Federal State Autonomous Educational Institution
for the Higher Education
National Research University "Higher School of Economics"
Faculty of Mathematics

Vorobiov Ivan Evgenievich

Specht Problem and Gelfand Conjecture

Bachelor's thesis

Field of study: 01.03.01 — Mathematics,

Degree programme: bachelor's educational programme "Mathematics"

Reviewer:
academic degree
full name

Scientific advisor:
Candidate of Sciences
Anton Sergeevich Khoroshkin
Co-advisor:
Doctor of Sciences, professor
Alexei Yakovlevich Kanel-Belov

Moscow 2024

Abstract

Let F be a free pro-p non-abelian group, and let Δ be a commutative Noetherian complete local ring with maximal ideal I such that $\mathrm{char}(\Delta/I) = p$.
Then consider the group

$$GL_2^1(\Delta) = \ker\left(GL_2(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_2(\Delta/I)\right)$$

A.N. Zubkov showed that F cannot be continuously embedded in $GL_2^1(\Delta)$ for $p \neq 2$.
D. Ben-Ezra and E. Zelmanov showed that F cannot be continuously embedded in $GL_2^1(\Delta)$ for $p = 2$ and $\mathrm{char}(\Delta) = 2$.
In this paper we are going to prove the same result for $\mathrm{char}(\Delta) = 4$.
In the second part we will investigate the connection between PI-theory and the old-standing Gelfand conjecture.

1. Introduction

1.1. On non-linearity of free non-abelian pro-p groups

The problem of linearity of topological groups is natural and has studied for many years. It is well known that discrete free groups are linear ([?]. Furthermore, they can be embedded in $GL_2(\mathbb{Z})$.
So it's also quite natural to ask whether a free pro-p non-abelian group is linear.

Definition 1.1.1. *Commutative Noetherian complete local ring Δ with a maximal ideal I is called pro-p ring if Δ/I is a finite field of characteristic p.*

Consider the congruence subgroup:

$$GL_2^1(\Delta) = \ker\left(GL_2(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_2(\Delta/I)\right)$$

One can see that $GL_2^1(\Delta)$ is a pro-p-group. So the main conjecture of this theory can be formulated as following:

Conjecture. *Non-abelian free pro-p group cannot be continuously embedded in $GL_d^1(\Delta)$ for any pro-p ring Δ.*

There are a lot of partial results for certain Δ, d, p. Let us list them out.

- In 1987, A.N Zubkov showed ([?] that for $d = 2, p \neq 2$ the conjecture holds true.

- In 1999, using the deep results of Pink ([?], Y. Barnea, M. Larsen ([?] proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})\,[[t]]$
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([?] proved conjecture for $\Delta = \mathbb{Z}_p$, $GL_d^1(\mathbb{Z}_p) = \ker\left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p)\right)$
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([?] that for $d = 2, p = 2$ and $\operatorname{char}(\Delta) = 2$ the conjecture holds true.
- In ..., E. Zelmanov ([?] announced that conjecture holds true for $p \gg d$.

One can see that this subject has be researched by many mathematicians. So first of all we are going to give a review of their methods.
We will focus mostly on Zubkov's and Ben-Ezra, Zelmanov's methods.
Zubkov's proof based on standard approaches of commutative algebra and the idea of generic matrices. Zelmanov and Ben-Ezra adopted Zubkov's method for the case $p = 2$ using the trace identities which dates back to polynomial identities theory (PI-theory for short).
Additionally, we intend to extend Zelmanov and Ben-Ezra's approach for $d = 2, p = 2$ and $\operatorname{char}(\Delta) = 4$. We also hope that it will be quite easy to extend it for $\operatorname{char}(\Delta) = 2^l$, and maybe even for $\operatorname{char}(\Delta) = 0$.

1.2. Gelfand conjecture

In 2022 the remarkable connection between PI-theory (to be more precisely Grishin's methods) and Gelfand Conjecture stated at ICM'70 (see [17]) was found.

Conjecture (Gelfand)**.** *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

This interesting connection was found during joint conversation between A.S. Khoroshkin, A.Ya. Kanel-Belov and with a little help of author. One can find Khoroshkin's sketch in [19], [18]. Also we will use results from author's last year's coursework about finitely based T-spaces of commutative polynomials.

2. Preliminaries

2.1. Profinite objects

Let us remind the reader of the classical definitions of the profinite groups theory.

Definition 2.1.1. *Inverse (projective) limit of finite groups is called a profinite group. In the case of finite p-groups we get pro-p group.*

It is clear that profinite groups can be equipped with the topology induced by the Tikhonov's product topology.

