Exercise 3 Imperfect bifurcations

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May 27, 2015

1 The equation

$$-\frac{1}{2}u^3 + ru + h = 0. \tag{1}$$

2 Bifurcation analysis

2.1 Manual analysis

Bifurcation analysis means looking at fixed point movement and the evolution of their stability. In this report equation 1 will be analyzed thoroughly. It contains the information of the fixed point locations for the system:

$$\dot{u} = -\frac{1}{2}u^3 + ru + h. \tag{2}$$

`matcont` may be used to do this kind of analysis. However to put the results into perspective. A manual analysis has been performed beforehand [1]. Figures 1,2 3 show each a graphical approach to find the zeros of 1 in the first column. The second column shows the evolution of the position of the zeros. For different values for the imperfection parameter $h = 5, 0$ and 1 as well as $r \in [-30, 30]$ with a step size of 0.1. Finally the last columns do the same for different values of h with r constant. Linear stability analysis may be employed to learn more about the system dynamics. The first derivative of $f(u) = -\frac{1}{2}u^3 + ru + h$ is:

$$f'(u) = -\frac{3}{2}u^2 + r. \tag{3}$$

At the extreme points the derivative is zero, which leads to $u_{max} = \sqrt{\frac{2}{3}r}$. Thus leading to the maximum values $f(u_{max}) = \frac{1}{3}\sqrt{\frac{2}{3}}r^{3/2}$. As bifurcations occur when h hits the local maximum values one obtains $h_c = \pm f(u_{max})$. In the second column of the figures 1,2 3 the effect of the imperfection parameter may be observed. $h = 0$ leads to a textbook pitchfork bifurcation. However if $h \neq 0$ the fork separates and a saddle node bifurcation occurs which one part moving up or down with respect to x depending on the parameter signs. Figure 4 is a three dimensional data representation. It contains all the other plots either as cross sections or projections [2]. The saddle node bifurcation, which leads to the folded shape of the r, h plane.

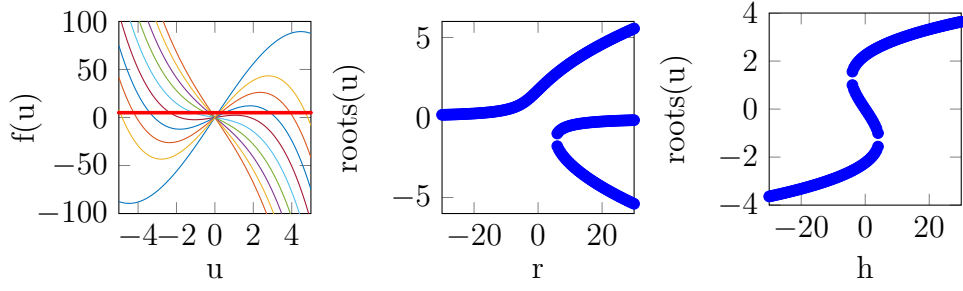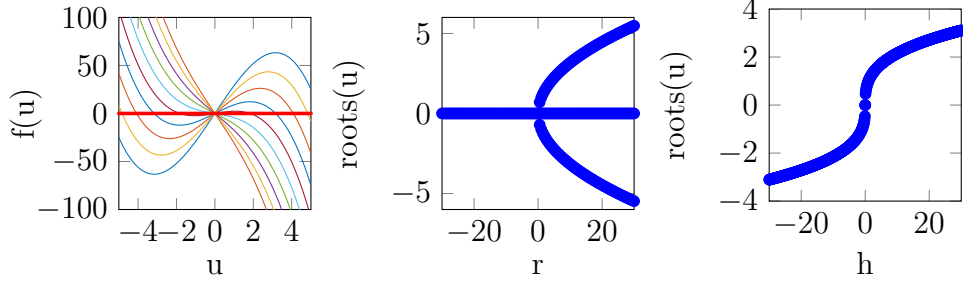[1] see pages 70-73 in Strogatz' book

