NUMERICAL SIMULATION OF PARTIAL DIFFERENTIAL EUQATIONS IN TWO DIMENSIONS.

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1 Implementation of explicit methods

In this report we are going to implement explicit methods to solve three different partial differential equations in two dimensions and analyze the results.

1.1 Heat Equation

We will begin with the numerical solution of the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

From the lecture we know that this problem may be solved by extension of the one dimensional explicit scheme:

$$\frac{U^{n+1} - U^n}{\triangle t} = b[\frac{\delta_x^2 U^n}{(\triangle x)^2} + \frac{\delta_y^2 U^n}{(\triangle y)^2}]. \tag{2}$$

with b = 1 in our case. By expanding the central differences we arrive at:

$$U_{r,x}^{n+1} = U_{r,s}^n(1 - 2\mu_x - 2\mu_y) + \mu_x U_{r+1,s}^n + \mu_x U_{r-1,s}^n + \mu_y U_{r,s+1}^n + \mu_y U_{r,s-1}^n. \tag{3}$$

Equation 3 my be implemented in matlab. With $\mu_x = \frac{\triangle t}{(\triangle x)^2}$ and $\mu_y = \frac{\triangle t}{(\triangle y)^2}$. As we are using a symmetric grid we have $\mu_x = \mu_y$ which leads to the implementation in listing 1:

```
tend = 3;
dt = 0.0001;
J = 30;
dx = 1/J;
dy = 1/J;
mu = dt/dx^2;

%Set up a mesh.
[x,y] = meshgrid(linspace(0,1,J));
%Initial solution.
```

```
U = sin(pi*x).*sin(pi*y);
U1 = zeros(J);
U2 = zeros(J);
for t = 1:(tend/dt)
    elements = 2:J-1;
    for i = 1:1:J
        %compute the columns where x is const.
        U1(elements,i) = mu*U(elements+1,i) + mu*U(elements-1,i);
        %compute the columns where y is const.
        U2(i,elements) = mu*U(i,elements+1) + mu*U(i,elements-1);
    end
    Unew = (1 - 4*mu) .* U + U1 + U2;
    U = Unew;
end
surf(x,y,U); axis([0 1 0 1 -1 1 -1 1]);
```

Listing 1: Explicit solution of the heat equation in two dimensions.

1.2 Wave equation

Next we are going to implement an explicit scheme to solve the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{4}$$

Approximating the second derivatives with central differences and solving for $U_{r,s}^{n+1}$ we obtain the scheme:

$$\begin{aligned} U_{r,s}^{n+1} = U_{r,s}(2 - 2\mu_x - 2\mu_y) - U_{r,s}^{n-1} \\ +\mu_x U_{r-1,s}^{n} + \mu_x U_{r+1,s}^{n} \\ +\mu_y U_{r,s-1}^{n} + \mu_y U_{r,s+1}. \end{aligned}$$

With $\mu_x = \frac{(\triangle t)^2}{(\triangle x)^2}$ and $\mu_y = \frac{(\triangle t)^2}{(\triangle y)^2}$. In comparison to listing 1 when listing only changes to the for loop. We got:

```
for t = 1:(tend/dt)
    elements = 2:J-1;
    for i = 1:1:J
        %compute the columns where x is const.
        U1(elements,i) = mu*U(elements+1,i) + mu*U(elements-1,i);
        %compute the columns where y is const.
        U2(i,elements) = mu*U(i,elements+1) + mu*U(i,elements-1);
    end
    Unew = (2 - 4*mu) .* U - Uold + U1 + U2;
    Uold = U;
    U = Unew;
end
```

Listing 2: Code for solving the wave equation.

Obviously different initial parameters will have to be chosen here.

1.3 Transport equation

Before we are going to consider the numerical stability and accuracy of these methods we will implement a final scheme to solve the transport equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}. \tag{5}$$

Using exclusively forward differences to approximate the first order differentials and solving for $U_{r,s}^{n+1}$ once more we obtain:

$$U_{r,s}^{n+1} = U_{r,s}^{n}(1 - \mu_x - \mu_y) + \mu_x U_{r+1,s} + \mu_y U_{r,s+1}. \tag{6}$$

With $\mu_x = \frac{\triangle t}{\triangle x}$ and $\mu_y = \frac{\triangle t}{\triangle y}$ .Which leads to the modified for-loop for the transport case:

```
elements = 2:J-1;
for i = 1:1:J
    %compute the columns where x is const.
    U1(elements,i) = mu*U(elements+1,i);
    %compute the columns where y is const.
    U2(i,elements) = mu*U(i,elements+1);
end
Unew = (1 - 2*mu) .* U + U1 + U2;
U = Unew;
```

Listing 3: Code for solving the two dimesional transport equation.

