Exercise 2 Bridge Oscillations

Moritz Wolter

May 25, 2015

In this report the effects of Wind on a poorly designed bridge will be explored.

1 Equation of Motion

The model for the structure leads to the following equation of motion:

$$0 = -F_I - F_d - F_e + F_{dr} \tag{1}$$

$$0 = -m\ddot{y} - r\dot{y} - ky + \frac{1}{2}\rho V^2 a C(\alpha). \tag{2}$$

Where $C(\alpha)$ is a nonlinear function.

2 Linear Analysis

$C(\alpha)$ is defined as a sum of several odd powers of α:

$$C(\alpha) = A_1\alpha - \underbrace{A_3\alpha^3 + A_5\alpha^5 - A_7\alpha^7.}_{\approx 0 \text{ for small } \alpha.} \tag{3}$$

for small α additionally the approximation $\alpha = \frac{\dot{y}}{V}$ is given. Plugging into 2 yields:

$$0 = -m\ddot{y} + (\frac{1}{2}\rho V a A_1 - r)\dot{y} - ky. \tag{4}$$

Which may be rewritten in terms of two first order equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\frac{1}{2}\rho V a A_1 - r}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Setting the derivatives to zero the fixed point $\mathbf{x}_1^* = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ is obtained. As a linear approximation as already taken place the Jacobi-matrix is identical to the system matrix given above. Thus for the trace τ_1 and determinant $\triangle_1$ at the fixed point the following equations are obtained:

$$\tau_1 = \frac{\frac{1}{2}\rho V a A_1 - r}{m} - \frac{k}{m} \tag{6}$$

$$\triangle_1 = \frac{k}{m}. \tag{7}$$

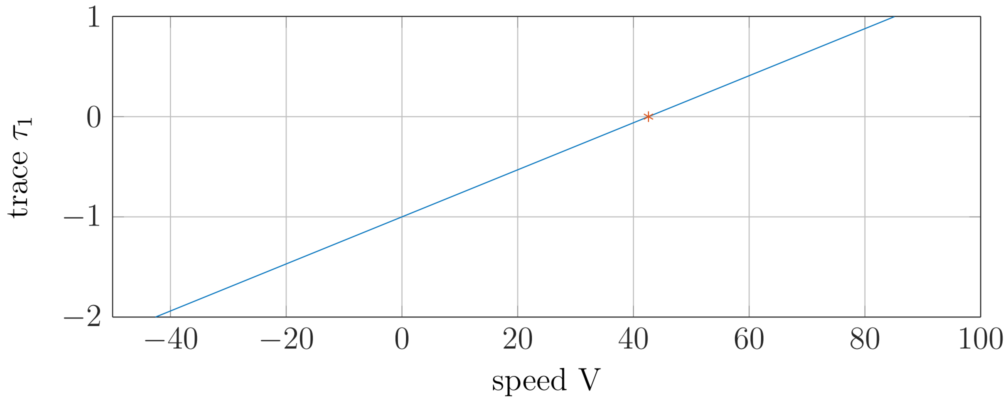

Figure 1: Plot of tau_1 for different speed values. The critical wind speed V_c is marked with an red asterisk.

Assuming $(k \wedge m) > 0$ the nature of the fixed point is determined by the trace. The critical value will occur for $tau_1 = 0$, therefore it my be found from:

$$0 = \frac{1}{2}\frac{\rho V^2 a A_1}{m V_c} - \frac{r}{m}. \tag{8}$$

Solving for V_c leads to:

$$V_c = \frac{2r}{\rho a A_1} = 42.5985. \tag{9}$$

When $m = 1,\ \rho = 1,\ r = 1\ k = 100,\ a = 1$ and $A_1 = 100$. A plot for τ_1 with respect for different values for V is given in figure 1. As the determinant remains positive at all times the fixed point at the center changes from a stable to an unstable spiral at the critical wind speed V_c.

