# Assignments
# "Non-linear systems and bifurcation analysis"

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This course is evaluated by means of a set of homework assignments (the student is required to make an individual report), which involve the study of a number of dynamical systems via analytical derivations, numerical simulations and bifurcation calculations. In order to help you getting through these assignments, a number of guided sessions are organized.

Each session is focused around one exercise, and is meant to get you started. It is impossible to finish the exercise within the session; instead, you should make sure that you know how to proceed independently. Also, at the beginning of each session, we will take a little time to deal with questions regarding the previous exercise. You are therefore encouraged not to leave the assignments until the end of the semester. Note that we cannot (and do not intend to) give all the answers to the assignments, since this is the examination.

**Session 1:** Wed 25/02 week 09
Laser (Sections 1.1, 1.2)

**Session 2:** Wed 11/03 week 11
Aero-elastic galloping (Section 2: 1-5)

**Session 3:** Wed 25/03 week 13

a) Intro to software: `PPLANE` , `PHASER` , `MATCONT`

b) Numerical bifurcation analysis
(Sections 3, 2.6, 2.7)

**Session 4:** Wed 01/04 week 14
Predator-Prey Model (Section 4.1) and
Numerical bifurcation analysis (Section 4.2)

**Session 5:** Wed 29/04 week 18
Lorenz Attractor, Chua and Duffing (Section 5)

**Session 6:** Wed 13/05 week 20
Pattern formation (Section 6)

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# 1 Model for a laser

Denote an algebraic equation relating $N$, the number of excited atoms, to $n$, the number of laser photons. It can be shown that after certain reasonable approximations, quantum mechanics leads to the system

$$\begin{aligned}\dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p.\end{aligned}$$

Here $G$ is the gain coefficient for stimulated emission, $k$ is the decay rate due to loss of photons by mirror transmissions, scattering, etc., $f$ is the decay rate for spontaneous emission, and $p$ is the pump strength. All parameters are positive, except $p$, which can have either sign.

## 1.1 Analysis of the first order system

1. Convert the above system to a one-dimensional system. Proceed as follows: suppose that $N$ relaxes much more rapidly than $n$. Then, we make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in term of $n(t)$ and derive a first order system for $n$. Such a procedure is often called an *adiabatic elimination.*

2. Show that the fixed point $n^\star = 0$ becomes unstable for $p > p_c$, where $p_c$ is to be determined. Also, determine the characteristic time scale for this fixed point and explain its meaning.

3. What type of bifurcation occurs at the laser threshold $p_c$?

## 1.2 Analysis for the two-dimensional system

1. Nondimensionalize the system.

2. Find and classify all fixed points. (Summarize the classification in a tabular form.)

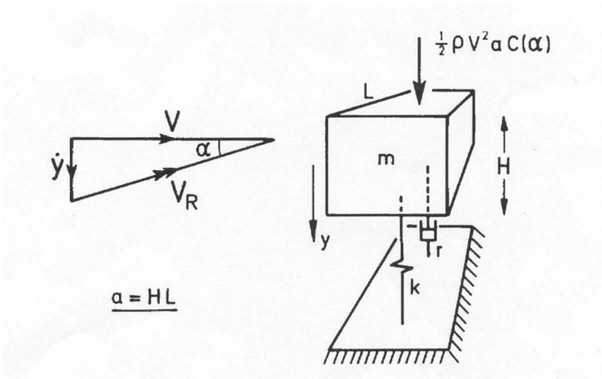
3. Sketch all the qualitatively different phase portraits that occur as the dimensionless parameters are varied. You can use PPLANE.

4. Plot the stability diagram for the system. What types of bifurcation occur?

**Note:** There is no need to repeat the information provided during the guided session. When possible, use tabular presentation rather than full sentences. The *page limit* for this exercise is *7 printed pages.*

## 2 Aero-elastic galloping

A constant wind over a flexible, elastic structure can produce or sustain large amplitude oscillations, as is testified by the Tacoma bridge disaster. One of the mechanisms that can cause this phenomenon is aero-elastic galloping.




