## Specht Problem and Gelfand Conjecture

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### Strucure of the work

There are two non-trivial applications of PI-theory those will be presented:

- Non-linearity of free pro-p groups
- Gelfand's conjecture

#### Structure:

- Preliminaries for pro-p structures
- A brief historical review
- Problem statement (on non-linearity of free pro-p group over 2-by-2 matrices)
- Review of Zubkov's approach
- Review of Zelmanov, Ben-Ezra's approach
- **1** Modification of Zelmanov, Ben-Ezra's approach for  $\mathrm{char}\Delta=4$
- Gelfand conjecture, statement and reformulation: whether some module is Noetherian
- Connection between Gelfand conjecture and PI-theory

## **Preliminaries**

#### **Definition**

The inverse (projective) limit of the projective system of finite groups (rings) is called a profinite group (ring).

### **Definition**

The inverse limit of the projective system of p-groups is called pro-p group.

### **Definition**

Commutative Noetherian complete local ring  $\Delta$  with a maximal ideal I is called pro-p ring if  $\Delta/I$  is a finite field of characteristic p.

In that case:

$$\Delta = \varprojlim \Delta/I^n$$



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### **Preliminaries**

#### Definition

Let F be a free group generated by alphabet S. Consider the completion  $\widetilde{F}_p$  of F with respect to topology, defined by all normal subgroups of a finite index  $p^I$ ,  $\forall I \in \mathbb{N}$  which have almost all generators from S. Then  $\widetilde{F}_p$  is called a free pro-p-group.

### Remark

Here and later the completion with respect to normal subgroups means the inverse limit of the system of factorgroups.

Let  $\Delta$  be pro-p ring

$$GL_d^1(\Delta) = \ker \left( GL_d(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_d(\Delta/I) \right)$$

is a pro-p-group.



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# Main problem

### Conjecture

A non-abelian free pro-p group  $\widetilde{F}_p$  cannot be continuously embedded in  $GL^1_d(\Delta)$  for any pro-p ring  $\Delta$ .

### Historical review

There are several partial results for certain  $\widetilde{F}_p, \Delta, p$ , that let us suppose that answer is positive for the general case:

- In 1987, A.N Zubkov ([1]) demonstrated that for  $d=2, p \neq 2$  the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([4]) established the conjecture for  $\Delta = \mathbb{Z}_p$ ,

$$GL^1_d(\mathbb{Z}_p) = \ker \left( GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$$

- In 1999, utilizing the profound results of Pink ([2]), Y. Barnea, M. Larsen ([3]) proved the conjecture for  $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 2005, E. Zelmanov ([6]) announced that conjecture holds true for  $p\gg d$ .
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([5]) that for d=2, p=2 and  $\operatorname{char}(\Delta)=2$  the conjecture holds true.



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# Zubkov's approach

## Theorem (Zubkov, 1987)

Let F be a free non-abelian pro-p group,  $\Delta$  is a pro-p ring. F cannot be continuously embedded in  $GL_2^1(\Delta)$ , when p > 2.

The first non-trivial idea is to introduce the following definition:

#### Definition

Let F be a free pro-p group, and G be a pro-p group. Then every  $1 \neq w \in F$  such that  $w \in Ker(\varphi)$  for all continuous homomorphisms  $\varphi : F \to G$  is called a pro-p identity of G.

Zubkov consider the natural homomorphism to algebra of generic matrices: Let  $x,y\in \widetilde{F}_p$  — generators,  $\pi:x\mapsto 1+x^*,y\mapsto 1+y^*$ , where  $x^*,y^*$  are the generic matrices over  $\mathbb{Z}_p$ . And one can continue  $\pi$  to the completion  $\langle\langle x,y\rangle\rangle$ , and it will map on closure of  $\langle 1+x^*,1+y^*\rangle$ .

# Zubkov's approach

Homomorphism  $\pi$  is called the universal representation, and that's why:

#### Lemma

Each  $1 \neq z \in \ker \pi$  is a pro-p identity of  $GL^1_d(\Delta)$  for all pro-p rings  $\Delta$ .

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

### <u>Theorem</u>

The universal representation of the degree 2 is not injective for  $p \neq 2$ .

So we need to construct the pro-p identity for generic matrices.

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# Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.

And he encountered a contradiction with the classical Witt's formula:

## Formula (Witt)

Rank of r-th factor of the lower central series of  $F_p$  (as a  $\mathbb{Z}_p$  – module):

$$\frac{1}{r}\sum_{m|r}\mu(m)\cdot 2^{\frac{r}{m}}$$

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# Ben-Ezra, Zelmanov's approach

## Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group,  $\Delta$  is a pro-2 ring. F cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $\operatorname{char}\Delta=2$ .

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case p=2: analogous homomorphism to generic matrices over  $\mathbb{Z}/2\mathbb{Z}$  (instead of  $\mathbb{Z}_2$ ).

Then the following lemma still holds and has a pretty simple proof:

### Lemma

Each  $1 \neq z \in \ker \pi$  is a pro-2 identity of  $GL_2^1(\Delta)$  for all pro-2 rings  $\Delta$  with  $\operatorname{char} \Delta = 2$ .

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

#### Theorem

The universal representation of the degree 2 is not injective.
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 $char \Delta = 4$ 

### Conjecture

Let F be a free non-abelian pro-2 group,  $\Delta$  is a pro-2 ring. F cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $\operatorname{char}\Delta=4$ .

We intend to prove it using the similar approaches, and believe that one can prove it even for the case  $\mathrm{char}\Delta=2^{I}.$ 

Furthermore, maybe the case  ${\rm char}\Delta=0$  can be investigated if the above statement will be proved.

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## PI-theory, preliminaries

Let T be the endomorphism (substitution) semigroup of the free algebra  $F = k\langle x_1, \ldots, x_i, \ldots \rangle$ .

#### Definition

An endomorphism  $\tau$  of F defined by the rule  $x_i \mapsto g_i, g_i \in F$ , is called a substitution of type  $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$ .

#### Definition

T-space in F is a vector subspace of F, that is closed under substitutions.

#### **Definition**

T-ideal in F is an ideal of F that is at the same time a T-space.

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# PI-theory, preliminaries

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

#### Theorem

Any T-space in algebra  $k[x_1, \ldots, x_n]$  is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T-spaces are finitely based using some  $\widetilde{T} \subset T$ The main idea is to use substitutions:

$$f(x_1,\ldots,x_i,\ldots,x_n)\mapsto f(x_1,\ldots,1+\alpha_iP(x_i),\ldots,x_n)$$

And then we linearize it on  $\alpha_i$ .



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# Gelfand conjecture

## Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by  $W_n$  the Lie algebra of formal vector fields on an n-dimensional plane V.

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras  $\prod_{k=d}^{\infty} S^k V \otimes V^*$  of a finite codimension are denoted by  $L_d(n)$ .

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# Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

#### Lemma

Any finitely generated  $L_d(n)$ -module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.

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