Проблема Шпехта и Гипотеза Гельфанда

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Strucure of the work

Мы рассмотрим два применения PI-теории:

- Нелинейность свободных про-p групп
- Гипотеза Гельфанда

Структура:

1. Предварительные сведения
2. Историческая справка
3. Постановка задачи (о нелинейности свободной про-p группы)
4. Обзор подхода А.Н. Зубкова ($d = 2, p > 2$)
5. Обзор подхода Бена-Эзры—Зельманова ($p = 2, d = 2, \mathrm{char}(\Delta) = 2$)
6. Случай $p = 2, d = 2, \mathrm{char}(\Delta) = 4$
7. Методы Гришина
8. Гипотеза Гельфанда
9. Связь гипотезы Гельфанда с методами А.В. Гришина

Preliminaries

Definition

Обратный (проективный) предел проективной системы конечных групп называется проконечной группой.

Definition

Обратный (проективный) предел проективной системы конечных p-групп называется про-p группой.

Definition

Коммутативное нетерово I-полное локальное кольцо Δ с максимальным идеалом I называется про-p кольцом, если Δ/I конечное поле характеристики p.

$$\Delta = \varprojlim \Delta/I^n$$

Preliminaries

Definition

Пусть F свободная группа порожденная алфавитом $\mathcal{S}$. Рассмотрим пополнение $\widetilde{F}_p$ группы F относительно топологии, определенной всеми нормальными подгруппами индекса p^l, $\forall l \in \mathbb{N}$. Тогда $\widetilde{F}_p$ называется свободной про-p группой.

Remark

Здесь и далее под подобным пополнением мы имеем в виду обратный предел факторгрупп.

Пусть Δ про-p кольцо.

$$GL_d^1(\Delta) = \ker\left(GL_d(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_d(\Delta/I)\right)$$

является про-p группой.

Основная гипотеза

Conjecture

Некоммутативная свободная про-p группа $\widetilde{F}_p$ не может быть непрерывно вложена в $GL^1_d(\Delta)$ для любого про-p кольца Δ.

Историческая справка

Существует множество частичных результатов для различных $\widetilde{F}_p, \Delta, p$, которые дают надежду на положительный результат и в общем случае:

- В 1987, А.Н. Зубков ([3]) доказал гипотезу для $d = 2, p \neq 2$.
- В 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([6]) доказали гипотезу для $\Delta = \mathbb{Z}_p$,
  $$GL_d^1(\mathbb{Z}_p) = \ker \left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$$
- В 1999, используя глубокие результаты Пинка ([4]), Y. Barnea, M. Larsen ([5]) доказали гипотезу для $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$.
- В 2005, E. Zelmanov ([?]) анонсировал доказательство гипотезы для $p \gg d$.
- В 2020, D. Ben-Ezra, E. Zelmanov доказали ([7]) гипотезу для $d = 2, p = 2$ и $\mathrm{char}(\Delta) = 2$.

Zubkov's approach

Theorem (Zubkov, 1987)

Let F be a free non-abelian pro-p group, Δ is a pro-p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.

The first non-trivial idea is to introduce the following definition:

Definition

Let F be a free pro-p group, and G be a pro-p group. Then every $1 \neq w \in F$ such that $w \in Ker(\varphi)$ for all continuous homomorphisms $\varphi : F \to G$ is called a pro-p identity of G.

Zubkov consider the natural homomorphism to algebra of generic matrices: Let $x, y \in \widetilde{F}_p$ — generators, $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$, where x^*, y^* are the generic matrices over $\mathbb{Z}_p$. And one can continue π to the completion $\langle\langle x, y\rangle\rangle$, and it will map on closure of $\langle 1 + x^*, 1 + y^*\rangle$.

Zubkov's approach

Homomorphism π is called the universal representation, and that's why:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-p identity of $GL_d^1(\Delta)$ for all pro-p rings Δ.

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

Theorem

The universal representation of the degree 2 is not injective for $p \neq 2$.

So we need to construct the pro-p identity for generic matrices.

Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.
And he encountered a contradiction with the classical Witt's formula:

Formula (Witt)

Rank of r-th factor of the lower central series of $\widetilde{F}_p$ (as a $\mathbb{Z}_p$ − module):

$$\frac{1}{r}\sum_{m|r}\mu(m)\cdot 2^{\frac{r}{m}}$$

Ben-Ezra, Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 2$.

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case $p = 2$: analogous homomorphism to generic matrices over $\mathbb{Z}/2\mathbb{Z}$ (instead of $\mathbb{Z}_2$).

Then the following lemma still holds and has a pretty simple proof:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ with $\mathrm{char}\Delta = 2$.

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

Theorem

The universal representation of the degree 2 is not injective.

$\mathrm{char}\Delta = 4$

Conjecture

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 4$.

We intend to prove it using the similar approaches, and believe that one can prove it even for the case $\mathrm{char}\Delta = 2^l$.
Furthermore, maybe the case $\mathrm{char}\Delta = 0$ can be investigated if the above statement will be proved.

Let T be the endomorphism (substitution) semigroup of the free algebra $F = k\langle x_1, \ldots, x_i, \ldots\rangle$.

Definition

An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$.

Definition

T-space in F is a vector subspace of F, that is closed under substitutions.

Definition

T-ideal in F is an ideal of F that is at the same time a T-space.

PI-theory, preliminaries

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

Theorem

Any T-space in algebra $k[x_1, \ldots, x_n]$ is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T-spaces are finitely based using some $\widetilde{T} \subset T$
The main idea is to use substitutions:

$$f(x_1, \ldots, x_i, \ldots, x_n) \mapsto f(x_1, \ldots, 1 + \alpha_i P(x_i), \ldots, x_n)$$

And then we linearize it on α_i.

Gelfand conjecture

Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by $\mathcal{W}_n$ the Lie algebra of formal vector fields on an n-dimensional plane V.

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod\limits_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.

Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

Lemma

Any finitely generated $L_d(n)$-module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.

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