3 Simulation of the non-linear System

In this sections simulation will be attempted without linearization. Including all terms the following system of first order ordinary differential equations is obtained:

$$\dot{y} = z \tag{10}$$

$$\dot{z} = -\frac{ky}{m} - \frac{rz}{m} + \frac{1}{2}\frac{\rho V^2 a}{m}[A_1\frac{z^1}{V^1} - A_3\frac{z^3}{V^3} + A_5\frac{z^5}{V^5} - A_7\frac{z^7}{V^7}]. \tag{11}$$

Equation 11 may be simulated in matlab using an explicit Runge-Kutta type solver (`ode45`). Results are given in figure.

4 Comparison of linear and nonlinear dynamics around V_c

Setting $m = \rho = r = a = 1$ yields the simplified version of equation 4 for the linear case at critical speed V_c:

$$0 = -\ddot{y} + (\frac{1}{2}V_c A_1 - 1)\dot{y} - ky. \tag{12}$$

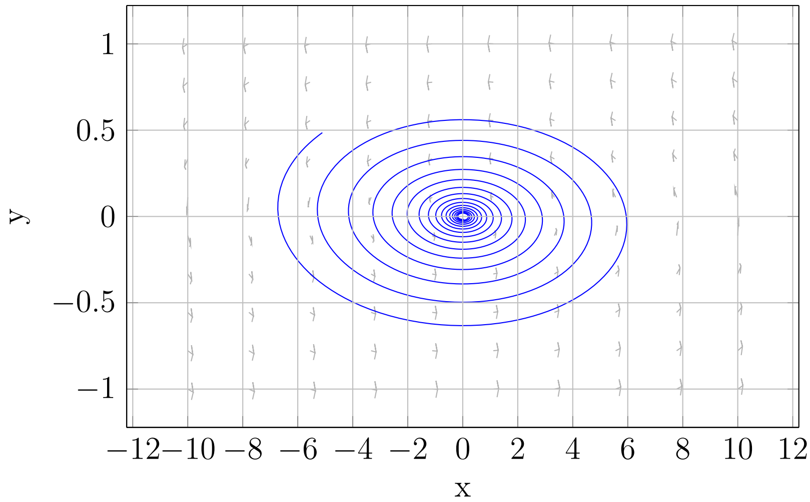

Figure 2: Plot of the two dimensional linearized system with $V = 10 < V_c$.

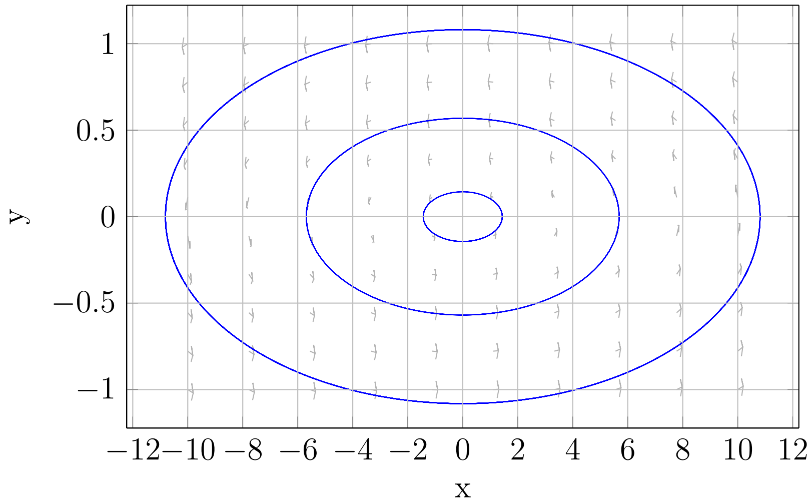

Figure 3: Plot of the two dimensional linearized system with $V = 42.5985 = V_c$.

