

Study of a predator prey model.

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1 The equation

$$\dot{x} = x(x - a)(1 - x) - bxy \quad (1)$$

$$\dot{y} = xy - cy - d. \quad (2)$$

With $a = 0.4$, $b = 0.3$, and $c \in [0.65, 0.75]$. x represents prey and y predators. The xy products of the system govern the interaction of the two species.

2 Analysis of a simplified model $d = 0$

2.1 One-dimensional approach

Setting d and y equal to zero turns the system into:

$$\dot{x} = x(x - a)(1 - x). \quad (3)$$

For this simplified case the fixed points may be read off easily. $\dot{x} = 0$ yields $x_1 = 0$, $x_2 = a$, $x_3 = 1$. Linear analysis will lead to further insight into the nature of these fixed points reading of $f(x) = x(x - a)(1 - x)$ and computing $f'(x)$ leads to:

$$f'(x) = -3x^2 + 2x + 2xa - a. \quad (4)$$

Substituting x with the fixed points yields:

$$f'(x_1) = -a \quad (5)$$

$$f'(x_2) = -a^2 + a = -0.4^2 + 0.4 > 0 \quad (6)$$

$$f'(x_3) = -3 + 2 + 2a - a = -1 + a = -0.6 < 0 \quad (7)$$

Thus it may be concluded, that x_2 is unstable and $x_3 \wedge x_1$ are stable. Figure 1 shows simulation results produced by a Runge-Kutta type numerical integration routine. The fixed point positions that were read off from the simplified system equation are confirmed by the results to be at $x_1 = 0$, $x_2 = a = 0.4$, $x_3 = 1$. Furthermore the fixed points show the predicted characteristics.

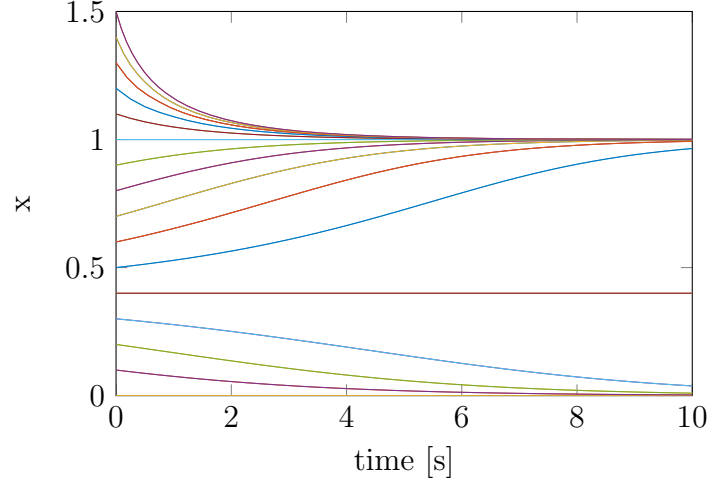


Figure 1: Simulation of the simplified system described by equation 3.

2.2 Two-dimensional approach

Once more the analysis starts with the computation of the fixed point locations. Setting the system equations to zero leads to:

$$0 = x(x - a)(1 - x) - bxy \quad (8)$$

$$0 = xy - cy. \quad (9)$$

Starting from the top equation 9 first x may be factored out:

$$0 = x[(x - a)(1 - x) - by]. \quad (10)$$

Therefore $x_1 = 0$. In order to obtain the remaining zeros the equation:

$$0 = (x - a)(1 - x) - by \quad (11)$$

has to be solved. After factoring out the brackets the pq-Formula is applicable thus the following expression is obtained:

$$x_{2,3} = \frac{1+a}{2} \pm \sqrt{\frac{(1+a)^2}{4} - (a+by)}. \quad (12)$$

Which will be simplified further once more is known about y . To finish the quest for the fixed points x values y is factored out in the second equation:

$$0 = y(x - c). \quad (13)$$

The equation 13 is zero when $x_4 = c$. Which is the missing x component. Looking at y , $y_1 = 0$ is quickly read off from 13. Turning back to equation 9 and solving for y while assuming $x \neq 0$ gives:

$$y_2 = \frac{(x - a)(1 - x)}{b} \quad (14)$$

$\mathbf{x}_1^* = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{x}_2^* = \begin{pmatrix} x_4 \\ y_2 \end{pmatrix} = \begin{pmatrix} c \\ \frac{(c-a)(1-c)}{b} \end{pmatrix}$
$\mathbf{x}_3^* = \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{x}_4^* = \begin{pmatrix} x_3 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

Table 1: Fixed point positions.