The square prism in the figure above represents an infinitesimal element of a bridge. The forces that act on the element are:

- *Inertia* : $m\ \ddot{y}$
- *Linear damping*: $r\ \dot{y}$
- *Elastic force*: $k\ y$
- *Driving force*: The wind speed $V_R$, relative to the prism, is composed of the prism's speed $\dot{y}$ and the constant absolute wind speed $V$. The resulting force depends of the angle $\alpha$ between the relative wind speed $V_R$ and the line perpendicular to the prisms direction of motion. For small $\alpha$, or equivalently large $V$, $\alpha$ can be approximated by $\alpha = \frac{\dot{y}}{V}$. The force along the *direction of the speed* $\dot{y}$ is then given by

  $$\frac{1}{2}\ \rho\ V^2\ a\ C(\alpha)$$

  (note the difference between $a$ and $\alpha$). We approximate $C(\alpha)$ by the 7th order polynomial

  $$C(\alpha) = A_1\ \alpha - A_3\ \alpha^3 + A_5\ \alpha^5 - A_7\ \alpha^7$$

  where $\alpha$ is expressed in radials and $A_1 = 0.04695$, $A_3 = 8.932 \times 10^{-4}$, $A_5 = 1.015 \times 10^{-5}$ and $A_7 = 2.955 \times 10^{-8}$. Since the driving force is a function of the speed $\dot{y}$, it can be seen as a non-linear damping.

In the subsequent analysis, use as numeric values (each time in SI units) for example: $m = 1, \rho = 1, r = 1, k = 100, a = 1$.

1. Write down the equations of the prism's motion.
2. Make a linear approximation of the model. Analyze the stability of eventual fixed points as a function of the parameter $V$. At which critical wind speed $V_C$ do we have a bifurcation? Identify the type of bifurcation of this linear model. (Provide a tabular classification of the fixed point.)
3. Make a model in Matlab of the non-linear model (using the `ode45` solver) whereby you can adapt the parameter $\frac{V}{V_C}$ during simulation.
4. When looking at the structure of $C(\alpha)$, use your intuition to describe what will happen in the neighbourhood of the bifurcation if the non-linear terms are not neglected ? Verify using your Matlab–model.
5. When $V$ grows larger than $V_C$ another bifurcation happens. Identify this type. (**Hint:** plot $y$ as a function of $\dot{y}$.) Lowering back $V$ shows the existence of a third bifurcation. Which one is this?
6. Use `MATCONT` to obtain:
   (a) a bifurcation diagram, plotting $\frac{A}{V_C}$ as a function of $\frac{V}{V_C}$ (where $A$ is the maximal value of $y$ during a period).
   (b) a plot of periodic orbits in three dimensions (parameter $\frac{V}{V_C}$ and two phase space coordinates, e.g., $\frac{y}{V_C}$ and $\frac{\dot{y}}{V_C}$).

   Implementation details:
   see `MATCONT` short reference (by Frank Schilder)

   - Initialising limit cycle continuation,
   - Computing branches of limit cycles,
   - Plotting Results (limit cycles): the function `plotcycle` and computing solution measures.
7. Show how the phase portrait evolves when varying $V$. To this end, plot $\frac{\dot{y}}{V_C}$ with respect to $\frac{y}{V_C}$ for relevant values of $\frac{V}{V_C}$.

**Note:** There is no need to repeat the information provided during the guided session. When possible, use tabular presentation rather than full sentences. The *page limit* for this exercise is *10 printed pages.*

## 3 Imperfect bifurcations – Numerical continuation

**Note:** Use graphical and tabular presentation rather than full sentences. The *page limit* for this exercise is *7 printed pages*, including figures.

Consider a simplified equilibrium equation with two parameters $r$ and $h$

$$-\frac{1}{2}u^3 + ru + h = 0,$$

in which $r \in [-1, 1]$.

**Note** that the terms *fold point*, *limit point*, *turning point* and *saddle node bifurcation* are all synonyms, and are used interchangeably in the literature (and this assignment).

1. Draw solution branches using $h$ as parameter for different values of $r$. For which values of $r$ can we observe turning points w.r.t. $h$? When do we encounter a non-generic turning point?
2. Draw solution branches using $r$ as parameter for different values of $h$ (e.g. $h \in \{-0.1, 0, 0.1\}$). Identify the branching points (turning points, bifurcation points). Show the connection with the disappearance of bifurcation points in 1-parameter problems under perturbation of the model.
3. Determine for the 2-parameter problem the fold curve (branch of turning points). Draw and discuss the projection of the fold curve in the $(u, r), (u, h)$ and $(r, h)$ plane. Compare the latter with your previous analysis. How does the fold curve evolve in the $(u, r, h)$-space?
4. **The parameters of the continuation strategy in `MATCONT` can influence the computed results significantly**.
   Determine the solution branch of the simplified equilibrium equation for $h \in \{-0.0025, -0.0005\}$, using $r$ as parameter.
   Experiment with the parameters (maximum number of Newton iterations, continuation steplength, etc.) of the continuation process.
   When does the numerical procedure lead to wrong results?

