# MONOTONICITY AND ASYMPTOTIC SLOPES FOR AIRY WANDERER LINE ENSEMBLES

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ABSTRACT. Insert abstract here.

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## 1. INTRODUCTION AND MAIN RESULTS

### 1.1. Preface.

### 1.2. The Airy wanderer line ensembles.

The goal of this section is to give a formal definition of the Airy wanderer line ensembles constructed in [4, 5]. Our exposition here closely follows that of [5, Section 1.2]. We begin by fixing the parameters of the model and some notation.

**Definition 1.1.** We assume that we are given four sequences of non-negative real numbers $\{a_i^+\}_{i\geq 1}$, $\{a_i^-\}_{i\geq 1}$, $\{b_i^+\}_{i\geq 1}$, $\{b_i^-\}_{i\geq 1}$ such that

$$\sum_{i=1}^{\infty}(a_i^+ + a_i^- + b_i^+ + b_i^-) < \infty \text{ and } a_i^{\pm} \geq a_{i+1}^{\pm}, b_i^{\pm} \geq b_{i+1}^{\pm} \text{ for all } i \geq 1, \tag{1.1}$$

as well as two real parameters $c^+, c^-$. We let $J_a^{\pm} = \inf\{k \geq 1 : a_k^{\pm} = 0\} - 1$ and $J_b^{\pm} = \inf\{k \geq 1 : b_k^{\pm} = 0\} - 1$. In words, $J_a^{\pm}$ is the largest index $k$ such that $a_k^{\pm} > 0$, with the convention that $J_a^{\pm} = 0$ if all $a_k^{\pm} = 0$ and $J_a^{\pm} = \infty$ if all $a_k^{\pm} > 0$, and analogously for $J_b^{\pm}$. For future reference, we denote the set of parameters satisfying the above conditions by $\mathcal{P}_{\mathsf{all}}$. We denote the subset of $\mathcal{P}_{\mathsf{all}}$ such that $c^- = 0$ and $J_a^- + J_b^- < \infty$ by $\mathcal{P}_{\mathsf{fin}}$, and the subset of $\mathcal{P}_{\mathsf{fin}}$ such that $c^+ = J_a^- = J_b^- = 0$ by $\mathcal{P}_{\mathsf{pos}}$.

Lastly, we define

$$\underline{a} = \begin{cases} 0 & \text{if } a_1^- + b_1^- > 0 \text{ or } c^- \neq 0, \\ \infty & \text{if } a_1^- + b_1^- = c^- = 0 \text{ and } a_1^+ = 0, \\ 1/a_1^+ & \text{if } a_1^- + b_1^- = c^- = 0 \text{ and } a_1^+ > 0, \end{cases} \text{ and } \underline{b} = \begin{cases} 0 & \text{if } a_1^- + b_1^- > 0 \text{ or } c^- \neq 0, \\ -\infty & \text{if } a_1^- + b_1^- = c^- = 0 \text{ and } b_1^+ = 0, \\ -1/b_1^+ & \text{if } a_1^- + b_1^- = c^- = 0 \text{ and } b_1^+ > 0. \end{cases}$$

Observe that $\underline{a} \in [0, \infty]$ and $\underline{b} \in [-\infty, 0]$.

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For $z \in \mathbb{C} \setminus \{0\}$ we define the function

$$\Phi_{a,b,c}(z) = e^{c^+ z + c^-/z} \cdot \prod_{i=1}^{\infty} \frac{(1+b_i^+ z)(1+b_i^-/z)}{(1-a_i^+ z)(1-a_i^-/z)}. \tag{1.2}$$

From (1.1) and [9, Chapter 5, Proposition 3.2], we have that the above defines a meromorphic function on $\mathbb{C}\setminus\{0\}$ whose zeros are at $\{-(b_i^+)^{-1}\}_{i=1}^{J_b^+}$ and $\{-b_i^-\}_{i=1}^{J_b^-}$, while its poles are at $\{(a_i^+)^{-1}\}_{i=1}^{J_a^+}$ and $\{a_i^-\}_{i=1}^{J_a^-}$. We also observe that $\Phi_{a,b,c}(z)$ is analytic in $\mathbb{C} \setminus [\underline{a}, \infty)$, and its inverse is analytic in $\mathbb{C} \setminus (-\infty, \underline{b}]$, where $\underline{a}, \underline{b}$ are as in Definition 1.1.

The following definitions present the Airy wanderer kernel, introduced in [2], starting with the contours that appear in it.

**Definition 1.2.** Fix $a \in \mathbb{R}$. We let $\Gamma_a^+$ denote the union of the contours $\{a + ye^{\pi \mathsf{i}/4}\}_{y \in \mathbb{R}_+}$ and $\{a + ye^{-\pi \mathsf{i}/4}\}_{y \in \mathbb{R}_+}$, and $\Gamma_a^-$ the union of the contours $\{a + ye^{3\pi \mathsf{i}/4}\}_{y \in \mathbb{R}_+}$ and $\{a + ye^{-3\pi \mathsf{i}/4}\}_{y \in \mathbb{R}_+}$. Both contours are oriented in the direction of increasing imaginary part.

