The Tortoise and the Hare, Restart GMRES

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Numerical linear algebra

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Recap GMRES(m)

1. Start: Choose x_0 and compute $r_0 = f - Ax_0$ and $v_1 = r_0/\|r_0\|$
2. Iterate: For j = 1,2,. . . ,m do
   $h_{i,j} = (Av_j, v_i), i = 1, \dots, j$
   $\hat{v}_{j+1} = Av_j - \sum_{i=1}^{j} h_{i,j}v_i$
   $h_{j+1,j} = \|\hat{v}_{j+1}\|$, and
   $v_{j+1} = \hat{v}_{j+1}/h_{j+1,j}$
3. Form the approximate solution: $x_m = x_0 + V_m y_m$ where y_m minimizes $\|\beta e_1 - H_m y\|, y \in \mathbb{R}^m$
4. Restart: Compute $r_m = f - Ax_m$; if satisfied stop, else $x_0 = x_m, v_1 = r_m/\|r_m\|$

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GMRES($\mathtt{m}$) convergence

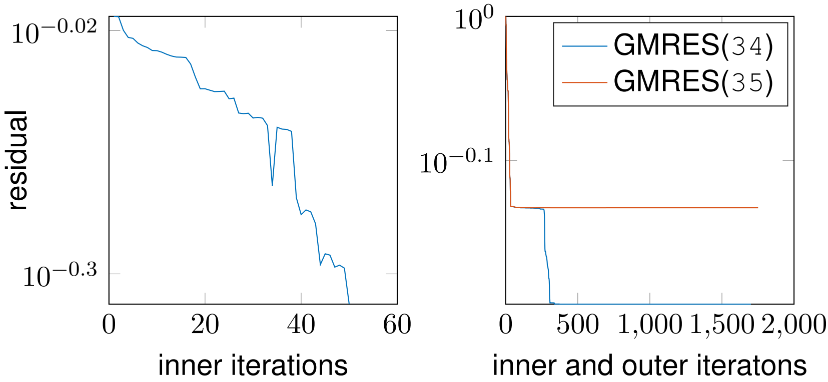

Figure : GMRES($\mathtt{m}$) convergence on large random matrix $\mathbf{A}$ of dimension 60 with $\mathbf{b} = [0 \ldots 1 \ldots 0]^T$ in blocks of 20 each.

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{b}_1 = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \tag{1}$$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{b}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

Gmres convergence

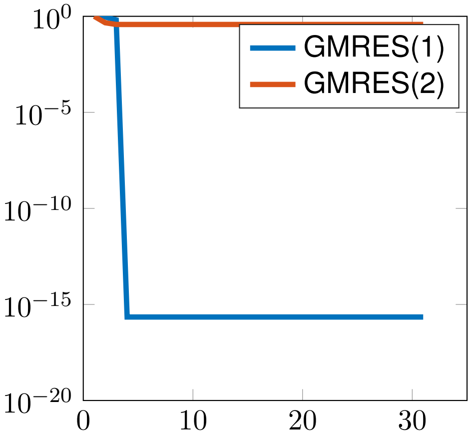

Figure : `GMRES` convergence on matrix $\mathbf{A}_1$.

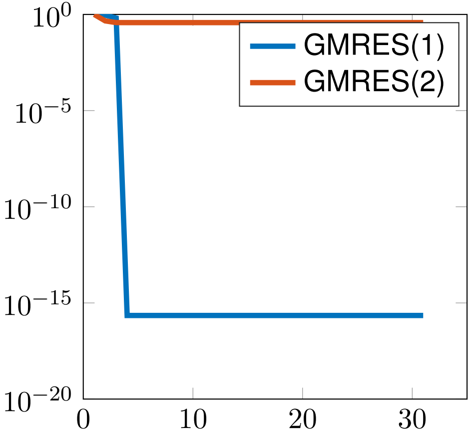

Figure : `GMRES` convergence on matrix $\mathbf{A}_2$.

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Convergence in a two dimesional plane

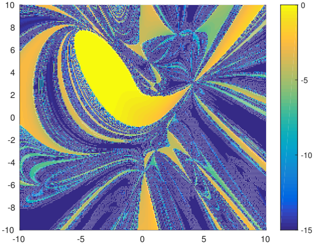

Figure : Residual magnitude Plot for matrix $\mathbf{A}_2, \mathbf{b}_2$, as seen in the Embree paper for GMRES(1)(left) and GMRES(2)(right) in figure 4.

