A NOTE ON LAST PASSAGE PERCOLATION AND SCHUR PROCESSES

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Abstract. In this note we provide a short proof of the distributional equality between last passage percolation with geometric weights along a general down-right path and Schur processes. We do this in both the full-space and half-space settings, and for general parameters. The main inputs for our arguments are generalizations of the Robinson-Schensted-Knuth correspondence and Greene's theorem due to Krattenthaler, which are based on Fomin's growth diagrams.

Contents

1. Introduction

Last passage percolation (LPP) and the related models known as the *polynuclear growth* (PNG), and the *totally asymmetric simple exclusion process* (TASEP), are some of the most well-studied stochastic systems in the *Kardar-Parisi-Zhang* (KPZ) universality class. Their analysis has revealed many remarkable connections between symmetric functions, increasing subsequences of random permutations, the combinatorics of Young tableaux, and related topics, see e.g. [18, Chapters 4-5].

There are different versions of LPP depending on the nature of the background noise ($w_{ij} : i, j \geq 1$) that defines the model, which could be independent Bernoulli, geometric or exponential random variables, and in this paper we focus on the geometric case. In the seminal paper [12] Johansson proved the convergence of geometric LPP to the Tracy-Widom distribution, and hence demonstrated that it belongs to the KPZ universality class. In [4], building up on the earlier works [2, 3], Baik and Rains considered a symmetrized version of geometric LPP, corresponding to conditioning on $w_{ij} = w_{ji}$ for $i \neq j$, and also established various convergence results to analogues of the Tracy-Widom distribution. Following [1], we will refer to the LPP with independent weights as *full-space*, and the one with symmetrized weights as *half-space*.

An important reason behind the success in the asymptotic analysis of LPP is that it has the structure of a *determinantal point process* in the full-space and a *Pfaffian point process* in the half-space setting. These structural properties come as consequences from the statements that there is a distributional equality between the full-space LPP and the *Schur processes* from [17] and between the half-space LPP and the *Pfaffian Schur processes* from [6]. The connections between LPP and Schur processes have been well-known to experts, and come from two deep results in combinatorics: the *Robinson-Schensted-Knuth* (RSK) *correspondence*, see [14] for the original result and [19, Section 7.11] for a textbook treatment, and *Greene's theorem* [11]. Despite being recognized

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over two decades ago, the connections between RSK and LPP, and their various generalizations, are still actively being studied and used to uncover many remarkable structural properties for a large family of stochastic systems, see [7] and the references therein.

The purpose of this note is to provide a short derivation of the distributional equality between LPP and Schur processes. For the sake of completeness we do this in both the full-space and half-space settings. While the full-space result, see Theorem 2.4, has been proved by other means previously, we believe that the half-space result, see Theorem 2.7, and its proof are new. We have tried to make our results under the most general conditions that we are aware of, considering general down-right paths and parameters. One of our main goals has been to make our arguments easy to follow even for non-experts and essentially self-contained – the only inputs we require are the generalizations of the RSK algorithm and Greene's theorem from [15, Section 4.1].

2. Definitions and main results

2.1. **Full-space geometric LPP.** The full-space model depends on two sequences of real parameters $\{x_i\}_{i\geq 1}$, $\{y_i\}_{i\geq 1}$, such that $x_i, y_j \geq 0$ and $x_i y_j \in [0,1)$ for all $i, j \geq 1$. The background noise is given by an array $W = (w_{i,j} : i, j \geq 1)$ of independent geometric variables $w_{i,j} \sim \mathsf{Geom}(x_i y_j)$, i.e.

$$\mathbb{P}(w_{i,j} = k) = (x_i y_j)^k \cdot (1 - x_i y_j) \text{ for } k \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

We visualize the weight $w_{i,j}$ as being associated with the point (i,j) on the lattice $\mathbb{Z}^2$, see Figure [Something]. An *up-right path* π in $\mathbb{Z}^2$ is a (possibly empty) sequence of vertices $\pi = (v_1, \dots, v_r)$ with $v_i \in \mathbb{Z}^2$ and $v_i - v_{i-1} \in \{(0,1),(1,0)\}$. For an up-right path π in $\mathbb{Z}^2_{\geq 1}$, we define its *weight* by

$$W(\pi) = \sum_{v \in \pi} w_v, \tag{2.2}$$

and for any $(m,n) \in \mathbb{Z}^2_{\geq 1}$ we define the *last passage time* $G_1(m,n)$ by

$$G_1(m,n) = \max_{\pi} W(\pi), \tag{2.3}$$

where the maximum is over all up-right paths that connect $(1,1)$ with (m,n). It turns out that $G_1(m,n)$ can naturally be embedded into a sequence $G(m,n) = (G_k(m,n) : k \geq 1)$ of higher-rank last passage times. There are two equivalent ways of introducing these, which we describe next.