Implementation details:
see `MATCONT` short reference (by Frank Schilder)

- Initialising equilibrium point continuation,
- Computing branches of equilibria,
- Plotting Results (equilibrium points).

# 4 Study of a predator-prey model

**Note:** There is no need to repeat the information provided during the guided session. When possible, use tabular presentation rather than full sentences. The *page limit* for this exercise is *8 pages text* (not including the figures).

We consider a predator-prey model

$$\dot{x} = x(x-a)(1-x) - bxy,$$
$$\dot{y} = xy - cy - d,$$

in which $a, b, c$ and $d$ are physical parameters.

| |
|---|
| The values of the parameters $a$ and $b$ are fixed and given to you (see Toledo). |

Give an ecological interpretation to the terms and parameters in the equations (not for the third-order term).

For this problem, all necessary simulations can be done using `MATLAB` and the packages `PPLANE` and `MATCONT` .

## 4.1 A qualitative study for $d = 0$

We consider the parameter $d = 0$, and confine ourselves to parameter values $c \in [0, 1.5]$.

1. First consider the case $y = 0$. Perform a numerical simulation for a number of initial values $x_0 \in [0, 1.5]$ and describe qualitatively the behaviour as a function of time.

2. For the two-dimensional problem, compute analytically the steady state solutions and determine for each steady state its stability and the topological structure of the phase diagram in the neighbourhood. For the equilibria in which $y = 0$, give the slow eigendirection of the nodes, and determine analytically the stable and the unstable manifolds for the saddles.

   **Hint:** Compute the Jacobian matrix, and evaluate the matrix for each fixed point. Use the trace and determinant of the Jacobian matrix to find the topological structure around an equilibrium as a function of the parameter $c$.

**Example:** For the equilibrium $(c, (c-a)(1-c)/b)$, the Jacobian matrix is given by

$$J = \begin{bmatrix} c(1+a-2c) & -bc \\ (c-a)(1-c)/b & 0 \end{bmatrix}$$

The trace $\tau = c(1+a-2c)$ is zero if $c = 0$ or $c = (1+a)/2$. The determinant $\Delta = c(c-a)(1-c)$ is zero if $c = 0$, $c = a$ or $c = 1$.

3. Confirm your analysis by drawing phase diagrams for some (relevant) values of $c$. Make sure your results are consistent with your analysis in item 2. How can you find the separatrices using a combination of analytical and numerical techniques? Determine the regions of attraction of the attractors.

4. What is the relation between (stable and unstable) manifolds of saddle points and separatrices? Indicate this on your figures.

5. Use the results of the previous questions to give a qualitative overview of the changes in the phase diagram when $c$ varies in $[0, 1.5]$. Give a qualitative picture of the evolution of the eigenvalues of the Jacobian in the complex plane for the steady state $(c, (1-c)(c-a)/b)$ as a function of the parameter $c$ and indicate which bifurcations occur. Make sure that your results are consistent with the bifurcation analysis in the next section.

6. Given the interval $[c_1, c_2]$ (see Toledo): between $c_1$ and $c_2$, an important global bifurcation phenomenon occurs. Monitor how the periodic solutions change as the parameter $c$ changes; also look at the period. Which bifurcation occurs? At which value of $c$? Draw the phase diagram for the critical value of $c$.

7. Compare the evolution of $x$ and $y$ as a function of time for

   - a limit cycle corresponding to an almost harmonic oscillation;
   - a limit cycle close to a heteroclinic cycle.

   Briefly discuss the qualitative difference between these two orbits.

## 4.2 Bifurcation analysis

For this part of the assignment, we confine ourselves to $c \in [0.1, 1.5]$.

1. Draw the bifurcation diagram for $d = 0$. Consider the $(x, c)$ projection. Discuss the branches, and discuss how the stability changes at all relevant bifurcation points. Also compute the periodic solutions. Make sure that your bifurcation diagram is consistent with the phase diagrams and analysis of the first part of the assignment.