**Definition 1.3.** Assume the same notation as in Definition 1.1. For $t_1, t_2, x_1, x_2 \in \mathbb{R}$ we define

$$\begin{aligned} &K_{a,b,c}(t_1, x_1; t_2, x_2) = K^1_{a,b,c}(t_1, x_1; t_2, x_2) + K^2_{a,b,c}(t_1, x_1; t_2, x_2) + K^3_{a,b,c}(t_1, x_1; t_2, x_2), \text{ with} \\ &K^1_{a,b,c}(t_1, x_1; t_2, x_2) = \frac{1}{2\pi \mathsf{i}} \int_{\gamma} dw \cdot e^{(t_2 - t_1)w^2 + (t_1^2 - t_2^2)w + w(x_2 - x_1) + x_1 t_1 - x_2 t_2 - t_1^3/3 + t_2^3/3} \\ &K^2_{a,b,c}(t_1, x_1; t_2, x_2) = -\frac{\mathbf{1}\{t_2 > t_1\}}{\sqrt{4\pi(t_2 - t_1)}} \cdot e^{-\frac{(x_2 - x_1)^2}{4(t_2 - t_1)} - \frac{(t_2 - t_1)(x_2 + x_1)}{2} + \frac{(t_2 - t_1)^3}{12}}; \\ &K^3_{a,b,c}(t_1, x_1; t_2, x_2) = \frac{1}{(2\pi \mathsf{i})^2} \int_{\Gamma_\alpha^+} dz \int_{\Gamma_\beta^-} dw \frac{e^{z^3/3 - x_1 z - w^3/3 + x_2 w}}{z + t_1 - w - t_2} \cdot \frac{\Phi_{a,b,c}(z + t_1)}{\Phi_{a,b,c}(w + t_2)}. \end{aligned} \tag{1.3}$$

In (1.3) $\alpha, \beta \in \mathbb{R}$ are such that $\alpha + t_1 < \underline{a}$ and $\beta + t_2 > \underline{b}$, the function $\Phi_{a,b,c}$ is as in (1.2) and the contours of integration in $K^3_{a,b,c}$ are as in Definition 1.2. If $\Gamma^+_{\alpha+t_1} (= t_1 + \Gamma^+_\alpha)$ and $\Gamma^-_{\beta+t_2} (= t_2 + \Gamma^+_\beta)$ have zero or one intersection points, we take $\gamma = \emptyset$ and then $K^1_{a,b,c} \equiv 0$. Otherwise, $\Gamma^+_{\alpha+t_1}$ and $\Gamma^-_{\beta+t_2}$ have exactly two intersection points, which are complex conjugates, and $\gamma$ is the straight vertical segment that connects them with the orientation of increasing imaginary part. See Figure 1.

The following proposition summarizes some basic properties of the kernel $K_{a,b,c}$ in Definition 1.3. It was proved as [5, Lemma 1.4] under the assumption that $c^+, c^- \geq 0$; however, the proof remains unchanged for all $c^+, c^- \in \mathbb{R}$, which is how we state the result below.

**Proposition 1.4.** *Assume the same notation as in Definition 1.1. For each $t_1, t_2, x_1, x_2 \in \mathbb{R}$ we have that the double integral in the definition of $K^3_{a,b,c}$ in (1.3) is convergent. The value of $K_{a,b,c}(t_1, x_1; t_2, x_2)$ does not depend on the choice of $\alpha$ and $\beta$ as long as $\alpha + t_1 < \underline{a}$ and $\beta + t_2 > \underline{b}$. Moreover, for each fixed $t_1, t_2 \in \mathbb{R}$ we have that $K_{a,b,c}(t_1, \cdot; t_2, \cdot)$ is continuous in $(x_1, x_2) \in \mathbb{R}^2$.*

## 2. Weak convergence and affine shifts

## 3. Schur processes

3.1. **Definitions.** In this section we define the Schur processes that we work with in the present paper. Our exposition follows [5, Section 3.1], which in turn goes back to [3, 7].

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers, called *parts*, such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and all but finitely many elements are zero. We denote the set of all partitions by $\mathbb{Y}$. The *weight* of a partition $\lambda$ is given by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ . There is a single partition of weight 0, which we denote by $\emptyset$. We say that two partitions $\lambda, \mu$ *interlace*, denoted $\lambda \succeq \mu$ or $\mu \preceq \lambda$, if

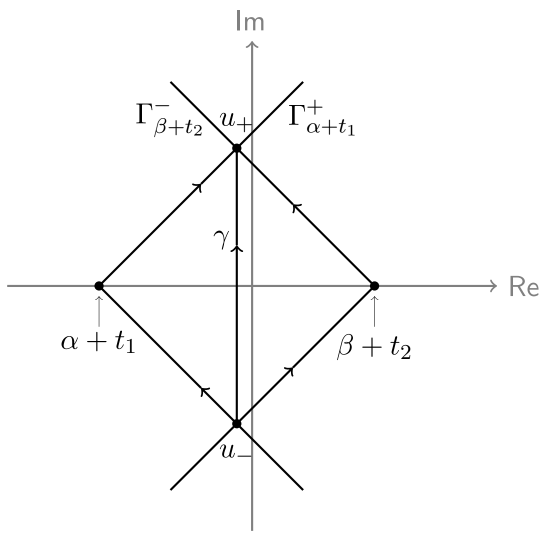


FIGURE 1. The figure depicts the contours $\Gamma^+_{\alpha+t_1}, \Gamma^-_{\beta+t_2}$ when they have two intersection points, denoted by $u_-$ and $u_+$. The contour $\gamma$ is the segment from $u_-$ to $u_+$.

