Study of a predator prey model.

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1 The equation

$$\dot{x} = x(x-a)(1-x) - bxy \tag{1}$$
$$\dot{y} = xy - cy - d. \tag{2}$$

With $a = 0.4,\ b = 0.3,$ and $c \in [0.650.75]$. x represents prey and y predators. The xy products of the system govern the interaction of the two species.

2 Analysis of a simplified model $d = 0$

2.1 One-dimensional approach

Setting d and y equal to zero turns the system into:

$$\dot{x} = x(x-a)(1-x). \tag{3}$$

For this simplified case the fixed points may be read off easily. $\dot{x} = 0$ yields $x_1 = 0,\ x_2 = a,\ x_3 = 1$. Linear analysis will lead to further insight into the nature of these fixed points reading of $f(x) = x(x-a)(1-x)$ and computing $f'(x)$ leads to:

$$f'(x) = -3x^2 + 2x + 2xa - a. \tag{4}$$

Substituting x with the fixed points yields:

$$f'(x_1) = -a \tag{5}$$
$$f'(x_2) = -a^2 + a = -0.4^2 + 0.4 > 0 \tag{6}$$
$$f'(x_3) = -3 + 2 + 2a - a = -1 + a = -0.6 < 0 \tag{7}$$

Thus it may be concluded, that x_2 is unstable and $x_3 \wedge x_1$ are stable. Figure 1 shows simulation results produced by a Runge-Kutta type numerical integration routine. The fixed point positions that where read off from the simplified system equation are confirmed by the results to be at $x_1 = 0,\ x_2 = a = 0.4,\ x_3 = 1$. Furthermore the fixed points show the predicted characteristics.

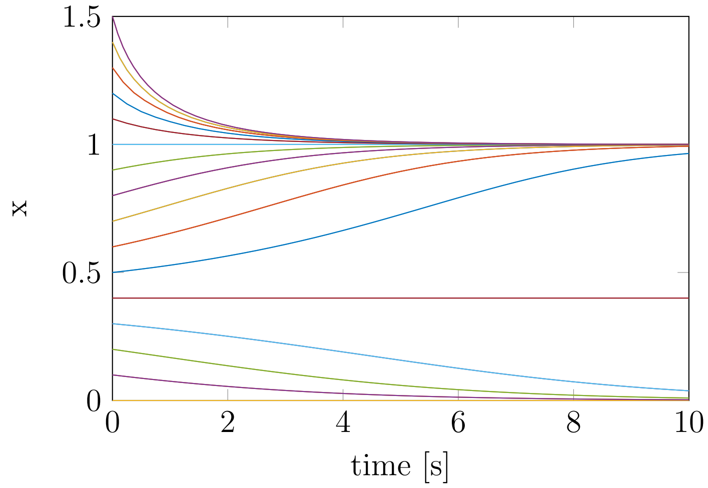

Figure 1: Simulation of the simplified system described by equation 3.

2.2 Two-dimensional approach

Once more the analysis starts with the computation of the fixed point locations. Setting the system equations to zero leads to:

$$0 = x(x-a)(1-x) - bxy \tag{8}$$
$$0 = xy - cy. \tag{9}$$

Starting from the top equation 9 first x may be factored out:

$$0 = x[(x-a)(1-x) - by]. \tag{10}$$

Therefore $x_1 = 0$. In order to obtain the remaining zeros the equation:

$$0 = (x-a)(1-x) - by \tag{11}$$

has to be solved. After factoring out the brackets the pq-Formula is applicable thus the following expression is obtained:

$$x_{2,3} = \frac{1+a}{2} \pm \sqrt{\frac{(1+a)^2}{4} - (a+by)}. \tag{12}$$

Which will be simplified further once more is known about y. To finish the quest for the fixed points x values y is factored out in the second equation:

$$0 = y(x-c). \tag{13}$$

The equation 13 is zero when $x_4 = c$. Which is the missing x component. Looking at y, $y_1 = 0$ is quickly read off from 13. Turning back to equation 9 and solving for y while assuming $x \neq 0$ gives:

$$y_2 = \frac{(x-a)(1-x)}{b} \tag{14}$$

$\mathbf{x}_1^* = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{x}_2^* = \begin{pmatrix} x_4 \\ y_2 \end{pmatrix} = \begin{pmatrix} c \\ \frac{(c-a)(1-c)}{b} \end{pmatrix}$
$\mathbf{x}_3^* = \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{x}_4^* = \begin{pmatrix} x_3 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

Table 1: Fixed point positions.

