Study of a predator prey model.

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1 The equation

$$\dot{x} = x(x-a)(1-x) - bxy \quad (1)$$
$$\dot{y} = xy - cy - d. \quad (2)$$

With $a = 0.4,\ b = 0.3,$ and $c \in [0.650.75]$. x represents prey and y predators. The xy products of the system govern the interaction of the two species.

2 Analysis of a simplified model $d = 0$

2.1 One-dimensional approach

Setting d and y equal to zero turns the system into:

$$\dot{x} = x(x-a)(1-x). \quad (3)$$

For this simplified case the fixed points may be read off easily. $\dot{x} = 0$ yields $x_1 = 0,\ x_2 = a,\ x_3 = 1$. Linear analysis will lead to further insight into the nature of these fixed points reading of $f(x) = x(x-a)(1-x)$ and computing $f'(x)$ leads to:

$$f'(x) = -3x^2 + 2x + 2xa - a. \quad (4)$$

Substituting x with the fixed points yields:

$$f'(x_1) = -a \quad (5)$$
$$f'(x_2) = -a^2 + a = -0.4^2 + 0.4 > 0 \quad (6)$$
$$f'(x_3) = -3 + 2 + 2a - a = -1 + a = -0.6 < 0 \quad (7)$$

Thus it may be concluded, that x_2 is unstable and $x_3 \wedge x_1$ are stable. Figure 1 shows simulation results produced by a Runge-Kutta type numerical integration routine. The fixed point positions that where read off from the simplified system equation are confirmed by the results to be at $x_1 = 0,\ x_2 = a = 0.4,\ x_3 = 1$. Furthermore the fixed points show the predicted characteristics.

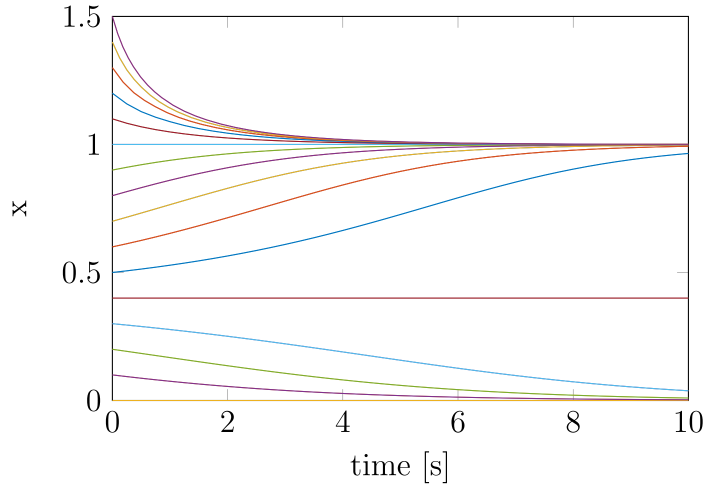

Figure 1: Simulation of the simplified system described by equation 3.

2.2 Two-dimensional approach

Once more the analysis starts with the computation of the fixed point locations. Setting the system equations to zero leads to:

$$0 = x(x-a)(1-x) - bxy \tag{8}$$
$$0 = xy - cy. \tag{9}$$

Starting from the top equation 8 first x may be factored out:

$$0 = x[(x-a)(1-x) - by]. \tag{10}$$

Therefore $x_1 = 0$. In order to obtain the remaining zeros the equation:

$$0 = (x-a)(1-x) - by \tag{11}$$

has to be solved. After factoring out the brackets the pq-Formula is applicable thus the following expression is obtained:

$$x_{2,3} = \frac{1+a}{2} \pm \sqrt{\frac{(1+a)^2}{4} - (a+by)}. \tag{12}$$

Which will be simplified further once more is known about y. To finish the quest for the fixed points x values y is factored out in the second equation:

$$0 = y(x-c). \tag{13}$$

The equation 13 is zero when $x_4 = c$. Which is the missing x component. Looking at y, $y_1 = 0$ is quickly read off from 13. Turning back to equation 8 and solving for y while assuming $x \neq 0$ gives:

$$y_2 = \frac{(c-a)(1-c)}{b} \tag{14}$$

$\mathbf{x}_1^* = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{x}_2^* = \begin{pmatrix} x_4 \\ y_2 \end{pmatrix} = \begin{pmatrix} c \\ \frac{(c-a)(1-c)}{b} \end{pmatrix}$
$\mathbf{x}_3^* = \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{x}_4^* = \begin{pmatrix} x_3 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

Table 1: Fixed point positions.

At this point two steady state solutions at $\begin{pmatrix} x_1 & y_1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} x_4 & y_2 \end{pmatrix}^T = \begin{pmatrix} c & \frac{(c-a)(1-c)}{b} \end{pmatrix}^T$ are already known. Using y_1 again equation 12 can be simplified further after plugging in and factoring out one obtains:

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{\frac{1-2a+a^2}{4}} \tag{15}$$

$$x_{2/3} = \frac{1+a}{2} \pm \sqrt{(\frac{1-a}{2})^2} \tag{16}$$

$$x_{2/3} = \frac{1+a}{2} \pm \frac{1-a}{2} \tag{17}$$

$$\Rightarrow x_2 = 1 \wedge x_3 = a \tag{18}$$

Now two more fixed points are known $\begin{pmatrix} x_2 & y_1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} x_3 & y_1 \end{pmatrix}^T = \begin{pmatrix} a & 0 \end{pmatrix}^T$.