2 Analysis

2.1 Stability

Figure 1 shows stable and unstable solutions for the three equations. We will proceed with taking a close look at the numerical properties of the methods we described so far. In order to obtain stable solutions in two dimensions we have to satisfy the condition:

$$\mu_x + \mu_y \leq 0.5 \tag{7}$$

For equally spaced with $\mu_x = \mu_y = \mu$ grids we get:

$$\mu \leq 0.25 \tag{8}$$

Where μ_x and μ_y are different for every method, as derived in the previous section. For the following computations we kept $J = 30$, defined $\triangle x = \triangle y = 1/J$ and computed dt from mu. For the heat equation for the solution in figure 1 in the top left is stable here we have $\mu = 0.25 = \mu_x = \mu_y$ at $t = 1$. However to compute the image in the bottom left we used a grid ratio of $\mu = 0.252$ and we observe instability at the same point in time. The solution of the wave equation is shown in the top middle position for time t=1 with $\mu = 0.25$. We found the wave equation to be surprisingly stable. In our computations first instabilities appeared at time $t = 10$ with a grid ratio of $\mu = 0.51$. The transport problem also turned out to be more stable then the heat equation. In the top right corner of figure 1 we show the solution of the transport equation for the given initial condition at time 0.5. First instabilities at this time appeared with a grid ratio of $\mu = 0.6$, which are shown below.

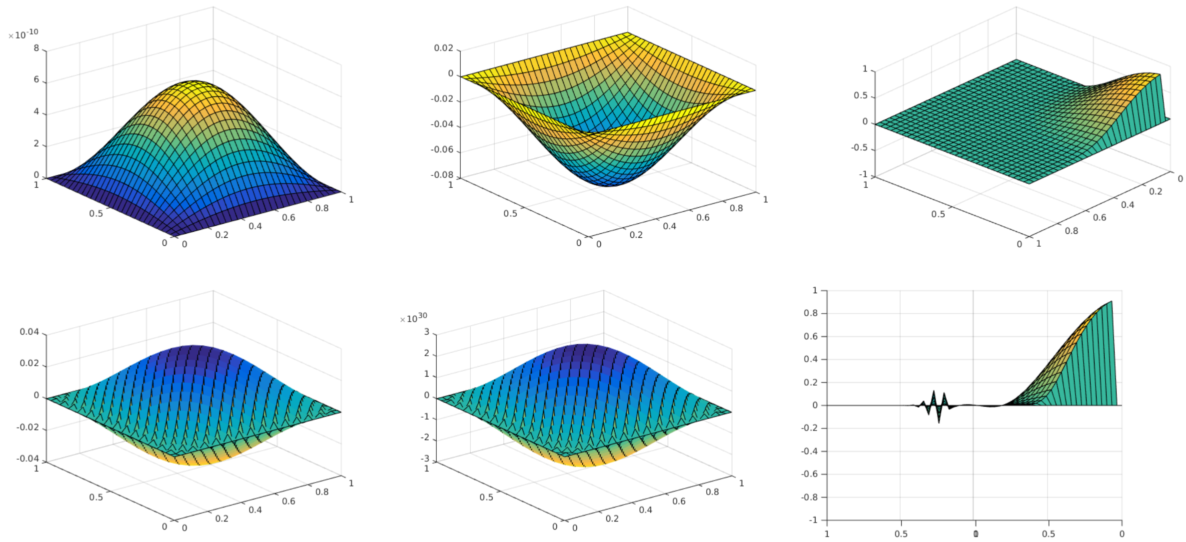

Figure 1: Numerical solutions computed using the schemes described above for stable (top row) and unstable (bottom row) grid ratios μ.

2.2 Accuracy

2.2.1 Exact Solution of the Heat Equation

By using the ansatz $u = T(t)Y(y)X(x)$ followed by separation of variables and using the initial conditions we arrived at the exact solution:

$$U(x, y, t) = e^{-2\pi^2 t} sin(\pi x) sin(\pi y). \tag{9}$$

Which indeed satisfies the differential equation.

2.2.2 Exact Solution of the Wave Equation

Again using the same Idea we propose the solution:

$$u(x, y, t) = cos(\sqrt{2}\pi t) sin(\pi x) sin(\pi y). \tag{10}$$

which indeed satisfies the partial differential equation.