[2] book strogatz page 73

Figure 1: Solution branches with the function $-\frac{1}{2}u^3 + ru$ numerous colors for different $r \in [-30, 30]$ and the constant function $h = 5$ in red (left). Root locus plot for the same r values (center). Solution for different h with constant $r = 5$ (right).
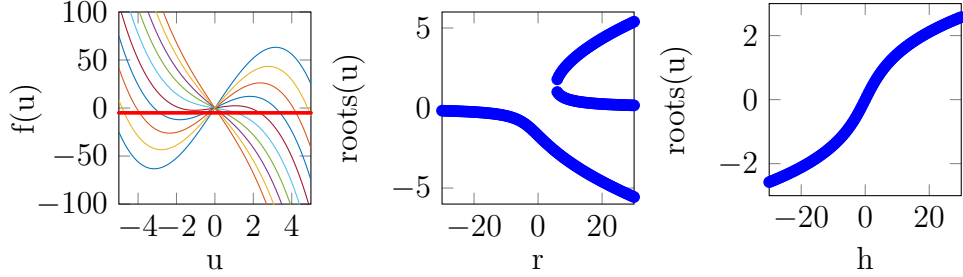

Figure 2: Solution branches with the function $-\frac{1}{2}u^3 + ru$ numerous colors for different $r \in [-30, 30]$ and the constant function $h = 0$ in red (left). Root locus plot for the same r values (center). Solution for different h with constant $r = 0$ (right).

Figure 3: Solution branches with the function $-\frac{1}{2}u^3 + ru$ numerous colors for different $r \in [-30, 30]$ and the constant function $h = -5$ in red (left). Root locus plot for the same r values (center). Solution for different h with constant $r = -5$ (right).
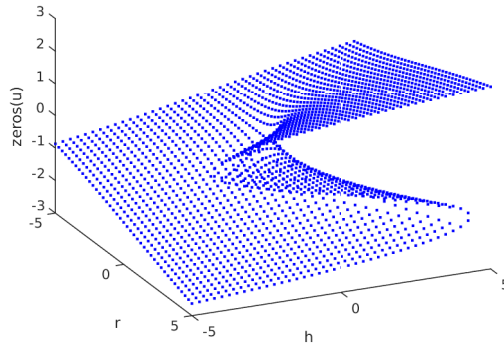
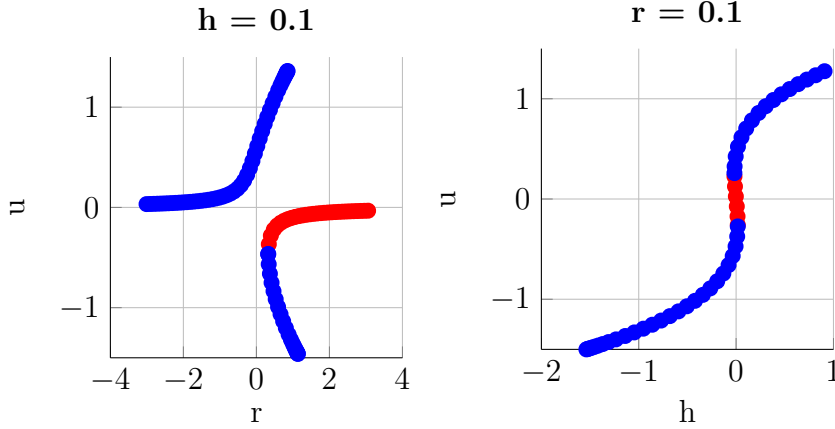

Figure 4: r, h, u plot of the polynomial roots.