Next the obtained points will be classified according to their properties. Starting from the system equations after factoring out the Jacobian is computed:

$$J = \begin{pmatrix} -3x^2 + 2x + 2xa - a - yb & -bx \\ y & x - c \end{pmatrix} \tag{19}$$

Linear analysis proceeds by plugging the fixed points into the Jacobian and compute the trace τ as well as the determinant $\triangle$. For the first fixed point $\mathbf{x}_1^*$ this gives:

$$J(\mathbf{x}_1^*) = \begin{pmatrix} -a & 0 \\ 0 & -c \end{pmatrix}. \tag{20}$$

Therefore the trace and determinant are $\tau_1 = -a - c \wedge \triangle_1 = ac$. Thus this node is a saddle point if $c < 0$, if $c > 0$ it is stable if $-a < c$. The spiral condition $\tau_1 - 4\triangle = (a-c)^2 < 0$, therefore this point should never spiral. The second fixed point $\mathbf{x}_2^*$ has the Jacobian:

$$J(\mathbf{x}_2^*) = \begin{pmatrix} c(1+a-2c) & -bc \\ (c-a)(1-c)/b & 0 \end{pmatrix} \tag{21}$$

With the determinant and trace $\tau_2 = c(1+a-2c) \wedge \triangle_2 = c(c-a)(1-c)$. From these two expressions it is possible to deduce, that if $c > 0$, $\mathbf{x}_2^*$ is a saddle point if additionally, $c > a \wedge c < 1$. If that is not the case then the determinant is positive, now the trace determines stability. $\tau_2 < 0$ is the case if $\frac{1+a}{2} < c$. However it $c < 0$ then the determinant

	$c \in [0, 0.4]$	$c \in [0.4, 0.45]$	$c \in [0.45, 0.7]$	$c \in [0.7, 0.9]$	$c \in [0.9, 1]$	$c \in [1, 1.5]$
$\mathbf{x}_1^*$	stable node	stable node	stable node	stable node	stable node	stable node
$\mathbf{x}_2^*$	saddle point	unstable node	unstable spiral	stable spiral	stable node	saddle point
$\mathbf{x}_3^*$	saddle point	saddle point	saddle point	saddle point	saddle point	stable node
$\mathbf{x}_4^*$	unstable node	saddle point	saddle point	saddle point	saddle point	saddle point

Table 2: Fixed point classification for various intervals of c

will always be negative, making $\mathbf{x_2^*}$ a saddle point. If the third fixed point is plugged into the Jacobian-matrix it changes to:

$$J(\mathbf{x_3^*}) = \begin{pmatrix} -1+a & -b \\ 0 & 1-c \end{pmatrix} \tag{22}$$

This matrix has the trace $\tau_3 = -c + a$ and the determinant $\triangle_3 = (a-1)(1-c) = a - ac - 1 + c$. Therefore the determinant if positive if $c > 1$ assuming that $(1-a) > 0$, which is known to be true since $a = 0.4$. If the determinant is positive the node is stable if $\tau_3 < 0 \Rightarrow a < c$. The node has spirals is $\tau^2 - 4\triangle < 0$. ¡ All that remains is the Jacobian of the fourth fixed point:

$$J(\mathbf{x_4^*}) = \begin{pmatrix} -a^2+a & -ba \\ 0 & a-c \end{pmatrix} \tag{23}$$

$\tau_4 = a^2 + 2a - c$, $\triangle_4 = (-a^2 + a)(a - c) = -a^3 + a^2c + a^2 - ac$.

2.2.1 Topological analysis

In this section the topology of the first interval $c \in [0, 0.4]$ will be deduced from the eigenvalues and eigenvectors. For $\mathbf{x}_1^*$ reading off from 20 the eigenvalues are found to be:

$$\lambda_{1,1} = -a \tag{24}$$
$$\lambda_{1,2} = -c \tag{25}$$

with the eigenvectors:

$$\mathbf{v}_{1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{26}$$
$$\mathbf{v}_{1,2} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \tag{27}$$

Thus around the stable node $(1,1)$ all trajectories are drawn towards this stable node.

For $\mathbf{x}_2^*$ no general expressions could be found from 21, the ones given are for $c = 0.2, a = 0.4$ and $b = 0.3$:

$$\lambda_{2,1} = 0.305 \tag{28}$$
$$\lambda_{2,2} = -1.04 \tag{29}$$
$$\mathbf{v}_{2,1} = \begin{pmatrix} -0.571 & 1 \end{pmatrix}^T \tag{30}$$
$$\mathbf{v}_{2,2} = \begin{pmatrix} 0.1967 & 1 \end{pmatrix}^T. \tag{31}$$

We have an unstable eigenvalue, therefore from this node the trajectories will leave along $\pm\mathbf{v}_{2,1}$ towards zero and infinity. While $\pm\mathbf{v}_{2,2}$ guides orbits toward the node.

