Проблема Шпехта и Гипотеза Гельфанда

Воробьев Иван Евгеньевич

Научный руководитель, доктор физ.-мат. наук, профессор
Алексей Яковлевич Канель-Белов

Соруководитель, кандидат физ.-мат. наук, доцент
Антон Сергеевич Хорошкин

Рецензент, доктор физ.-мат. наук, профессор
Сергей Олегович Горчинский

19 июня 2024 г.

Strucure of the work

Мы рассмотрим два применения PI-теории:

- Нелинейность свободных про-p групп
- Гипотеза Гельфанда

Structure:

1. Предварительные сведения
2. Историческая справка
3. Постановка задачи (о нелинейности свободной про-p группы)
4. Обзор подхода А.Н. Зубкова ($d = 2, p > 2$)
5. Обзор подхода Бена-Эзры—Зельманова ($p = 2, d = 2, \mathrm{char}(\Delta) = 4$)
6. Случай $p = 2, d = 2, \mathrm{char}(\Delta) = 4$
7. Гипотеза Гельфанда
8. Связь гипотезы Гельфанда с методами А.В. Гришина

Preliminaries

Definition

Обратный (проективный) предел проективной системы конечных групп называется проконечной группой.

Definition

Обратный (проективный) предел проективной системы конечных p-групп называется про-p группой.

Definition

Коммутативное нетерово I-полное локальное кольцо Δ с максимальным идеалом I называется про-p кольцом, если Δ/I конечное поле характеристики p.

$$\Delta = \varprojlim \Delta/I^n$$

Definition

Пусть F свободная группа порожденная алфавитом $\mathcal{S}$. Рассмотрим пополнение $\widetilde{F}_p$ группы F относительно топологии, определенной всеми нормальными подгруппами индекса p^l, $\forall l \in \mathbb{N}$. Тогда $\widetilde{F}_p$ называется свободной про-p группой.

Remark

Здесь и далее под подобным пополнением мы имеем в виду обратный предел факторгрупп.

Пусть Δ про-p кольцо.

$$GL_d^1(\Delta) = \ker\left(GL_d(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_d(\Delta/I)\right)$$

is a pro-p-group.

Основная гипотеза

Conjecture

Некоммутативная свободная про-p группа $\widetilde{F}_p$ *не может быть непрерывно вложена в* $GL_d^1(\Delta)$ *для любого про-p кольца* Δ.

Историческая справка

Существует множество частичных результатов для различных $\widetilde{F}_p, \Delta, p$, которые дают надежду на положительный результат и в общем случае:

- In 1987, A.N Zubkov ([3]) demonstrated that for $d = 2, p \neq 2$ the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([6]) established the conjecture for $\Delta = \mathbb{Z}_p$, $GL_d^1(\mathbb{Z}_p) = \ker\left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p)\right)$
- In 1999, utilizing the profound results of Pink ([4]), Y. Barnea, M. Larsen ([5]) proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 2005, E. Zelmanov ([?]) announced that conjecture holds true for $p \gg d$.
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([7]) that for $d = 2, p = 2$ and $\mathrm{char}(\Delta) = 2$ the conjecture holds true.

Zubkov's approach

Theorem (Zubkov, 1987)

Let F be a free non-abelian pro-p group, Δ is a pro-p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.

The first non-trivial idea is to introduce the following definition:

Definition

Let F be a free pro-p group, and G be a pro-p group. Then every $1 \neq w \in F$ such that $w \in Ker(\varphi)$ for all continuous homomorphisms $\varphi : F \to G$ is called a pro-p identity of G.

Zubkov consider the natural homomorphism to algebra of generic matrices: Let $x, y \in \widetilde{F}_p$ — generators, $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$, where x^*, y^* are the generic matrices over $\mathbb{Z}_p$. And one can continue π to the completion $\langle\langle x, y\rangle\rangle$, and it will map on closure of $\langle 1 + x^*, 1 + y^*\rangle$.