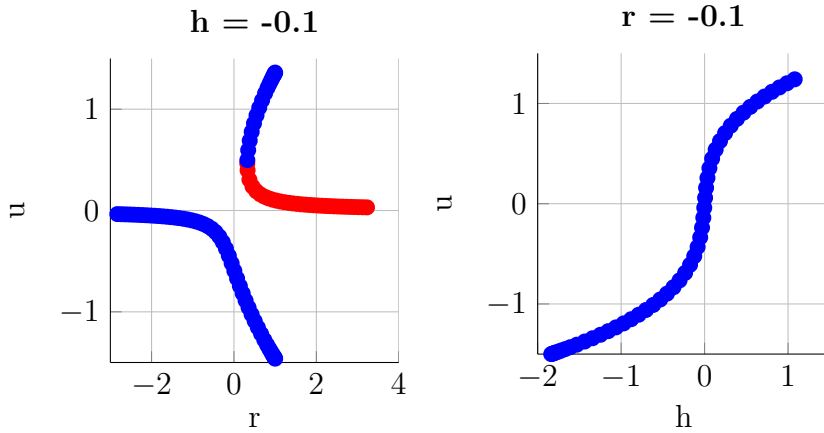Figure 5: `matcont` output for $h = 0.1$

Figure 6: `matcont` output for $h = -0.1$

2.2 `matcont` analysis

Figures 5 and 6, repeat the previous experiments using the `matcont` command line package. The computed figures are similarly shaped as those found previously. Thus it is assumed that computations are correct. The `matcont` package has the added benefit, that unstable points are distinguished from stable ones and drawn in red.

Figure 7 shows the position of the polynomials roots in the r, h-plane. The plot is called a stability diagram.