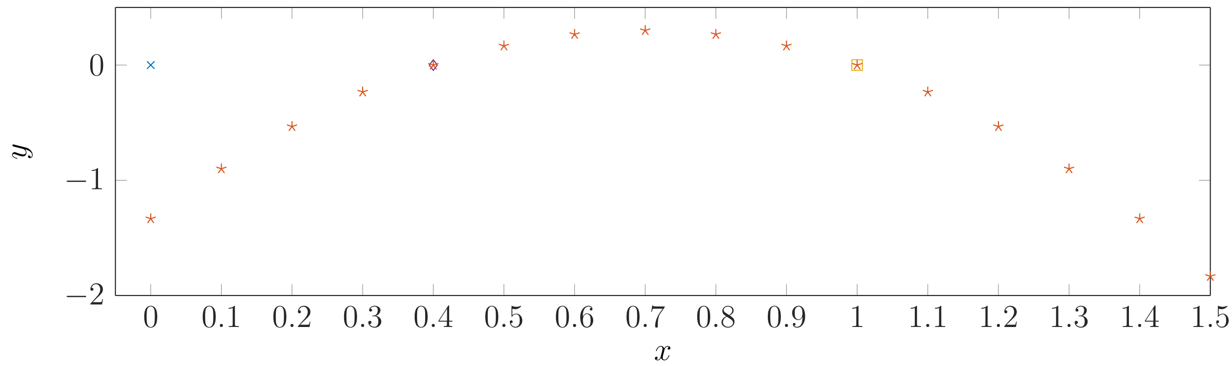

Figure 2: Fixed point position for varying c. The constant position of $\mathbf{x}_1^*$ is marked with an **x** at $0, 0$. The variable position of $\mathbf{x}_2^*$ is marked with a series of stars. Finally $\mathbf{x}_3^*$ and $\mathbf{x}_4^*$ are always at $0.4, 0$ and $1, 0$ marked with a square and a diamond.

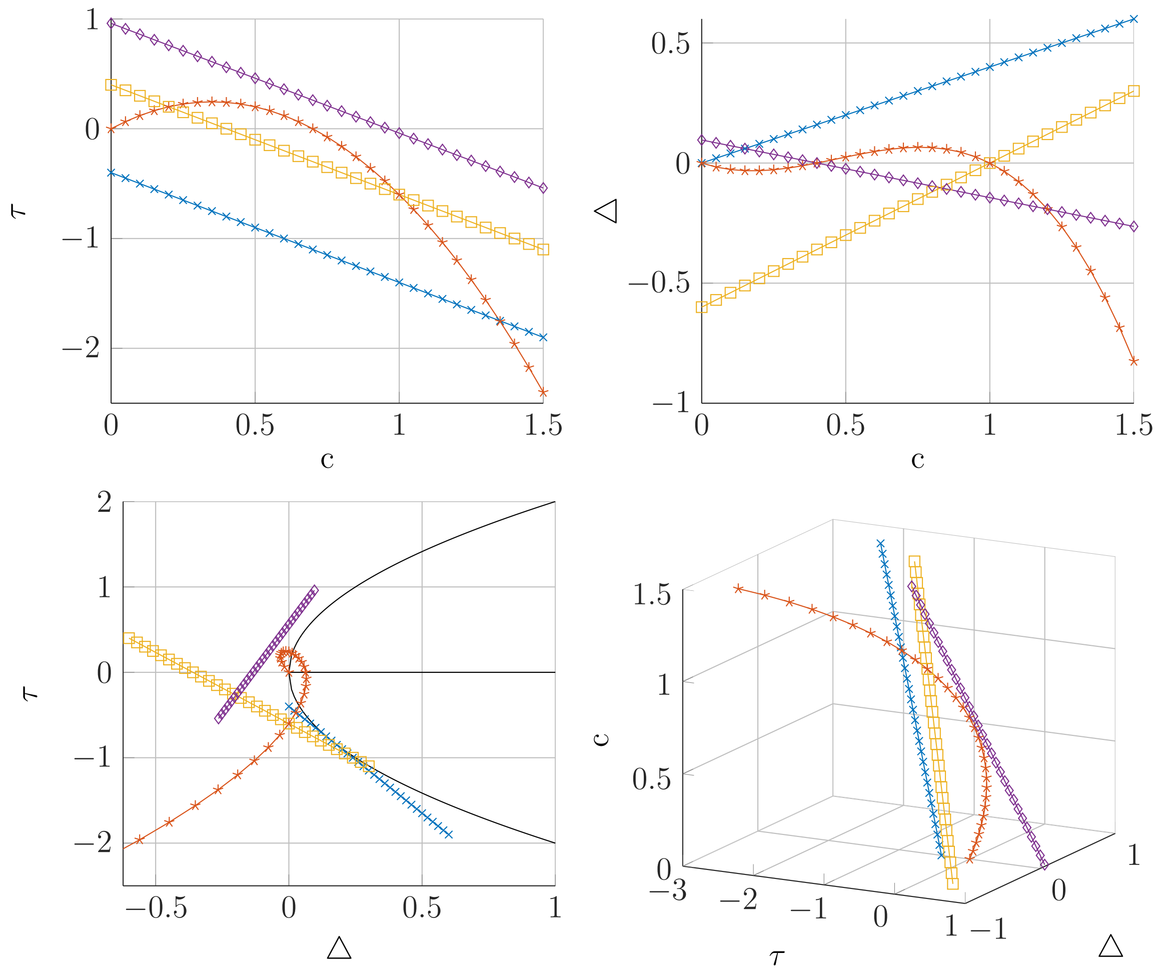

Figure 3: Jacobian trace and determinant for the four fixed points for increasing c. Values associated with $\mathbf{x}_1^*$ are marked with an x and shown in blue. Plots connected to $\mathbf{x}_2^*$ are marked with stars and graphed in orange. Representations of the trace and determinant of $\mathbf{x}_3^*$ have squares on each line and are colored in yellow. Finally values connected to $\mathbf{x}_4^*$ are marked with a diamonds and a drawn in purple.

