Exercise 6 Pattern formation

Moritz Wolter

June 5, 2015

1 The Brusselator

$$u_t = D_u u_{xx} + A - (B+1)u + u^2 v, \tag{1}$$
$$v_t = D_v v_{xx} + Bu - u^2. \tag{2}$$

The above equations describe molecule concentrations during a coupled reaction. They are known to exhibit patterns.

2 Stability of the steady state

At the steady state $u_t = v_t = u_{xx} = v_{xx} = 0$. Thus following equations remain:

$$0 = A - (B+1)u + u^2 v, \tag{3}$$
$$0 = u(B - uv). \tag{4}$$
$$\tag{5}$$

From which $u_0 = A$ and $v_0 = B/A$ is deduced. Considering the linearized system evaluated at u_0,v_0 in Matrix form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} D_u u_{xx} \\ D_v v_{xx} \end{pmatrix} + \underbrace{\begin{pmatrix} B-1 & A^2 \\ -B & -A^2 \end{pmatrix}}_{\text{Jacobian evaluated at } (u_0, v_0)} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6}$$

Using $\begin{pmatrix} u & v \end{pmatrix}^T = \begin{pmatrix} u_1 & v_1 \end{pmatrix}^T \cdot \exp(\lambda t + ikx)$ leads to:

$$u_{xx} = (ik)^2 u_1 \exp(st + ikx) = -k^2 u \tag{7}$$
$$u_{yy} = (ik)^2 v_1 \exp(st + ikx) = -k^2 v \tag{8}$$

When substituting this into the linearized equation 6. The expression:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (B-1) - k^2 D_u & A^2 \\ -B & -A^2 - k^2 D_v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{9}$$

The new matrix above has trace τ and determinant $\triangle$:

$$\tau = B - 1 - A^2 - k^2(D_u + D_v) \tag{10}$$
$$\triangle = [(B-1) - k^2 D_u][-A^2 - k^2 D_v] + BA^2 \tag{11}$$
$$= A^2 - BA^2 + A^2 k^2 D_u + k^2 D_v B - k^2 D_v + k^2 D_u D_v + BA^2 \tag{12}$$
$$= A^2 + k^2(A^2 D_u + (1-B)D_v) + k^4 D_u D_v. \tag{13}$$

Now linear algebra has a nice relationship for the eigenvalues:

$$s_{\pm} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\triangle}). \tag{14}$$

Turing instability can occur when the ordinary differential and the partial differential part of the system work against each other.