Exercise 1 Laser Model

Moritz Wolter

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1 The equations

$$\dot{n} = GnN - kn. \tag{1}$$
$$\dot{N} = -GnN - fN + p. \tag{2}$$

2 First Order Analysis

Assume that the number of excited atoms remains quasi-static.

$$\dot{N} \approx 0. \tag{3}$$
$$\Rightarrow \dot{n} = Gn \cdot \frac{p}{Gn + f} - kn. \tag{4}$$
$$\tag{5}$$

2.1 Linear Stability analysis

In the two dimensional case fixed points are at the intersections of the first derivative with the real axis. Thus the can be found by solving $\dot{n} = 0$. Leading to the problem:

$$0 = n(\frac{Gp}{Gn + f} - k) \tag{6}$$

Which has the zeros 0 and $\frac{p}{k} - \frac{f}{G}$. To learn more about the nature of the fixed points one has to set the derivative of 4 to zero. Using the Quotient rule the expression:

$$\ddot{n} = \frac{Gfp}{(Gn + f)^2} - k \tag{7}$$

is obtained. A fixed point is stable if the second derivative is negative. Likewise it is unstable if the second derivative is positive [1]. Evaluating the second derivative for the fixed point a zero leads to $\ddot{n}(0) = \frac{Gp}{f} - k$ which leads to a values of $p_c = \frac{fk}{G}$. Thus the fixed points becomes unstable if $p > p_c$.

Figure 1 indicates a trans-critical bifurcation.

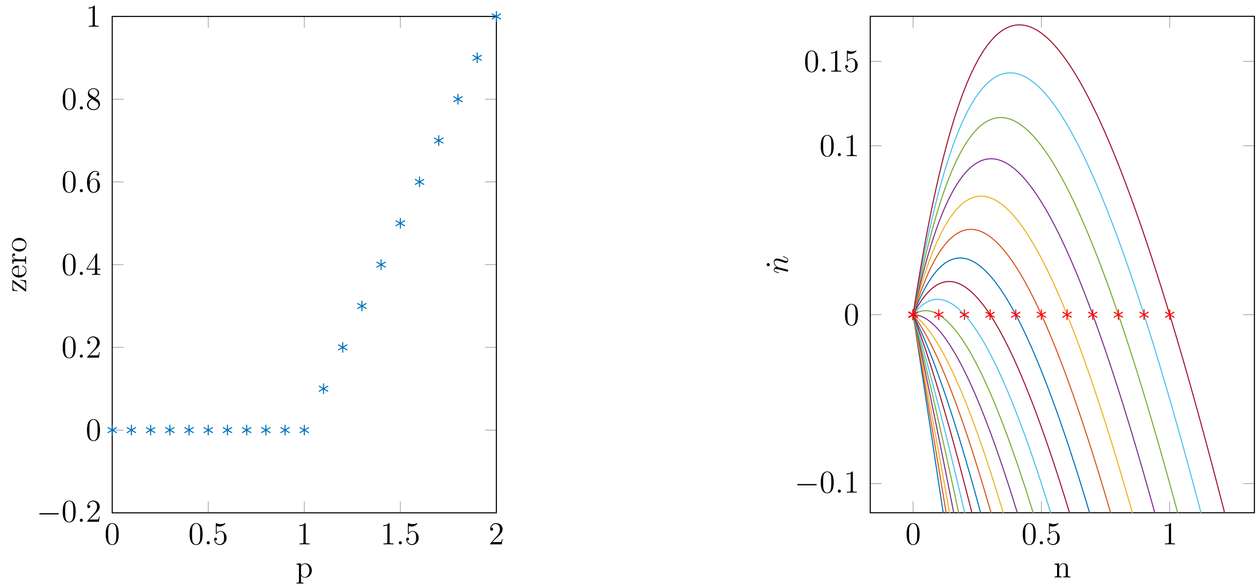

Figure 1: Plot of the position of the right zero and the parameter p (left). Plot of $\dot{n}$ for different p values with the zeros marked with red stars. $G = f = k = 1$ is assumed, leading to $p_c = 1$.