Suppose that $A = (a_{i,j} : i = 1, \dots, m, j = 1, \dots, n)$ is an $n \times m$ matrix with *non-negative* real entries. To make our notation consistent with the one for the array W above we will number the rows of A by $j = 1, \dots, n$ from bottom to top, and the columns of A by $i = 1, \dots, m$ from left to right. For $k = 1, \dots, \min(m,n)$ we define

$$g_k(A) = \max_{\pi_1, \dots, \pi_k} [A(\pi_1) + \cdots + A(\pi_k)], \text{ where } A(\pi) = \sum_{v \in \pi} a_v, \tag{2.4}$$

and the maximum is over k-tuples of disjoint up-right paths $(\pi_1, \dots, \pi_k)$ with π_i connecting the points $(1,i)$ with $(m, n-k+i)$. Note that $g_{\min(m,n)}(A) = \sum_{i=1}^m \sum_{j=1}^n a_{i,j}$ as one can find $\min(m,n)$ disjoint paths as above that contain all the vertices, see Figure [Something]. When $k \geq \min(m,n)+1$ we cannot find k disjoint paths as above, and so the convention is to set

$$g_k(A) = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \text{ for } k \geq \min(m,n). \tag{2.5}$$

A *north-east*(NE)-*chain* χ is a (possibly empty) sequence $\chi = ((a_1,b_1), \dots, (a_r,b_r))$ with $a_i, b_i \in \mathbb{Z}$, and $a_{i-1} \leq a_i$, $b_{i-1} \leq b_i$. Analogously to (2.4) we define for $k \geq 1$

$$h_k(A) = \max_{\chi_1, \dots, \chi_k} [A(\chi_1) + \cdots + A(\chi_k)], \text{ where } A(\chi) = \sum_{v \in \chi} a_v, \tag{2.6}$$

and the maximum is over k-tuples of disjoint NE-chains. As up-right paths are in particular NE-chains we have the following trivial inequalities for any real matrix A and $k \geq 1$

$$g_k(A) \leq h_k(A). \tag{2.7}$$

When the entries of A are non-negative we have the following stronger statement.

Lemma 2.1. *Suppose that A is a real $n \times m$ matrix with non-negative entries. If $g_k(A)$ are as in (2.4) and (2.5), and $h_k(A)$ are as in (2.6), then*

$$g_k(A) = h_k(A) \text{ for all } k \geq 1. \tag{2.8}$$

Remark 2.2. Lemma 2.1 appears to be known to experts; however, as we could not find its proof in the literature, we provide it in Appendix A.

With the above notation, we can now define the higher-rank last passage times by

$$G_k(m,n) := g_k(W[m,n]) \text{ or, equivalently by Lemma 2.1, } G_k(m,n) := h_k(W[m,n]), \tag{2.9}$$

where $W[m,n] = (w_{i,j} : i = 1, \dots, m, j = 1, \dots, n)$.

2.2. **Half-space geometric LPP.** The half-space model depends on a sequence of real parameters $\{x_i\}_{i\geq 1}$, and a parameter c, such that $x_i \geq 0$, $c \geq 0$, $x_ix_j \in [0,1)$ and $cx_i \in [0,1)$. The background noise is given by an array $W = (w_{i,j} : i,j \geq 1)$, where $(w_{i,j} : 1 \leq i \leq j)$ are independent geometric variables with $w_{i,j} \sim \mathsf{Geom}(x_ix_j)$ when $i \neq j$ and $w_{i,i} \sim \mathsf{Geom}(cx_i)$, and $w_{i,j} = w_{j,i}$ for all $i,j \geq 1$. In other words, W has independent geometric entries, conditioned on the matrix being symmetric.

In this setup we introduce the same higher-rank last passage times as in (2.9). One thing to notice here is that for the half-space model

$$G_k(m,n) = G_k(n,m) \text{ for all } k,m,n \geq 1. \tag{2.10}$$

The latter follows from the statement $h_k(A^T) = h_k(A)$ for any real matrix A (here A^T denotes the transpose of A), which is immediate from the definition of NE-chains.

