# Exercise 2 Bridge Oscillations

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In this report the effects of Wind on a poorly designed bridge will be explored.

## 1 Equation of Motion

The model for the structure leads to the following equation of motion:

$$0 = -F_I - F_d - F_e + F_{dr} \tag{1}$$

$$0 = -m\ddot{y} - r\dot{y} - ky + \frac{1}{2}\rho V^2 a C(\alpha). \tag{2}$$

Where $C(\alpha)$ is a nonlinear function.

## 2 Linear Analysis

$C(\alpha)$ is defined as a sum of several odd powers of $\alpha$:

$$C(\alpha) = A_1\alpha - \underbrace{A_3\alpha^3 + A_5\alpha^5 - A_7\alpha^7.}_{\approx 0 \text{ for small } \alpha.} \tag{3}$$

for small $\alpha$ additionally the approximation $\alpha = \frac{\dot{y}}{V}$ is given. Plugging into 2 yields:

$$0 = -m\ddot{y} + (\frac{1}{2}\rho V a A_1 - r)\dot{y} - ky. \tag{4}$$

Which may be rewritten in terms of two first order equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\frac{1}{2}\rho V a A_1 - r}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Setting the derivatives to zero the fixed point $\mathbf{x}_1^* = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ is obtained. As a linear approximation as already taken place the Jacobi-matrix is identical to the system matrix given above. Thus for the trace $\tau_1$ and determinant $\triangle_1$ at the fixed point the following equations are obtained:

$$\tau_1 = \frac{\frac{1}{2}\rho V a A_1 - r}{m} - \frac{k}{m} \tag{6}$$

$$\triangle_1 = \frac{k}{m}. \tag{7}$$

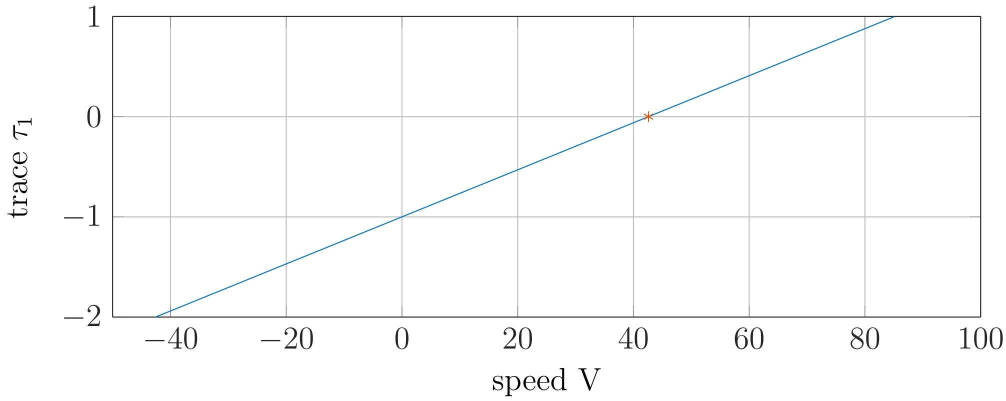


Figure 1: Plot of $tau_1$ for different speed values. The critical wind speed $V_c$ is marked with an red asterisk.

Assuming $(k \wedge m) > 0$ the nature of the fixed point is determined by the trace. The critical value will occur for $tau_1 = 0$, therefore it my be found from:

$$0 = \frac{1}{2}\frac{\rho V^2 a A_1}{m V_c} - \frac{r}{m}. \tag{8}$$

Solving for $V_c$ leads to:

$$V_c = \frac{2r}{\rho a A_1} = 42.5985. \tag{9}$$

When $m = 1,\ \rho = 1,\ r = 1\ k = 100,\ a = 1$ and $A_1 = 100$. A plot for $\tau_1$ with respect for different values for $V$ is given in figure 1. As the determinant remains positive at all times the fixed point at the center changes from a stable to an unstable spiral at the critical wind speed $V_c$.

# 3 Simulation of the non-linear System

In this sections simulation will be attempted without linearization. Including all terms the following system of first order ordinary differential equations is obtained:

$$\dot{y} = z \tag{10}$$

$$\dot{z} = -\frac{ky}{m} - \frac{rz}{m} + \frac{1}{2}\frac{\rho V^2 a}{m}[A_1\frac{z^1}{V^1} - A_3\frac{z^3}{V^3} + A_5\frac{z^5}{V^5} - A_7\frac{z^7}{V^7}]. \tag{11}$$

Equation 11 may be simulated in matlab using an explicit Runge-Kutta type solver (`ode45`). Results are given in figure.

