

Проблема Шпехта и Гипотеза Гельфанда

Воробьев Иван Евгеньевич

Научный руководитель, доктор физ.-мат. наук, профессор
Алексей Яковлевич Канель-Белов

Соруководитель, кандидат физ.-мат. наук, доцент
Антон Сергеевич Хорошкин

Рецензент, доктор физ.-мат. наук, профессор
Сергей Олегович Горчинский

19 июня 2024 г.

Structure of the work

Мы рассмотрим два применения PI-теории:

- Нелинейность свободных про- p групп
- Гипотеза Гельфанда

Структура:

- 1 Предварительные сведения
- 2 Историческая справка
- 3 Постановка задачи (о нелинейности свободной про- p группы)
- 4 Обзор подхода А.Н. Зубкова ($d = 2, p > 2$)
- 5 Обзор подхода Бена-Эзры—Зельманова ($p = 2, d = 2, \text{char}(\Delta) = 4$)
- 6 Случай $p = 2, d = 2, \text{char}(\Delta) = 4$
- 7 Методы Гришина
- 8 Гипотеза Гельфанда
- 9 Связь гипотезы Гельфанда с методами А.В. Гришина

Definition

Обратный (проективный) предел проективной системы конечных групп называется проконечной группой.

Definition

Обратный (проективный) предел проективной системы конечных p -групп называется про- p группой.

Definition

Коммутативное нетерово I -полное локальное кольцо Δ с максимальным идеалом I называется про- p кольцом, если Δ/I конечное поле характеристики p .

$$\Delta = \varprojlim \Delta/I^n$$

Definition

Пусть F свободная группа порожденная алфавитом \mathcal{S} . Рассмотрим пополнение \tilde{F}_p группы F относительно топологии, определенной всеми нормальными подгруппами индекса p^l , $\forall l \in \mathbb{N}$. Тогда \tilde{F}_p называется свободной про- p группой.

Remark

Здесь и далее под подобным пополнением мы имеем в виду обратный предел факторгрупп.

Пусть Δ про- p кольцо.

$$GL_d^1(\Delta) = \ker \left(GL_d(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_d(\Delta/I) \right)$$

is a pro- p -group.

Conjecture

Некоммутативная свободная про- p группа \tilde{F}_p не может быть непрерывно вложена в $GL_d^1(\Delta)$ для любого про- p кольца Δ .

Существует множество частичных результатов для различных \tilde{F}_p, Δ, p , которые дают надежду на положительный результат и в общем случае:

- In 1987, A.N Zubkov ([3]) demonstrated that for $d = 2, p \neq 2$ the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([6]) established the conjecture for $\Delta = \mathbb{Z}_p$,

$$GL_d^1(\mathbb{Z}_p) = \ker \left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \rightarrow \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$$

- In 1999, utilizing the profound results of Pink ([4]), Y. Barnea, M. Larsen ([5]) proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 2005, E. Zelmanov ([?]) announced that conjecture holds true for $p \gg d$.
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([7]) that for $d = 2, p = 2$ and $\text{char}(\Delta) = 2$ the conjecture holds true.

Zubkov's approach

Theorem (Zubkov, 1987)

Let F be a free non-abelian pro- p group, Δ is a pro- p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.

The first non-trivial idea is to introduce the following definition:

Definition

Let F be a free pro- p group, and G be a pro- p group. Then every $1 \neq w \in F$ such that $w \in \text{Ker}(\varphi)$ for all continuous homomorphisms $\varphi : F \rightarrow G$ is called a pro- p identity of G .

Zubkov consider the natural homomorphism to algebra of generic matrices: Let $x, y \in \tilde{F}_p$ — generators, $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$, where x^*, y^* are the generic matrices over \mathbb{Z}_p . And one can continue π to the completion $\langle\langle x, y \rangle\rangle$, and it will map on closure of $\langle 1 + x^*, 1 + y^* \rangle$.

Zubkov's approach

Homomorphism π is called the universal representation, and that's why:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro- p identity of $GL_d^1(\Delta)$ for all pro- p rings Δ .

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

Theorem

The universal representation of the degree 2 is not injective for $p \neq 2$.

So we need to construct the pro- p identity for generic matrices.

Zubkov investigated the lower central series of Lie algebra of generic matrices.

And he encountered a contradiction with the classical Witt's formula:

Formula (Witt)

Rank of r -th factor of the lower central series of \tilde{F}_p (as a \mathbb{Z}_p – module):

$$\frac{1}{r} \sum_{m|r} \mu(m) \cdot 2^{\frac{r}{m}}$$

Ben-Ezra, Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\text{char}\Delta = 2$.

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case $p = 2$: analogous homomorphism to generic matrices over $\mathbb{Z}/2\mathbb{Z}$ (instead of \mathbb{Z}_2).

Then the following lemma still holds and has a pretty simple proof:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ with $\text{char}\Delta = 2$.

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

Theorem

The universal representation of the degree 2 is not injective.

$$\text{char}\Delta = 4$$

Conjecture

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\text{char}\Delta = 4$.

We intend to prove it using the similar approaches, and believe that one can prove it even for the case $\text{char}\Delta = 2^l$.

Furthermore, maybe the case $\text{char}\Delta = 0$ can be investigated if the above statement will be proved.

Let T be the endomorphism (substitution) semigroup of the free algebra $F = k\langle x_1, \dots, x_i, \dots \rangle$.

Definition

An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \dots, x_i, \dots) \mapsto (g_1, \dots, g_i, \dots)$.

Definition

T -space in F is a vector subspace of F , that is closed under substitutions.

Definition

T -ideal in F is an ideal of F that is at the same time a T -space.