2.2.3 Exact Solution of Transport Equation

As a solution for the transport equation of form

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} \tag{11}$$

with a given initial conditions, $u_0(x, y)$, and homogenous dirichet boundary conditions we propose a solution of the following form:

$$u(x, y, t) = u_0(x - vt, y - vt) \tag{12}$$

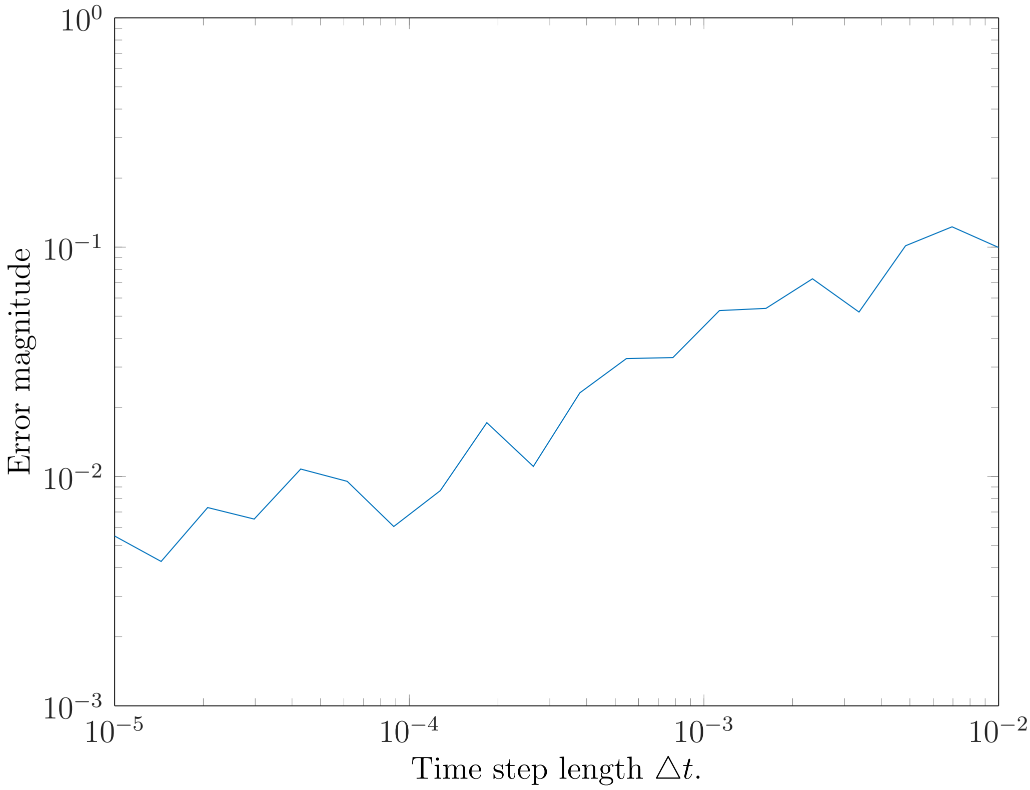

Figure 2: The error magnitude of the heat equation with constant $\mu = \frac{\triangle t}{(\triangle x)^2} = 0.2$. At time $t = 0.1$.

Calculating the partial derivatives found in the transport equation we get

$$\frac{\delta u}{\delta t} = (-v)\frac{\delta u_0(x,y)}{\delta x} + (-v)\frac{\delta u_0(x,y)}{\delta y} \tag{13}$$

$$\frac{\delta u}{\delta x} = \frac{\delta u_0(x,y)}{\delta x}, \frac{\delta u}{\delta y} = \frac{\delta u_0(x,y)}{\delta y} \tag{14}$$

Filling this then in the transport equation gives

$$-v\frac{\delta u_0}{\delta x} - v\frac{\delta u_0}{\delta y} = \frac{\delta u_0}{\delta x} + \frac{\delta u_0}{\delta y} \tag{15}$$

which fits when $v = -1$ and so our solution is

$$u(x,y,t) = u_0(x+t, y+t) \tag{16}$$

2.3 Error analysis of our scheme for the heat equation

Analog to its one dimensional counterpart we find the two dimensional heat equation truncation error to be:

$$T(x,t) = \frac{1}{2}u_{tt}\triangle t - \frac{1}{12}u_{xxxx} - \frac{1}{12}u_{yyyy} + \dots \tag{17}$$

Which is of first order. In figure 2 the error of the heat equation with constant grid ratio at time $t = 0.1$ is shown with logarithmic scaling. We observe first order convergence to the exact result in the plot for decreasing step sizes.