At this point two steady state solutions at $\begin{pmatrix} x_1 & y_1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} x_4 & y_2 \end{pmatrix}^T = \begin{pmatrix} c & \frac{(c-a)(1-c)}{b} \end{pmatrix}^T$ are already known. Using y_1 again equation 12 can be simplified further after plugging in and factoring out one obtains:

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{\frac{1-2a+a^2}{4}} \tag{15}$$

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{(\frac{1-a}{2})^2} \tag{16}$$

$$x_{2/3} = \frac{1+a}{2} \pm \frac{1-a}{2} \tag{17}$$

$$\Rightarrow x_2 = 1 \wedge x_3 = a \tag{18}$$

Now two more fixed points are known $\begin{pmatrix} x_2 & y_1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} x_3 & y_1 \end{pmatrix}^T = \begin{pmatrix} a & 0 \end{pmatrix}^T$.

Next the obtained points will be classified according to their properties. Starting from the system equations after factoring out the Jacobian is computed:

$$J = \begin{pmatrix} -3x^2 + 2x + 2xa - a - yb & -bx \\ y & x - c \end{pmatrix} \tag{19}$$

Linear analysis proceeds by plugging the fixed points into the Jacobian and compute the trace τ as well as the determinant $\triangle$. For the first fixed point $\mathbf{x}_1^*$ this gives:

$$J(\mathbf{x}_1^*) = \begin{pmatrix} -a & 0 \\ 0 & -c \end{pmatrix}. \tag{20}$$

Therefore the trace and determinant are $\tau_1 = -a - c \wedge \triangle_1 = ac$. Thus this node is a saddle point if $c < 0$, if $c > 0$ it is stable if $-a < c$. The spiral condition $\tau_1 - 4\triangle = (a-c)^2 < 0$, therefore this point should never spiral. The second fixed point $\mathbf{x}_2^*$ has the Jacobian:

$$J(\mathbf{x}_2^*) = \begin{pmatrix} c(1+a-2c) & -bc \\ (c-a)(1-c)/b & 0 \end{pmatrix} \tag{21}$$

With the determinant and trace $\tau_2 = c(1+a-2c) \wedge \triangle_2 = c(c-a)(1-c)$. From these two expressions it is possible to deduce, that if $c > 0$, $\mathbf{x}_2^*$ is a saddle point if additionally, $c > a \wedge c < 1$. If that is not the case then the determinant is positive, now the trace determines stability. $\tau_2 < 0$ is the case if $\frac{1+a}{2} < c$. However it $c < 0$ then the determinant

	$c \in [0, 0.4]$	$c \in [0.4, 0.45]$	$c \in [0.45, 0.7]$	$c \in [0.7, 1]$	$c \in [0.9, 1]$	$c \in [1, 1.5]$
$\mathbf{x}_1^*$	stable node	stable node	stable node	stable node	stable node	stable node
$\mathbf{x}_2^*$	saddle point	unstable node	unstable spiral	stable spiral	stable node	saddle point
$\mathbf{x}_3^*$	saddle point	saddle point	saddle point	saddle point	saddle point	stable node
$\mathbf{x}_4^*$	unstable node	saddle point	saddle point	saddle point	saddle point	saddle point

Table 2: Fixed point classification for various intervals of c

will always be negative, making $\mathbf{x_2^*}$ a saddle point. If the third fixed point is plugged into the Jacobian-matrix it changes to:

$$J(\mathbf{x_3^*}) = \begin{pmatrix} -1+a & -b \\ 0 & 1-c \end{pmatrix} \tag{22}$$

This matrix has the trace $\tau_3 = -c + a$ and the determinant $\triangle_3 = (a-1)(1-c) = a - ac - 1 + c$. Therefore the determinant if positive if $c > 1$ assuming that $(1-a) > 0$, which is known to be true since $a = 0.4$. If the determinant is positive the node is stable if $\tau_3 < 0 \Rightarrow a < c$. The node has spirals is $\tau^2 - 4\triangle < 0$. ¡ All that remains is the Jacobian of the fourth fixed point:

$$J(\mathbf{x_4^*}) = \begin{pmatrix} -a^2+a & -ba \\ 0 & a-c \end{pmatrix} \tag{23}$$

$\tau_4 = a^2 + 2a - c$, $\triangle_4 = (-a^2 + a)(a - c) = -a^3 + a^2c + a^2 - ac$.

