Exercise 1 Laser Model

Moritz Wolter

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1 The equations

$$\dot{n} = GnN - kn. \tag{1}$$
$$\dot{N} = -GnN - fN + p. \tag{2}$$

2 First Order Analysis

Assume that the number of excited atoms remains quasi-static.

$$\dot{N} \approx 0. \tag{3}$$
$$\Rightarrow \dot{n} = Gn \cdot \frac{p}{Gn + f} - kn. \tag{4}$$
$$\tag{5}$$

2.1 Linear Stability analysis

In the two dimensional case fixed points are at the intersections of the first derivative with the real axis. Thus the can be found by solving $\dot{n} = 0$. Leading to the problem:

$$0 = n(\frac{Gp}{Gn + f} - k) \tag{6}$$

Which has the zeros 0 and $\frac{p}{k} - \frac{f}{G}$. To learn more about the nature of the fixed points one has to set the derivative of 4 to zero. Using the Quotient rule the expression:

$$\ddot{n} = \frac{Gfp}{(Gn + f)^2} - k \tag{7}$$

is obtained. A fixed point is stable if the second derivative is negative. Likewise it is unstable if the second derivative is positive [1]. Evaluating the second derivative for the fixed point a zero leads to $\ddot{n}(0) = \frac{Gp}{f} - k$ which leads to a values of $p_c = \frac{fk}{G}$. Thus the fixed points becomes unstable if $p > p_c$.

Figure 1 indicates a trans-critical bifurcation.

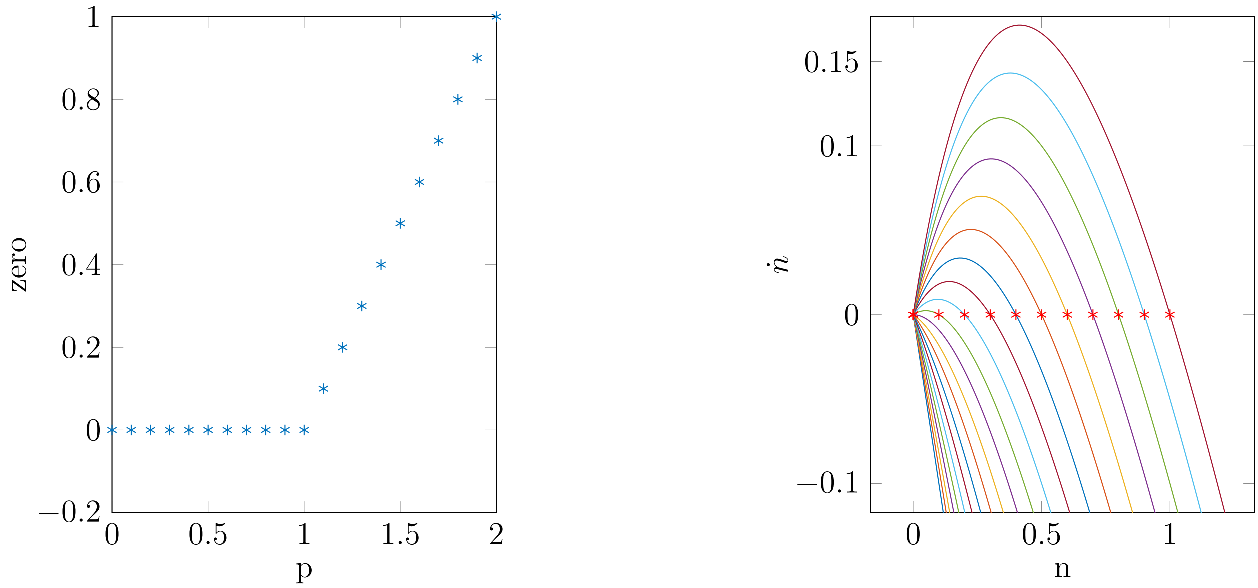

Figure 1: Plot of the position of the right zero and the parameter p (left). Plot of $\dot{n}$ for different p values with the zeros marked with red stars. $G = f = k = 1$ is assumed, leading to $p_c = 1$.

