - D. $\forall x(\exists y : y - x = 0)$
2. (Girraween 2025 Q2) The statement “For all positive integers n , there exists a positive integer p such that $p = n^2$ ” is represented in mathematical symbols by:
- A) $\in n \ni \mathbb{Z}^+ \ni p \in \mathbb{Z}^+$ such that $p = n^2$
 - B) $\exists n \in \mathbb{Z} \ni p \in \mathbb{Z}$ such that $p = n^2$
 - C) $\forall n \in \mathbb{Z}^+ \exists p \in \mathbb{Z}^+$ such that $p = n^2$
 - D) $\forall n \in \mathbb{Z} \exists p \in \mathbb{Z}$ such that $p = n^2$
3. (Sydney Tech 2020 Q15a) If xyz represents a three digit number (not the product of x, y and z), show that if $x + z = y$ then the number is divisible by 11. (x, y and z are positive integers)

4. (Sydney Tech 2024 Q15a)i)) Show that 1, 2 and 3 are the only positive integers that satisfy

$$a^2 + b^2 + c^2 = 14$$

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5. (Sydney Tech 2025 Q13) Prove by contradiction that there are an infinite number of prime numbers.

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