2. Draw the bifurcation diagrams for $d = 0.01$ and $d = -0.01$; Consider both $(x, c)$ and $(y, c)$ projections. Compare the results with the bifurcation diagram for $d = 0$, and indicate how the transcritical bifurcations disappear. (**Note:** it might be useful to extend your figures to include negative values of $x$, $y$ and $c$ to see all branches.)

## NOTE: Drawing phase diagrams

Several computer programs for drawing phase diagrams exist.
We recommend

- `PPLANE` for `MATLAB` , see http://math.rice.edu//∼dfield;
  also available as Java applet, see http://math.rice.edu/∼dfield/dfpp.html
- `PHASER` , see http://www.phaser.com/ (a copy valid for 30 days can be downloaded for free)

Guidelines to compute a phase diagram of the predator-prey model using `PHASER` :

- choose in menu: 'Equation'; 'Add custom equation': fill in the predator-prey model and the parameters $a$, $b$, $c$ and $d$. Give initial values to the parameters.
- choose in menu: 'Phaser'; 'Numerics'; 'Current view':
  - click on item 'Window size' (under 'Graphics'): change the rectangle in the phase plane that is shown in the main Phaser window: e.g., X-axis: -0.1 ; 1.5; Y-axis: -0.1; 1.5
  - click on item 'Flow' : enable 'Flow through grid points'
  - click on 'Apply'
  - still within window 'Numerics editor' : click on 'Time' (top of window): change 'stop plotting' to '30' (or higher)
  - go to main window: click on 'Go'.

  Now trajectories in the phase plane are drawn by time integration of the equations starting from a 'grid' of initial conditions. Each dot represent a computed point. The evolution of the red dots (current point on the trajectory) gives an idea about the dynamics.
  By clicking again on 'Go' you can recompute the trajectories and get more insight in the dynamics. If you go to the 'Numerics editor' window, click on 'Time', enable 'Plot time extender' (extend by 5),click on 'Apply', then clicking in the main window on 'Go' will continue the time integration (starting form the last computed points).

# 5 Lyapunov exponents, Chua's circuit and Duffing oscillator with periodic forcing

## 5.1 Lyapunov exponents of the Lorenz Equations

The Matlab source code for this part of the assignment will be distributed during the session. You will get the files `rhs_lorenz.m` and `lyapunov.m`. The file `rhs_lorenz.m` contains the righthand side of the famous Lorenz equations, as well as the corresponding variational equations. The file `lyapunov.m` contains a function that computes and plots the Lyapunov exponents; it takes as input a function such as `rhs_lorenz.m` and a number of method parameters.

Take a careful look at this code and explain how/why it computes the Lyapunov coefficients. What do the variables `st` and `kkmax` mean? For the Lorenz equations, the Lyapunov exponents are approximately 0.9, 0 and -14.57. Verify this using the Matlab code. Experiment with the method parameters and discuss how they influence the accuracy of the obtained results.

## 5.2 The Duffing oscillator with periodic forcing

### 5.2.1 Simulation

The Duffing oscillator with periodic forcing can be described by the following differential equation

$$\ddot{x} + k\dot{x} + x^3 = B\cos t, \tag{1}$$

in which you are asked to choose the model parameters $k$ and $B$ such that the oscillator is in a chaotic regime (see figure). Make sure not to take exactly the same parameter values as a different group. Discuss the system behaviour and the sensitivity with respect to the initial conditions. Illustrate your claims by making use of time plots and phase diagrams.

Repeat this experiment for parameter values in the subharmonic regime and show how this case differs from the chaotic case.

**Hint:** Transform the Duffing equation to a state space representation. You can then use Matlab's `ode45` solver to simulate the time behaviour. This function returns a solution structure `sol`. The solution can be evaluated afterwards in a number of specified points using the method `deval`. See the Matlab help for details.