# 4 Comparison of linear and nonlinear dynamics around $V_c$

Setting $m = \rho = r = a = 1$ yields the simplified version of equation 4 for the linear case at critical speed $V_c$:

$$0 = -\ddot{y} + (\frac{1}{2}V_c A_1 - 1)\dot{y} - ky. \tag{12}$$

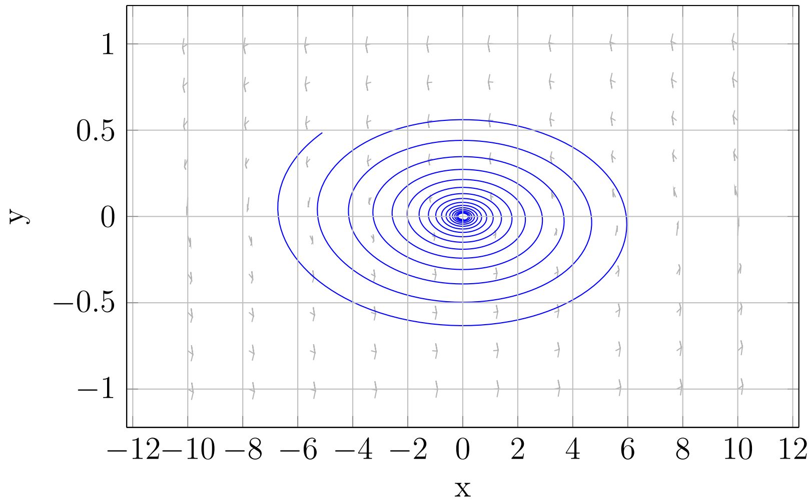


Figure 2: Plot of the two dimensional linearized system with $V = 10 < V_c$.

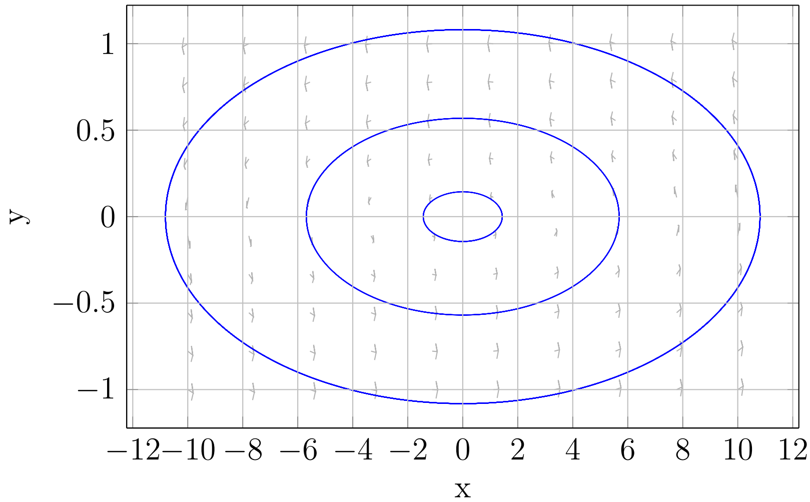


Figure 3: Plot of the two dimensional linearized system with $V = 42.5985 = V_c$.

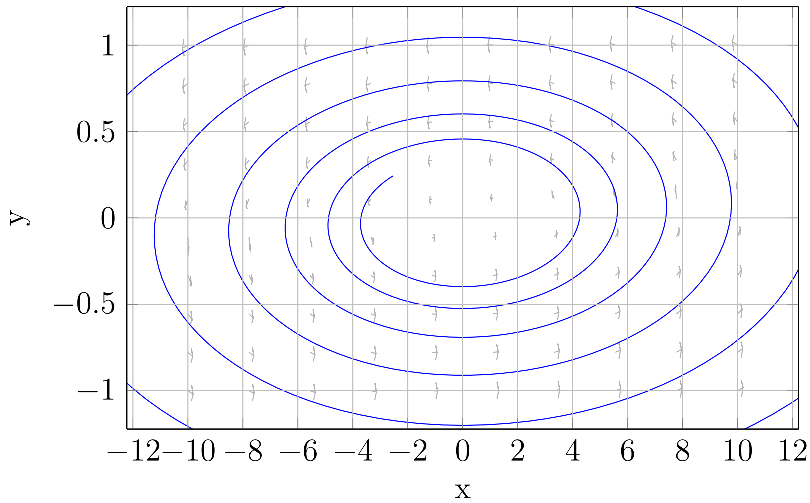


Figure 4: Plot of the two dimensional linearized system with $V = 80 > V_c$.

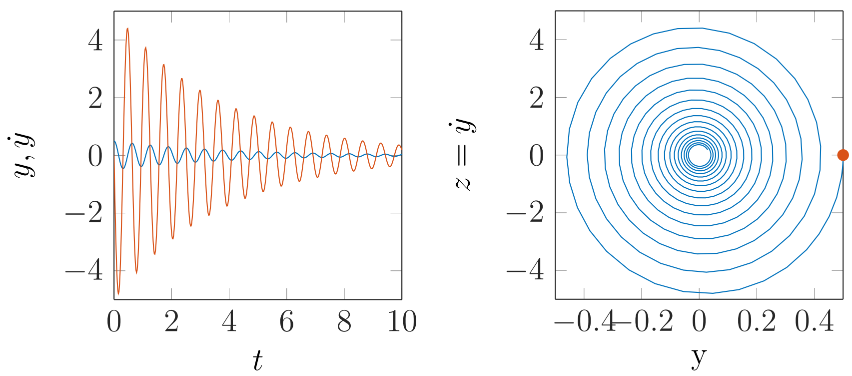


Figure 5: Nonlinear simulation results shown as time plot (left) and in their phase plane representation (right) for $V = 20 < V_c$. In the left plot bridge position is shown in blue. The first derivative is depicted in red. In the right plot the initial condition is depicted as a red dot.