2.3. **Main results.** From the definition of NE-chains and (2.6) one readily observes that if we set

$$\lambda_1(m,n) = G_1(m,n) \text{ and } \lambda_k(m,n) = G_k(m,n) - G_{k-1}(m,n) \text{ for } k \geq 2, \tag{2.11}$$

then $\lambda_k(m,n)$ are non-negative integers, $\lambda_k(m,n) = 0$ for $k > \min(m,n)$ and $\sum_{i=1}^{\infty} \lambda_i(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{i,j}$. In fact, one has that

$$\lambda_k(m,n) \geq \lambda_{k+1}(m,n) \text{ for all } k \geq 1, \tag{2.12}$$

which means that $\lambda(m,n) := (\lambda_k(m,n) : k \geq 1)$ is a *partition* (i.e. a decreasing sequence of non-negative integers that is eventually zero). The latter follows from the following more general statement, which will be established in Section 3.1 as an immediate consequence of the RSK correspondence and Greene's theorem in that section.

Lemma 2.3. *Suppose that A is a real $n \times m$ matrix with non-negative entries. If $h_k(A)$ are as in (2.6) and $h_0(A) = 0$, then*

$$h_k(A) - h_{k-1}(A) \geq h_{k+1}(A) - h_k(A) \text{ for all } k \geq 1. \tag{2.13}$$

Our main results describe the joint distribution of $(\lambda(v_0), \dots, \lambda(v_r))$ for several $v_0, \dots, v_r \in \mathbb{Z}^2_{\geq 0}$, and in order to state them we require a bit more notation.

A *down-right path* γ is a sequence $\gamma = (v_0, \dots, v_r)$ with $v_i - v_{i-1} \in \{(0,-1),(1,0)\}$. Note that any down-right path γ can be identified with a word of length r, where the i-th letter is D if $v_i - v_{i-1} = (0,-1)$ and is R if $v_i - v_{i-1} = (1,0)$, see Figure [Something].

Given two partitions λ, μ, we say that they *interlace* (denoted $\lambda \succeq \mu$ or $\mu \preceq \lambda$) if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$. For two partitions λ, μ the *skew Schur polynomial* in a single variable x, is given by

$$s_{\lambda/\mu}(x) = \mathbf{1}\{\lambda \succeq \mu\} \cdot x^{\sum_{i\geq 1}(\lambda_i - \mu_i)}. \tag{2.14}$$

With the above notation in place we can state our main result about the full-space model.

Theorem 2.4. *Suppose that W is as in Section 2.1, and $(\lambda(m,n) : m,n \geq 1)$ are as in (2.11) for $G_k(m,n)$ as in (2.9). In addition, set $\lambda(m,n)$ to be the zero partitions when $m = 0$ or $n = 0$. Finally, fix any $M, N \geq 0$ and a down-right path $\gamma = (v_0, \dots, v_{M+N})$ from $(0,N)$ to $(M,0)$, whose corresponding word is $s_1 s_2 \cdots s_{M+N}$. Then, for any partitions $\lambda^0, \dots, \lambda^{M+N}$ we have that*

$$\mathbb{P}(\lambda(v_i) = \lambda^i \text{ for } i = 0, \dots, M+N) = Z \cdot \mathbf{1}\left\{\lambda^0 = \lambda^{M+N} = \emptyset\right\} \cdot \prod_{i=1}^{M+N} Q(\lambda^{i-1}, \lambda^i), \text{ where} \tag{2.15}$$

$\emptyset$ is the empty partition (with all parts equal to zero), Z is a normalization constant, and

$$Q(\lambda^{i-1}, \lambda^i) = \begin{cases} s_{\lambda^{i-1}/\lambda^i}(y_i) & \text{if } s_i = D, \\ s_{\lambda^i/\lambda^{i-1}}(x_i) & \text{if } s_i = R. \end{cases} \tag{2.16}$$

Remark 2.5. (ED: Explain how Theorem 2.4 is related to the Schur processes from [17] and what is the formula for Z.)

Remark 2.6. Theorem 2.4 has appeared previously under various different assumptions on γ and the parameters. The closest statement we could find is [13, Theorem 5.2], which with substantial effort can be seen to be equivalent to Theorem 2.4 although it is written with a different notation and proved by different means.

We next turn to the half-space setting, where in view of (2.10), we have that $\lambda(m,n) = \lambda(n,m)$ for all $m,n \geq 1$. Consequently, it suffices to describe the joint distribution of $\lambda(m,n)$ along a down-right path contained in the region $m \geq n$. The exact statement is as follows.