At this point two steady state solutions at $(x_1 \ y_1)^T = (0 \ 0)^T$ and $(x_4 \ y_2)^T = \left(c \ \frac{(c-a)(1-c)}{b}\right)^T$ are already known. Using y_1 again equation 12 can be simplified further after plugging in and factoring out one obtains:

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{\frac{1-2a+a^2}{4}} \quad (15)$$

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{\left(\frac{1-a}{2}\right)^2} \quad (16)$$

$$x_{2/3} = \frac{1+a}{2} \pm \frac{1-a}{2} \quad (17)$$

$$\Rightarrow x_2 = 1 \wedge x_3 = a \quad (18)$$

Now two more fixed points are known $(x_2 \ y_1)^T = (1 \ 0)^T$ and $(x_3 \ y_1)^T = (a \ 0)^T$.

Next the obtained points will be classified according to their properties. Starting from the system equations after factoring out the Jacobian is computed:

$$J = \begin{pmatrix} -3x^2 + 2x + 2xa - a - yb & -bx \\ y & x - c \end{pmatrix} \quad (19)$$

Linear analysis proceeds by plugging the fixed points into the Jacobian and compute the trace τ as well as the determinant Δ . For the first fixed point \mathbf{x}_1^* this gives:

$$J(\mathbf{x}_1^*) = \begin{pmatrix} -a & 0 \\ 0 & -c \end{pmatrix}. \quad (20)$$

Therefore the trace and determinant are $\tau_1 = -a - c \wedge \Delta_1 = ac$. Thus this node is a saddle point if $c < 0$, if $c > 0$ it is stable if $-a < c$. The spiral condition $\tau_1 - 4\Delta = (a - c)^2 < 0$, therefore this point should never spiral. The second fixed point \mathbf{x}_2^* has the Jacobian:

$$J(\mathbf{x}_2^*) = \begin{pmatrix} c(1+a-2c) & -bc \\ (c-a)(1-c)/b & 0 \end{pmatrix} \quad (21)$$

With the determinant and trace $\tau_2 = c(1+a-2c) \wedge \Delta_2 = c(c-a)(1-c)$. From these two expressions it is possible to deduce, that if $c > 0$, \mathbf{x}_2^* is a saddle point if additionally, $c > a \wedge c < 1$. If that is not the case then the determinant is positive, now the trace determines stability. $\tau_2 < 0$ is the case if $\frac{1+a}{2} < c$. However if $c < 0$ then the determinant

	$c \in [0, 0.4]$	$c \in [0.4, 0.45]$	$c \in [0.45, 0.7]$	$c \in [0.7, 1]$	$c \in [0.9, 1]$	$c \in [1, 1.5]$
\mathbf{x}_1^*	stable node	stable node	stable node	stable node	stable node	stable node
\mathbf{x}_2^*	saddle point	unstable node	unstable spiral	stable spiral	stable node	saddle point
\mathbf{x}_3^*	saddle point	saddle point	saddle point	saddle point	saddle point	stable node
\mathbf{x}_4^*	unstable node	saddle point	saddle point	saddle point	saddle point	saddle point

Table 2: Fixed point classification for various intervals of c

will always be negative, making \mathbf{x}_2^* a saddle point. If the third fixed point is plugged into the Jacobian-matrix it changes to:

$$J(\mathbf{x}_3^*) = \begin{pmatrix} -1 + a & -b \\ 0 & 1 - c \end{pmatrix} \quad (22)$$

This matrix has the trace $\tau_3 = -c + a$ and the determinant $\Delta_3 = (a - 1)(1 - c) = a - ac - 1 + c$. Therefore the determinant is positive if $c > 1$ assuming that $(1 - a) > 0$, which is known to be true since $a = 0.4$. If the determinant is positive the node is stable if $\tau_3 < 0 \Rightarrow a < c$. The node has spirals if $\tau^2 - 4\Delta < 0$. All that remains is the Jacobian of the fourth fixed point:

$$J(\mathbf{x}_4^*) = \begin{pmatrix} -a^2 + a & -ba \\ 0 & a - c \end{pmatrix} \quad (23)$$

$$\tau_4 = a^2 + 2a - c, \Delta_4 = (-a^2 + a)(a - c) = -a^3 + a^2c + a^2 - ac.$$

2.2.1 Topological analysis

In this section the topology of the first interval $c \in [0, 0.4]$ will be deduced from the eigenvalues and eigenvectors. For \mathbf{x}_1^* reading off from 20 the eigenvalues are found to be $\lambda_{1,1} = -a$, $\lambda_{1,2} = -c$, with the eigenvectors $\mathbf{v}_{1,1} = (1 \ 0)^T$, $\mathbf{v}_{1,2} = (0 \ 1)^T$.