2.3 Step-size Experiment

Figure 8 shows computations results for two h values close to zero with the same `matcont` `parameters`. The second figure is presumably a correct computation as the results agree with those found in figure 3 and 6. The top plot is not correct. Instead of making the correct left turn the solution continues incorrectly onto the upper branch. This problem can be fixed by choosing a smaller step size and a higher point maximum. Incorrect solutions appear if input and stepsize differ too much in magnitude. Furthermore computations fails if the to little newton iterations are done if the upper limit for the computation points is set too low.
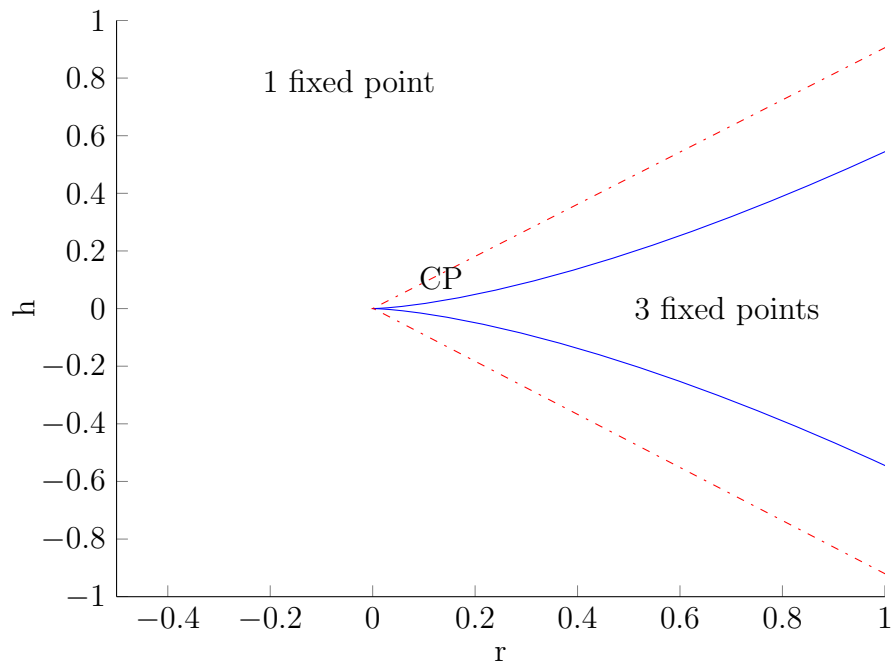
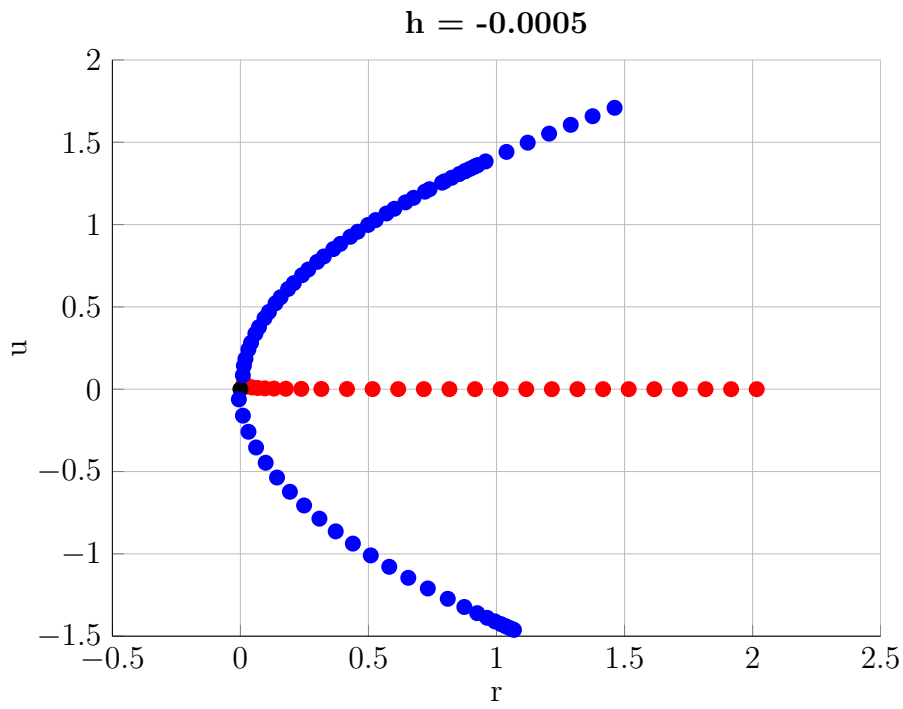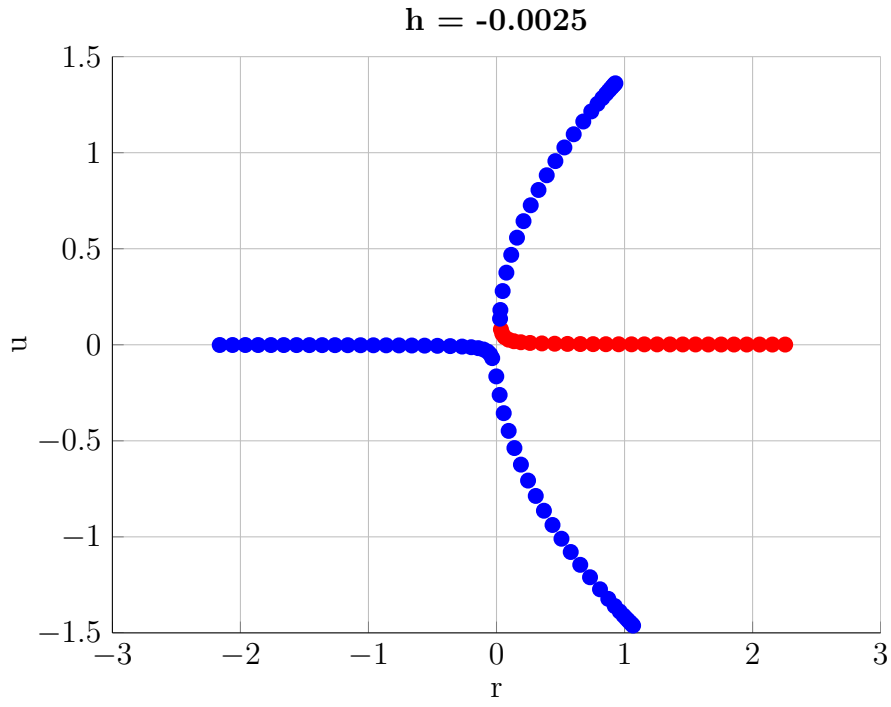Figure 7: Evolution of the zeros within the r, h plane.

Figure 8: Solution branch of $h = -0.0025 \wedge -0.0005$. With
`opt=contset(opt,'MinStepsize',0.01); opt=contset(opt,'MaxStepsize',0.1);`
`opt=contset(opt,'MaxNewtonIters',4);`