Theorem 2.7. *Suppose that W is as in Section 2.2, and $(\lambda(m,n) : m,n \geq 1)$ are as in (2.11) for $G_k(m,n)$ as in (2.9). In addition, set $\lambda(m,n)$ to be the zero partitions when $m = 0$ or $n = 0$. Finally, fix any $M, N \geq 0$ and a down-right path $\gamma = (v_0, \dots, v_{M+N})$ from (N,N) to $(M+N,0)$, whose corresponding word is $s_1 s_2 \cdots s_{M+N}$. Then, for any partitions $\lambda^0, \dots, \lambda^{M+N}$ we have that*

$$\mathbb{P}(\lambda(v_i) = \lambda^i \text{ for } i = 0, \dots, M+N) = Z \cdot \tau_{\lambda^0}(c) \cdot \mathbf{1}\left\{\lambda^{M+N} = \emptyset\right\} \cdot \prod_{i=1}^{M+N} Q(\lambda^{i-1}, \lambda^i), \tag{2.17}$$

where $\emptyset$ is the empty partition, $Q(\lambda^{i-i}, \lambda^i)$ are as in (2.16) with $y_i = x_i$, $\tau_\lambda(c) = c^{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots}$, and Z is a normalization constant.

Remark 2.8. (ED: Explain how Theorem 2.4 is related to the Pfaffian Schur processes from [6] and what is the formula for Z.)

Theorem 2.7 generalizes [1, Proposition 3.10], which establishes the distributional equality between $(\lambda_1(v))_{v\in\gamma}$ (or equivalently $(G_1(v))_{v\in\gamma}$) and the Pfaffian Schur process. It is worth mentioning that for the special case when the path γ goes straight down, [1, Remark 3.14] sketches an argument of how to prove the distributional equality in Theorem 2.7. Our proof formalizes that outline and generalizes it to arbitrary down-right paths.

While the proofs of Theorems 2.4 and 2.7, which are given in Section 4, rely on a simple inductive argument, at a high level they can be viewed as applying a certain Markovian dynamics that preserves Schur processes. This dynamics is of what is called *RSK-type* and is different from the *push-block* dynamics that was used to prove [1, Proposition 3.10]. We refer the interested reader to [16] for background on RSK-type and push-block dynamics for Schur processes and their generalizations.

3. RSK fillings and Greene's theorem

In this section we recall the generalizations of the RSK correspondence and Greene's theorem as defined by Krattenthaler in [15, Section 4.1], which are based on Fomin's growth diagrams [8–10]. In the general setup, described in Section 3.1, the RSK is a one-to-one correspondence between integer fillings of Ferres shapes and sequences of interlacing partitions, and Greene's theorem provides a global description of this bijection in terms of NE-chains. We also explain how these results specialize to the setting of symmetric fillings of symmetric Ferres shapes in Section 3.2.

3.1. **General setup.** We recall that partitions, interlacing and down-right paths were defined in Section 2.3. We start with the following algorithms, which for given partitions μ, ν define the forward and backward maps of a bijection between pairs (ρ, m) and λ. Here, ρ is a partition such that $\rho \preceq \mu, \rho \preceq \nu$ and m is a non-negative integer, and λ is a partition such that $\lambda \succeq \mu, \lambda \succeq \nu$.

Algorithm $\mathbf{F}^1$ in [15, Section 4.1].

Step 0. Set $CARRY := m$ and $i := 1$.
Step 1. Set $\lambda_i := \max(\mu_i, \nu_i) + CARRY$.
Step 2. If $\lambda_i = 0$, then stop. The output is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1}, 0, \dots)$. If not, then set $CARRY := \min(\mu_i, \nu_i) - \rho_i$ and $i := i + 1$. Go to Step 1.

Algorithm $\mathbf{B}^1$ in [15, Section 4.1].

Step 0. Set $i := \max\{j : \lambda_j > 0\}$ and $CARRY := 0$.
Step 1. Set $\rho_i := \min(\mu_i, \nu_i) - CARRY$.
Step 2. Set $CARRY := \lambda_i - \max(\mu_i, \nu_i)$ and $i := i - 1$. If $i = 0$, then stop. The output of the algorithm is $\rho = (\rho_1, \rho_2, \dots)$ and $m = CARRY$. If not, go to Step 1.