$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$. For two partitions $\lambda, \mu$, we define the (skew) Schur polynomial in a single variable by

$$s_{\lambda/\mu}(x) = \mathbf{1}\{\lambda \succeq \mu\} \cdot x^{|\lambda|-|\mu|}, \tag{3.1}$$

and in several variables by

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{\lambda^1, \dots, \lambda^{n-1} \in \mathbb{Y}} \prod_{i=1}^{n} s_{\lambda^i/\lambda^{i-1}}(x_i), \tag{3.2}$$

where $\lambda^0 = \mu$ and $\lambda^n = \lambda$. If $\mu = \emptyset$ in (3.2), we drop it from the notation and simply write $s_\lambda$ in place of $s_{\lambda/\emptyset}$. With the above notation in place we can define our measures.

**Definition 3.1.** Fix $M, N \in \mathbb{N}$ and suppose that $X = (x_1, \dots, x_M)$, $Y = (y_1, \dots, y_N)$ are such that $x_i, y_j \geq 0$ and $x_i y_j < 1$ for all $i = 1, \dots, M$, and $j = 1, \dots, N$. With this data we define the measure

$$\mathbb{P}_{X,Y}(\lambda^1, \dots, \lambda^M) = \prod_{i=1}^{M} \prod_{j=1}^{N} (1 - x_i y_j) \cdot \prod_{i=1}^{M} s_{\lambda^i/\lambda^{i-1}}(x_i) \cdot s_{\lambda^M}(y_1, \dots, y_M), \tag{3.3}$$

where $\lambda^0 = \emptyset$ and $\lambda^i \in \mathbb{Y}$. We call the measure $\mathbb{P}_{X,Y}$ the *Schur process* with parameters $X, Y$.

*Remark* 3.2. Note that, in view of (3.1), $\mathbb{P}_{X,Y}$ is supported on sequences $(\lambda^1, \dots, \lambda^M)$ such that

$$\emptyset \preceq \lambda^1 \preceq \lambda^2 \preceq \cdots \preceq \lambda^{M-1} \preceq \lambda^M.$$

3.2. **Schur dynamics.** In this section we describe a sampling algorithm for the Schur processes in Definition 3.1. This algorithm was introduced in [1], and it is based on certain Markov dynamics on the Schur processes that are based on sequential update. At a high level, the sampling algorithm allows us to sequentially build $(\lambda(n,1), \dots, \lambda(n,M))$, distributed according to $\mathbb{P}_{X,Y_n}$ for $n = 1, \dots, N$, where $Y_n = (y_1, \dots, y_n)$, by evolving the parts of $\lambda(n,j)$. This evolution of the parts is based on certain truncated geometric random variables, which we introduce next.

**Definition 3.3.** Fix $q \in [0,1)$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z} \cup \{\infty\}$ such that $a \le b$. With this data we define the probability mass function

$$p(x|a,b,q) = \frac{q^x}{\sum_{y=a}^{b} q^y} \text{ for } x \in \mathbb{Z}, a \le x \le b, \text{ and } p(x|a,b,q) = 0 \text{ otherwise.} \tag{3.4}$$

We also define the corresponding cumulative distribution function (cdf) by

$$F(x|a,b,q) = \sum_{y \le x} p(y|a,b,q) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x \ge b \\ \frac{1-q^{\lfloor x \rfloor - a + 1}}{1-q^{b-a+1}} & \text{if } x \in (a,b). \end{cases} \tag{3.5}$$

*Remark* 3.4. In words, the distribution $p(x|a,b,q)$ in Definition 3.3 is just a geometric distribution with parameter $q$, conditioned to be inside $[0, b-a]$, and translated by $a$.

We next introduce the the sampling algorithm.

**Sampling algorithm.** Assume $M, N, X, Y$ are as in Definition 3.1. We proceed to sample $\{\lambda(n,m) : m = 0, \dots, M, n = 0, \dots, N\}$ as follows.

1. Initialization. We set $\lambda(0,m) = \emptyset = \lambda(n,0)$ for $m = 0, \dots, M$ and $n = 0, \dots, N$.
2. Update $n \to n+1$. Assume that $(\lambda(n,1), \dots, \lambda(n,M))$ have already been sampled. We proceed to sequentially sample $\lambda(n+1,m)$ for $m = 1, \dots, M$ as follows.
3. Sampling $\lambda(n+1,m)$. For $i \in \mathbb{N}$, we define $a_i = \max(\lambda_i(n+1,m-1), \lambda_i(n,m))$ and $b_i = \min(\lambda_{i-1}(n+1,m-1), \lambda_{i-1}(n,m))$, with the convention $b_1 = \infty$. Let $X_i(n+1,m)$ be independent random variables with distributions $p(\cdot|a_i, b_i, x_m y_{n+1})$ as in Definition 3.3, and set $\lambda_i(n+1,m) = X_i(n+1,m)$.