3 Two dimensional analysis

3.1 Nondimensionalization

To reduce the amount of parameters and simplify the analysis the dimensions are non-dimensionalized. In a first step scaling parameters are introduced:

$$\tau = \frac{t}{\gamma} \quad x = \frac{n}{\alpha} \quad y = \frac{N}{\beta}. \tag{8}$$

$$\Rightarrow t = n\tau \quad n = x\alpha \quad N = y\beta \tag{9}$$

Using the chain rule for the derivatives leads to $\dot{n} = \frac{dn}{dt} = \frac{dn}{d\tau} \cdot \frac{d\tau}{dn} = \frac{dn}{d\tau} \cdot \frac{1}{\gamma} = \frac{dx}{d\tau} \cdot \frac{\alpha}{\gamma} = \hat{x}\frac{\alpha}{\gamma}$. Following a similar derivation for $\hat{y}$ and substituting leads to:

$$\hat{x} = Gxy\beta\gamma - kx\gamma \tag{10}$$

$$\hat{y} = -Gx\alpha y\gamma - fy\gamma + \frac{p\gamma}{\beta}. \tag{11}$$

Now the equations may be simplified by setting groups of coefficients to one. This choice determines the position of the constants. Several options exist:

3.2 The naïve approach

Ensuring the non-dimensionality of the system the last group in equation 11 is set to one. Next the last group in 10 is set to one. Now the remaining options are only the first terms

[1] Strogatz, p.25

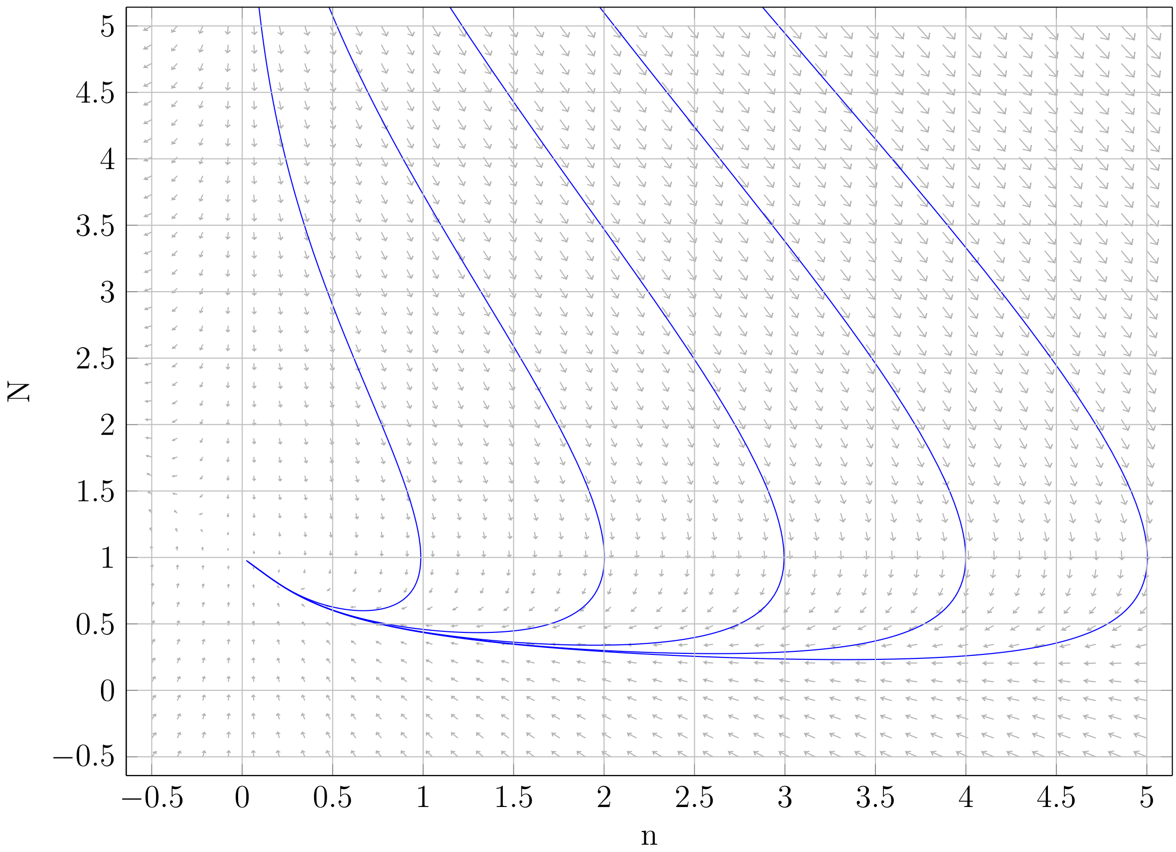

Figure 2: Phase plane of the original system $n' = GnN - kn \quad N' = -GnN - fN + p$, which an uncritical choice for the parameters $p < p_c$. $(f = k = G = p = 1)$