Convergence in a two dimesional plane

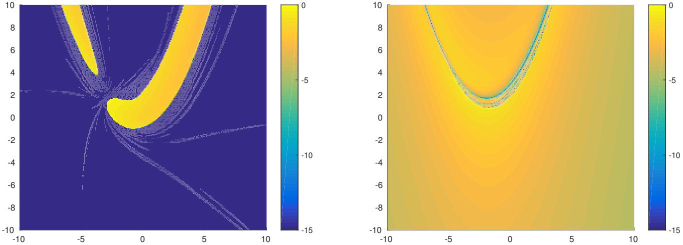

Figure : Residual magnitude Plot for the $\mathbf{A}_1, \mathbf{b}_1$ pair, as seen in the Embree paper GMRES(1)(left) and GMRES(2)(right) in figure two.

Conditioning makes a difference.

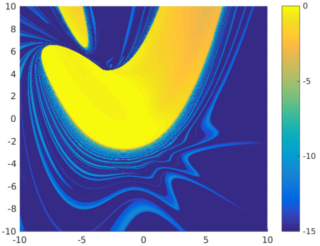

Figure : GMRES(1) and GMRES(2) convergence on $\mathbf{A} - 0.35 \cdot \mathbf{I}$.
$\kappa(\mathbf{A}) = 14.2950$ changes to $\kappa(\mathbf{A} - 0.35 \cdot \mathbf{I}) = 39.1873$.

A random matrix.

$$\mathbf{R} = \begin{pmatrix} -0.3034 & 0.8884 & -0.8095 \\ 0.2939 & -1.1471 & -2.9443 \\ -0.7873 & -1.0689 & 1.4384 \end{pmatrix} \quad (3)$$

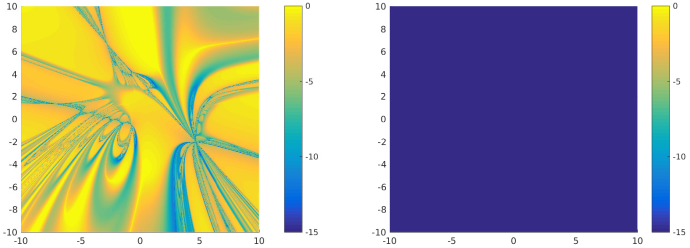

Figure : Convergence of `GMRES(1)` (left) and `GMRES(2)` (right) for the random matrix $\mathbf{R}$. $\kappa(\mathbf{R}) = 5.1025$

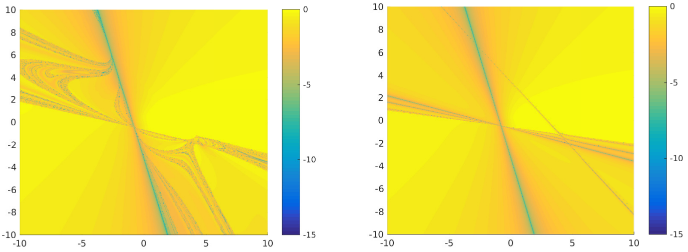

Figure : Convergence results for $\mathbf{R} + 2\mathbf{I}$. $\kappa(\mathbf{R} + 2\mathbf{I}) = 77.5446$.

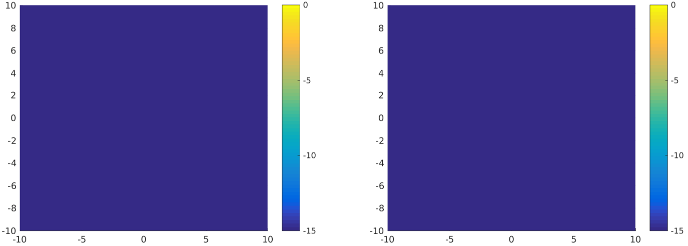

Figure : Convergence results for $\mathbf{R} + 3\mathbf{I}$. $\mathbf{R} + 3\mathbf{I} = 6.4262$.

$$\mathbf{C} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \mathbf{r_0} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{4}$$

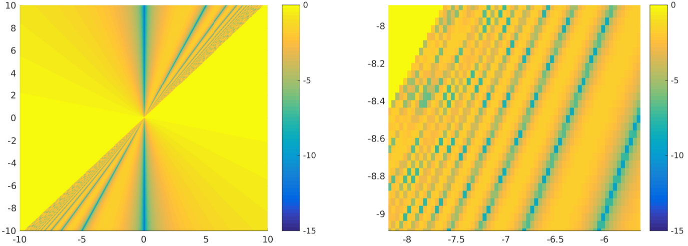

Figure : A plot of the convergence plain for the matrix $\mathbf{C}$ proposed at the end of Embree's paper. With zoom on an interesting region.