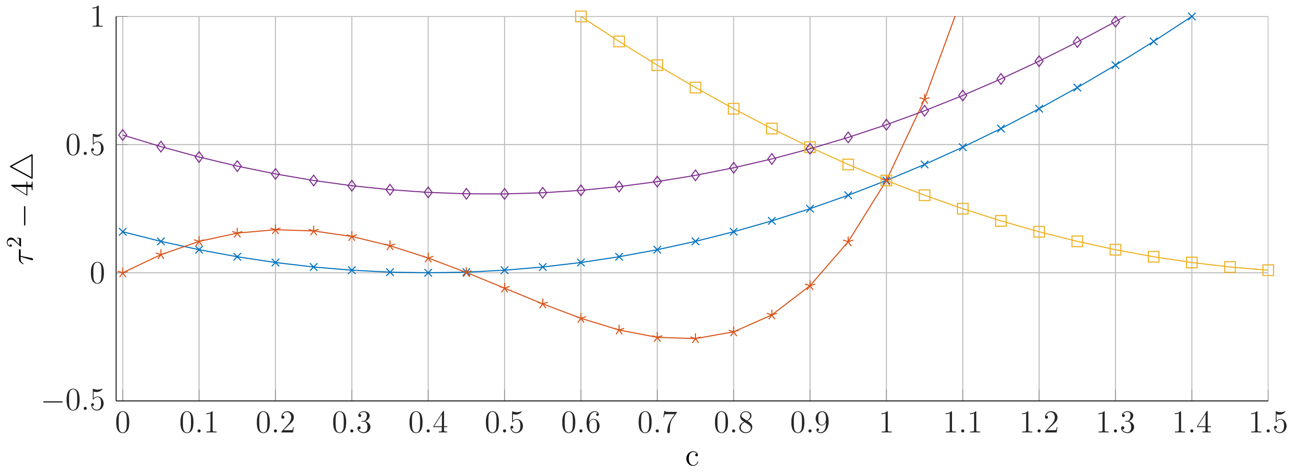

Figure 4: Plot of the spiral condition for all four fixed points. If $\tau^2 - 4\triangle < 0$ a node turns into a spiral.

For $\mathbf{x}_3^*$ the eignevalues of the Jacobian 22 may be read of the diagonal:

$$\lambda_{3,1} = -1 + a \tag{32}$$
$$\lambda_{3,2} = 1 - c. \tag{33}$$

With the eigenvalues known the eigenvectors may be computed, they turn out to be:

$$\mathbf{v}_{3,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{34}$$
$$\mathbf{v}_{3,2} = \begin{pmatrix} \frac{b}{-2+a+c} & 1 \end{pmatrix}^T. \tag{35}$$

When a, b, c are replaced with their numerical values this leads to $\lambda_{3,1} = -0.6,$ and $\lambda_{3,2} = 0.8$. Similarly for the eigenvectors, $\mathbf{v}_{3,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_{3,2} = \begin{pmatrix} -2.1428 & 1 \end{pmatrix}^T$ is obtained.

Finally for $\mathbf{x}_4^*$ is very similar from 23;

$$\lambda_{4,1} = -a^2 + a \tag{36}$$
$$\lambda_{4,2} = a - c, \tag{37}$$

are the read off eigenvalues, thus the eigenvectors are:

$$\mathbf{v}_{4,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{38}$$
$$\mathbf{v}_{4,2} = \begin{pmatrix} \frac{-ab}{a^2-c} & 1 \end{pmatrix}^T \tag{39}$$

One last time the numerical values have to be fed into the expressions, this yields $\lambda_1 = 0.24$ and $\lambda_2 = 0.2$. And for the vectors, $\mathbf{v}_{4,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_{4,2} = \begin{pmatrix} 3 & 1 \end{pmatrix}^T$. At this point numerical values for all eigenvalue, eigenvector pairs are known. If the eigenvectors are added and subtracted from their corresponding fixed points location. The image shown in figure 5 is obtained. It is important to note that most eigenvectors have been scaled by either six or three to address scaling problems. The eigendirection has been deduced from the eigenvalues. $\lambda > 0$ means unstable $\lambda < 0$ means stable or attracting.