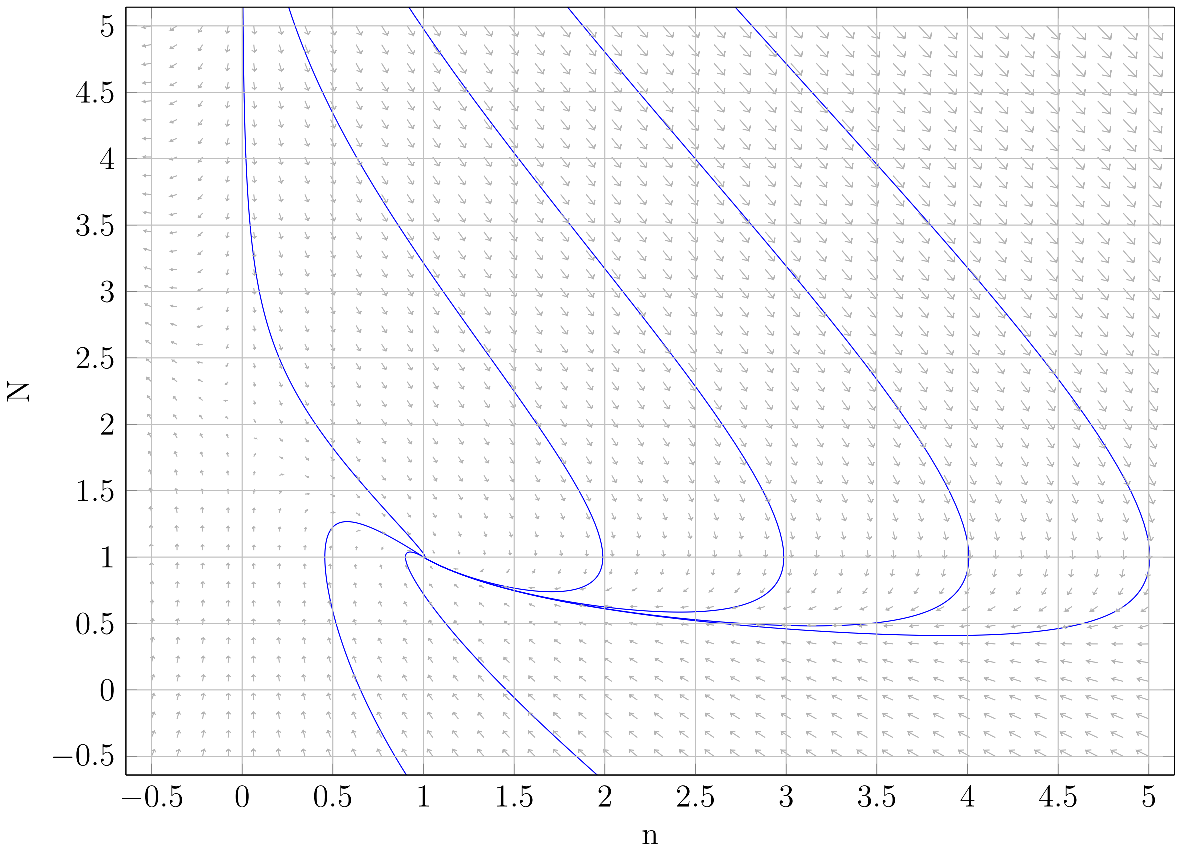

Figure 3: Phase plane of the original system $n' = GnN - kn \quad N' = -GnN - fN + p$, with $p < p_c$.$(f = k = G = 1 \;\; p = 2)$

$\mathbf{x}_1^* = \begin{pmatrix} 0 \\ \frac{1}{c_2} \end{pmatrix}$	$\tau_1 = \frac{c_1}{c_2} - c_2 - 1$	$\triangle_1 = c_2 - c_1$
$\mathbf{x}_2^* = \begin{pmatrix} -c_2 + c_1 \\ \frac{1}{c_1} \end{pmatrix}$	$\tau_2 = -c_1$	$\triangle_2 = c_1 - c_2$

Table 1: Fixed points, Jacobi trance and determinant as found by using the naïve approach.