3 Two dimensional analysis

3.1 Nondimensionalization

To reduce the amount of parameters and simplify the analysis the dimensions are non-dimensionalized. In a first step scaling parameters are introduced:

$$\tau = \frac{t}{\gamma} \quad x = \frac{n}{\alpha} \quad y = \frac{N}{\beta}. \tag{8}$$

$$\Rightarrow t = n\tau \quad n = x\alpha \quad N = y\beta \tag{9}$$

Using the chain rule for the derivatives leads to $\dot{n} = \frac{dn}{dt} = \frac{dn}{d\tau} \cdot \frac{d\tau}{dn} = \frac{dn}{d\tau} \cdot \frac{1}{\gamma} = \frac{dx}{d\tau} \cdot \frac{\alpha}{\gamma} = \hat{x}\frac{\alpha}{\gamma}$. Following a similar derivation for $\hat{y}$ and substituting leads to:

$$\hat{x} = Gxy\beta\gamma - kx\gamma \tag{10}$$

$$\hat{y} = -Gx\alpha y\gamma - fy\gamma + p\frac{\gamma}{\beta}. \tag{11}$$

Now the equations may be simplified by setting groups of coefficients to one. Starting with $p\frac{\gamma}{\beta} = 1$ to ensure non-dimensionality. For the other terms it is beneficial to have the coefficients in front of the cross terms xy since these will end up on the diagonal of the Jacobi matrix. Thus $k\gamma = 1$ and

[1]Strogatz, p.25

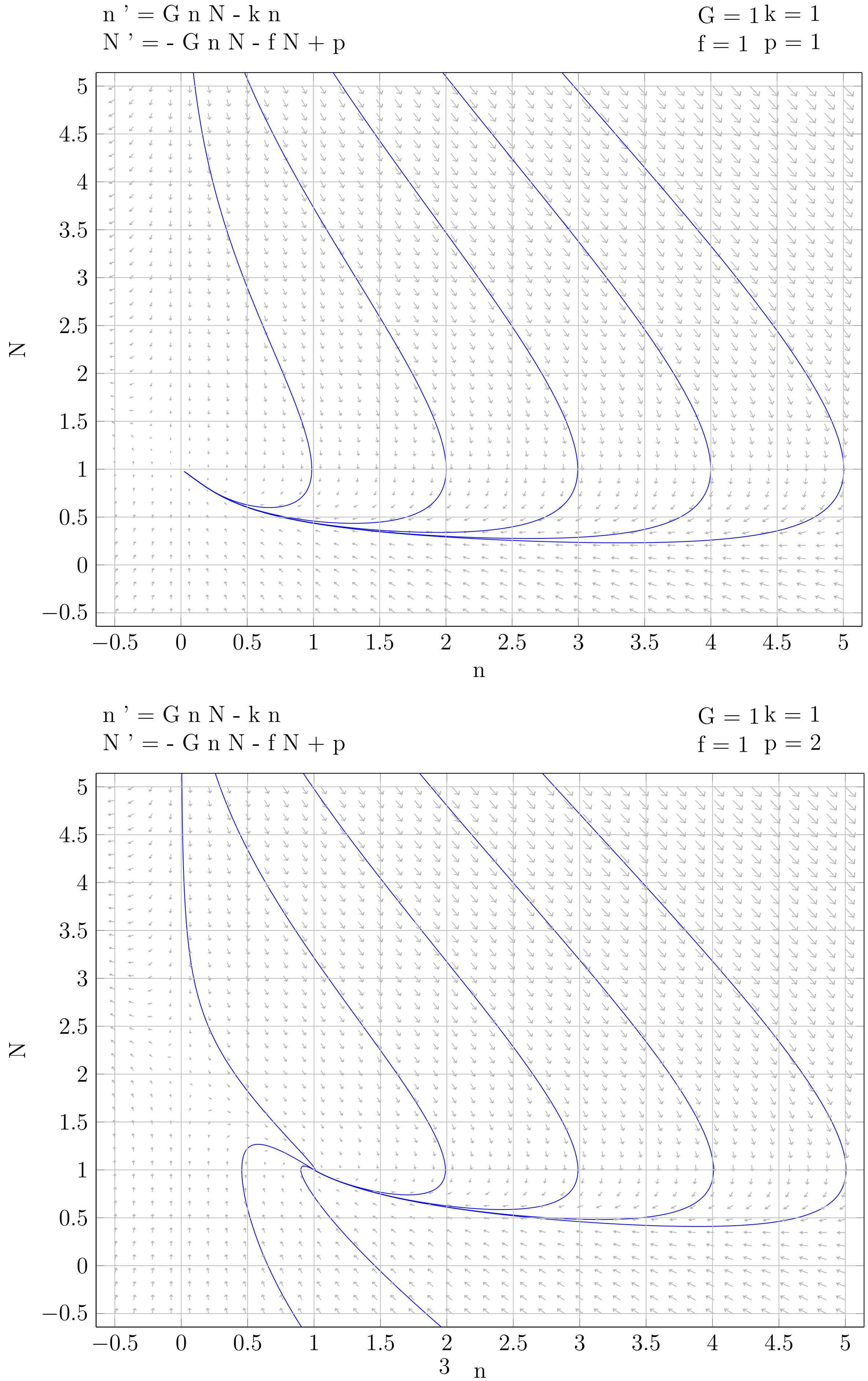

Figure 2: Phase plane of the original system. With two different states predicted from the linear analysis. A plot of the system for uncritical p is given in the top plot. The bottom plot shows the result for a critical $p > p$