*Remark* 3.5. By induction on $m+n$, one directly shows that $\lambda_i(n,m) = 0$ for all $i > \min(m,n)$. In particular, in the third step of the sampling algorithm we only need to generate finitely many variables $X_i(n+1,m)$ for $i = 1, \dots, \min(m,n)$, making the algorithm implementable on a computer.

*Remark* 3.6. One directly observes from the above algorithm that if $0 \le m_1 \le m_2 \le M$, and $0 \le n_1 \le n_2 \le N$, then $\lambda(n_1,m_1) \subseteq \lambda(n_2,m_2)$ in the sense that $\lambda_i(m_1,n_1) \le \lambda_i(m_2,n_2)$ for $i \ge 1$.

The key result we require about the above sampling algorithm is contained in the following proposition, which is a special case of [1, Theorem 10 and Remark 12] (applied to the specializations in [1, Example 9]).

**Proposition 3.7.** *Assume the same notation as in Definition 3.1. Suppose that $\{\lambda(n,m) : m = 0, \dots, M, n = 0, \dots, N\}$ are as in the above sampling algorithm. Then, $(\lambda(N,1), \dots, \lambda(N,M))$ is distributed according to $\mathbb{P}_{X,Y}$.*

3.3. **Quantile functions.** Suppose that $F$ is a cdf on $\mathbb{R}$. We define the quantile function $Q$ associated with $F$ via

$$Q(u) = \inf\{t : u \le F(t)\} \text{ for all } u \in (0,1). \tag{3.6}$$

The following proposition summarizes the properties we require about $Q$. Each statement below is either proved in [8, Sections 11.4, 11a] or follows immediately from statements there.

**Proposition 3.8.** *Suppose that $F$ is a cdf on $\mathbb{R}$ and $Q$ is as in (3.6). Then, the following all hold.*

- *P1. The function $Q$ is increasing on $(0,1)$.*
- *P2. $Q(u) \le t$ if and only if $u \le F(t)$ for each $u \in (0,1)$ and $t \in \mathbb{R}$.*
- *P3. If $U$ is a uniform random variable on $(0,1)$, then $X = Q(U)$ has distribution function $F$.*
- *P4. Let $F_1, F_2$ be two cdfs on $\mathbb{R}$, with associated quantile functions $Q_1, Q_2$. If $F_2(t) \le F_1(t)$ for all $t \in \mathbb{R}$, then $Q_1(u) \le Q_2(u)$ for all $u \in (0,1)$.*

The key result we show in this section is that for certain choices of parameters, we can monotonically couple two distributions as in Definition 3.3.

**Lemma 3.9.** *Suppose that $q_1, q_2 \in [0,1)$, $a_1, a_2 \in \mathbb{Z}$, $b_1, b_2 \in \mathbb{Z} \cup \{\infty\}$ satisfy $a_1 \leq b_1$, $a_2 \leq b_2$, $q_1 \leq q_2$, $a_1 \leq a_2$, and $b_1 \leq b_2$. For $i = 1,2$ we let $F_i(x) = F(x|a_i, b_i, q_i)$ be the cdfs from (3.5), and let $Q_i$ denote the corresponding quantile functions. Let $U$ be a uniform $(0,1)$ random variable and set $X_i = Q_i(U)$. Then, $X_i$ has distribution $F_i$ for $i = 1,2$ and $X_1 \leq X_2$.*

*Proof.* The fact that $X_i$ has distribution $F_i$ for $i = 1,2$ follows from property P3 in Proposition 3.8. From property P4 in Proposition 3.8 and the fact that $F_i$ are constant on each interval $(a, a+1)$ for $a \in \mathbb{Z}$, we see that to show that $X_1 \leq X_2$, it suffices to prove that for each $x \in \mathbb{Z}$

$$F_2(x) \leq F_1(x). \tag{3.7}$$

If $x < a_2$, we see that (3.7) trivially holds as the left side is 0, and if $x \geq b_1$, then it trivially holds as the right side is 1. We may thus assume that $a_2 \leq x \leq b_1$, in which case from (3.5) we see that (3.7) is equivalent to

$$\frac{1 - q_2^{x-a_2+1}}{1 - q_2^{b_2-a_2+1}} \leq \frac{1 - q_1^{x-a_1+1}}{1 - q_1^{b_1-a_1+1}}.$$

The latter is clear if $q_1 = 0$ (as then the right side is equal to 1), and so we may assume $q_1 > 0$. Clearing denominators and multiplying both sides by $q_1^{a_1} q_2^{a_2}$, the above becomes equivalent to

$$0 \leq G(x|a_1, b_1, q_1, a_2, b_2, q_2) \text{ for } a_2 \leq x \leq b_1, \tag{3.8}$$

where

$$G(x|a_1, b_1, q_1, a_2, b_2, q_2) = q_1^{a_1} q_2^{x+1} - q_1^{b_1+1} q_2^{x+1} + q_1^{b_1+1} q_2^{a_2} - q_1^{x+1} q_2^{a_2} - q_1^{a_1} q_2^{b_2+1} + q_1^{x+1} q_2^{b_2+1}.$$