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

Theorem

Any T -space in algebra $k[x_1, \dots, x_n]$ is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T -spaces are finitely based using some $\tilde{T} \subset T$

The main idea is to use substitutions:

$$f(x_1, \dots, x_i, \dots, x_n) \mapsto f(x_1, \dots, 1 + \alpha_i P(x_i), \dots, x_n)$$

And then we linearize it on α_i .

Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by \mathcal{W}_n the Lie algebra of formal vector fields on an n -dimensional plane V .

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.

Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

Lemma

Any finitely generated $L_d(n)$ -module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.



I. Sanov, *The property of one free group representation*, *Doklady Akademii Nauk USSR*, vol. 57, no. 7, pp. 657–659, 1947.



A. Kanel-Belov, *Local finite basability and local representability of varieties of associative rings*, *Doklady Akademii Nauk*, vol. 432, no. 6, pp. 727–731, 2010.



A. Zubkov, *Non-abelian free pro- p -groups cannot be represented by 2-by-2 matrices*, *Siberian Mathematical Journal*, vol. 28, pp. 742–747, 1987.



R. Pink, *Compact subgroups of linear algebraic groups*, *Journal of Algebra*, vol. 206, pp. 438–504, 1998.



Y. Barnea and M. Larsen, *A non-abelian free pro- p group is not linear over a local field*, *Journal of Algebra*, vol. 214, pp. 338–341, 1999.



J. Dixon, A. Mann, M. du Sautoy, and D. Segal, *Analytic pro- p -groups*, *London Mathematical Society Lecture Note Series*, Cambridge University Press, 1991.



D. Ben-Ezra and E. Zelmanov, *On Pro-2 Identities of 2×2 Linear Groups*, *arXiv:1910.05805v2*, 2020.



E. Zelmanov, *Infinite algebras and pro- p groups*, *Infinite groups: geometric, combinatorial and dynamical aspects*, *Progr. Math.*, vol. 248, pp. 403–413, 2005.



E. Zelmanov, *Groups with identities*, *Note. Mat.*, vol. 36, pp. 101–113, 2016.



I.M. Gelfand, *The cohomology of infinite dimensional Lie algebras; Some questions of integral geometry*, *Proceedings of ICM*, vol. T.1, p. 106, 1970.



B. Feigin, A. Kanel-Belov, and A. Khoroshkin, *On finite dimensionality of homology of subalgebras of vector fields*, *arXiv:2211.08510v1*, 2022.



L. Centrone, A. Kanel-Belov, A. Khoroshkin, and I. Vorobiov, *Specht property for systems of commutative polynomials and Gelfand conjecture*, https://www.researchgate.net/publication/355916110_Gelfand_conjecture_and_the_method_of_proof_of_Specht_problem, 2022.



A. Kemer, *Finite basability of identities of associative algebras*, *Algebra and Logics*, vol. 26, no. 5, pp. 597–641, 1987.



C. Procesi, *The geometry of polynomial identities*, *Izv. Math.*, vol. 80, no. 5, pp. 910–953, 2016.



A. Grishin, *On finitely based systems of generalized polynomials*, *Math. USSR-Izv.*, vol. 37, no. 2, pp. 243–272, 1991.



V. Shchigolev, *Finite-basis property of T-spaces over fields of characteristic zero*, *Izv. Ross. Akad. Nauk, Ser. Mat.*, vol. 65, no. 5, pp. 1041–1071, 2001.



A. Lubotzky, *Combinatorial group theory for PRO-p groups*, *Pure and Applied Algebra*, vol. 25, pp. 311–325, 1982.



E. Aljadeff, A. Kanel-Belov, and Y. Karasik, *Kemer's theorem for affine PI algebras over a field of characteristic zero*, *Pure and Applied Algebra*, vol. 220, pp. 2771–2808, 2016.



A. Grishin, *On finitely based systems of generalized polynomials*, *Math. USSR-Izv.*, vol. 37, no. 2, pp. 243–272, 1991.



A. Grishin and V. Shchigolev, *T-spaces and their applications*, *Math. Sci., New York*, vol. 134, no. 1, pp. 1799–1878, 2004.



I. Benediktovich and A. Zalesskii, *T-ideals of free Lie algebras with polynomial growth of a sequence of codimensions*, *Proceedings of the National Academy of Sciences of Belarus. Series of Physical-Mathematical Sciences*, vol. 3, pp. 5–10, 1980.



A. Vais and E. Zelmanov, *Kemer's theorem for finitely generated Jordan algebras*, *Izv. Vyssh. Uchebn. Zved. Mat.*, vol. 33, no. 6, pp. 42–51, 1989. Note: Translation: *Soviet Math. (Iz. VUZ)* 33(6) (1989), 38–47.



L. Centrone, A. Estrada, and A. Ioppolo, *On PI-algebras with additional structures: rationality of Hilbert series and Specht's problem*, *J. Algebra*, vol. 592, pp. 300–356, 2022.



A. Kanel-Belov, *Counterexamples to the Specht problem*, *Sb. Math.*, vol. 191, no. 3, pp. 13–24, 2000. Note: Translation: *Sb. Math.* 131(3-4) (2000), 329–340.



A. Grishin, *Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property*, *Fundam. Prikl. Mat.*, vol. 5, no. 1, pp. 101–118, 1999.



V. Shchigolev, *Examples of infinitely based T-ideals*, *Fundam. Prikl. Mat.*, vol. 5, no. 1, pp. 307–312, 1999.



E. Aljadeff and A. Kanel-Belov, *Representability and Specht problem for G-graded algebras*, *Adv. Math.*, vol. 225, no. 5, pp. 2391–2428, 2010.



I. Sviridova, *Identities of pi-algebras graded by a finite abelian group*, *Comm. Algebra*, vol. 39, no. 9, pp. 3462–3490, 2011.



D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Springer Science & Business Media, 2012.