Exercise 2 Bridge Oscillations

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In this report the effects of Wind on a poorly designed bridge will be explored.

1 Equation of Motion

The model for the structure leads to the following equation of motion:

$$0 = -F_I - F_d - F_e + F_{dr} \tag{1}$$

$$0 = -m\ddot{y} - r\dot{y} - ky + \frac{1}{2}\rho V^2 a C(\alpha). \tag{2}$$

Where $C(\alpha)$ is a nonlinear function.

2 Linear Analysis

$C(\alpha)$ is defined as a sum of several odd powers of α:

$$C(\alpha) = A_1\alpha - \underbrace{A_3\alpha^3 + A_5\alpha^5 - A_7\alpha^7.}_{\approx 0 \text{ for small } \alpha.} \tag{3}$$

for small α additionally the approximation $\alpha = \frac{\dot{y}}{V}$ is given. Plugging into 2 yields:

$$0 = -m\ddot{y} + (\frac{1}{2}\rho V a A_1 - r)\dot{y} - ky. \tag{4}$$

Which may be rewritten in terms of two first order equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\frac{1}{2}\rho V a A_1 - r}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Setting the derivatives to zero the fixed point $\mathbf{x}_1^* = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ is obtained. As a linear approximation as already taken place the Jacobi-matrix is identical to the system matrix given above. Thus for the trace τ_1 and determinant $\triangle_1$ at the fixed point the following equations are obtained:

$$\tau_1 = \frac{\frac{1}{2}\rho V a A_1 - r}{m} - \frac{k}{m} \tag{6}$$

$$\triangle_1 = \frac{k}{m}. \tag{7}$$

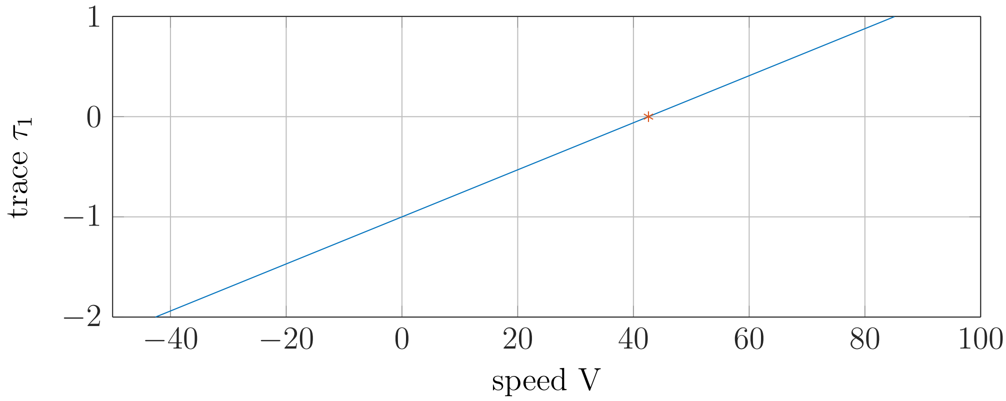

Figure 1: Plot of tau_1 for different speed values. The critical wind speed V_c is marked with an red asterisk.

Assuming $(k \wedge m) > 0$ the nature of the fixed point is determined by the trace. The critical value will occur for $tau_1 = 0$, therefore it my be found from:

$$0 = \frac{1}{2}\frac{\rho V^2 a A_1}{m V_c} - \frac{r}{m}. \tag{8}$$

Solving for V_c leads to:

$$V_c = \frac{2r}{\rho a A_1} = 42.5985. \tag{9}$$

When $m = 1,\ \rho = 1,\ r = 1\ k = 100,\ a = 1$ and $A_1 = 100$. A plot for τ_1 with respect for different values for V is given in figure 1. As the determinant remains positive at all times the fixed point at the center changes from a stable to an unstable spiral at the critical wind speed V_c.