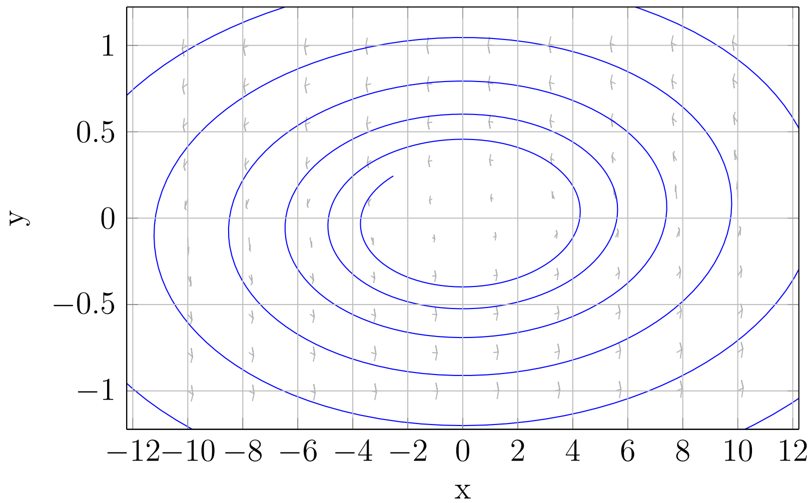

Figure 4: Plot of the two dimensional linearized system with $V = 80 > V_c$.

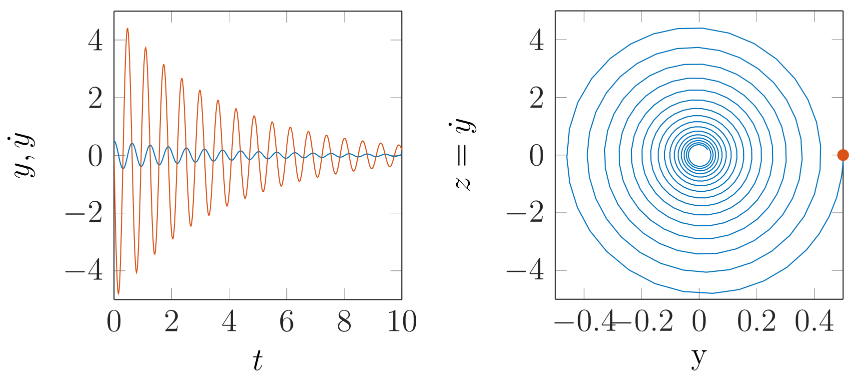

Figure 5: Nonlinear simulation results shown as time plot (left) and in their phase plane representation (right) for $V = 20 < V_c$. In the left plot bridge position is shown in blue. The first derivative is depicted in red. In the right plot the initial condition is depicted as a red dot.

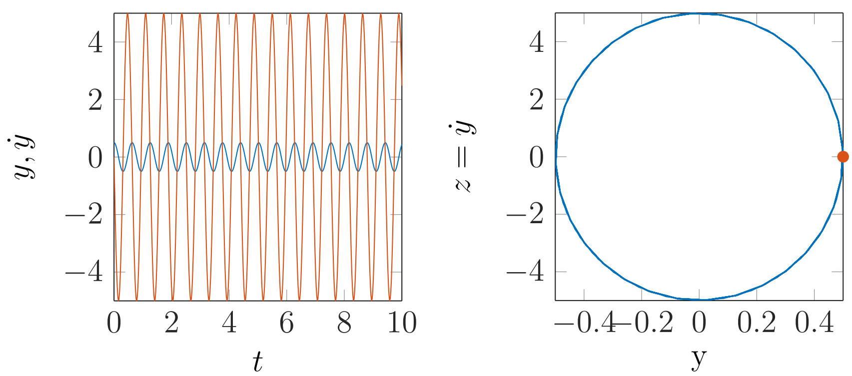

Figure 6: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = V_c$.

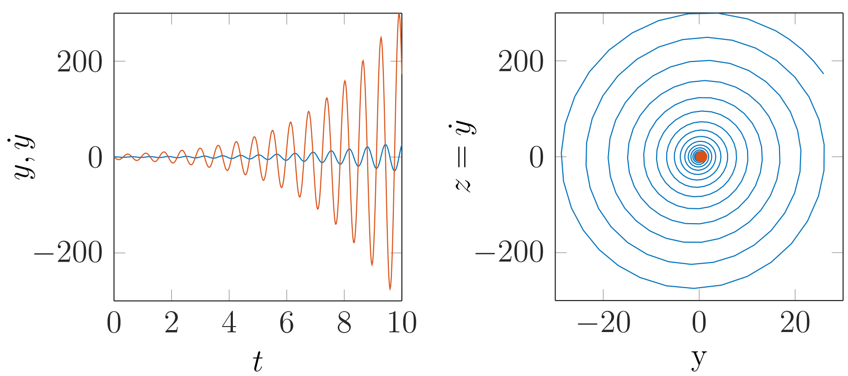

Figure 7: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = 80 > V_c$.

