Exercise 1 Laser Model

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1 The equations

$$\dot{n} = GnN - kn. \tag{1}$$
$$\dot{N} = -GnN - fN + p. \tag{2}$$

2 First Order Analysis

Assume that the number of excited atoms remains quasi-static.

$$\dot{N} \approx 0. \tag{3}$$
$$\Rightarrow \dot{n} = Gn \cdot \frac{p}{Gn + f} - kn. \tag{4}$$
$$\tag{5}$$

2.1 Linear Stability analysis

In the two dimensional case fixed points are at the intersections of the first derivative with the real axis. Thus the can be found by solving $\dot{n} = 0$. Leading to the problem:

$$0 = n(\frac{Gp}{Gn + f} - k) \tag{6}$$

Which has the zeros 0 and $\frac{p}{k} - \frac{f}{G}$. To learn more about the nature of the fixed points one has to set the derivative of 4 to zero. Using the Quotient rule the expression:

$$\ddot{n} = \frac{Gfp}{(Gn + f)^2} - k \tag{7}$$

is obtained. A fixed point is stable if the second derivative is negative. Likewise it is unstable if the second derivative is positive [1]. Evaluating the second derivative for the fixed point a zero leads to $\ddot{n}(0) = \frac{Gp}{f} - k$ which leads to a values of $p_c = \frac{fk}{G}$. Thus the fixed points becomes unstable if $p > p_c$.

Figure 1 indicates a trans-critical bifurcation.

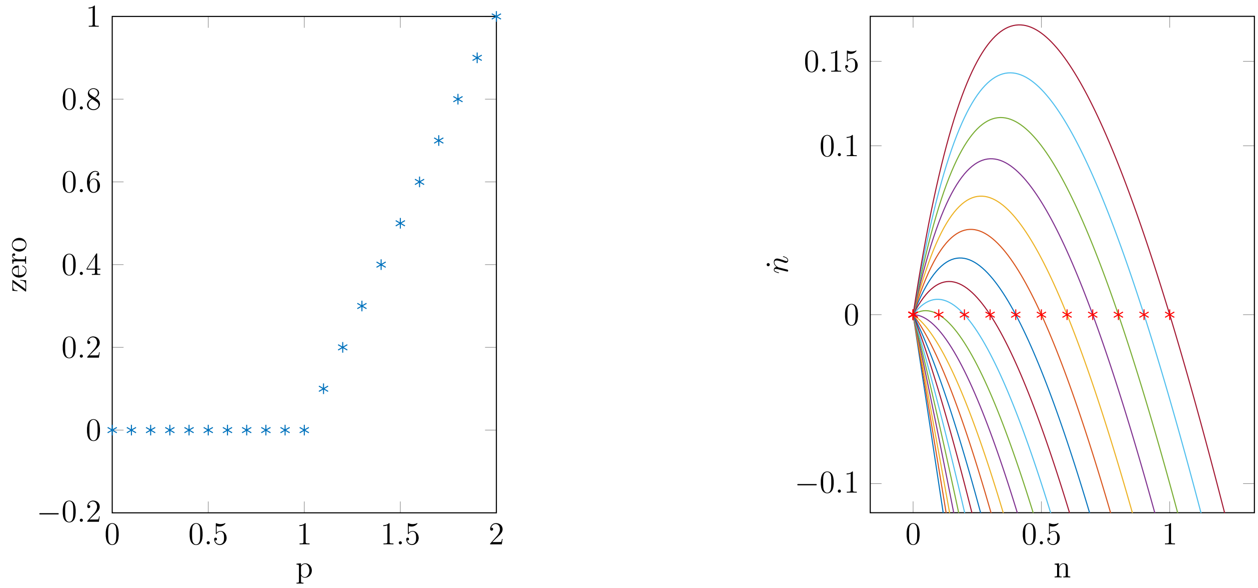
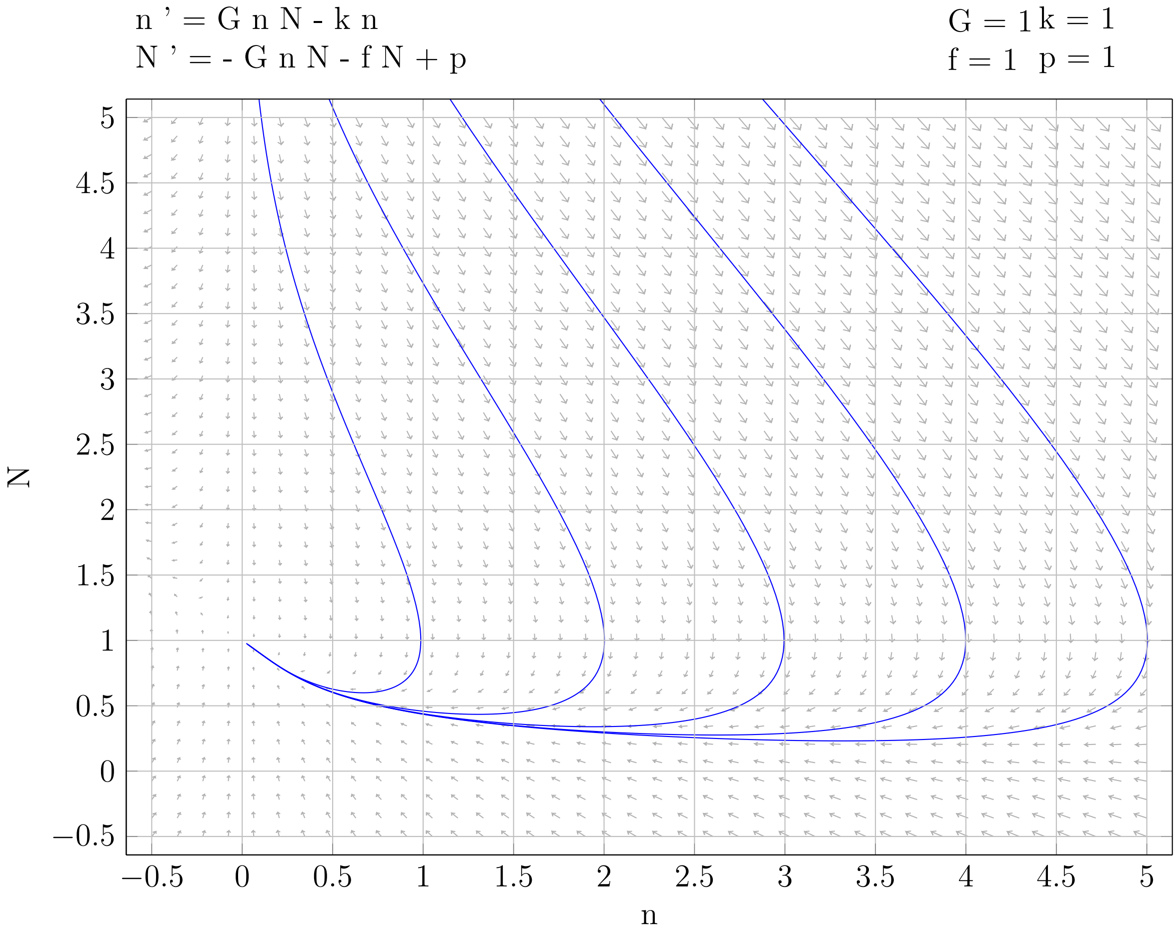