By a direct computation we have

$$\begin{aligned} &G(x|a_1, b_1, q_1, a_2, b_2, q_2) - G(x|a_2, b_1, q_1, a_2, b_1, q_2), \\ &= (q_1^{a_1} - q_1^{a_2})(q_2^{x+1} - q_2^{b_2+1}) + (q_2^{b_1+1} - q_2^{b_2+1})(q_1^{a_2} - q_1^{x+1}) \geq 0, \end{aligned}$$

where in the last inequality we used that $q_1, q_2 \in (0,1)$, $a_1 \leq a_2 \leq x \leq b_1 \leq b_2$. In particular, we conclude that to show (3.8), it suffices to prove that for $a \leq x \leq b$ and $0 < q_1 \leq q_2 < 1$, we have

$$0 \leq G(x|a, b, q_1, a, b, q_2). \tag{3.9}$$

If $b = \infty$, we note that

$$G(x|a, \infty, q_1, a, \infty, q_2) = q_1^a q_2^{x+1} - q_1^{x+1} q_2^a = q_1^a q_2^{x+1}(1 - (q_1/q_2)^{x+1-a}),$$

which clearly implies (3.9), as $q_2 \geq q_1$, and so we may assume that $b < \infty$.

We finally observe that

$$\Delta(x) = G(x|a, b, q_1, a, b, q_2) - G(x-1|a, b, q_1, a, b, q_2) = (1-q_2) q_2^x (q_1^{b+1} - q_1^a) + (1-q_1) q_1^x (q_2^a - q_2^{b+1}).$$

The latter shows that $\Delta(x) \leq 0$ if and only if

$$(1-q_2) q_2^x (q_1^a - q_1^{b+1}) \geq (1-q_1) q_1^x (q_2^a - q_2^{b+1}) \iff (q_2/q_1)^x \geq \frac{(1-q_1)(q_2^a - q_2^{b+1})}{(1-q_2)(q_1^a - q_1^{b+1})}.$$

As $q_2 \geq q_1$, we see that $(q_2/q_1)^x$ is increasing in $x$, and so we either have

$$\Delta(x) \leq 0 \text{ for all } x = a, a+1, \dots, b, \text{ or}$$

$$\Delta(x) > 0 \text{ for } x = a, a+1, \dots, x^* \text{ and } \Delta(x) \leq 0 \text{ for } x = x^* + 1, \dots, b,$$

for some $x^* \in [a, b] \cap \mathbb{Z}$. In both cases, we see that $G(x|a, b, q_1, a, b, q_2)$ is minimal on $[a-1, b]$ when $x = a-1$, or $x = b$. Since by direct computation

$$G(a-1|a, b, q_1, a, b, q_2) = 0 = G(b|a, b, q_1, a, b, q_2),$$

we conclude (3.9) and hence the lemma. $\square$

3.4. **Monotone couplings of Schur processes.** In this section we establish two types of monotone couplings for the Schur processes in Definition 3.1. They are both established by appropriately coupling the Schur dynamics in Section 3.2 using Lemma 3.9. Our first result is as follows.

**Proposition 3.10.** *Fix $M, N \in \mathbb{N}$, and let*

$$\mathcal{P}_{M,N} = \{(\vec{x}, \vec{y}) \in [0,\infty)^M \times [0,\infty)^N : x_i y_j < 1 \text{ for } i = 1, \dots, M, j = 1, \dots, N\}. \tag{3.10}$$

*There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of random sequences*

$$(\lambda^1[X,Y], \dots, \lambda^M[X,Y]) \in \mathbb{Y}^M,$$

*indexed by $(X,Y) \in \mathcal{P}_{M,N}$, so that the following hold. Under $\mathbb{P}$ the distribution of the sequence $(\lambda^1[X,Y], \dots, \lambda^M[X,Y])$ is $\mathbb{P}_{X,Y}$ as in Definition 3.1. In addition, we have for each $(X,Y) \in \mathcal{P}_{M,N}, (\tilde{X}, \tilde{Y}) \in \mathcal{P}_{M,N}$, $\omega \in \Omega$, $k \in \mathbb{N}$, and $j \in \{1, \dots, M\}$ that*

$$\lambda^j_k[X,Y](\omega) \leq \lambda^j_k[\tilde{X}, \tilde{Y}](\omega), \tag{3.11}$$

*provided that $x_i y_j \leq \tilde{x}_i \tilde{y}_j$ for all $i = 1, \dots, M$ and $j = 1, \dots, N$.*

*Remark* 3.11. Observe that from (3.11), we have in particular

$$\sum_{i=1}^{k} \lambda^j_i[X,Y](\omega) \leq \sum_{i=1}^{k} \lambda^j_i[\tilde{X}, \tilde{Y}](\omega). \tag{3.12}$$

In [5, Proposition 3.7] we established a monotone coupling of Schur processes, which ensures (3.12), and is based on the *Robinson-Schensted-Knuth (RSK)* correspondence and *Greene's theorem*. It is worth pointing out that the coupling in Proposition 3.10 is different from the one in [5, Proposition 3.7], for which (3.11) may fail for general $k$, see [5, Remark 3.9].