Zubkov's approach

Homomorphism π is called the universal representation, and that's why:

Lemma

Each $1 \neq z \in \ker\pi$ is a pro-p identity of $GL_d^1(\Delta)$ for all pro-p rings Δ.

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

Theorem

The universal representation of the degree 2 is not injective for $p \neq 2$.

So we need to construct the pro-p identity for generic matrices.

Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.
And he encountered a contradiction with the classical Witt's formula:

Formula (Witt)

Rank of r-th factor of the lower central series of $\widetilde{F}_p$ (as a $\mathbb{Z}_p-$module):

$$\frac{1}{r}\sum_{m|r}\mu(m)\cdot 2^{\frac{r}{m}}$$

Ben-Ezra, Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 2$.

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case $p = 2$: analogous homomorphism to generic matrices over $\mathbb{Z}/2\mathbb{Z}$ (instead of $\mathbb{Z}_2$).
Then the following lemma still holds and has a pretty simple proof:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ with $\mathrm{char}\Delta = 2$.

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

Theorem

The universal representation of the degree 2 is not injective.

$\mathrm{char}\Delta = 4$

Conjecture

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 4$.

We intend to prove it using the similar approaches, and believe that one can prove it even for the case $\mathrm{char}\Delta = 2^l$.
Furthermore, maybe the case $\mathrm{char}\Delta = 0$ can be investigated if the above statement will be proved.

PI-theory, preliminaries

Let T be the endomorphism (substitution) semigroup of the free algebra $F = k\langle x_1, \ldots, x_i, \ldots \rangle$.

Definition

An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$.

Definition

T-space in F is a vector subspace of F, that is closed under substitutions.

Definition

T-ideal in F is an ideal of F that is at the same time a T-space.

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

Theorem

Any T-space in algebra $k[x_1, \ldots, x_n]$ is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T-spaces are finitely based using some $\widetilde{T} \subset T$
The main idea is to use substitutions:

$$f(x_1, \ldots, x_i, \ldots, x_n) \mapsto f(x_1, \ldots, 1 + \alpha_i P(x_i), \ldots, x_n)$$

And then we linearize it on α_i.

Gelfand conjecture

Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by $\mathcal{W}_n$ the Lie algebra of formal vector fields on an n-dimensional plane V.

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

Lemma

Any finitely generated $L_d(n)$-module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.

Библиография

- I. Sanov, *The property of one free group representation*, *Doklady Akademii Nauk USSR*, vol. 57, no. 7, pp. 657–659, 1947.
- A. Kanel-Belov, *Local finite basability and local representability of varieties of associative rings*, *Doklady Akademii Nauk*, vol. 432, no. 6, pp. 727–731, 2010.
- A. Zubkov, *Non-abelian free pro-p-groups cannot be represented by 2-by-2 matrices*, *Siberian Mathematical Journal*, vol. 28, pp. 742–747, 1987.
- R. Pink, *Compact subgroups of linear algebraic groups*, *Journal of Algebra*, vol. 206, pp. 438–504, 1998.
- Y. Barnea and M. Larsen, *A non-abelian free pro-p group is not linear over a local field*, *Journal of Algebra*, vol. 214, pp. 338–341, 1999.
- J. Dixon, A. Mann, M. du Sautoy, and D. Segal, *Analytic pro-p-groups*, *London Mathematical Society Lecture Note Series*, Cambridge University Press, 1991.
- D. Ben-Ezra and E. Zelmanov, *On Pro-2 Identities of 2$\times$2 Linear Groups*, *arXiv:1910.05805v2*, 2020.
- E. Zelmanov, *Infinite algebras and pro-p groups*, *Infinite groups: geometric, combinatorial and dynamical aspects*, Progr. Math., vol. 248, pp. 403–413, 2005.
- E. Zelmanov, *Groups with identities*, *Note. Mat.*, vol. 36, pp. 101–113, 2016.
- I.M. Gelfand, *The cohomology of infinite dimensional Lie algebras; Some questions of integral geometry*, *Proceedings of ICM*, vol. T.1, p. 106, 1970.
- B. Feigin, A. Kanel-Belov, and A. Khoroshkin, *On finite dimensionality of homology of subalgebras of vector fields*, *arXiv:2211.08510v1*, 2022.