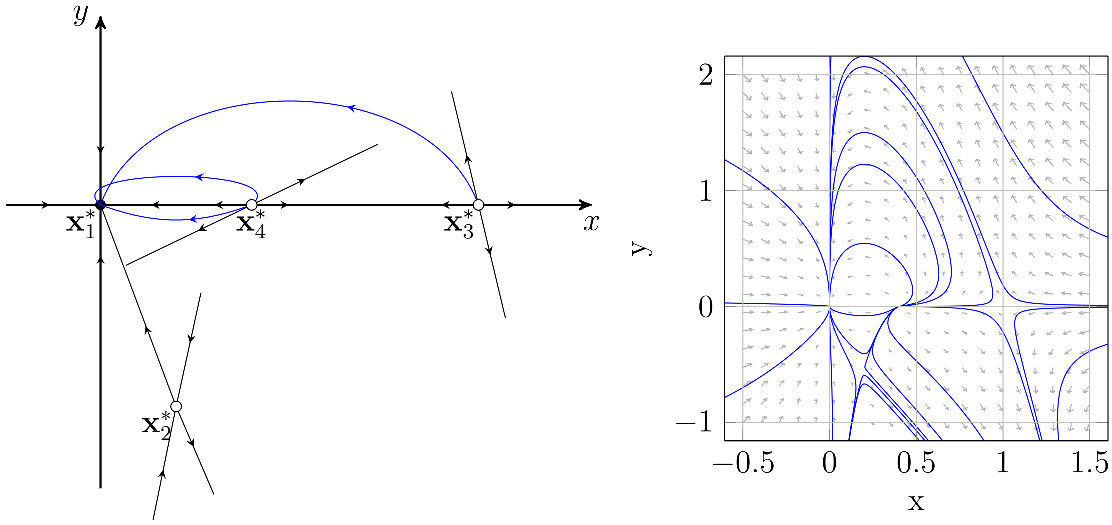

Figure 5: Prediction of the topology of the predator prey model for $c = 0.2$. Eigenvector based predictions are shown in black, others in blue (left). Numerical simulation results using `pplane` for the same set of parameters (right).

2.3 Numerical Analysis

In order to fully graph the topology the topological analysis outlined above for $c = 0.2$ would have to be repeated five times. As this is a very tedious and error prone endeavor pplane will be used instead. Results are shown in figure 7. A quick comparison with table 2 and figure 2 confirms the predictions made earlier. When c is chosen greater then 0.7, two attracting nodes will exist. Which raises the question, which areas of the phase plane will be attracted to which attractor. The separatrices which separate these areas are the answer to this question. These lines are trajectories, that run along the border of the basins of attraction of two different attractors separating them. Typically these trajectories end at a saddle point [1]. For $c > 0.7$ $\mathbf{x}_4^*$ remains a saddle point at all times. Furthermore it lies between the stable nodes, for which the basin of attraction is to be determined. The stable eigenvectors of $\mathbf{x}_4^*$ are thus likely to point along the separatrices. $\mathbf{x}_4^*$ is located at:

$$\mathbf{x}_4^* = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \end{pmatrix} \tag{40}$$

From equations 37 and 39 the stable eigenvector is determined to be:

$$\mathbf{v}_{4,2} = \begin{pmatrix} \frac{-ab}{a^2-c} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1875 \\ 1 \end{pmatrix} \tag{41}$$

Thus one would expect a separatrice that is pointing upwards ($y = 1$) and slightly to the right ($x = 0.1875$). If the eigenvector found above is added and subtracted from $\mathbf{x}_4^*$ figure 6 is obtained. Here the stable and unstable manifolds meet. The situation depicted comes form the one for $c = 0.8$ in figure 7 here only the top separatrices is drawn in red, however the idealization drawn earlier turns out to be true around $\mathbf{x}_4^*$.

[1]Strogatz p.158

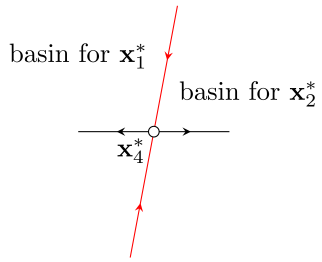

Figure 6: Idealized depiction of the separatrices at $\mathbf{x}_4^*$

3 Bifurcation Analysis

Figure 8 shows the evolution of the real and imaginary part of $J(\mathbf{x}_2^*)$'s two eigenvalues. Originally both eigenvalues are real with one being positive and one negative. Therefore $\mathbf{x}_2^*$ is initially a saddle point. The situation changes when the second eigenvalues becomes positive. The node now turns into an unstable one. When the eigenvalues' imaginary parts start to have nonzero values spiraling will start to occur. With the spirals stability determined by the real parts. At 0.7 the both real parts turn negative causing a reverse Hopf-bifurcation[2] which ultimately results in stable spiral. When the eigenvalues turn real again at 0.9 the spiraling stops and a stable node is created. Finally as the real parts take opposing signs $\mathbf{x}_2^*$ turns into a saddle point again, just like it was in the beginning.

As c increases from 0.65 to 0.75 , most importantly at 0.7 the reverse Hopf-bifurcation occurs. For $c = 0.7$ initial conditions close the the fixed points $\mathbf{x}_2^*$ will keep oscillating almost undamped see figures 9 and 7 . The period of the solutions decreases as orbits come closer to the separatrice, which can be observed in figure **??**. Perhaps not very surprisingly before the bifurcation both species die out as the node at $(0, 0)$ is the only fixed point, that can be reached from positive initial conditions. At $c = 0.7$ the two animal races have at least the option of eternal oscillation instead of mutual extinction. Finally after the bifurcation all three initial conditions tried out here converge to the stable spiral at $\mathbf{x}_2^*$. However it is important that $\mathbf{x}_1^*$ at $(0, 0)$ remains stable and is indeed approached from other initial conditions. Figures 10 and 11 show bifurcation diagrams for $d = 0$, $d = 0.01$ and finally $d = -0.01$. For the $d = 0$ case `matcont` correctly detects the Hopf-bifurcation, that was already identified from the eigenvalues. Additionally it marks the points where $\mathbf{x}_2^*$ changes its stability as branch points. The parameter d changes the nature of the bifurcations drastically like an imperfection parameter it introduces symmetric distortions with respect to the sign of d.