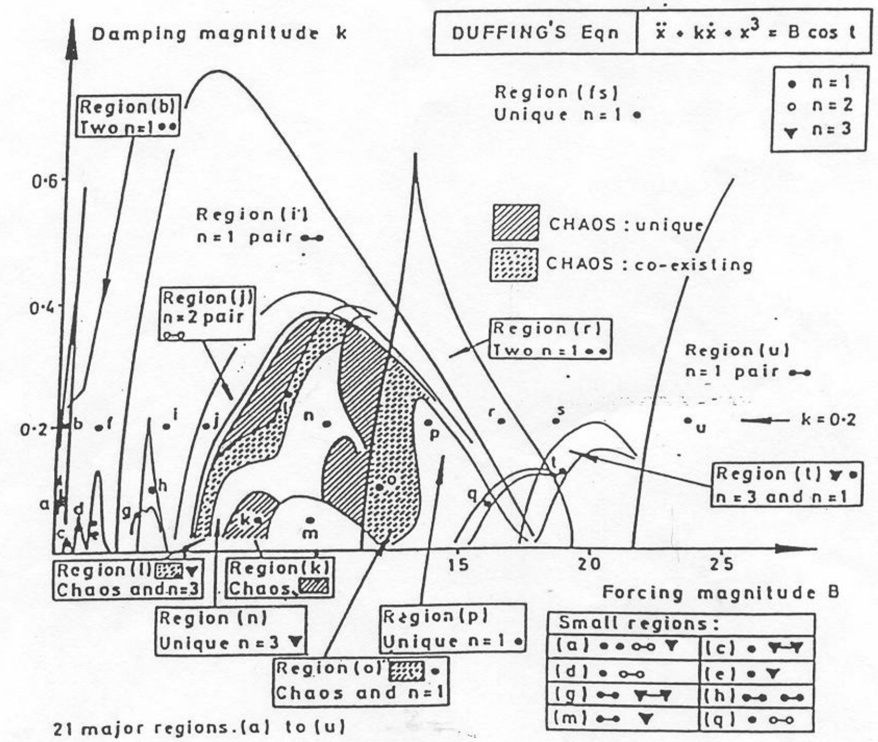



**Figure 1.10** Regimes of the various long-term behaviours of Duffing's equation as mapped by Ueda, as a function of damping $k$ and forcing $B$. *From Ueda (1980a), with permission of SIAM*

## 5.3 Chua's circuit

Consider the following non-linear (but piecewise linear) circuit.

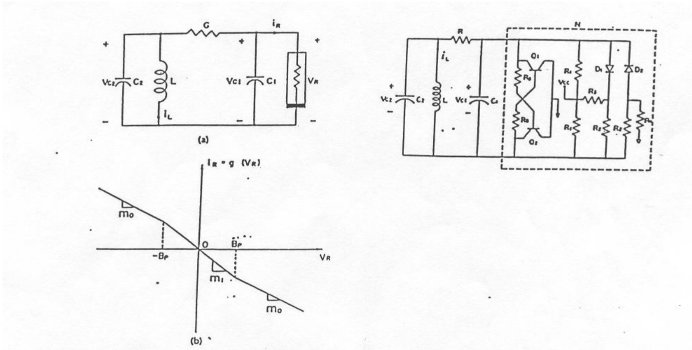


Fig. 1. A simple autonomous circuit with a chaotic attractor. (a) The circuitry. (b) v-i characteristic of the nonlinear resistor.

The dynamics of this circuit are given by

$$\begin{cases} C_1 \frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\ C_2 \frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2} + i_t \\ L_{di_L} dt &= -v_{C_2} \end{cases}$$

with $g(\cdot)$ the piecewise linear characteristic that is depicted in the figure. The parameter values are chosen to be

$$\tfrac{1}{C_1} = 9 \quad \tfrac{1}{C_2} = 1 \quad \tfrac{1}{L} = 7 \quad G = 0.7 \quad m_0 = -0.5 \quad m_1 = -0.8 \quad B_p = 1$$

The system can be written as

$$\begin{cases} \frac{dx}{d\tau} &= \alpha(y - h(x)) \\ \frac{dy}{d\tau} &= x - y + z \\ \frac{dz}{d\tau} &= -\beta y \end{cases} \qquad h(x) = \begin{cases} bx + a - b & x > 1 \\ ax & |x| \leq 1 \\ bx - a + b & x < -1 \end{cases}$$

via

$$\begin{array}{ccc} x = \frac{v_{C_1}}{B_p} & y = \frac{v_{C_2}}{B_p} & z = \frac{i_L}{B_p G} \\ \tau = \frac{tG}{C_2} & a = \frac{m_1}{G} + 1 & b = \frac{m_0}{G} + 1 \\ \alpha = \frac{C_2}{C_1} & \beta = \frac{C_2}{LG^2} & \end{array}$$

1. Simulate the system behaviour. Show phase portraits in $x, y$ and $z$, as well as individual trajectories starting from two sets of nearby initial conditions. Illustrate your suspicion about the (chaotic) behaviour of the system.