For \mathbf{x}_2^* no general expressions could be found from 21, the ones given are for $c = 0.2$, $a = 0.4$ and $b = 0.3$. $\lambda_{2,1} = 0.305$, $\lambda_{2,2} = -1.04$, $\mathbf{v}_{2,1} = (-0.571 \ 1)^T$, $\mathbf{v}_{2,2} = (0.1967 \ 1)^T$.

For \mathbf{x}_3^* the eigenvalues of the Jacobian 22 may be read off the diagonal: $\lambda_{3,1} = a + 1$, $\lambda_{3,2} = 1 - c$. With the eigenvalues known the eigenvectors may be computed, they turn out to be: $\mathbf{v}_{3,1} = (1 \ 0)$, $\mathbf{v}_{3,2} = (1 \ -\frac{a+c}{b})$.

Finally for \mathbf{x}_4^* is very similar from 23, $\lambda_{4,1} = -a^2 + a$ and $\lambda_{4,2} = a - c$ are the read off eigenvalues, thus the eigenvectors are $\mathbf{v}_{4,1} = (1 \ 0)^T$, $\mathbf{v}_{4,2} = (1 \ -\frac{-a^2+c}{ba})^T$

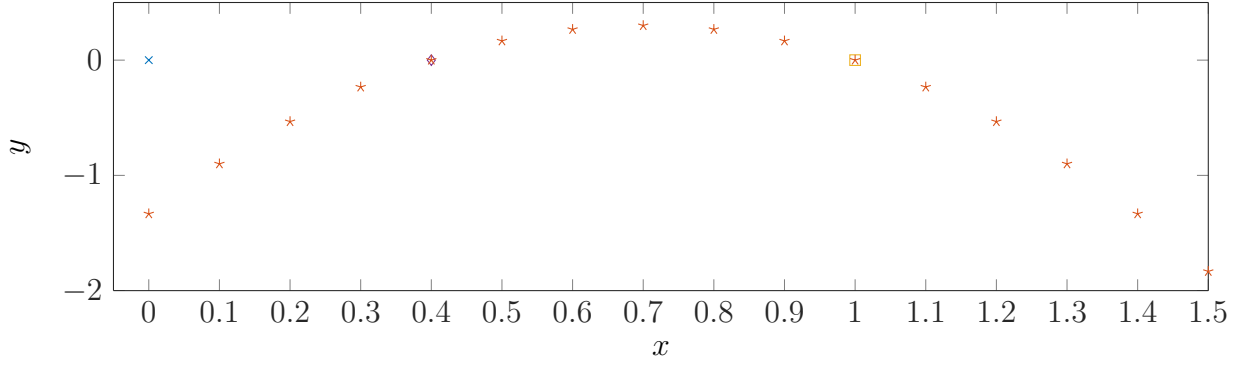


Figure 2: Fixed point position for varying c . The constant position of \mathbf{x}_1^* is marked with an \times at 0,0. The variable position of \mathbf{x}_2^* is marked with a series of stars. Finally \mathbf{x}_3^* and \mathbf{x}_4^* are always at 0.4,0 and 1,0 marked with a square and a diamond.