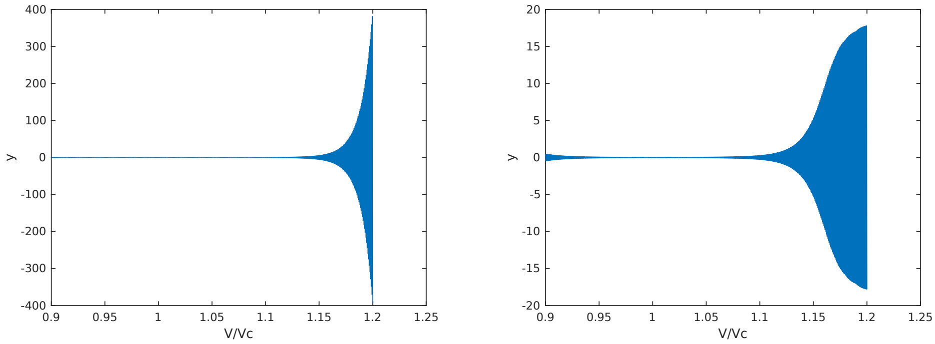

Figure 8: Linear(left) and nonlinear(right) model simulation results for wind speed values close to V_c

Given that V_c simplifies to $V_c = \frac{2}{A_1}$ the linear approximation predicts the term for the first derivative to be zero. However in the nonlinear case one obtains:

$$0 = -\ddot{y} + \underbrace{(\frac{1}{2}V_c A_1 - 1)}_{=0}\dot{y} - ky + \underbrace{\frac{1}{2}V_c^2(-A_3\frac{z^3}{V_c^3} + A_5\frac{z^5}{V_c^5} - A_7\frac{z^7}{V_c^7})}_{\neq 0} \quad (13)$$

The additional nonlinear terms will move the position of the bifurcation points slightly. Furthermore this polynomial will exhibit quite different growth behavior in comparison to it's linear counter part. An expectation, which can indeed be confirmed by simulation. Figure 8 shows the linear and nonlinear model dynamics for wind speed values close to V_c.

5 Limit cycle and bifurcation analysis

So far only a bifurcation with respect to the stability of the fixed point at the origin has been discovered. However a closer look at the system's eigenvalues will reveal another interesting property. Considering the expression derived in 5 the eigenvalues are:

$$\lambda_{1,2} = -\frac{1}{2} + \frac{1}{4}VA_1 \pm \sqrt{(\frac{1}{2} - \frac{1}{4}VA_1)^2 - 1} \quad (14)$$

The interesting part is the one underneath the square root. If this part turns negative the eigenvalues of the Jacobian become complex, which would lead to a Hopf-bifurcation

[1]. Thus the following condition is derived:

$$(\frac{1}{2} - \frac{1}{4}VA_1)^2 - 1 < 0 \tag{15}$$

$$(\frac{1}{2} - \frac{1}{4}VA_1)^2 < 1 \tag{16}$$

$$(\frac{1}{2} - \frac{1}{4}VA_1) < 1 \tag{17}$$

$$\frac{1}{2} < \frac{1}{4}VA_1 \tag{18}$$

$$\frac{2}{A_1} < V \tag{19}$$

$$\tag{20}$$

This new inequality is true when V surpasses V_c. Using equation 9:

$$V_c < V. \tag{21}$$

Thus the first bifurcation is a Hopf-bifurcation. Figures and explore what happens after this first Hopf bifurcation.