[2]Strogatz p.252

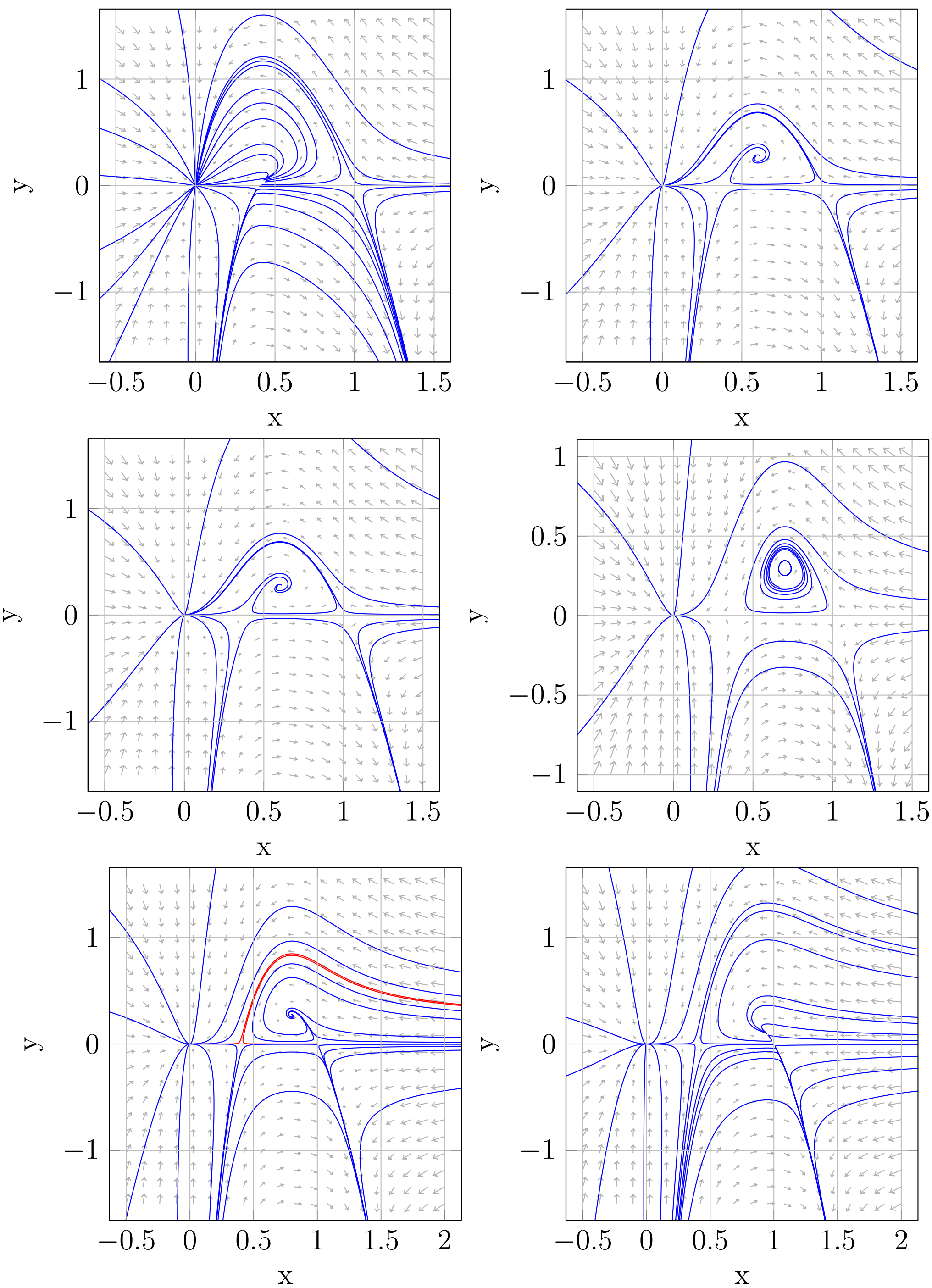

Figure 7: Phase plane topology found by `pplane` for $c = 0.425, 0.6, 0.7, 0.8, 0.95$. In the plot for $c = 0.8$ the separatrices have been drawn in red.