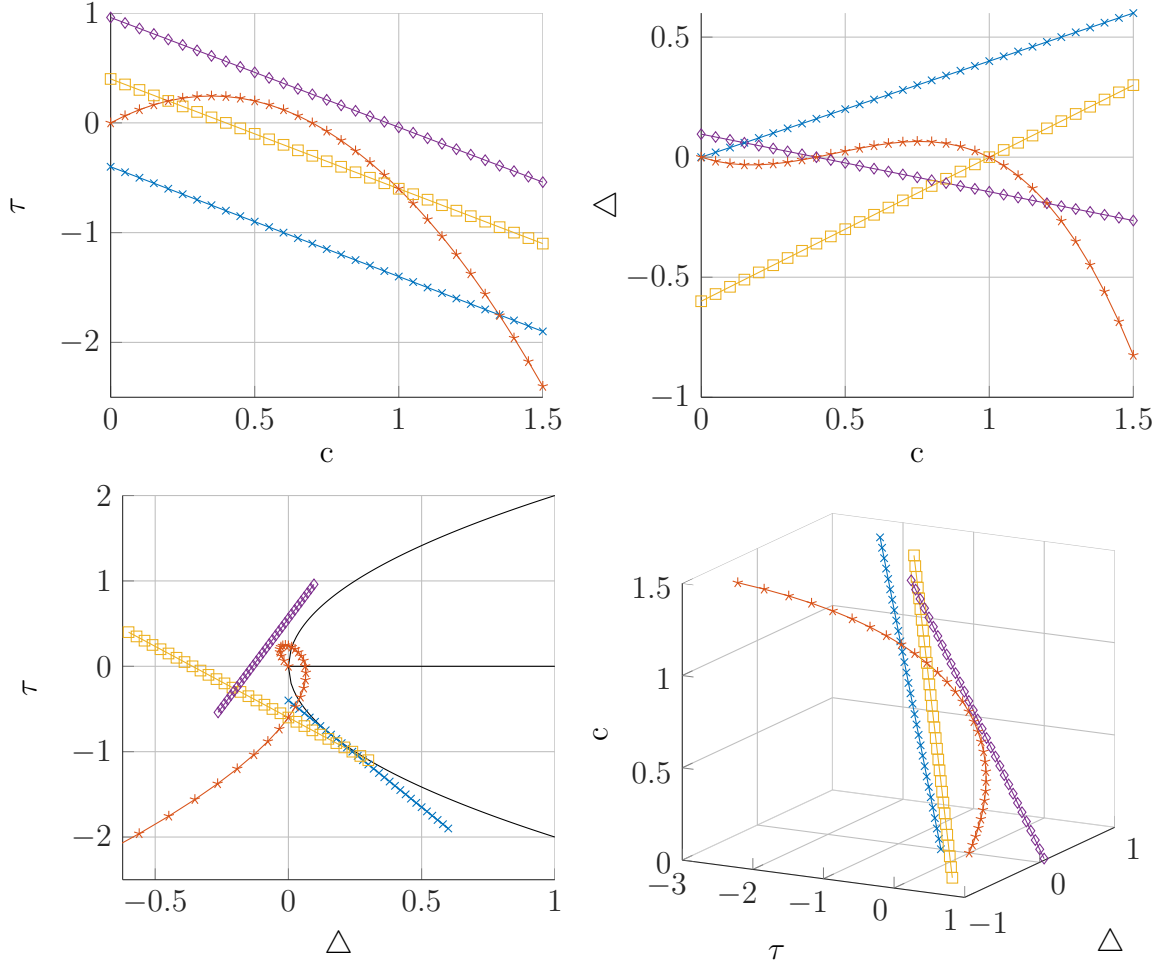


Figure 3: Jacobian trace and determinant for the four fixed points for increasing c . Values associated with \mathbf{x}_1^* are marked with an \times and shown in blue. Plots connected to \mathbf{x}_2^* are marked with stars and graphed in orange. Representations of the trace and determinant of \mathbf{x}_3^* have squares on each line and are colored in yellow. Finally values connected to \mathbf{x}_4^* are marked with a diamonds and a drawn in purple.

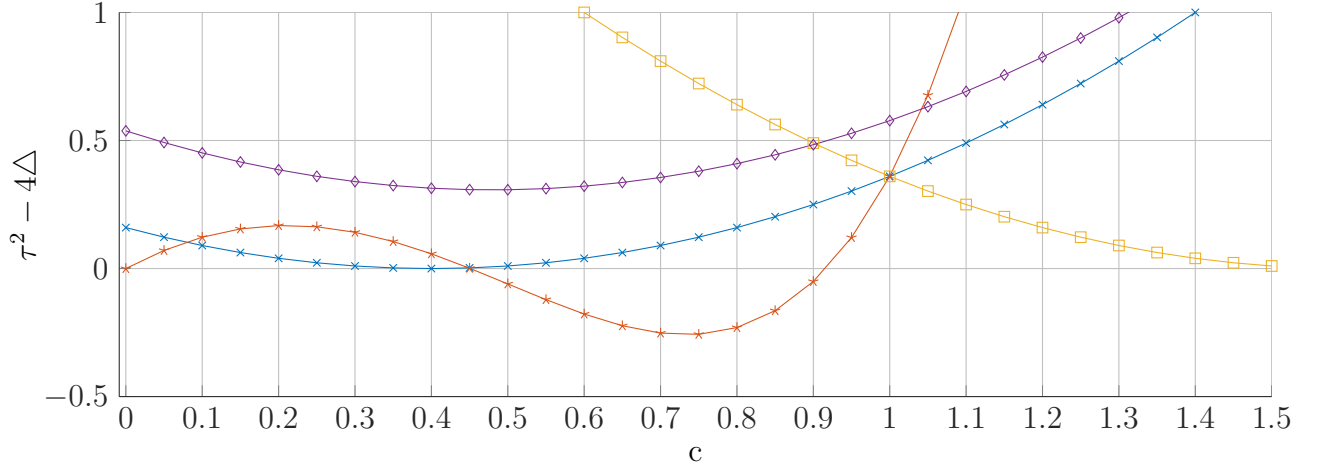


Figure 4: Plot of the spiral condition for all four fixed points. If $\tau^2 - 4\Delta < 0$ a node turns into a spiral.