of the two equations. Somewhat arbitrarily the first group of the second equation is set to one. This leads to:

$$\frac{p\gamma}{\beta} = 1 \tag{12}$$
$$k\gamma = 1 \tag{13}$$
$$G\alpha\gamma = 1. \tag{14}$$

Solving for the scaling parameters and substituting the remaining system parameters with c_1 and c_2 leads to:

$$\hat{x} = c_1 xy - x. \tag{15}$$
$$\hat{y} = -xy - c_2 y + 1. \tag{16}$$

Now the fixed points can be determined by solving for the zero vector $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$:

$$\mathbf{x}_1^* = \begin{pmatrix} 0 \\ \frac{1}{c_2} \end{pmatrix} \tag{17}$$
$$\mathbf{x}_2^* = \begin{pmatrix} -c_2 + c_1 \\ \frac{1}{c_1} \end{pmatrix}. \tag{18}$$

Above $c_1 = \frac{Gp}{k^2}$ and $c_2 = \frac{f}{k}$. To learn more about the nature of these points the Jacobi-matrix is computed, which is done by computing the partial derivative with respect to x in the first and to y in the second column.

$$J = \begin{pmatrix} yc_1 - 1 & xc_1 \\ -y & -x - c_2. \end{pmatrix} \tag{19}$$

This matrix my now be evaluated at the fixed points, so $J(\mathbf{x}_1^*)$ and $J(\mathbf{x}_2^*)$. Are computed, from here the nature of the fixed points can be classified by looking at the trace τ and the determinant $\triangle$. The results may be found in table 1.

3.3 The smart approach

In an attempt to simplify the expressions obtained above a second way to setting the scaling parameters will be considered. From the Jacobi matrix given in equation 19 it follows that the factors before the xy terms show up twice in the matrix. Coefficients that scale a single x or y terms appear once. While the constant disappears fully. However to keep the equations dimensionless the constant term has to remain one. Thus the remaining two groups before the high-frequent xy terms will be set to one. Starting from

$x_1^* = \begin{pmatrix} 0 \\ \frac{1}{c_2} \end{pmatrix}$	$\tau_1 = \frac{1}{c_2} - c_1 - c_2$	$\triangle_1 = c_1 c_2 - 1$
$x_2^* = \begin{pmatrix} \frac{1-c_1c_2}{c_1} \\ c_1 \end{pmatrix}$	$\tau_2 = -\frac{1}{c_1}$	$\triangle_2 = -c_1c_2 + 1$

Table 2: Results obtained by using the smart method.

equations 10 and 11, a similar process as the one outlined above is followed, which leads to the system:

$$\hat{x} = xy - c_1 x \tag{20}$$
$$\hat{y} = -xy - c_2 y + 1 \tag{21}$$

With the two constants $c_1 = \sqrt{\frac{k^2}{Gp}}$ and $c_2 = \sqrt{\frac{f^2}{Gp}}$. And the fixed points turn out to be $x_1^* = \begin{pmatrix} 0 & \frac{1}{c_2} \end{pmatrix}^T$ and $x_2^* = \begin{pmatrix} \frac{1-c_1c_2}{c_1} & c_1 \end{pmatrix}^T$. This new system leads indeed to a simpler Jacobi matrix, which is given by:

$$J = \begin{pmatrix} y - c_1 & x \\ -y & -x - c_2 \end{pmatrix} \tag{22}$$

Again the Jacobi matrix is evaluated at the fixed points, results are given in table 2. Unfortunately, the obtained expressions are not simpler then those computed earlier. However the add another angle to the analysis of the results.

3.4 Interpretation

3.4.1 Fixed point position

Due to the rescaling of the axes that has been performed naturally the position of the fixed points changes, however their properties will remain unchanged. In stead of the original position at

$$\mathbf{n}_1^* = \begin{pmatrix} 0 \\ \frac{p}{f} \end{pmatrix} \tag{23}$$
$$\mathbf{n}_2^* = \begin{pmatrix} -\frac{f}{G} + \frac{p}{k} \\ \frac{k}{G} \end{pmatrix} \tag{24}$$

The fixed points will appear at:

$$\mathbf{x}_1^* = \begin{pmatrix} 0 \\ \frac{p}{f\beta} \end{pmatrix} \tag{25}$$
$$\mathbf{x}_2^* = \begin{pmatrix} -\frac{f}{G\alpha} + \frac{p}{k} \\ \frac{k}{G\beta} \end{pmatrix} \tag{26}$$

after the rescaling process. The new position depends on the scaling parameters α and β.