2.2.1 Topological analysis

In this section the topology of the first interval $c \in [0, 0.4]$ will be deduced from the eigenvalues and eigenvectors. For $\mathbf{x}_1^*$ reading off from 20 the eigenvalues are found to be:

$$\lambda_{1,1} = -a \tag{24}$$

$$\lambda_{1,2} = -c \tag{25}$$

with the eigenvectors:

$$\mathbf{v}_{1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{26}$$

$$\mathbf{v}_{1,2} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \tag{27}$$

Thus around the stable node $(1, 1)$ all trajectories are drawn towards this stable node.

For $\mathbf{x}_2^*$ no general expressions could be found from 21, the ones given are for $c = 0.2, a = 0.4$ and $b = 0.3$:

$$\lambda_{2,1} = 0.305 \tag{28}$$

$$\lambda_{2,2} = -1.04 \tag{29}$$

$$\mathbf{v}_{2,1} = \begin{pmatrix} -0.571 & 1 \end{pmatrix}^T \tag{30}$$

$$\mathbf{v}_{2,2} = \begin{pmatrix} 0.1967 & 1 \end{pmatrix}^T. \tag{31}$$

We have an unstable eigenvalue, therefore from this node the trajectories will leave along $\pm\mathbf{v}_{2,1}$ towards zero and infinity. While $\pm\mathbf{v}_{2,2}$ guides orbits toward the node.

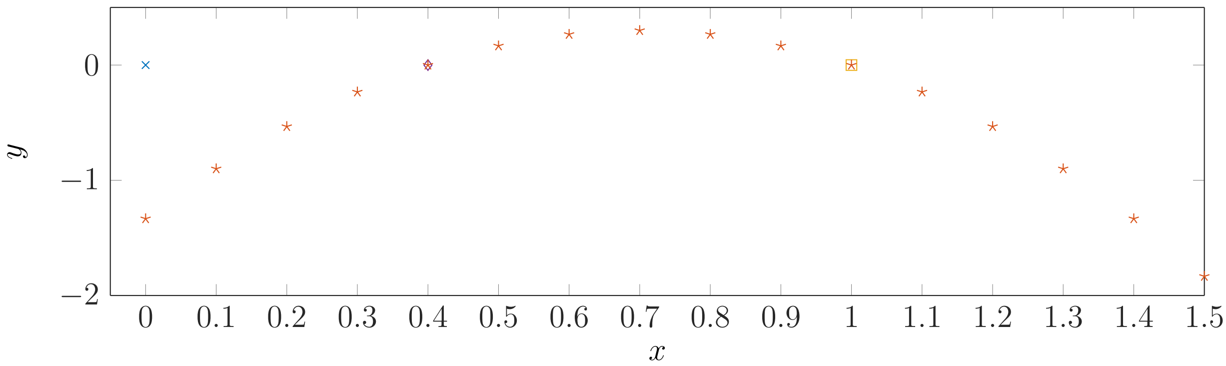

Figure 2: Fixed point position for varying c. The constant position of $\mathbf{x}_1^*$ is marked with an $\mathbf{x}$ at $0, 0$. The variable position of $\mathbf{x}_2^*$ is marked with a series of stars. Finally $\mathbf{x}_3^*$ and $\mathbf{x}_4^*$ are always at $0.4, 0$ and $1, 0$ marked with a square and a diamond.