Suppose that $m, n \geq 0$, and we are given a down-right path $\gamma = (v_0, \dots, v_{m+n})$, where $v_0 = (0, n)$ and $v_{m+n} = (m, 0)$. Recall that any such path is encoded by a word $s_1 s_2 \cdots s_{m+n}$ of n letters D and m letters R. We define the set

$$Y(\gamma) = \left\{(i, j) \in \mathbb{Z}^2_{\geq 1} : (i + d, j + d) \in \gamma \text{ for some } d \geq 0\right\},$$

which can be visualized as the Ferres shape, or equivalently Young diagram, in the first quadrant enclosed by the path γ, see Figure [Something]. Note that in this interpretation, the Ferres shape is in the so-called French notation. We remark that in Appendix A we will use the English notation for Young diagrams, which should cause no confusion with the present discussion as it is unrelated.

A *filling* of $Y(\gamma)$ is a map $f : Y(\gamma) \to \mathbb{Z}_{\geq 0}$. Given a filling f of $Y(\gamma)$, we can use Algorithm F^1 to construct partitions $\{\lambda^{(u,v)} : (u, v) \in Y(\gamma)\}$ as follows. We take any sequence of down-right paths $\gamma_0, \gamma_1, \dots, \gamma_r$ connecting $(0, n)$ to $(m, 0)$, such that $Y(\gamma_0) = \emptyset$, $\gamma_r = \gamma$, and

$$Y(\gamma_i) \subset Y(\gamma_{i+1}), \quad |Y(\gamma_{i+1}) \setminus Y(\gamma_i)| = 1.$$

In words, we are taking a sequence of paths whose enclosed Ferres shapes are growing by one box at each step. If (m_i, n_i) is the point $Y(\gamma_i)$ that is not in $Y(\gamma_{i-1})$, we let $\lambda^{(m_i,n_i)}$ be the output of applying Algorithm F^1 to $\rho = \lambda^{(m_i-1,n_i-1)}$, $\nu = \lambda^{(m_i-1,n_i)}$, $\mu = \lambda^{(m_i,n_i-1)}$, and $m = f(m_i, n_i)$. This construction is initiated by setting $\lambda^{(m,n)} = \emptyset$ if $m = 0$ or $n = 0$. It is not hard to see from the local nature of Algorithm F^1 that the obtained $\{\lambda^{(u,v)} : (u, v) \in Y(\gamma)\}$ does not depend on the sequence of down-right paths $\gamma_0, \gamma_1, \dots, \gamma_r$ as long as they satisfy the above conditions. We denote the sequence $(\lambda^{v_0}, \dots, \lambda^{v_{m+n}})$ by $\mathsf{RSK}_\gamma(f)$.

With the above notation we can state the RSK correspondence from [15, Theorem 7].

Proposition 3.1. *The map RSK_γ defines a bijection between fillings of $Y(\gamma)$ and sequences of partitions $(\emptyset = \lambda^0, \dots, \lambda^{m+n} = \emptyset)$, such that $\lambda^i \succeq \lambda^{i-1}$ if $s_i = R$, whereas $\lambda^i \preceq \lambda^{i-1}$ if $s_i = D$.*

We next state the following generalization of Greene's theorem from [15, Theorem 8] .

Proposition 3.2. *For a filling f of $Y(\gamma)$ define $\{\mu_i^{(u,v)} : i \geq 1, (u,v) \in Y(\gamma)\}$ through the equations*

$$\sum_{i=1}^{k} \mu_i^{(u,v)} = h_k(f[u,v]), \text{ where } f[u,v] = (f(i,j) : 1 \leq i \leq u, 1 \leq j \leq v), \tag{3.1}$$

and h_k are as in (2.6). Then, $(\mu_i^{(u,v)} : i \geq 1) = \lambda^{(u,v)}$ for all $(u,v) \in Y(\gamma)$. In particular, (3.1) defines a bijection between fillings of $Y(\gamma)$ and sequences of partitions $(\emptyset = \mu^{v_0}, \ldots, \mu^{v_{m+n}} = \emptyset)$, such that $\mu^{v_i} \succeq \mu^{v_{i-1}}$ if $s_i = R$, whereas $\mu^{v_i} \preceq \mu^{v_{i-1}}$ if $s_i = D$.

Remark 3.3. Proposition 3.2 appears with h_k replaced with g_k as in (2.4) and (2.5) in [7, Section 8.2] for the special case when

$$s_1 \ldots s_{m+n} = \underbrace{R, \ldots, R}_{m \text{ times}} \underbrace{D, \ldots, D}_{n \text{ times}},$$

and for general down-right paths in [5, Theorem 5]. In both papers the result is attributed to [15], and the distinction between h_k and g_k, i.e. between NE-chains and up-right paths, appears to be overlooked. Of course, Proposition 3.2 holds with h_k replaced with g_k in view of Lemma 2.1.