Figure 1: Plot of the position of the right zero and the parameter p (left). Plot of $\dot{n}$ for different p values with the zeros marked with red stars. $G = f = k = 1$ is assumed, leading to $p_c = 1$.

3 Two dimensional analysis

3.1 Nondimensionalization

[1]Strogatz, p.25



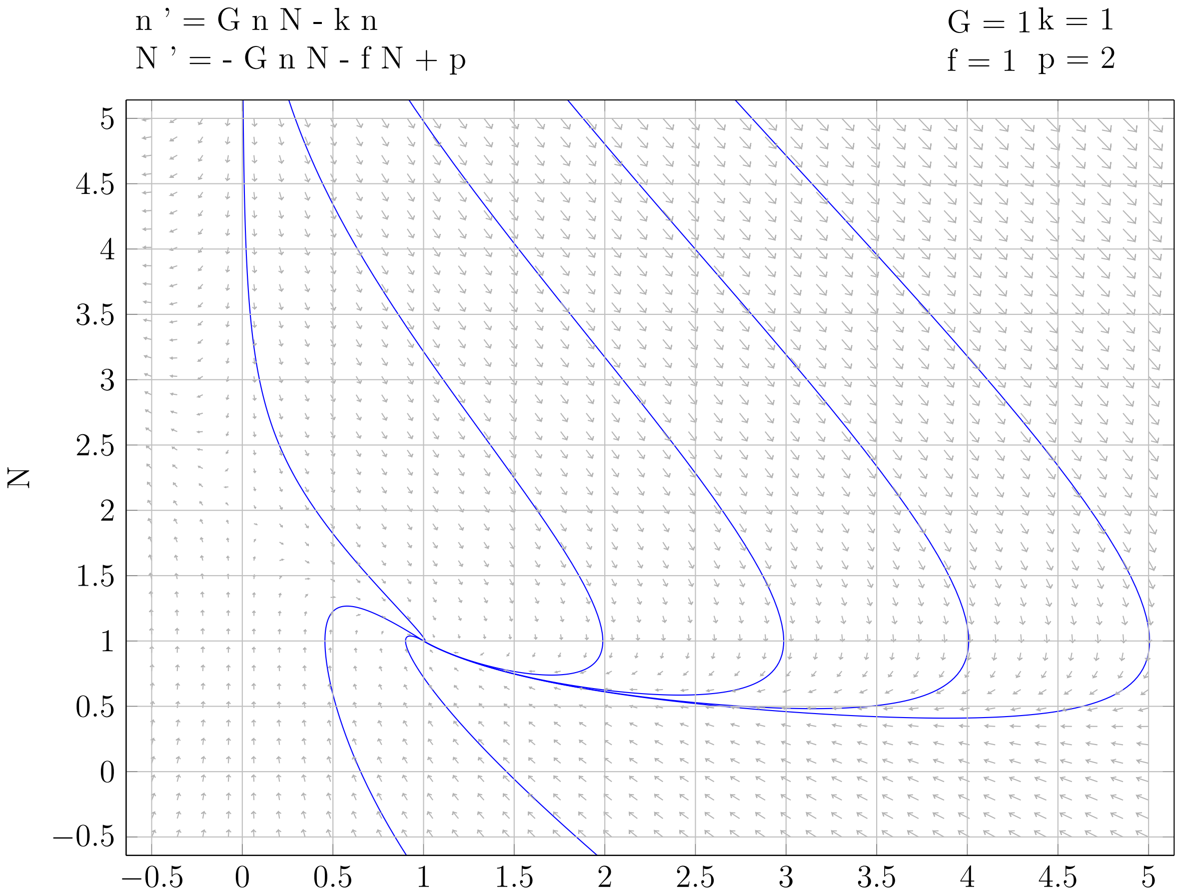


Figure 2: Phase plane of the original system. With two different states predicted from the linear analysis. A plot of the system for uncritical p is given in the top plot. The bottom plot shows the result for a critical $p > p$