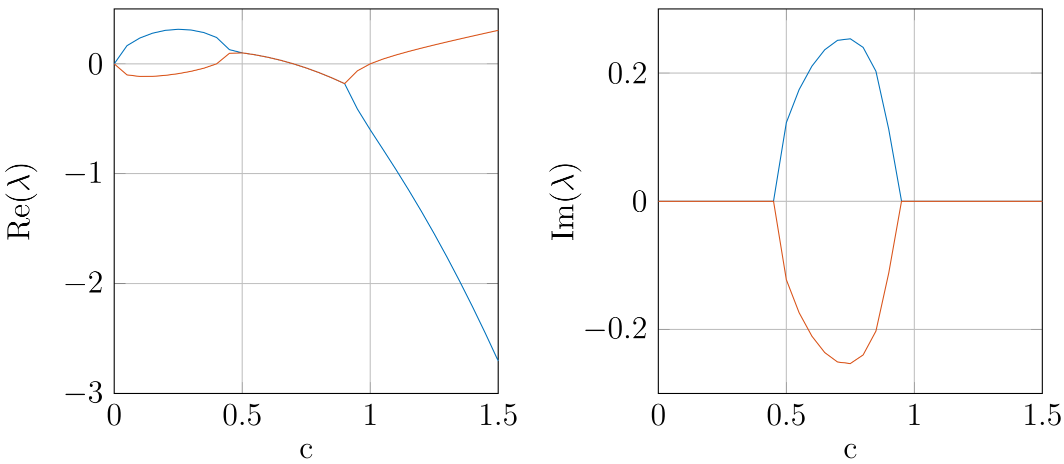

Figure 8: Evolution of real and complex part of $\texttt{eig}(J(\mathbf{x}_2^*))$

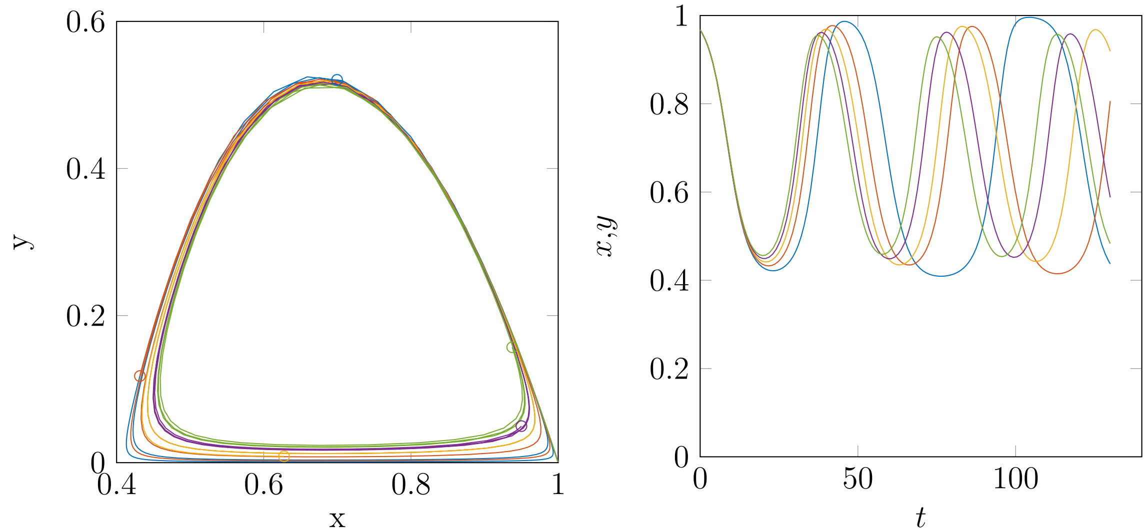

Figure 9: System solutions for $c = 0.677, 0.678, 0.679, 0.68, 0.681$ in time and phase plane representation.