Definition 2.1.2. *The free pro-p group $F_p(X)$ is the completion of the discrete free group $F(X)$ with respect to a topology defined by all normal subgroups $N \subseteq F(X)$ whose indices are equal to the order of p and which contain almost all generators of $F(X)$.*

One can define the free pro-p group in a classical way using universal property in the category of pro-p groups.
Also note that if Δ is a pro-p ring as defined in introduction, and I is a maximal ideal. Then

$$\Delta = \varprojlim \Delta/I^n$$

2.2. PI-theory

Now let us give some basic definitions of PI-theory.
Let k be a field of characteristic zero and $F = k\langle x_1, \ldots, x_i, \ldots \rangle$ be a free, countably generated, associative algebra over a field k and T be the endomorphism (substitution) semigroup of F. $X = \{x_1, \ldots, x_i, \ldots\}$
Now let us give some classical definitions.

Definition 2.2.1. *An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$.*

Definition 2.2.2. *T-space in F is a vector subspace of F, that is closed under substitutions.*

Definition 2.2.3. *T-ideal in F is an ideal of F that is at the same time a T-space.*

Definition 2.2.4. *We say that T-space M is finitely based if there is a finite subset $B \subset M$ such that T-space generated by B coincides with M.*

During the 1980s, A.R. Kemer's resolution of Specht problem was a significant breakthrough in the PI-theory ([1], see also simplified version of Kemer's proof in [2], [3]):
Here is a well known reformulation of Kemer's theorem:

Theorem. *Any T-ideal of the algebra F is finitely based.*

It's natural to ask the same question for T-spaces.
In 2001, V.V. Shchigolev combined Grishin ([**?**] and Kanel-Belov's ([**?**] methods. Then Shchigolev noticed that methods similar to Kemer's can be applied to localisation of Specht problem for the T-spaces. And finally he proved ([7]):

Theorem (V.V. Shchigolev, 2001)**.** *Any T-space of the algebra F is finitely based.*

We will use one simple special case of Shchigolev's theorem, which was also proven in author's last year's coursework.

3. Main results

3.1. Zubkov's approach

Theorem (Zubkov, 1987)**.** *Let F be a free non-abelian pro-p group, Δ is a pro-p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.*

The first non-trivial idea is to give the following definition:

Definition 3.1.1. *Let F be a free pro-p group, and G be a pro-p group. Then every $1 \neq w \in F$ such that $w \in Ker(\varphi)$ for all continuous homomorphisms $\varphi : F \to G$ is called a pro-p identity of G.*

Then Zubkov defines a ring of generic matrices over p-adic numbers $\mathbb{Z}_p$ (the formal way to do it is quite long, see [**?**]
He defines a natural homomorphism π from a free pro-p group F generated by X, Y to the pro-p group generated by $1+x_*, 1+y_*$, where x_*, y_* are generic matrices.
This homomorphism is called a universal representation:

Theorem 3.1.1. *Each $1 \neq w(X, Y) \in \ker \pi$ is a pro-p identity of $GL_d^1(\Delta)$ for all pro-p rings Δ.*

This theorem has pretty simple proof using the standard commutative algebra approachs. Surprisingly, it remains to prove that

Theorem 3.1.2. *The universal representation is not injective for $p \neq 2$.*

Zubkov proved it by investigating the Lie algebra of generic 2-by-2 matrices by fairly long calculations.

3.2. Ben-Ezra and Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)**.** *Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring, such that* $\text{char}(\Delta) = 2$*. F cannot be continuously embedded in $GL_2^1(\Delta)$.*

The main ideas of Ben-Ezra and Zelmanov's proof are quite similar to Zubkov's ones. But they use a slightly different way to define the universal representation π. They define generic matrices over $\mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{Z}_2$ as Zubkov. Then the analogous theorem about the kernel of universal representation is still true (at least in the case $\text{char}(\Delta) = 2$):

Theorem 3.2.1. *Each $1 \neq w(X, Y) \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ, such that* $\text{char}(\Delta) = 2$.

Thus, the main result boils down to

Theorem 3.2.2. *The universal representation is not injective.*

This was proven using the 20 pages fairly difficult calculations.

3.3. An extension of Ben-Ezra and Zelmanov's approach

We are going to prove slightly different version of the theorem 3.2.1 for the case $\text{char}(\Delta) = 4$ and perhaps we will be able to do the similar thing for $\text{char}(\Delta) = 2^l$. Then we hope that the same calculations will complete the proof.
Also, we will discuss the case $\text{char}(\Delta) = 0$ and how it can be investigated with aforementioned approaches.