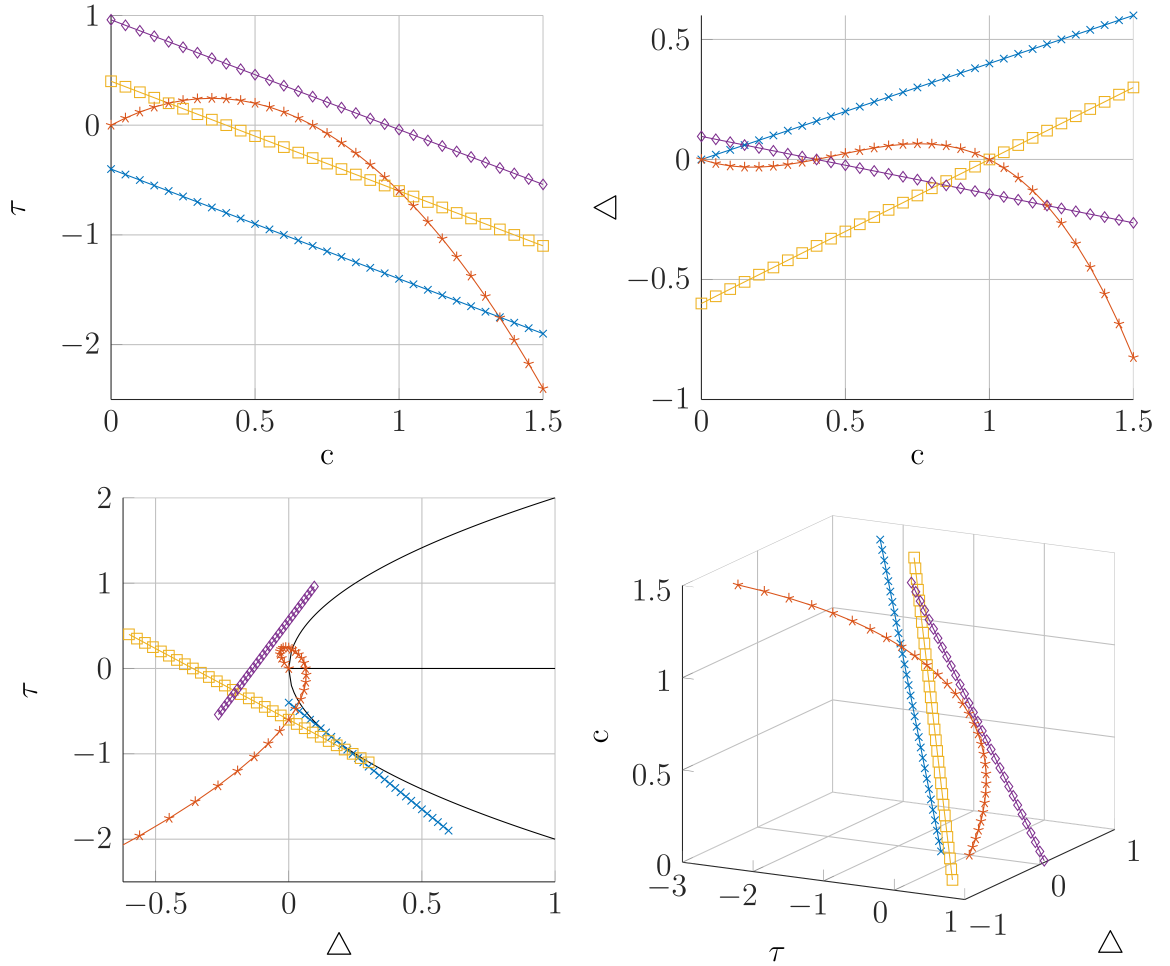

Figure 3: Jacobian trace and determinant for the four fixed points for increasing c. Values associated with $\mathbf{x}_1^*$ are marked with an x and shown in blue. Plots connected to $\mathbf{x}_2^*$ are marked with stars and graphed in orange. Representations of the trace and determinant of $\mathbf{x}_3^*$ have squares on each line and are colored in yellow. Finally values connected to $\mathbf{x}_4^*$ are marked with a diamonds and a drawn in purple.