condition	trace	determinant	fixed point classification
$p = p_c$	$\tau_1 < 0$	$\triangle_1 = 0$	$\mathbf{x}_1^*$ non-isolated stable fixed point
$c_1 = c_2$	$\tau_2 < 0$	$\triangle_2 = 0$	$\mathbf{x}_2^*$ non-isolated stable fixed point
$p > p_c$	$\tau_1 > -c_2$	$\triangle_1 < 0$	$\mathbf{x}_1^*$ saddle point
$c_1 > c_2$	$\tau_2 < 0$	$\triangle_2 > 0$	$\mathbf{x}_2^*$ stable fixed point
$p < p_c$	$\tau_1 < -c_2$	$\triangle_1 > 0$	$\mathbf{x}_1^*$ stable fixed point
$c_1 < c_2$	$\tau_2 < 0$	$\triangle_2 < 0$	$\mathbf{x}_2^*$ saddle point

Table 3: Classification of fixed points using the results from the naïve approach

3.4.2 Bifurcations

Since the results from the smart approach do not turn out to be simpler then those found with the first one. The formulations from table 1 will be used during the subsequent analysis. Starting from the condition for p_c:

$$p = p_c \tag{27}$$

$$p = \frac{fk}{G} \tag{28}$$

$$Gp = fk \tag{29}$$

$$\frac{Gp}{k} = f \tag{30}$$

$$\frac{Gp}{k^2} = \frac{f}{k} \tag{31}$$

$$\Leftrightarrow c_1 = c_2. \tag{32}$$

Using the same procedure for $p > p_c$ yields $c_1 > c_2$ and $p < p_c$ leads to $c_1 < c_2$. Armed with these new inequalities it is now possible to proceed with evaluate the expressions given in table 1. Results are shown in 3.

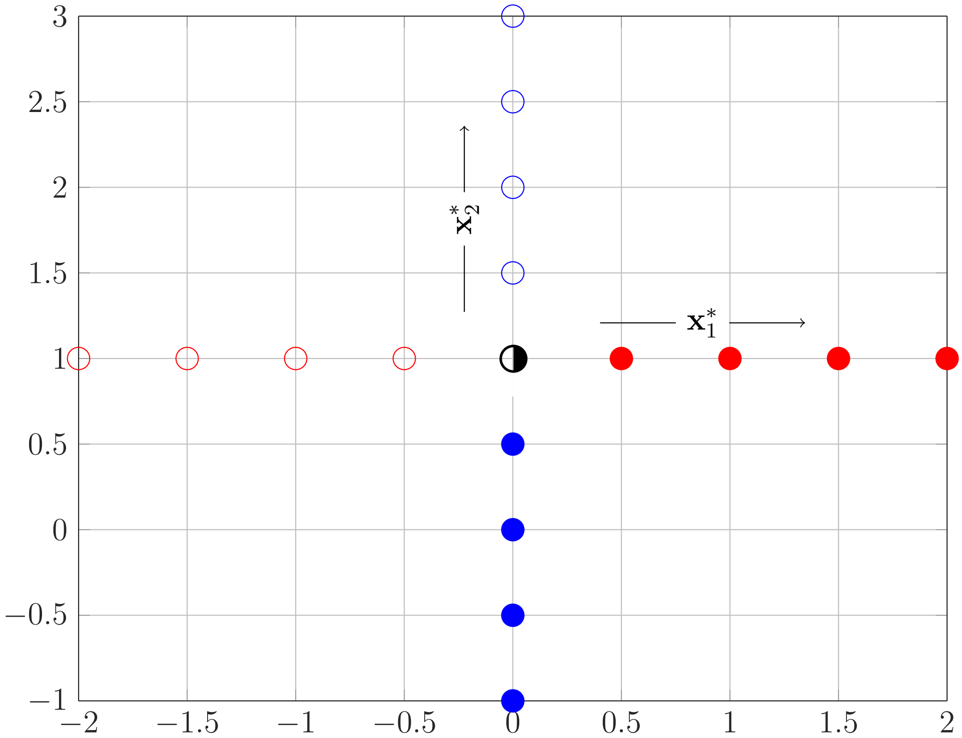

Figure 4: Bifurcation diagram for the given system with $f = k = G = 1$ and $p \in (-1, 3)$. The filled circles indicate stable attracting fixed points. The empty ones symbolize a saddle point. The half filled circle represents a bifurcation.