[1]book page 251

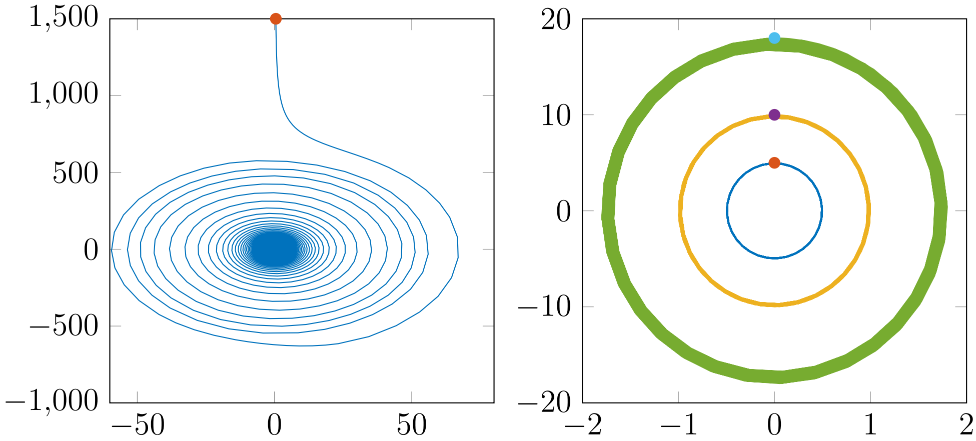

Figure 9: Initial situation with $V = 0.9 < V_c$ (left). The fixed point at the origin is stable.$V = 1 = V_c$ (right). Initial conditions within the green circle oscillate forever.

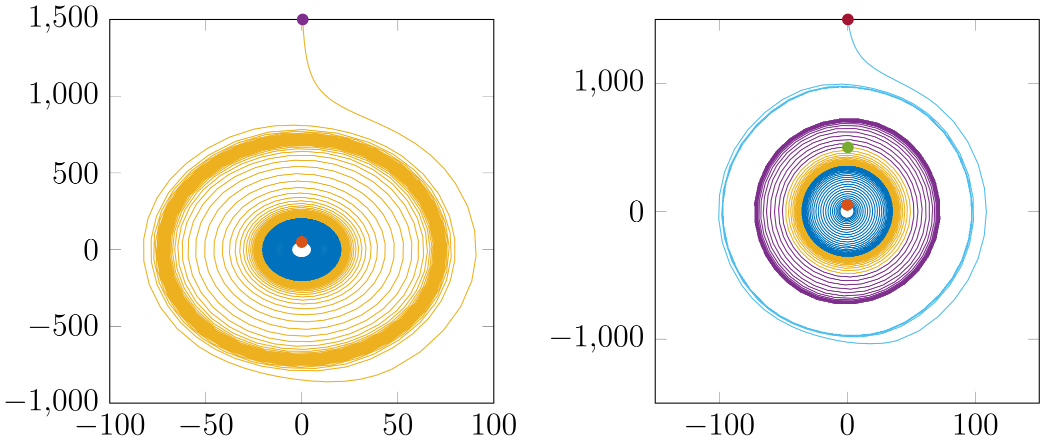

Figure 10: $V = 1.241 > V_c$ the Hopf bifurcation has occurred and a yellow ghost foreshadows another event (left). $V = 1.5 > V_c$ the ghost reveals a cyclic saddle node bifurcation, as the backward solution of the initial condition at $(0, 500)$ shown in purple hits another boundary (right).

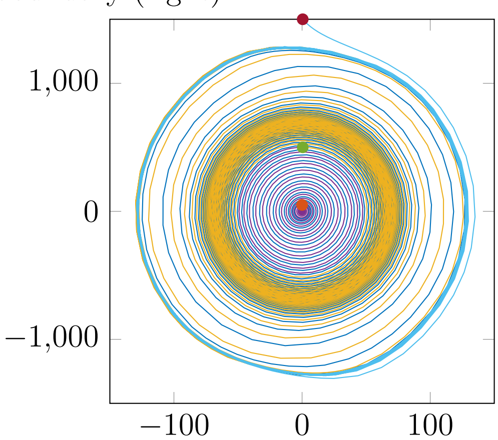

Figure 11: $V = 1.9 > V_c$ the two inner circles have collided and annihilated each other leaving behind another ghost that slows trajectories in its vicinity.

