

# NUMERICAL SIMULATION OF PARTIAL DIFFERENTIAL EQUATIONS IN TWO DIMENSIONS.

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## 1 Implementation of explicit methods

In this report we are going to implement explicit methods to solve three different partial differential equations in two dimensions and analyze the results.

### 1.1 Heat Equation

We will begin with the numerical solution of the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad (1)$$

From the lecture we know that this problem may be solved by extension of the one dimensional explicit scheme:

$$\frac{U^{n+1} - U^n}{\Delta t} = b \left[ \frac{\delta_x^2 U^n}{(\Delta x)^2} + \frac{\delta_y^2 U^n}{(\Delta y)^2} \right]. \quad (2)$$

with  $b = 1$  in our case. By expanding the central differences we arrive at:

$$U_{r,x}^{n+1} = U_{r,s}^n (1 - 2\mu_x - 2\mu_y) + \mu_x U_{r+1,s}^n + \mu_x U_{r-1,s}^n + \mu_y U_{r,s+1}^n + \mu_y U_{r,s-1}^n. \quad (3)$$

Equation 3 may be implemented in matlab. With  $\mu_x = \frac{\Delta t}{(\Delta x)^2}$  and  $\mu_y = \frac{\Delta t}{(\Delta y)^2}$ . As we are using a symmetric grid we have  $\mu_x = \mu_y$  which leads to the implementation in listing 1:

```
tend = 3;
dt = 0.0001;
J = 30;
dx = 1/J;
dy = 1/J;
mu = dt/dx^2;

%Set up a mesh.
[x,y] = meshgrid(linspace(0,1,J));
%Initial solution.
```

```

U = sin(pi*x).*sin(pi*y);
U1 = zeros(J);
U2 = zeros(J);
for t = 1:(tend/dt)
    elements = 2:J-1;
    for i = 1:1:J
        %compute the columns where x is const.
        U1(elements,i) = mu*U(elements+1,i) + mu*U(elements-1,i);
        %compute the columns where y is const.
        U2(i,elements) = mu*U(i,elements+1) + mu*U(i,elements-1);
    end
    Unew = (1 - 4*mu) .* U + U1 + U2;
    U = Unew;
end
surf(x,y,U); axis([0 1 0 1 -1 1 -1 1]);

```

Listing 1: Explicit solution of the heat equation in two dimensions.

## 1.2 Wave equation

Next we are going to implement an explicit scheme to solve the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad (4)$$

Approximating the second derivatives with central differences and solving for  $U_{r,s}^{n+1}$  we obtain the scheme:

$$\begin{aligned}
U_{r,s}^{n+1} = & U_{r,s}(2 - 2\mu_x - 2\mu_y) - U_{r,s}^{n-1} \\
& + \mu_x U_{r-1,s}^n + \mu_x U_{r+1,s}^n \\
& + \mu_y U_{r,s-1}^n + \mu_y U_{r,s+1}^n.
\end{aligned}$$

With  $\mu_x = \frac{(\Delta t)^2}{(\Delta x)^2}$  and  $\mu_y = \frac{(\Delta t)^2}{(\Delta y)^2}$ . In comparison to listing 1 when listing only changes to the for loop. We got:

```

for t = 1:(tend/dt)
    elements = 2:J-1;
    for i = 1:1:J
        %compute the columns where x is const.
        U1(elements,i) = mu*U(elements+1,i) + mu*U(elements-1,i);
        %compute the columns where y is const.
        U2(i,elements) = mu*U(i,elements+1) + mu*U(i,elements-1);
    end
    Unew = (2 - 4*mu) .* U - Uold + U1 + U2;
    Uold = U;
    U = Unew;
end

```

Listing 2: Code for solving the wave equation.

Obviously different initial parameters will have to be chosen here.

### 1.3 Transport equation

Before we are going to consider the numerical stability and accuracy of these methods we will implement a final scheme to solve the transport equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}. \quad (5)$$

Using exclusively forward differences to approximate the first order differentials and solving for  $U_{r,s}^{n+1}$  once more we obtain:

$$U_{r,s}^{n+1} = U_{r,s}^n(1 - \mu_x - \mu_y) + \mu_x U_{r+1,s} + \mu_y U_{r,s+1}. \quad (6)$$

With  $\mu_x = \frac{\Delta t}{\Delta x}$  and  $\mu_y = \frac{\Delta t}{\Delta y}$ . Which leads to the modified for-loop for the transport case:

```
elements = 2:J-1;
for i = 1:1:J
    %compute the columns where x is const.
    U1(elements,i) = mu*U(elements+1,i);
    %compute the columns where y is const.
    U2(i,elements) = mu*U(i,elements+1);
end
Unew = (1 - 2*mu) .* U + U1 + U2;
U = Unew;
```

Listing 3: Code for solving the two dimesional transport equation.