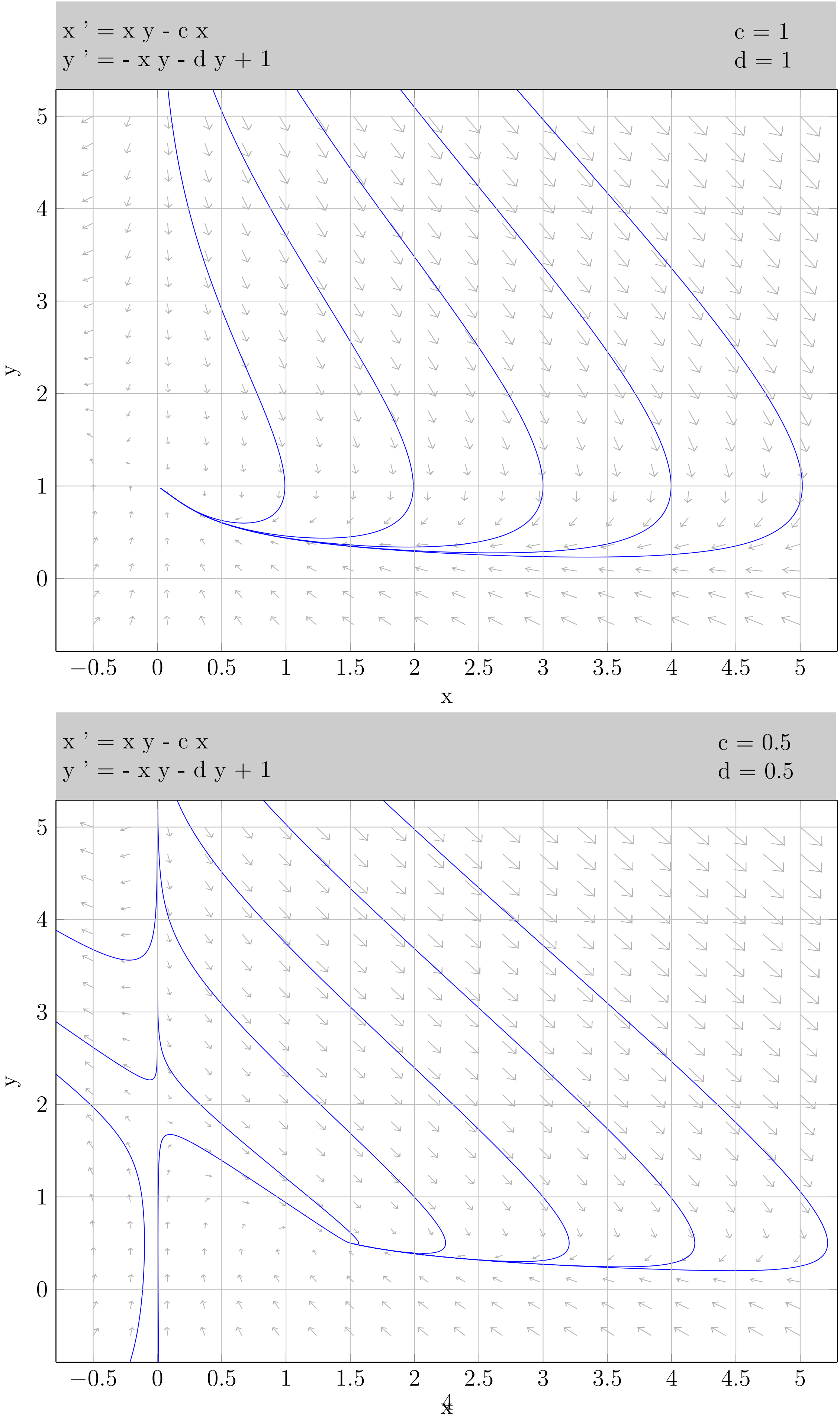

Figure 3: Phase plane of the nonDimensional system.