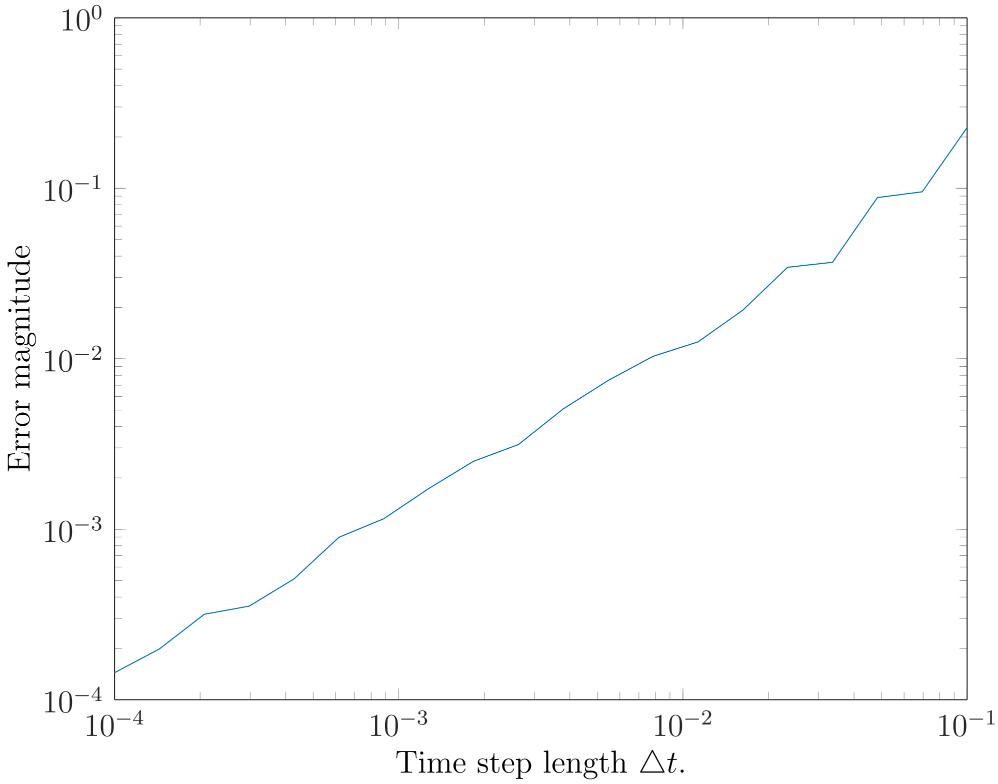

Figure 3: The error magnitude of the wave equation with constant $\mu = \frac{(\triangle t)^2}{(\triangle x)^2} = 0.2$. At time $t = 0.1$.

2.4 Error analysis of our scheme the wave equation

Figure 3 shows the magnitude of the error for different steps sizes. Again with logarithmic scaling. We observe that the solution of the scheme converges to the exact solution for smaller step sizes.

2.4.1 Error analysis of the transport problem

The truncation error for the upwind scheme with $a = 1$ can be found to be

$$T_j^n = -\frac{1}{2}(1-\nu)\Delta x u_{xx} + ... \tag{18}$$

Which is first order in Δx, and therefore, under constant ν, also first order in Δt.

We then now that the maximum error E^n at a point in time n is bound by a function of the same order. And this is what we see in figure 4

3 Another initial solution

Finally we are going to compute the solutions again with the initial solution:

$$u_0(x,y) = 15(x - x^2)(y - y^2)e^{-50(x-0.5)^2+(y-0.5)^2} \tag{19}$$

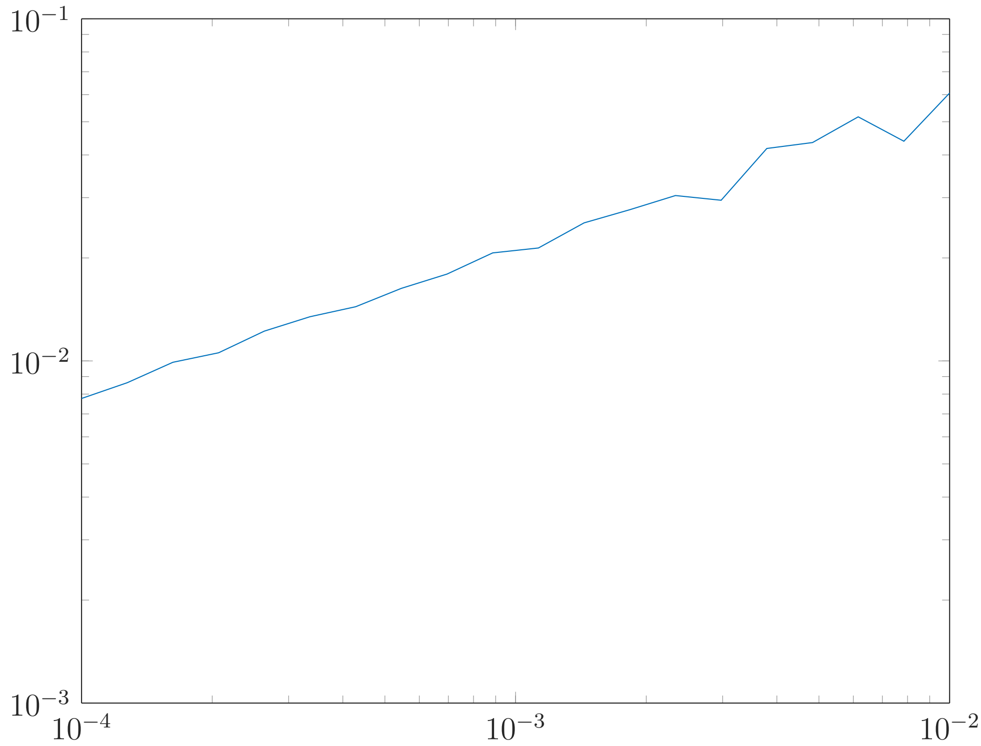

Figure 4: The maximum error at time = 0.1 in function of the time-step at a constant $\mu = \frac{\triangle t}{\triangle x}$ of 0.2

Figure 5 shows the results.[1]. It is important to note that the solution of the heat equation disappears extremely quickly. The maximum is of scale 10^{-2} at time $t = 0.1$ and decreases to 10^{-3} at $t = 0.2$.

[1]For solution of the wave equation, which we found exepctionally pretty a video may be downloaded from: `https://www.youtube.com/watch?v=gE13nlbE5ug&feature=youtu.be`

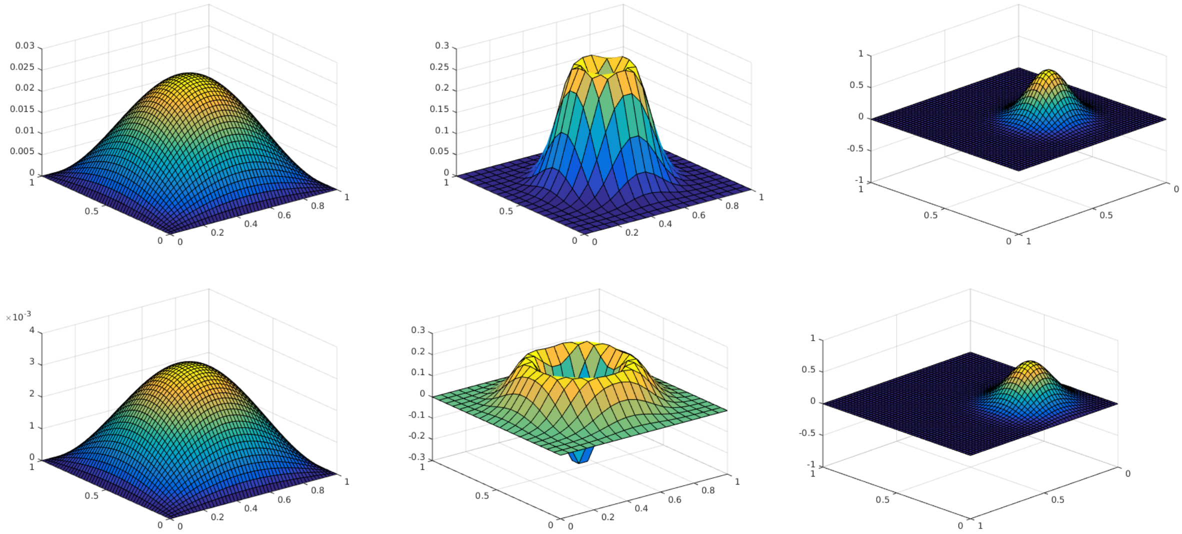

Figure 5: The tree equations with $\mu_x = \mu_y = 0.1$ at $t = 0.1$ (top row) and $t = 0.2$ (bottom row).