3 Simulation of the non-linear System

In this sections simulation will be attempted without linearization. Including all terms the following system of first order ordinary differential equations is obtained:

$$\dot{y} = z \tag{10}$$

$$\dot{z} = -\frac{ky}{m} - \frac{rz}{m} + \frac{1}{2}\frac{\rho V^2 a}{m}[A_1\frac{z^1}{V^1} - A_3\frac{z^3}{V^3} + A_5\frac{z^5}{V^5} - A_7\frac{z^7}{V^7}]. \tag{11}$$

Equation 11 may be simulated in matlab using an explicit Runge-Kutta type solver (`ode45`). Results are given in figure.

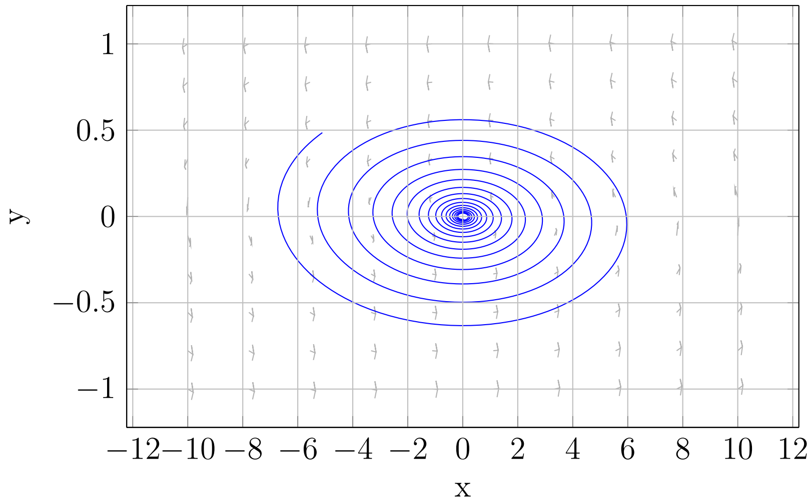

Figure 2: Plot of the two dimensional linearized system with $V = 10 < V_c$.

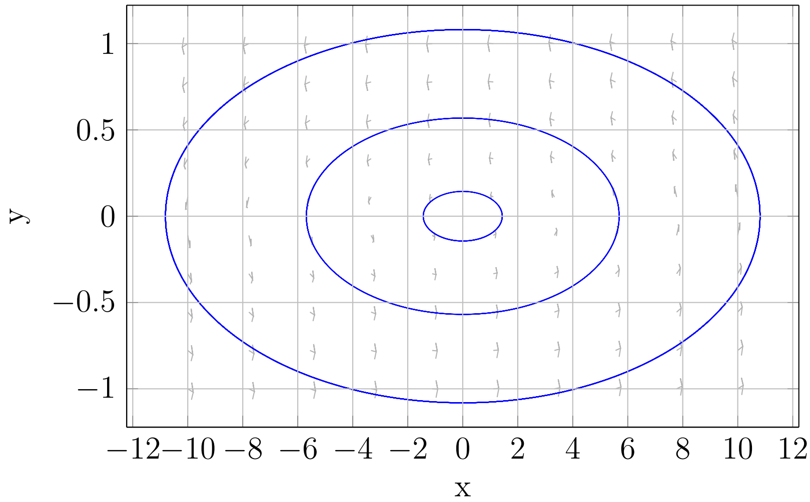

Figure 3: Plot of the two dimensional linearized system with $V = 42.5985 = V_c$.

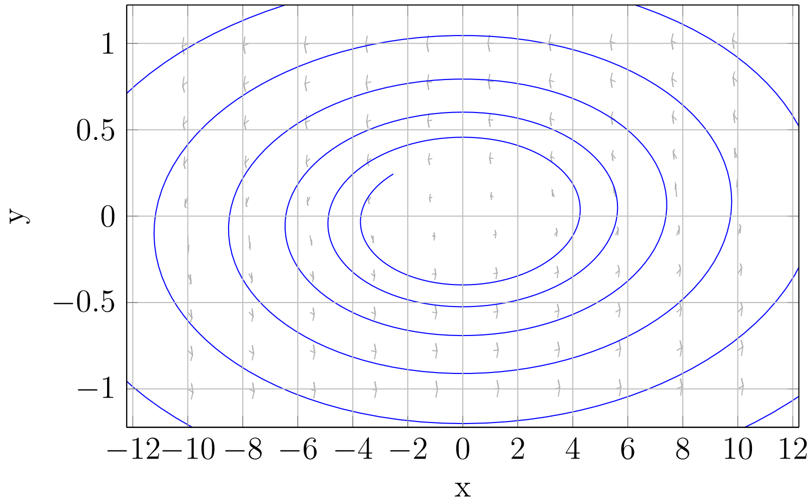

Figure 4: Plot of the two dimensional linearized system with $V = 80 > V_c$.