## 2 Analysis

### 2.1 Stability

Figure 1 shows stable and unstable solutions for the three equations. We will proceed with taking a close look at the numerical properties of the methods we described so far. In order to obtain stable solutions in two dimensions we have to satisfy the condition:

$$\mu_x + \mu_y \leq 0.5 \quad (7)$$

For equally spaced with  $\mu_x = \mu_y = \mu$  grids we get:

$$\mu \leq 0.25 \quad (8)$$

Where  $\mu_x$  and  $\mu_y$  are different for every method, as derived in the previous section. For the following computations we kept  $J = 30$ , defined  $\Delta x = \Delta y = 1/J$  and computed  $dt$  from  $\mu$ . For the heat equation for the solution in figure 1 in the top left is stable here we have  $\mu = 0.25 = \mu_x = \mu_y$  at  $t = 1$ . However to compute the image in the bottom left we used a grid ratio of  $\mu = 0.252$  and we observe instability at the same point in time. The solution of the wave equation is shown in the top middle position for time  $t=1$  with  $\mu = 0.25$ . We found the wave equation to be surprisingly stable. In our computations first instabilities appeared at time  $t = 10$  with a grid ratio of  $\mu = 0.51$ . The transport problem also turned out to be more stable then the heat equation. In the top right corner of figure 1 we show the solution of the transport equation for the given initial condition at time 0.5. First instabilities at this time appeared with a grid ratio of  $\mu = 0.6$ , which are shown below.

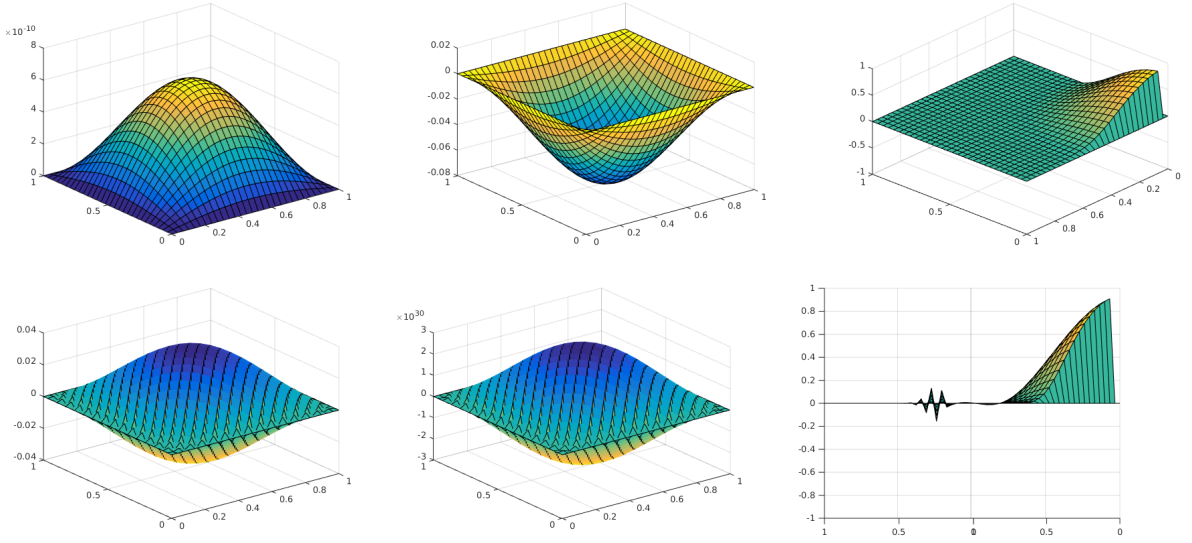


Figure 1: Numerical solutions computed using the schemes described above for stable (top row) and unstable (bottom row) grid ratios  $\mu$ .

## 2.2 Accuracy

### 2.2.1 Exact Solution of the Heat Equation

By using the ansatz  $u = T(t)Y(y)X(x)$  followed by separation of variables and using the initial conditions we arrived at the exact solution:

$$U(x, y, t) = e^{-2\pi^2 t} \sin(\pi x) \sin(\pi y). \quad (9)$$

Which indeed satisfies the differential equation.

### 2.2.2 Exact Solution of the Wave Equation

Again using the same Idea we propose the solution:

$$u(x, y, t) = \cos(\sqrt{2}\pi t) \sin(\pi x) \sin(\pi y). \quad (10)$$

which indeed satisfies the partial differential equation.

