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## **Specht Problem and Gelfand Conjecture**

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## Abstract

Let  $F$  be a free pro- $p$  non-abelian group, and let  $\Delta$  be a commutative Noetherian complete local ring with maximal ideal  $I$  such that  $\text{char}(\Delta/I) = p$ . Then consider the group

$$GL_2^1(\Delta) = \ker \left( GL_2(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_2(\Delta/I) \right)$$

A.N. Zubkov showed that  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$  for  $p \neq 2$ .

D. Ben-Ezra and E. Zelmanov showed that  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$  for  $p = 2$  and  $\text{char}(\Delta) = 2$ .

In this paper we are going to prove the same result for  $\text{char}(\Delta) = 4$ . In the second part we will investigate the connection between PI-theory and the old-standing Gelfand conjecture.

## 1. Introduction

### 1.1. On non-linearity of free non-abelian pro- $p$ groups

The problem of linearity of topological groups is natural and has studied for many years. It is well known that discrete free groups are linear ([?]. Furthermore, they can be embedded in  $GL_2(\mathbb{Z})$ .

So it's also quite natural to ask whether a free pro- $p$  non-abelian group is linear.

**Definition 1.** *Commutative Noetherian complete local ring  $\Delta$  with a maximal ideal  $I$  is called pro- $p$  ring if  $\Delta/I$  is a finite field of characteristic  $p$ .*

Consider the congruence subgroup:

$$GL_2^1(\Delta) = \ker \left( GL_2(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_2(\Delta/I) \right)$$

One can see that  $GL_2^1(\Delta)$  is a pro- $p$ -group. So the main conjecture of this theory can be formulated as following:

**Conjecture.** *Non-abelian free pro- $p$  group cannot be continuously embedded in  $GL_d^1(\Delta)$  for any pro- $p$  ring  $\Delta$ .*

There are a lot of partial results for certain  $\Delta, d, p$ . Let us list them out.

- In 1987, A.N Zubkov showed ([?]) that for  $d = 2, p \neq 2$  the conjecture holds true.

- In 1999, using the deep results of Pink ([?], Y. Barnea, M. Larsen ([?] proved the conjecture for  $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([?] proved conjecture for  $\Delta = \mathbb{Z}_p$ ,  $GL_d^1(\mathbb{Z}_p) = \ker \left( GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \rightarrow \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([?] that for  $d = 2, p = 2$  and  $\text{char}(\Delta) = 2$  the conjecture holds true.
- In ..., E. Zelmanov ([?] announced that conjecture holds true for  $p \gg d$ .

One can see that this subject has been researched by many mathematicians. So first of all we are going to give a review of their methods.

We will focus mostly on Zubkov's and Ben-Ezra, Zelmanov's methods.

Zubkov's proof based on standard approaches of commutative algebra and the idea of generic matrices. Zelmanov and Ben-Ezra adopted Zubkov's method for the case  $p = 2$  using the trace identities which dates back to polynomial identities theory (PI-theory for short).

Additionally, we intend to extend Zelmanov and Ben-Ezra's approach for  $d = 2, p = 2$  and  $\text{char}(\Delta) = 4$ . We also hope that it will be quite easy to extend it for  $\text{char}(\Delta) = 2^l$ , and maybe even for  $\text{char}(\Delta) = 0$ .

## 1.2. Gelfand conjecture

In 2022 the remarkable connection between PI-theory (to be more precisely Grishin's methods) and Gelfand Conjecture stated at ICM'70 (see [17]) was found.

**Conjecture** (Gelfand). *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

This interesting connection was found during joint conversation between A.S. Khoroshkin, A.Ya. Kanel-Belov and with a little help of author. One can find Khoroshkin's sketch in [19], [18]. Also we will use results from author's last year's coursework about finitely based  $T$ -spaces of commutative polynomials.

## 2. Preliminaries

### 2.1. Profinite objects

Let us remind the reader of the classical definitions of the profinite groups theory.

**Definition 2.** *Inverse (projective) limit of finite groups is called a profinite group. In the case of finite  $p$ -groups we get pro- $p$  group.*

It is clear that profinite groups can be equipped with the topology induced by the Tikhonov's product topology.

**Definition 3.** *The free pro- $p$  group  $F_p(X)$  is the completion of the discrete free group  $F(X)$  with respect to a topology defined by all normal subgroups  $N \subseteq F(X)$  whose indices are equal to the order of  $p$  and which contain almost all generators of  $F(X)$ .*

One can define the free pro- $p$  group in a classical way using universal property in the category of pro- $p$  groups.

