Gradient Theorem:

"The fundamental theorem of calculus in *n* dimensions"

$$\int_a^b \nabla F \cdot dr = F(b) - F(a)$$

"A line integral through a gradient field ∇F is the difference in end-point values of the associated scalar field F."

Note: If a vector field is the gradient of a function, aka scalar field, e.g. $\vec{G} = \nabla F$, this automatically implies the vector field $\vec{G}$ is *conservative*, i.e., the integral from one point to another point within that field depends only on those points – also called "path-independence". This also implies $\vec{G}$ is "irrotational" which means it has no curl anywhere, i.e. $\nabla \times \vec{G} \equiv 0$.

Divergence Theorem:

Also known as Gauss' Theorem (here n is a vector normal to the surface)

$$\oiiint (\nabla \cdot F) dV = \oiint (F \cdot n) dS$$

"The divergence of a field $\nabla \cdot F$ within a volume is equivalent to the flux $F \cdot n$ through its surface."

Note: To see how this implies a conservation, consider an example: If we imagine a positive charge contained within a volume, and if that charge were to increase, the divergence of its field would increase, and likewise, the amount the field permeates through the surface of our volume (the flux) would also increase. In simple terms, this means we know how much charge is within a given volume based on how much is pouring out.

Curl Theorem:

A generalization of Green's Theorem, yet not as general as Stokes' Theorem.

$$\oiint (\nabla \times F) dA = \oint F \cdot dS$$

"The curl of a vector field $\nabla \times F$ within an area is equivalent to 'how parallel' the vector field is to the boundary."

Note: This is obviously related to Divergence Theorem, but not quite as obviously related to the Gradient Theorem. What the Curl Theorem implies is that all the 'curls' within a given area will cancel out, leaving only the 'curls at the boundary', which can be interpreted as the degree to which the vector field lies parallel to the boundary itself. I highly suggest looking at images demonstrating this theorem. It has broad application, especially in magnetostatics.

Stokes' Theorem:

Most general statement relating the integral and differential forms of functions to manifolds

$$\oint_{\partial\Omega} \omega = \oint_{\Omega} d\omega$$

"The integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω."

Note: This is as powerful as it is abstract. With this tool, for example, one can simplify two of the four Maxwell's equations. It also has broad application in fluid dynamics. In general, the sentiment is a conservation of information, i.e., what goes in must come out. Or, more precisely, if what's inside changes, something entered or exited, and vice versa. Admittedly, the theorem says much more, and I myself don't understand it all.

Continuity Equations:

Crudely, these equations relate changes over space with the changes over time.

$$\frac{\partial\rho}{\partial t} + \nabla \cdot j = \sigma$$

$\partial\rho/\partial t$ is the change in density over time – this can be mass density, charge density, you name it.
$\nabla \cdot j$ is the divergence of the flux, which can be understood as 'change in current per area'.
σ is the *generation* of the quantity q whose density ρ is changing over time.

Note that the quantity q is not necessarily conserved, i.e., it can be generated or destroyed. Only when sigma = 0 is the quantity conserved:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot j = 0 \therefore \frac{\partial\rho}{\partial t} = -\nabla \cdot j$$

This is all to say, for conserved quantities; any change in density over time is always compensated for by a change in current over space. To see how involved these equations are with physics in general, I will list different continuity equations from different areas in physics. These are in differential form, though they can be written in integral form, like the theorems above.

$\partial\rho/\partial t = -\nabla \cdot j \Rightarrow$ Electromagnetism (ρ is charge density, j is electric current)

$\partial\rho/\partial t = -\nabla \cdot (\rho\boldsymbol{u}) \Rightarrow$ Fluid Dynamics (ρ is fluid density, $\boldsymbol{u}$ is a flow velocity vector field)

$\partial u/\partial t = -\nabla \cdot \boldsymbol{q} \Rightarrow$ Thermodynamics (u is energy density, $\boldsymbol{q}$ is energy flux)

$\partial|\psi|^2/\partial t = -\nabla \cdot j \Rightarrow$ Quantum Mechanics (ψ is the wave function, j is the probability flux)

As you can see, all these equations represent similar relationships, namely, the conservation of different quantities: charge, fluid, heat, or probability.