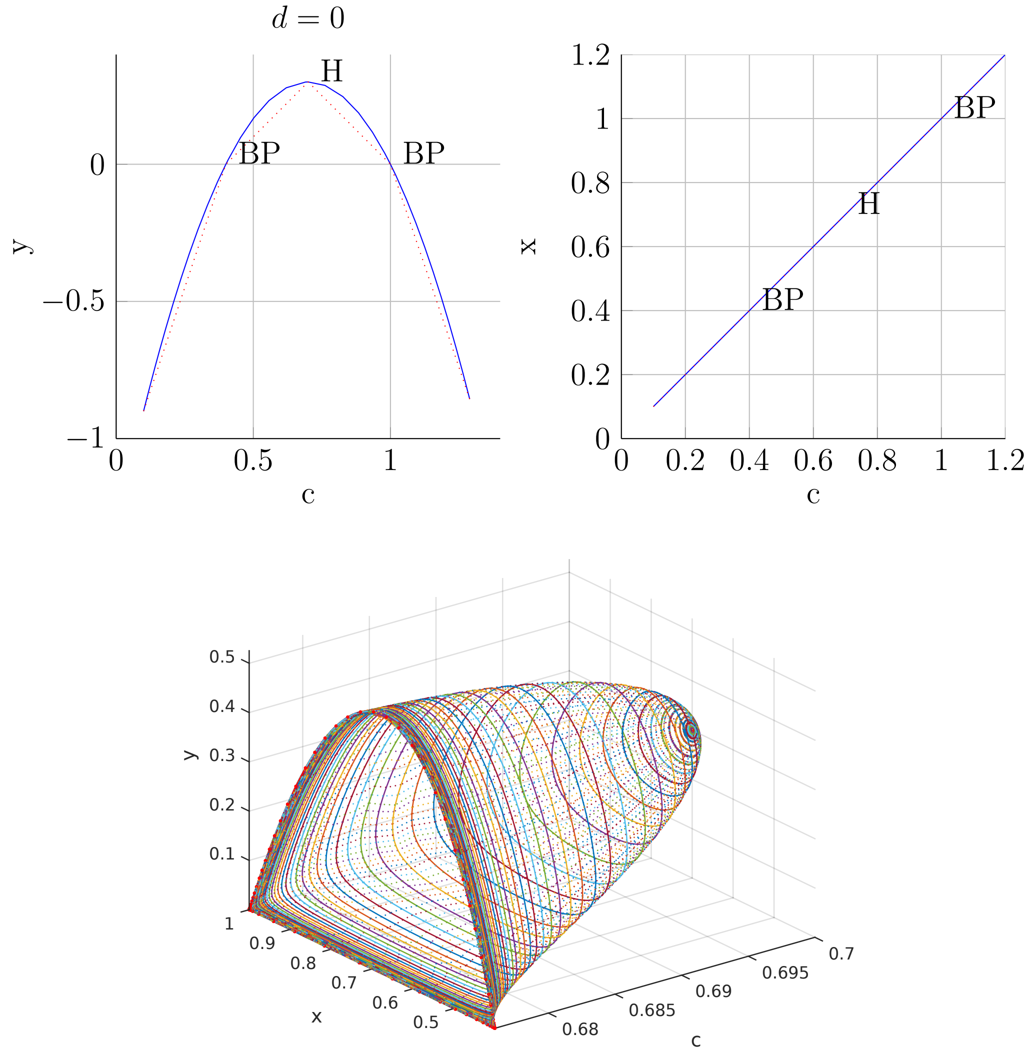

Figure 10: `matcont` continuation for $\mathbf{x}_2^*$ starting from $c = 0.1$ as projection on x, c and y, c. Limit cycle continuation troughout its short lifetime.

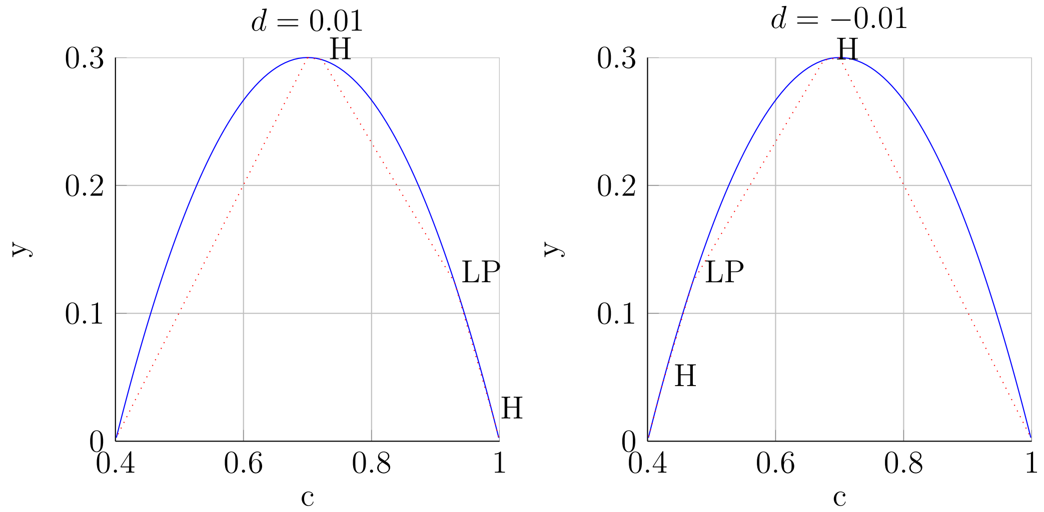

Figure 11: `matcont` made bifurcation diagrams. With $d = 0.01$ and $d = -0.01$

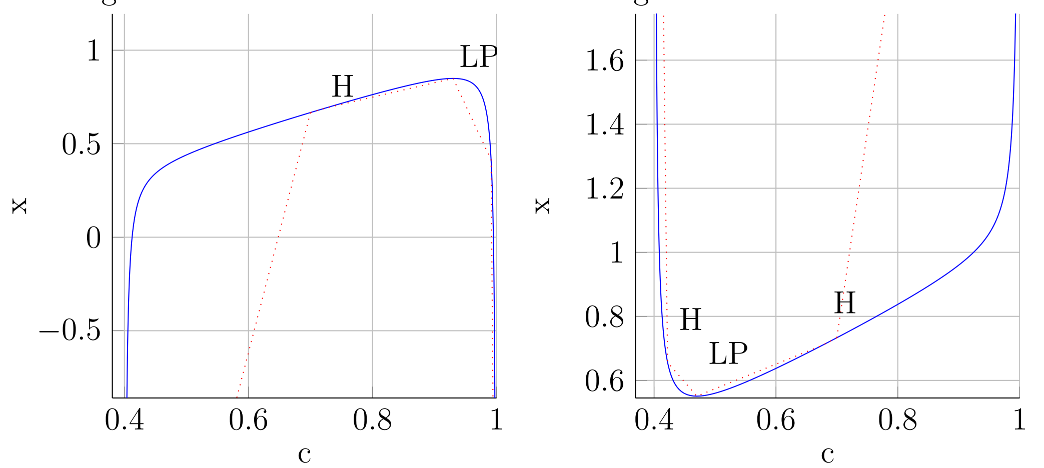

Figure 12: `matcont` made bifurcation diagrams. With $d = 0.01$ and $d = -0.01$, in c, x projection.