Библиография

L. Centrone, A. Kanel-Belov, A. Khoroshkin, and I. Vorobiov, *Specht property for systems of commutative polynomials and Gelfand conjecture*, https://www.researchgate.net/publication/355916110_Gelfand_conjecture_and_the_method_of_proof_of_Specht_problem, 2022.

A. Kemer, *Finite basability of identities of associative algebras*, *Algebra and Logics*, vol. 26, no. 5, pp. 597–641, 1987.

C. Procesi, *The geometry of polynomial identities*, *Izv. Math.*, vol. 80, no. 5, pp. 910–953, 2016.

A. Grishin, *On finitely based systems of generalized polynomials*, *Math. USSR-Izv.*, vol. 37, no. 2, pp. 243–272, 1991.

V. Shchigolev, *Finite-basis property of T-spaces over fields of characteristic zero*, *Izv. Ross. Akad. Nauk, Ser. Mat.*, vol. 65, no. 5, pp. 1041–1071, 2001.

A. Lubotzky, *Combinatorial group theory for PRO-p groups*, *Pure and Applied Algebra*, vol. 25, pp. 311–325, 1982.

E. Aljadeff, A. Kanel-Belov, and Y. Karasik, *Kemer's theorem for affine PI algebras over a field of characteristic zero*, *Pure and Applied Algebra*, vol. 220, pp. 2771–2808, 2016.

A. Grishin, *On finitely based systems of generalized polynomials*, *Math. USSR-Izv.*, vol. 37, no. 2, pp. 243–272, 1991.

A. Grishin and V. Shchigolev, *T-spaces and their applications*, *Math. Sci., New York*, vol. 134, no. 1, pp. 1799–1878, 2004.

I. Benediktovich and A. Zalesskii, *T-ideals of free Lie algebras with polynomial growth of a sequence of codimensions*, *Proceedings of the National Academy of Sciences of Belarus. Series of Physical-Mathematical Sciences*, vol. 3, pp. 5–10, 1980.

Библиография

A. Vais and E. Zelmanov, *Kemer's theorem for finitely generated Jordan algebras*, *Izv. Vyssh. Uchebn. Zved. Mat.*, vol. 33, no. 6, pp. 42–51, 1989. Note: Translation: Soviet Math. (Iz. VUZ) 33(6) (1989), 38–47.

L. Centrone, A. Estrada, and A. Ioppolo, *On PI-algebras with additional structures: rationality of Hilbert series and Specht's problem*, *J. Algebra*, vol. 592, pp. 300–356, 2022.

A. Kanel-Belov, *Counterexamples to the Specht problem*, *Sb. Math.*, vol. 191, no. 3, pp. 13–24, 2000. Note: Translation: Sb. Math. 131(3-4) (2000), 329–340.

A. Grishin, *Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property*, *Fundam. Prikl. Mat.*, vol. 5, no. 1, pp. 101–118, 1999.

V. Shchigolev, *Examples of infinitely based T-ideals*, *Fundam. Prikl. Mat.*, vol. 5, no. 1, pp. 307–312, 1999.

E. Aljadeff and A. Kanel-Belov, *Representability and Specht problem for G-graded algebras*, *Adv. Math.*, vol. 225, no. 5, pp. 2391–2428, 2010.

I. Sviridova, *Identities of pi-algebras graded by a finite abelian group*, *Comm. Algebra*, vol. 39, no. 9, pp. 3462–3490, 2011.

D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Springer Science & Business Media, 2012.