Also note that if  $\Delta$  is a pro- $p$  ring as defined in introduction, and  $I$  is a maximal ideal. Then

$$\Delta = \varprojlim \Delta/I^n$$

## 2.2. PI-theory

Now let us give some basic definitions of PI-theory.

Let  $k$  be a field of characteristic zero and  $F = k\langle x_1, \dots, x_i, \dots \rangle$  be a free, countably generated, associative algebra over a field  $k$  and  $T$  be the endomorphism (substitution) semigroup of  $F$ .  $X = \{x_1, \dots, x_i, \dots\}$

Now let us give some classical definitions.

**Definition 4.** *An endomorphism  $\tau$  of  $F$  defined by the rule  $x_i \mapsto g_i, g_i \in F$ , is called a substitution of type  $(x_1, \dots, x_i, \dots) \mapsto (g_1, \dots, g_i, \dots)$ .*

**Definition 5.**  *$T$ -space in  $F$  is a vector subspace of  $F$ , that is closed under substitutions.*

**Definition 6.**  *$T$ -ideal in  $F$  is an ideal of  $F$  that is at the same time a  $T$ -space.*

**Definition 7.** *We say that  $T$ -space  $M$  is finitely based if there is a finite subset  $B \subset M$  such that  $T$ -space generated by  $B$  coincides with  $M$ .*

During the 1980s, A.R. Kemer's resolution of Specht problem was a significant breakthrough in the PI-theory ([1], see also simplified version of Kemer's proof in [2], [3]):

Here is a well known reformulation of Kemer's theorem:

**Theorem.** *Any  $T$ -ideal of the algebra  $F$  is finitely based.*

It's natural to ask the same question for  $T$ -spaces.

In 2001, V.V. Shchigolev combined Grishin ([?]) and Kanel-Belov's ([?]) methods. Then Shchigolev noticed that methods similar to Kemer's can be applied to localisation of Specht problem for the  $T$ -spaces. And finally he proved ([7]):

**Theorem** (V.V. Shchigolev, 2001). *Any  $T$ -space of the algebra  $F$  is finitely based.*

We will use one simple special case of Shchigolev's theorem, which was also proven in author's last year's coursework.

### 3. Main results

#### 3.1. Zubkov's approach

The first non-trivial idea is using the following definition:

**Definition 8.** *Let  $F$  be a free pro- $p$  group, and  $G$  be a pro- $p$  group. Then every  $1 \neq w \in F$  such that  $w \in \text{Ker}(\varphi)$  for all continuous homomorphisms  $\varphi : F \rightarrow G$  is called a pro- $p$  identity of  $G$ .*

Then Zubkov defines a ring of generic matrices (the formal way to do it is quite long, see [?])

He defines a natural homomorphism  $\pi$  from a free pro- $p$  group  $F$  generated by  $X, Y$  to the pro- $p$  group generated by  $1+x_*, 1+y_*$ , where  $x_*, y_*$  are generic matrices.

This homomorphism is called a universal representation:

**Theorem 1.** *Each  $1 \neq w(X, Y) \in F$  is a pro- $p$  identity of  $GL_d^1(\Delta)$  for all pro- $p$  rings  $\Delta$ .*

This theorem has pretty simple proof using the standard commutative algebra approaches. Surprisingly, it remains to prove that

**Theorem 2.** *The universal representation is not injective.*

Zubkov proved it by investigating the Lie algebra of generic 2-by-2 matrices by fairly long calculations.

#### 3.2. Ben-Ezra and Zelmanov's approach

#### 3.3. Gelfand's conjecture

We are going to consider one special case of Shchigolev's result ([?]):

**Theorem 3.** *Any  $T$ -space in algebra  $k[x_1, \dots, x_n]$  is finitely based.*

Then we will discuss Gelfand's conjecture.

**Conjecture.** *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

We denote by  $\mathcal{W}_n$  the Lie algebra of formal vector fields on an  $n$ -dimensional plane  $V$ .

Well known that

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras  $\prod_{k=d}^{\infty} S^k V \otimes V^*$  of a finite codimension are denoted by  $L_d(n)$ .

Using the classical considerations of homological algebra (which will be omitted), one can reduce Gelfand's conjecture to the following lemma

**Lemma 1.** *Any finitely generated  $L_d(n)$ -module is Noetherian.*

Finally, we will notice that the methods from the theorem ... can be applied to prove this lemma.

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