*Proof.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space that supports $\{U(n,m,k) : m,n,k \in \mathbb{N}\}$, where the latter are i.i.d. uniform $(0,1)$ random variables. We proceed to define $\{\lambda[X,Y](n,m) : m = 0, \dots, M, n = 0, \dots, N\}$ using the sampling algorithm from Section 3.2. We start by setting $\lambda[X,Y](0,m) = \emptyset = \lambda[X,Y](n,0)$ for $m = 0, \dots, M$ and $n = 0, \dots, N$. If $\lambda[X,Y](n-1,m)$ and $\lambda[X,Y](n,m-1)$ have been constructed, we set for $k \geq 1$

$$\lambda_k[X,Y](n,m) = Q^{n,m}_k[X,Y](U(n,m,k)), \tag{3.13}$$

where $Q^{n,m}_k[X,Y]$ is the quantile function of the distribution $F(\cdot|a_k, b_k, q)$ from (3.5) with

$$\begin{aligned} a_k &= \max(\lambda_k[X,Y](n,m-1), \lambda_k[X,Y](n-1,m)), \\ b_k &= \min(\lambda_{k-1}[X,Y](n,m-1), \lambda_{k-1}[X,Y](n-1,m)), \;\; q = x_m y_n, \end{aligned} \tag{3.14}$$

where as before $b_1 = \infty$. Finally, we set $\lambda^i[X,Y] = \lambda[X,Y](N,i)$ for $i = 1, \dots, M$. From Lemma 3.9, we know that $\lambda_k[X,Y](n,m)$ has distribution $F(\cdot|a_k, b_k, q)$, and so from Proposition 3.7, we conclude that under $\mathbb{P}$ the distribution of the sequence $(\lambda^1[X,Y], \dots, \lambda^M[X,Y])$ is $\mathbb{P}_{X,Y}$.

What remains is to verify (3.11), for which it suffices to show that if $0 \leq m \leq M, 0 \leq n \leq N$

$$\lambda_k[X,Y](n,m)(\omega) \leq \lambda_k[\tilde{X}, \tilde{Y}](n,m)(\omega). \tag{3.15}$$

Note that (3.15) holds trivially when $m = 0$ or $n = 0$, as both sides are equal to zero. Assuming that (3.15) holds when $(n,m)$ is replaced by $(n-1,m)$ or $(n,m-1)$, we see that

$$a_k \leq \tilde{a}_k, \;\; b_k \leq \tilde{b}_k, \;\; q = x_m y_n \leq \tilde{x}_m \tilde{y}_n = \tilde{q}, \tag{3.16}$$

where $a_k, b_k$ are as in (3.14) and $\tilde{a}_k, \tilde{b}_k$ are as in (3.14) with $X, Y$ replaced with $\tilde{X}, \tilde{Y}$. From Lemma 3.9 we conclude that for $k \geq 1$

$$Q^{n,m}_k[X,Y](U(n,m,k)) \leq Q^{n,m}_k[\tilde{X}, \tilde{Y}](U(n,m,k)),$$

which together with (3.13) shows that (3.15) holds for $(n,m)$. The fact that (3.15) holds for all $0 \leq m \leq M, 0 \leq n \leq N$ now follows by induction on $m + n$. □

We end this section with our second monotone coupling result for Schur processes.

**Proposition 3.12.** *Fix $M, N \in \mathbb{N}$, $A, B \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{P}_{M,N}$ be as in (3.10). There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two families of random sequences*

$$(\lambda^1[X,Y], \dots, \lambda^M[X,Y]), (\tilde{\lambda}^1[X,Y], \dots, \tilde{\lambda}^M[X,Y]) \in \mathbb{Y}^M,$$

*indexed by $(X,Y) \in \mathcal{P}_{M,N}$, so that the following hold. Under $\mathbb{P}$ the distribution of the sequences $(\lambda^1[X,Y], \dots, \lambda^M[X,Y])$ and $(\tilde{\lambda}^1[X,Y], \dots, \tilde{\lambda}^M[X,Y])$ is $\mathbb{P}_{X,Y}$ as in Definition 3.1. In addition, we have for each $(X,Y) \in \mathcal{P}_{M,N}$, $(\tilde{X}, \tilde{Y}) \in \mathcal{P}_{M,N}$, $\omega \in \Omega$, $k \in \mathbb{N}$, and $j \in \{1, \dots, M\}$ that*

$$\lambda^j_{k+\max(A,B)}[X,Y](\omega) \leq \tilde{\lambda}^j_k[\tilde{X}, \tilde{Y}](\omega), \tag{3.17}$$

*provided that $x_{i+A}y_{j+B} \leq \tilde{x}_i \tilde{y}_j$ for all $i = 1, \dots, M-A$ and $j = 1, \dots, N-B$.*