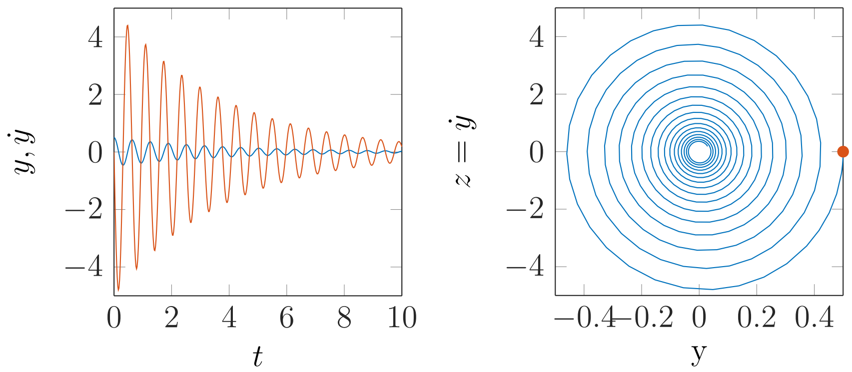

Figure 5: Nonlinear simulation results shown as time plot (left) and in their phase plane representation (right) for $V = 20 < V_c$. In the left plot bridge position is shown in blue. The first derivative is depicted in red. In the right plot the initial condition is depicted as a red dot.

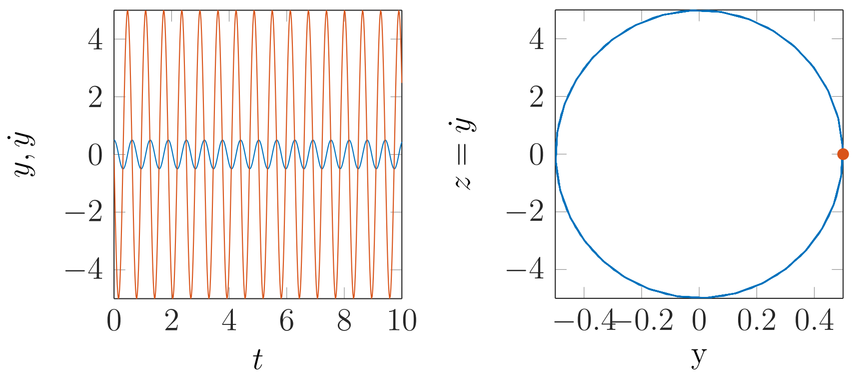

Figure 6: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = V_c$.

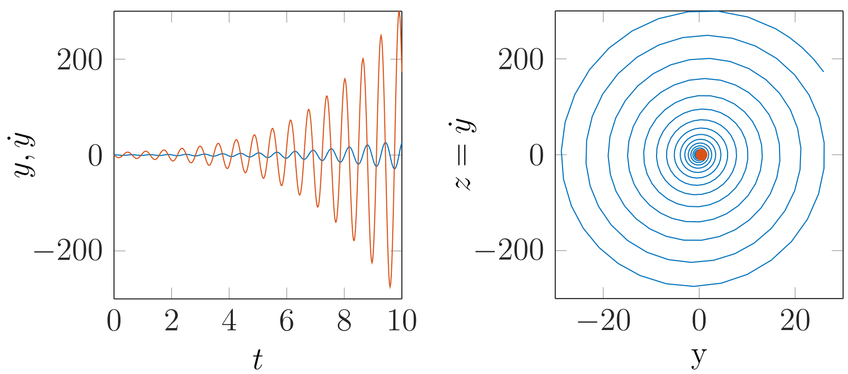

Figure 7: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = 80 > V_c$.