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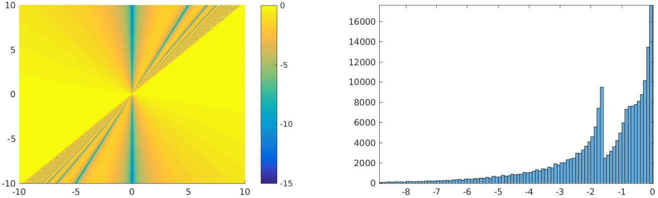

Figure : Convergence Plain and error histogram for matrix $\mathbf{C}$.

- $\mathcal{D}_1 = 10^{-1} + \|\mathcal{N}(0, 2.5 \cdot 10^{-3})\|$
- $\mathcal{D}_2 = 10^{-5} + \|\mathcal{N}(0, 2.5 \cdot 10^{-3})\|$

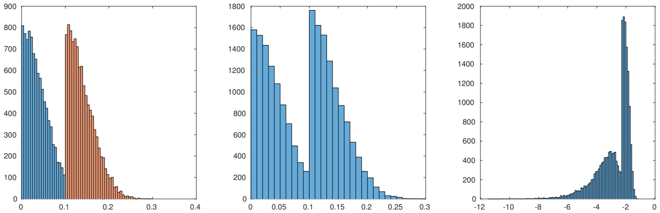

Figure : Plots of the distributions generated by `d1 = 0.00001 + 0.05*abs(randn(10000,1)); d2 = 0.1 + 0.05*abs(randn(10000,1));` . The last plot is on a semilogarithmic scale.

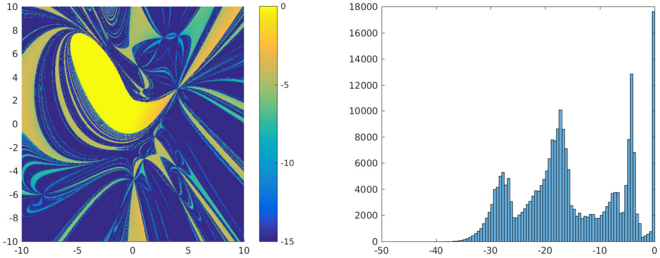

Figure : Convergence Plain and error histogram for matrix $\mathbf{B}$.

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Recurrence for `gmres(1)`.

$$\mathbf{r}_{k+1}^{(1)} = \mathbf{r}_k^{(1)} - \frac{\mathbf{r}_k^{(1)T}\mathbf{A}\mathbf{r}_k^{(1)}}{\mathbf{r}_k^{(1)T}\mathbf{A}^T\mathbf{A}\mathbf{r}_k^{(1)}}\mathbf{A}\mathbf{r}_k^{(1)} \qquad (5)$$

$\Rightarrow$ `gmres(1)` must stagnate if $\mathbf{r}_k^{(1)T}\mathbf{A}\mathbf{r}_k^{(1)} = 0$.

Matrix $\mathbf{A}$

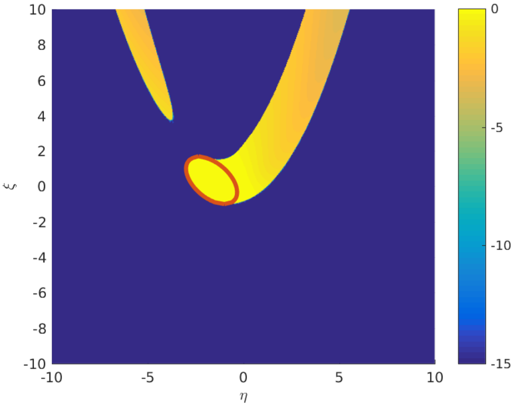

Figure : Region of guaranteed convergence with `gmres(1)` on matrix $\mathbf{A}$.

Matrix $\mathbf{B}$

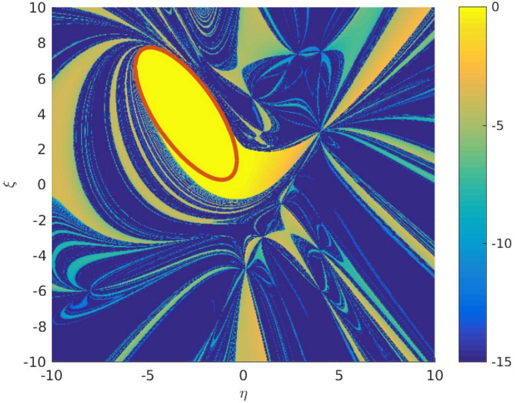

Figure : Region of guaranteed convergence with `gmres(1)` on matrix $\mathbf{B}$.