2. Use the code from the previous exercise to compute the Lyapunov exponents.

## 6 Pattern formation

**Note:** There is no need to repeat the information provided during the guided session. When possible, use tabular and graphical presentation rather than full sentences.
The *page limit* for this exercise is ± *5 printed pages* (including figures).

We consider a system of two interacting species, the famous *Brusselator* model, which is known to exhibit Turing patterns. The equations for a one-dimensional spatial domain are given by

$$\begin{aligned} u_t &= D_u u_{xx} + A - (B+1)u + u^2 v, \\ v_t &= D_v v_{xx} + Bu - u^2 v, \end{aligned} \tag{2}$$

where $D_u$ and $D_v$ are the diffusion constants of the two species $u$ and $v$, and $A$ and $B$ are concentrations which are kept constant by coupling them to a reservoir. Systems of this type are often called *activator-inhibitor* models due to the nonlinear coupling term. The system has a spatially uniform steady state, in which $u_0 = A$, $v_0 = B/A$.

1. To study the stability of the uniform steady state, write down the linearization of the Brusselator around $(u_0, v_0)$. Show that the linear evolution matrix of a Fourier mode $(u, v) = (u_1, v_1)\exp(\lambda t + ikx)$ is given by
$$\begin{bmatrix} (B-1) - D_u k^2 & A^2 \\ -B & -A^2 - D_v k^2 \end{bmatrix}$$

2. Calculate the eigenvalues of the linear evolution equation as a function of the wavenumber $k$. Show that they are given by the following *dispersion relation*,
$$s_\pm = \frac{1}{2}(\Sigma \pm \sqrt{\Sigma^2 - 4\Delta}),$$
where
$$\begin{aligned} \Sigma &= B - 1 - A^2 - k^2(D_u + D_v), \\ \Delta &= A^2 + k^2(A^2 D_u + (1-B)D_v) + k^4 D_u D_v. \end{aligned}$$

3. When writing $s_\pm = \sigma \pm i\omega$, a Turing instability occurs when $\sigma$ changes sign, and $\omega$ is zero. Explain this.

4. For $B$ very small, the spatially uniform solution is stable. We would like to increase $B$, and are interested in the minimal value for $B$ for which the Turing instability will set in. Compute the critical value of

$B$ and the corresponding critical wavenumber $k$. Show that the Turing instability sets in at the critical value

$$B_T = (1 + A\eta)^2,$$

and that the critical wavenumber is given by

$$k_T = \left(\frac{A^2}{D_u D_v}\right)^{1/4}.$$

Here, $\eta = \sqrt{D_u/D_v}$.

5. When ignoring the diffusion term, one can easily check that the resulting system of 2 ordinary differential equation exhibits a Hopf bifurcation when $B = 1 + A^2$. Show that the Turing instability sets in first when

$$\eta < \frac{\sqrt{A^2+1}-1}{A}.$$

6. Use the applet on `http://www.cmp.caltech.edu/~mcc/Patterns/Demo4_5.html` .
The applet performs time integration of the *two-dimensional* Brusselator model, defined on a unit square, i.e. Eq.(1) with $u_{xx}$ replaced by $u_{xx} + u_{yy}$ and $v_{xx}$ replaced by $v_{xx} + v_{yy}$. Periodic boundary conditions and a random initial condition are imposed. Note that in this model the diffusion coefficients are scaled by the actual size of the physical domain, i.e. $D_u = \overline{D_u}/L$ and $D_v = \overline{D_v}/L$ where $L$ is the length of each size of the physical square domain and $\overline{D_u}$ and $\overline{D_v}$ are the physical diffusion coefficients.

   (a) Start with the following values: $A$=4.5; $B = 7$; $D_u = 1$; $D_v = 8$. Use the analytical results to check whether the conditions for a Turing instability are satisfied. Describe what you observe.

   (b) Lower the value of $B$ so that the conditions for a Turing instability are not satisfied and the homogeneous steady state is stable.
   Note that the $u$ (or $v$) values are greycoded so that black and white always correspond to the minimal and maximal values. For a correct interpretation of the values you must check the actual minimal and maximal values (Min and Max).

   (c) Reset $B = 7$ and increase $L$. Describe what you observe.

   (d) Reset the diffusion coefficients and vary $B$ in the interval (6.5,9). Describe what you observe.