3.4. Gelfand's conjecture

We are going to consider one special case of Shchigolev's result ([**?**]:

Theorem 3.4.1. *Any T-space in algebra $k[x_1, \dots, x_n]$ is finitely based.*

Then we will discuss Gelfand's conjecture.

Conjecture. *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

We denote by $\mathcal{W}_n$ the Lie algebra of formal vector fields on an n-dimensional plane V.

Well known that

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod\limits_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$. Using the classical considerations of homological algebra (which will be omitted), one can reduce Gelfand's conjecture to the following lemma

Lemma 3.4.1. *Any finitely generated $L_d(n)$-module is Noetherian.*

Finally, we will notice that the methods from the theorem ... can be applied to prove this lemma.

References

[1] *A. Kemer*, *Finite basability of identities of associative algebras*, Algebra i Logika 26(5) (1987), 597-641, 650.

[2] *Eli Aljadeff, Alexei Kanel-Belov, Yaakov Karasik*, *Kemer's theorem for affine PI algebras over a field of characteristic zero*, Journal of Pure and Applied Algebra 220, No. 8, 2771–2808 (2016).

[3] *C. Procesi*, *"The geometry of polynomial identities"*, Izv. Math., 80:5 (2016), 910–953.

[4] *Grishin, A. V.; Shchigolev, V. V.*, *T-spaces and their applications.* (English. Russian original) J. Math. Sci., New York 134, No. 1, 1799–1878 (2006); translation from Sovrem. Mat. Prilozh. 18, 26–97 (2004).

[5] *A. V. Grishin*, *On the finite-basis property for systems of generalized polynomials*, Izv. Akad. Nauk SSSR, Ser. Mat., 37, No. 2, 243–272 (1991).

[6] *A. V. Grishin*, *"On finitely based systems of generalized polynomials", Math. USSR-Izv.*, 37:2 (1991), 243–272

[7] *V. V. Shchigolev*, *Finite-basis property of T-spaces over fields of characteristic zero,* Izv. Ross. Akad. Nauk, Ser. Mat., 65, No. 5, 1041–1071 (2001).

[8] *Kanel–Belov A., Eli Aljadeff*, *Representability and Specht problem for G-graded algebras*, Advances in Math., 225:5 (2010), 2391–2428 , arXiv: 0903.0362.

[9] *I. I. Benediktovich, A. E. Zalesskii*, *T-ideals of free Lie algebras with polynomial growth of a sequence of codimensions* (Russian), Vestsi Akad. Navuk BSSR Ser. Fiz-Math. Navuk 3 (1980), 5-10; translation: Proceedings of the National Academy of Sciences of Belarus. Series of Physical-Mathematical Sciences 3(3) (1980).

[10] *A. Ja. Vais, E. I. Zelmanov*, *Kemer's theorem for finitely generated Jordan algebras* (Russian), Izv. Vyssh. Uchebn. Zved. Mat. 33(6) (1989), 42-51; translation: Soviet Math. (Iz. VUZ) 33(6) (1989), 38-47.

[11] *L. Centrone, A. Estrada, A. Ioppolo*, *On PI-algebras with additional structures: rationality of Hilbert series and Specht's problem*, J. Algebra 592 (2022), 300-356.

[12] *A. Ya. Belov*, *Counterexamples to the Specht problem* (Russian), Sb. Math. 191(3) (2000), 13-24; translation: Sb. math. 131 (3-4) (2000), 329-340.

[13] *A. V. Grishin*, *Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property*, Fundam. Prikl. Mat. 5(1) (1999), 101-118.

[14] *V. V. Shchigolev*, *Examples of infinitely based T-ideals*, Fundam. Prikl. Mat. 5(1) (1999), 307-312.

[15] *E. Aljadeff, A. Kanel-Belov*, *Representability and Specht problem for G-graded algebras*, Adv. Math. 225(5) (2010), 2391- 2428.

[16] *I. Sviridova*, *Identities of pi-algebras graded by a finite abelian group*, Comm. Algebra 39(9) (2011), 3462–3490.

[17] *I. M. Gelfand*, *The cohomology of infinite dimensional Lie algebras; Some questions of integral geometry*, Proceedings of ICM 1970-Nice, T.1 pp.95-111.

[18] *Lucio Centrone, Alexei Kanel-Belov, Anton Khoroshkin, and Ivan Evgenievich Vorobiev*, *Specht property for systems of commutative polynomials and Gelfand conjecture*,

[19] *Boris Feigin, Alexei Kanel-Belov, Anton Khoroshkin*, *On finite dimensionality of homology of subalgebras of vector fields*, arXiv:2211.08510v1