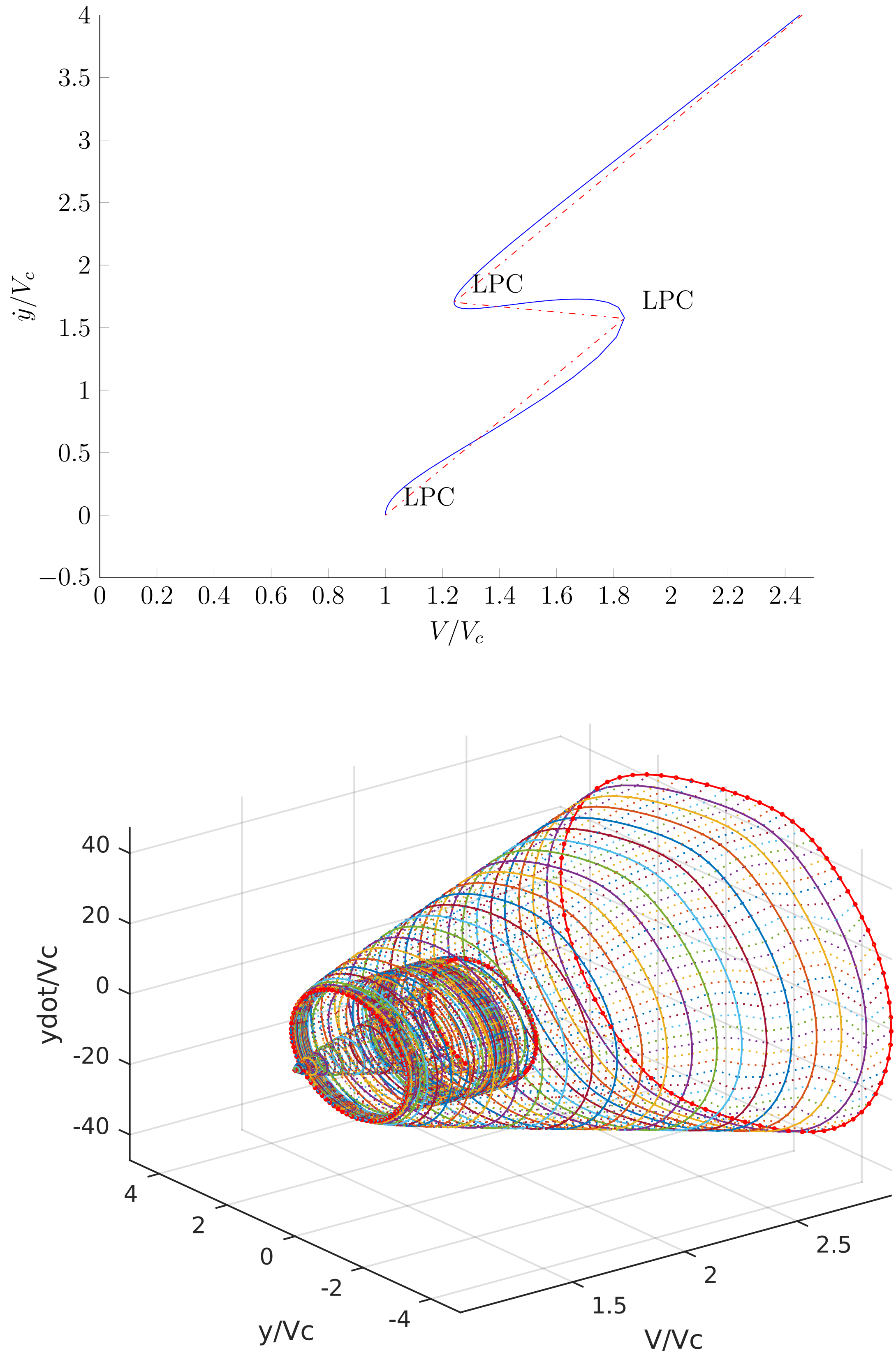

Figure 12: `matcont` output. The top plot an excerpt of the bottom 3D-graphic. The two additional bifurcation points 1.241 and 1.84 can be read off.