*Proof.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space that supports $\{U(n,m,k) : m,n,k \in \mathbb{N}\}$, where the latter are i.i.d. uniform $(0,1)$ random variables. Define further the random variables

$$\tilde{U}(n,m,k) = U(n+B, m+A, k+\max(A,B)), \tag{3.18}$$

which we observe are again i.i.d. uniform $(0,1)$. As in the proof of Proposition 3.10, we construct $\{\lambda[X,Y](n,m) : m = 0, \dots, M, n = 0, \dots, N\}$ and $\{\tilde{\lambda}[X,Y](n,m) : m = 0, \dots, M, n = 0, \dots, N\}$ using the sampling algorithm from Section 3.2. We start by setting

$$\lambda[X,Y](0,m) = \lambda[X,Y](n,0) = \tilde{\lambda}[X,Y](0,m) = \tilde{\lambda}[X,Y](n,0) = \emptyset$$

for $m = 0, \dots, M$ and $n = 0, \dots, N$. If $\lambda[X,Y](n-1,m)$, $\lambda[X,Y](n,m-1)$, $\tilde{\lambda}[X,Y](n-1,m)$, $\tilde{\lambda}[X,Y](n,m-1)$ have been constructed, we set for $k \geq 1$

$$\lambda_k[X,Y](n,m) = Q^{n,m}_k[X,Y](U(n,m,k)) \text{ and } \tilde{\lambda}_k[X,Y](n,m) = \tilde{Q}^{n,m}_k[X,Y](\tilde{U}(n,m,k)), \tag{3.19}$$

where $Q^{n,m}_k[X,Y]$ is the quantile function of the distribution $F(\cdot|a_k, b_k, q)$ from (3.5) with $a_k, b_k, q$ as in (3.14), and $\tilde{Q}^{n,m}_k[X,Y]$ is the quantile function of the distribution $F(\cdot|\tilde{a}_k, \tilde{b}_k, q)$ with $\tilde{a}_k, \tilde{b}_k$ as in (3.14) but with $\lambda$ replaced with $\tilde{\lambda}$. Finally, we set $\lambda^i[X,Y] = \lambda[X,Y](N,i)$ and $\tilde{\lambda}^i[X,Y] = \tilde{\lambda}[X,Y](N,i)$ for $i = 1, \dots, M$, and note that from Proposition 3.7, we have that under $\mathbb{P}$ the distribution of the sequences $(\lambda^1[X,Y], \dots, \lambda^M[X,Y])$ and $(\tilde{\lambda}^1[X,Y], \dots, \tilde{\lambda}^M[X,Y])$ is $\mathbb{P}_{X,Y}$.

What remains is to verify (3.17), for which it suffices to show that if $0 \leq m \leq M$, $0 \leq n \leq N$

$$\lambda_{k+\max(A,B)}[X,Y](n,m)(\omega) \leq \tilde{\lambda}_k[\tilde{X}, \tilde{Y}](m,n)(\omega). \tag{3.20}$$

Note that (3.20) trivially holds if $0 \leq n \leq B$ or $0 \leq m \leq A$ as then the left side is equal to zero, cf. Remark 3.5. Hence, we only need to show (3.20) when $A+1 \leq m \leq M$, and $B+1 \leq n \leq N$. Using that $\lambda[\tilde{X}, \tilde{Y}](n,m) \subseteq \lambda[\tilde{X}, \tilde{Y}](n', m')$, when $n \leq n'$ and $m \leq m'$, cf. Remark 3.6, we see that it suffices to show

$$\lambda_{k+\max(A,B)}[X,Y](n+B, m+A)(\omega) \leq \tilde{\lambda}_k[\tilde{X}, \tilde{Y}](n,m)(\omega), \tag{3.21}$$

for all $k \geq 1$, $n = 0, \dots, N-B$, $m = 0, \dots, M-A$. Note that (3.21) holds trivially when $m = 0$ or $n = 0$, as both sides are equal to zero. Assuming that (3.21) holds when $(n,m)$ is replaced by $(n-1,m)$ or $(n,m-1)$, we see that

$$a_{k+\max(A,B)} \leq \tilde{a}_k, \;\; b_{k+\max(A,B)} \leq \tilde{b}_k, \;\; q = x_{m+A}y_{n+B} \leq \tilde{x}_m \tilde{y}_n = \tilde{q}, \tag{3.22}$$

where $a_{k+\max(A,B)}, b_{k+\max(A,B)}$ are as in (3.14) with $m$ replaced with $m+A$ and $n$ replaced with $n+B$, and $\tilde{a}_k, \tilde{b}_k$ are as in (3.14) with $X, Y$ replaced with $\tilde{X}, \tilde{Y}$, and $\lambda$ replaced with $\tilde{\lambda}$. From Lemma 3.9 and (3.18) we conclude that for $k \geq 1$

$$\begin{aligned} &Q^{n+B,m+A}_{k+\max(A,B)}[X,Y](U(n+B, m+A, k+\max(A,B))) \\ &= Q^{n+B,m+A}_{k+\max(A,B)}[X,Y](\tilde{U}(n,m,k)) \leq \tilde{Q}^{n,m}_k[\tilde{X}, \tilde{Y}](\tilde{U}(n,m,k)), \end{aligned}$$

which together with (3.19) shows that (3.21) holds for $(n,m)$. The fact that (3.21) holds for all $0 \leq m \leq M - A, 0 \leq n \leq N - B$ now follows by induction on $m + n$. $\square$

3.5. **Convergence of Schur processes.** In this section we state a weak convergence result about the Schur processes in Definition 3.7, which was shown in [6]. We begin by explaining how we scale our parameters in the following definition.