### 2.2.3 Exact Solution of Transport Equation

As a solution for the transport equation of form

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} \quad (11)$$

with a given initial conditions,  $u_0(x, y)$ , and homogenous dirichet boundary conditions we propose a solution of the following form:

$$u(x, y, t) = u_0(x - vt, y - vt) \quad (12)$$

Calculating the partial derivatives found in the transport equation we get

$$\frac{\delta u}{\delta t} = (-v) \frac{\delta u_0(x, y)}{\delta x} + (-v) \frac{\delta u_0(x, y)}{\delta y} \quad (13)$$

$$\frac{\delta u}{\delta x} = \frac{\delta u_0(x, y)}{\delta x}, \frac{\delta u}{\delta y} = \frac{\delta u_0(x, y)}{\delta y} \quad (14)$$

Filling this then in the transport equation gives

$$-v \frac{\delta u_0}{\delta x} - v \frac{\delta u_0}{\delta y} = \frac{\delta u_0}{\delta x} + \frac{\delta u_0}{\delta y} \quad (15)$$

which fits when  $v = -1$  and so our solution is

$$u(x, y, t) = u_0(x + t, y + t) \quad (16)$$

## 2.3 Error analysis of our scheme for the heat equation

TODO...

## 2.4 Error analysis of our scheme the wave equation

TODO...

### 2.4.1 Error analysis of the transport problem

The truncation error for the upwind scheme with  $a = 1$  can be found to be

$$T_j^n = -\frac{1}{2}(1 - \nu)\Delta x u_{xx} + \dots \quad (17)$$

Which is first order in  $\Delta x$ , and therefore, under constant  $\nu$ , also first order in  $\Delta t$ .

We then now that the maximum error  $E^n$  at a point in time  $n$  is bound by a function of the same order. And this is what we see in figure 2

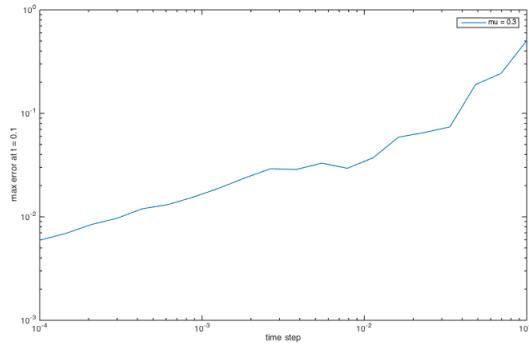


Figure 2: The maximum error at time = 0.1 in function of the time step at a constant  $\nu$  of 0.3

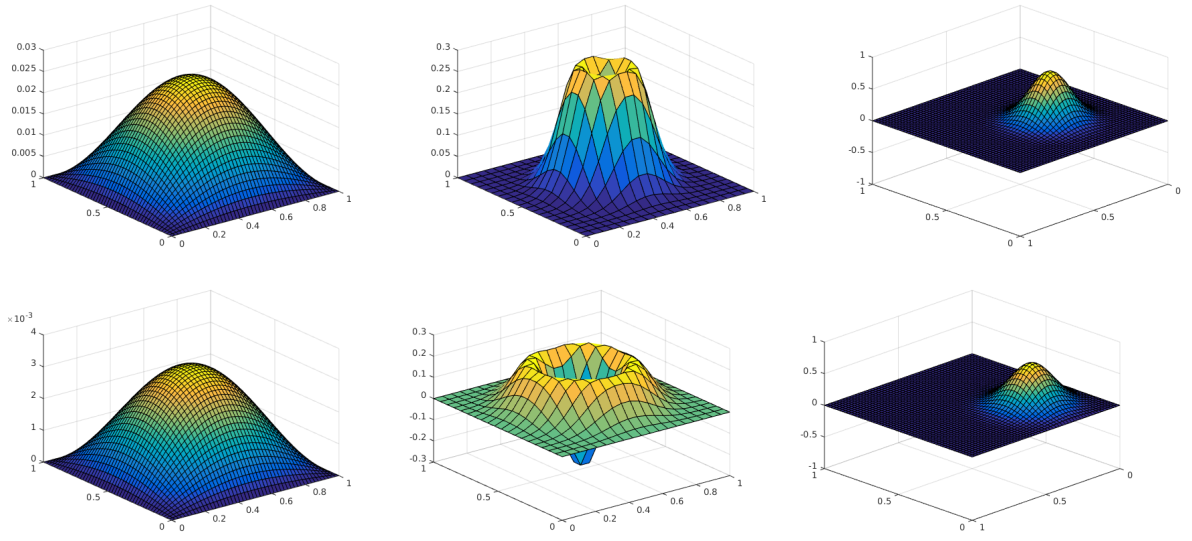


Figure 3: The tree equations with  $\mu_x = \mu_y = 0.1$  at  $t = 0.1$  (top row) and  $t = 0.2$  (bottom row).

### 3 Another initial solution

Finally we are going to compute the solutions again with the initial solution:

$$u_0(x, y) = 15(x - x^2)(y - y^2)e^{-50(x-0.5)^2 + (y-0.5)^2} \quad (18)$$

Figure 3 shows the results.<sup>1</sup> It is important to note that the solution of the heat equation disappears extremely quickly. The maximum is of scale  $10^{-2}$  at time  $t = 0.1$  and decreases to  $10^{-3}$  at  $t = 0.2$ .

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<sup>1</sup>For solution of the wave equation, which we found exepctionally pretty a video may be downloaded from: <https://www.youtube.com/watch?v=gE13n1bE5ug&feature=youtu.be>