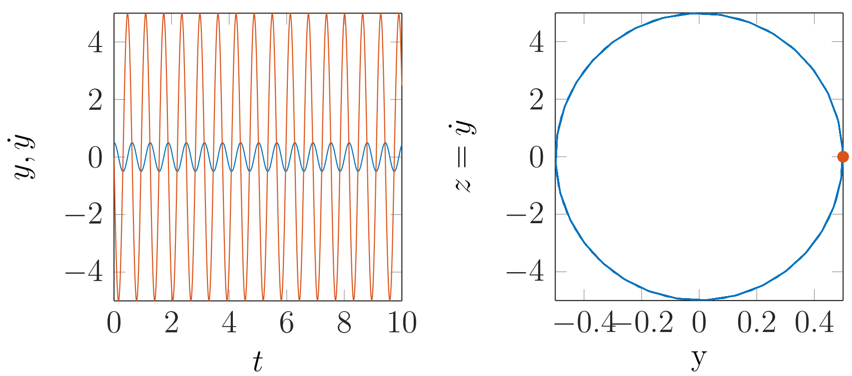


Figure 6: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = V_c$.

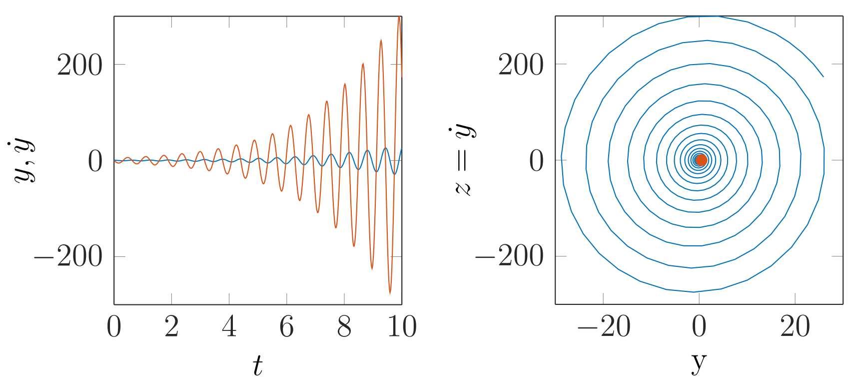


Figure 7: Nonlinear simulation results shown as time plot and in their phase plane representation for $V = 80 > V_c$.

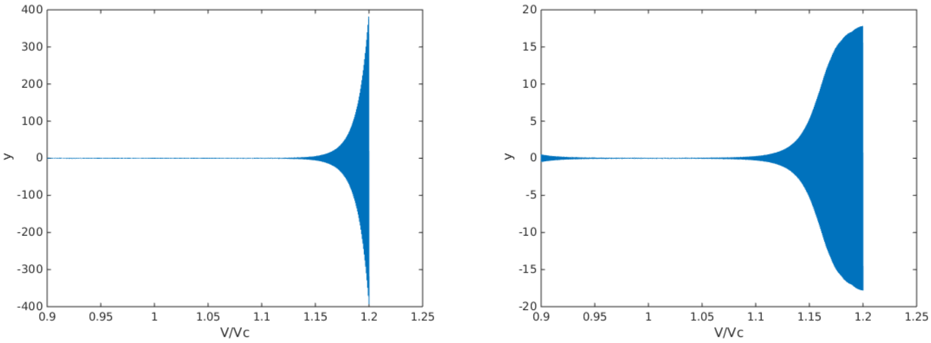


Figure 8: Linear(left) and nonlinear(right) model simulation results for wind speed values close to $V_c$

Given that $V_c$ simplifies to $V_c = \frac{2}{A_1}$ the linear approximation predicts the term for the first derivative to be zero. However in the nonlinear case one obtains:

$$0 = -\ddot{y} + \underbrace{(\frac{1}{2}V_c A_1 - 1)}_{=0}\dot{y} - ky + \underbrace{\frac{1}{2}V_c^2(-A_3\frac{z^3}{V_c^3} + A_5\frac{z^5}{V_c^5} - A_7\frac{z^7}{V_c^7})}_{\neq 0} \tag{13}$$

The additional nonlinear terms will move the position of the bifurcation points slightly. Furthermore this polynomial will exhibit quite different growth behavior in comparison to it's linear counter part. An expectation, which can indeed be confirmed by simulation. Figure 8 shows the linear and nonlinear model dynamics for wind speed values close to $V_c$.