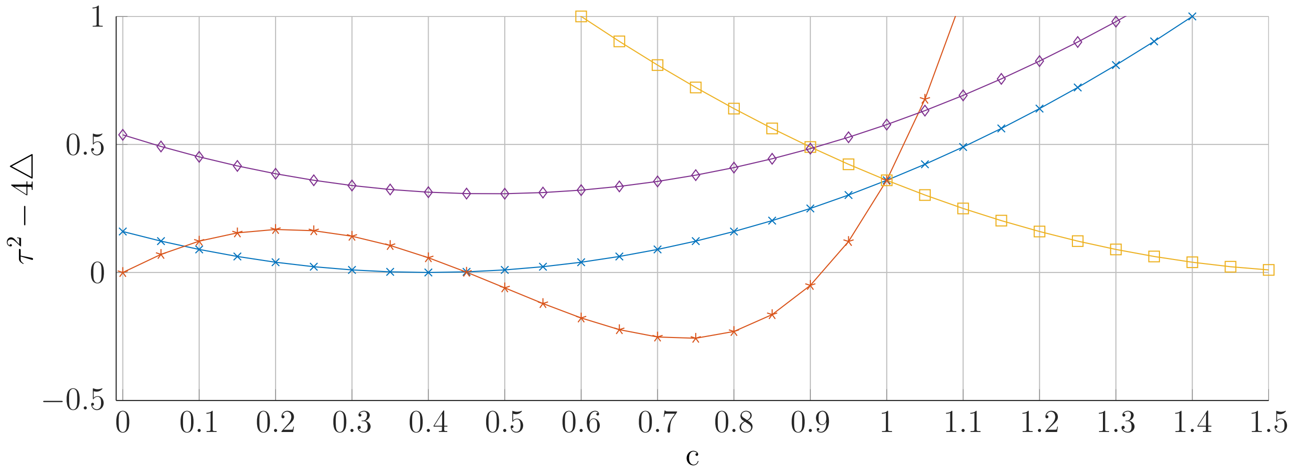

Figure 4: Plot of the spiral condition for all four fixed points. If $\tau^2 - 4\triangle < 0$ a node turns into a spiral.

For $\mathbf{x}_3^*$ the eignevalues of the Jacobian 22 may be read of the diagonal:

$$\lambda_{3,1} = -1 + a \tag{32}$$
$$\lambda_{3,2} = 1 - c. \tag{33}$$

With the eigenvalues known the eigenvectors may be computed, they turn out to be:

$$\mathbf{v}_{3,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{34}$$
$$\mathbf{v}_{3,2} = \begin{pmatrix} \frac{b}{-2+a+c} & 1 \end{pmatrix}^T. \tag{35}$$

When a, b, c are replaced with their numerical values this leads to $\lambda_{3,1} = -0.6,$ and $\lambda_{3,2} = 0.8$. Similarly for the eigenvectors, $\mathbf{v}_{3,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \mathbf{v}_{3,2} = \begin{pmatrix} -2.1428 & 1 \end{pmatrix}^T$ is obtained.

Finally for $\mathbf{x}_4^*$ is very similar from 23;

$$\lambda_{4,1} = -a^2 + a \tag{36}$$
$$\lambda_{4,2} = a - c, \tag{37}$$

are the read off eigenvalues, thus the eigenvectors are:

$$\mathbf{v}_{4,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{38}$$
$$\mathbf{v}_{4,2} = \begin{pmatrix} \frac{-ab}{a^2-c} & 1 \end{pmatrix}^T \tag{39}$$

One last time the numerical values have to be fed into the expressions, this yields $\lambda_1 = 0.24$ and $\lambda_2 = 0.2$. And for the vectors, $\mathbf{v}_{4,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \mathbf{v}_{4,2} = \begin{pmatrix} 3 & 1 \end{pmatrix}^T$. At this points numerical values for all eigenvalue, eigenvector pair are known. If the eigenvectors are added and subtracted from their corresponding fixed points location. The image shown in figure 5 is obtained. It is important to note that most eigenvectors have been scaled by either six or three to address scaling problems. The eigendirection has been deduced from the eigenvalues. $\lambda > 0$ means unstable $\lambda < 0$ means stable or attracting.

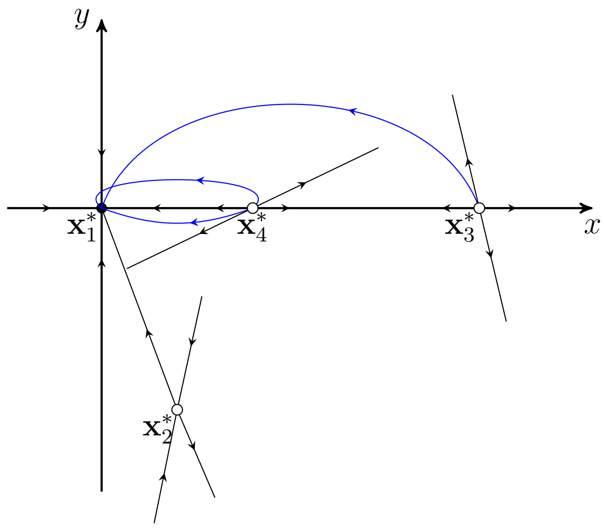

Figure 5: Prediction of the topology of the predator prey model for $c = 0.2$. Eigenvector based predictions are shown in black, others in blue.