We end this section with the proof of Lemma 2.3.

Proof of Lemma 2.3. Assume first that $a_{i,j} \in \mathbb{Z}_{\geq 0}$. Setting $\mu_k = h_k(A) - h_{k-1}(A)$ for $k \geq 1$, we have from Proposition 3.2 that $(\mu_k : k \geq 1)$ is a partition, which proves (2.13). If $a_{i,j} \in \mathbb{Q}_{\geq 0}$, then we can clear the denominators of the entries and deduce (2.13) from the integer case. Lastly, we can deduce (2.13) when $a_{i,j} \in \mathbb{R}_{\geq 0}$ by taking limits over $\mathbb{Q}_{\geq 0}$. □

3.2. **Symmetric setup.** In this section we restrict the map RSK_γ from Section 3.1 to symmetric fillings and deduce half-space analogues of Propositions 3.1 and 3.2. In the sequel we continue with the same notation as in Section 3.1.

We say that a down-right path γ is *symmetric* if $(i,j) \in \gamma$ implies $(j,i) \in \gamma$. Note that any symmetric down-right path γ contains an odd number of vertices $2r+1$, and that if $s_1 s_2 \cdots s_{2r}$ is the word corresponding to γ we have $s_i = D$ if and only if $s_{2r-i+1} = R$ for $i = 1, \ldots, 2r$. In addition, note that $(i,j) \in Y(\gamma)$ if and only if $(j,i) \in Y(\gamma)$. We say that a filling f of $Y(\gamma)$ is *symmetric* if $f(i,j) = f(j,i)$ for all $(i,j) \in Y(\gamma)$.

Suppose that $m, n \geq 0$, and we are given a symmetric down-right path $\gamma = (v_0, \ldots, v_{2m+2n})$, where $v_{m+n} = (n,n)$ and $v_{2m+2n} = (m+n, 0)$. By a straightforward inductive argument on $|Y(\gamma)|$ one readily checks that applying successively Algorithm F^1 to a symmetric filling of $Y(\gamma)$ results in a sequence $\mathsf{RSK}_\gamma(f) = (\lambda^{v_0}, \ldots, \lambda^{v_{2m+2n}})$ such that $\lambda^{v_i} = \lambda^{v_{2m+2n-i}}$ for $i = 0, \ldots, 2m+2n$. Of course, by Proposition 3.1 this sequence further satisfies

$$\lambda^{v_i} \succeq \lambda^{v_{i-1}} \text{ if } s_i = R, \text{ whereas } \lambda^{v_i} \preceq \lambda^{v_{i-1}} \text{ if } s_i = D.$$

Conversely, if we start from a sequence $\lambda^{v_0}, \ldots, \lambda^{v_{2m+2n}}$ that satisfies the above two conditions, and apply successively Algorithm B^1, we directly verify that we obtain a symmetric filling of $Y(\gamma)$. Combining the above observations, and setting $\mathsf{RSK}_\gamma^{\mathsf{sym}} = (\lambda^{v_{m+n}}, \ldots, \lambda^{v_{2m+2n}})$, we obtain the following corrolaries to Propositions 3.1 and 3.2.

Corollary 3.4. *Assume the same notation as in this section. The map $\mathsf{RSK}_\gamma^{\mathsf{sym}}$ defines a bijection between symmetric fillings of $Y(\gamma)$ and sequences of partitions $(\lambda^0, \ldots, \lambda^{m+n} = \emptyset)$, such that $\lambda^i \succeq \lambda^{i-1}$ if $s_{m+n+i} = R$, whereas $\lambda^i \preceq \lambda^{i-1}$ if $s_{m+n+i} = D$.*

Corollary 3.5. *Assume the same notation as in this section. Then, (3.1) defines a bijection between symmetric fillings of $Y(\gamma)$ and sequences of partitions $(\emptyset = \mu^{v_0}, \ldots, \mu^{v_{2m+2n}} = \emptyset)$, such that (1) $\mu^{v_i} = \mu^{v_{2m+2n-i}}$ for $i = 0, \ldots, 2m+2n$ and (2) $\mu^{v_i} \succeq \mu^{v_{i-1}}$ if $s_i = R$, whereas $\mu^{v_i} \preceq \mu^{v_{i-1}}$ if $s_i = D$.*

4. Proofs of Theorems 2.4 and 2.7

Appendix A. Up-right paths and NE-chains

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