**Definition 3.13.** We fix parameters $(a,b,c) \in \mathcal{P}_{\mathsf{pos}}$ as in Definition 1.1. We fix $q \in (0,1)$ and set

$$\sigma_q = \frac{q^{1/3}(1+q)^{1/3}}{1-q}, \quad p = \frac{q}{1-q}, \quad \sigma = \sqrt{p(1+p)}, \text{ and } f_q = \frac{q^{1/3}}{2(1+q)^{2/3}}. \tag{3.23}$$

For $N \in \mathbb{N}$ we consider two numbers $A_N, B_N$ and sequences $\{x_i^N\}_{i\geq 1}$ and $\{y_i^N\}_{i \geq 1}$ such that

$$x_i^N = 1 - \frac{1}{N^{1/3} b_i^+ \sigma_q} \text{ for } i = 1, \dots, B_N, \text{ and } y_i^N = 1 - \frac{1}{N^{1/3} a_i^+ \sigma_q} \text{ for } i = 1, \dots, A_N, \tag{3.24}$$

where $B_N \leq \min\left(\lfloor N^{1/12} \rfloor, J_b^+\right)$ is the largest integer such that $x_{B_N}^N \geq q$, and $A_N \leq \min\left(\lfloor N^{1/12} \rfloor, J_a^+\right)$ is the largest integer such that $y_{A_N}^N \geq q$. Here, we use the convention $x_0^N = y_0^N = 1$ so that $A_N = 0$ and $B_N = 0$ are possible. We also have

$$x_i^N = q \text{ for } i > B_N \text{ and } y_i^N = q \text{ for } i > A_N. \tag{3.25}$$

Note that if $M, N \in \mathbb{N}$ we can define the ascending Schur process in Definition 3.7 with parameters $X = (x_1^N, \dots, x_M^N)$ and $Y = (y_1^N, \dots, y_N^N)$ as above, since $x_i^N, y_i^N \in [q,1)$ for all $i \in \mathbb{N}$.

In the sequel we assume the same notation as in Definition 3.13. We also assume that $M_N \geq N + N^{3/4} + 1$ and let $\mathbb{P}_N$ be the measure $\mathbb{P}_{X,Y}$ from Definition 3.7 with $M = M_N$, $x_i = x_i^N$ for $i \in [\![1, M_N]\!]$, $y_i = y_i^N$ for $i \in [\![1, N]\!]$. If $\{\lambda_i^j : j = 1, \dots, M_N, i \geq 1\}$ has distribution $\mathbb{P}_N$, we define the sequence of $\mathbb{N}$-indexed line ensembles $\mathfrak{L}^N = \{L_i^N\}_{i \geq 1}$ on $\mathbb{R}$ via

$$L_i^N(s) = \begin{cases} \lambda_i^{N+s} & \text{if } s \in [-N+1, M_N - N] \cap \mathbb{Z}, \\ \lambda_i^1 & \text{if } s \leq -N, \\ \lambda_i^{M_N} & \text{if } s > M_N - N, \end{cases} \tag{3.26}$$

extended by linear interpolation for non-integer $s$. We finally define $\mathcal{L}^N = \{\mathcal{L}_i^N\}_{i \geq 1}$ via

$$\mathcal{L}_i^N(s) = \sigma^{-1} N^{-1/3} \cdot \left( L_i^N(sN^{2/3}) - psN^{2/3} - 2pN \right). \tag{3.27}$$

The following is a special case of [6, Propostion 6.2], corresponding to setting $c^+ = 0$.

**Proposition 3.14.** *For any $(a,b,c) \in \mathcal{P}_{\mathsf{pos}}$ the sequence $\mathcal{L}^N$ in (3.27) converges weakly to the line ensemble $\left\{ f_q^{-1/2} \mathcal{L}_i^{a,b,c}(f_q t) : i \geq 1, t \in \mathbb{R} \right\}$, where $\mathcal{L}^{a,b,c}$ is as in Proposition [Something].*

## 4. Monotone couplings of Airy wanderer line ensembles

## 5. Asymptotic slopes

## References


1. A. Borodin, *Schur dynamics of the Schur processes*, Adv. Math. **228** (2011), no. 4, 2268–2291.
2. A. Borodin and Péché, *Airy kernel with two sets of parameters in directed percolation and random matrix theory*, J. Stat. Phys. **132** (2008), 275–290.
3. A. Borodin and E.M. Rains, *Eynard–Mehta theorem, Schur process, and their Pfaffian analogs*, J. Stat. Phys. **121** (2005), 291–317.
4. I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Invent. Math. **195** (2014), 441–508.
5. E. Dimitrov, *Airy wanderer line ensembles*, arXiv:2408.08445 (2024).
6. ———, *Tightness for interlacing geometric random walk bridges*, arXiv:2410.23899 (2024).

7. A. Okounkov and N. Reshetikhin, *Correlation function of the Schur process with application to local geometry of random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), 581–603.
8. P. Pfeiffer, *Probability for Applications*, Springer, 1990.
9. E. Stein and R. Shakarchi, *Complex analysis*, Princeton University Press, Princeton, 2003.