

Specht Problem and Gelfand Conjecture

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Strucure of the work

There are two non-trivial applications of PI-theory those will be presented:

- Non-linearity of free pro- p groups
- Gelfand's conjecture

Structure:

- 1 Preliminaries for pro- p structures
- 2 A brief historical review
- 3 Problem statement (on non-linearity of free pro- p group over 2-by-2 matrices)
- 4 Review of Zubkov's approach
- 5 Review of Zelmanov, Ben-Ezra's approach
- 6 Modification of Zelmanov, Ben-Ezra's approach for $\text{char} \Delta = 4$
- 7 Gelfand conjecture, statement and reformulation: whether some module is Noetherian
- 8 Connection between Gelfand conjecture and PI-theory

Definition

The inverse (projective) limit of the projective system of finite groups (rings) is called a profinite group (ring).

Definition

The inverse limit of the projective system of p -groups is called pro- p group.

Definition

Commutative Noetherian complete local ring Δ with a maximal ideal I is called pro- p ring if Δ/I is a finite field of characteristic p .

In that case:

$$\Delta = \varprojlim \Delta/I^n$$

Definition

Let F be a free group generated by alphabet \mathcal{S} . Consider the completion \tilde{F}_p of F with respect to topology, defined by all normal subgroups of a finite index p^l , $\forall l \in \mathbb{N}$ which have almost all generators from \mathcal{S} . Then \tilde{F}_p is called a free pro- p -group.

Remark

Here and later the completion with respect to normal subgroups means the inverse limit of the system of factorgroups.

Let Δ be pro- p ring

$$GL_d^1(\Delta) = \ker \left(GL_d(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_d(\Delta/I) \right)$$

is a pro- p -group.

Main problem

Conjecture

A non-abelian free pro- p group \tilde{F}_p cannot be continuously embedded in $GL_d^1(\Delta)$ for any pro- p ring Δ .

Historical review

There are several partial results for certain $\widetilde{F}_p, \Delta, p$, that let us suppose that answer is positive for the general case:

- In 1987, A.N Zubkov ([1]) demonstrated that for $d = 2, p \neq 2$ the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([4]) established the conjecture for $\Delta = \mathbb{Z}_p$,
$$GL_d^1(\mathbb{Z}_p) = \ker \left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \rightarrow \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$$
- In 1999, utilizing the profound results of Pink ([2]), Y. Barnea, M. Larsen ([3]) proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 2005, E. Zelmanov ([6]) announced that conjecture holds true for $p \gg d$.
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([5]) that for $d = 2, p = 2$ and $\text{char}(\Delta) = 2$ the conjecture holds true.

Zubkov's approach

Theorem (Zubkov, 1987)

Let F be a free non-abelian pro- p group, Δ is a pro- p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.

The first non-trivial idea is to introduce the following definition:

Definition

Let F be a free pro- p group, and G be a pro- p group. Then every $1 \neq w \in F$ such that $w \in \text{Ker}(\varphi)$ for all continuous homomorphisms $\varphi : F \rightarrow G$ is called a pro- p identity of G .

Zubkov consider the natural homomorphism to algebra of generic matrices: Let $x, y \in \tilde{F}_p$ — generators, $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$, where x^*, y^* are the generic matrices over \mathbb{Z}_p . And one can continue π to the completion $\langle\langle x, y \rangle\rangle$, and it will map on closure of $\langle 1 + x^*, 1 + y^* \rangle$.

Zubkov's approach

Homomorphism π is called the universal representation, and that's why:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro- p identity of $GL_d^1(\Delta)$ for all pro- p rings Δ .

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

Theorem

The universal representation of the degree 2 is not injective for $p \neq 2$.

So we need to construct the pro- p identity for generic matrices.

Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.

And he encountered a contradiction with the classical Witt's formula:

Formula (Witt)

Rank of r -th factor of the lower central series of \tilde{F}_p (as a \mathbb{Z}_p - module):

$$\frac{1}{r} \sum_{m|r} \mu(m) \cdot 2^{\frac{r}{m}}$$

Ben-Ezra, Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\text{char}\Delta = 2$.

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case $p = 2$: analogous homomorphism to generic matrices over $\mathbb{Z}/2\mathbb{Z}$ (instead of \mathbb{Z}_2).

Then the following lemma still holds and has a pretty simple proof:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ with $\text{char}\Delta = 2$.

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

Theorem

The universal representation of the degree 2 is not injective.

$$\text{char}\Delta = 4$$

Conjecture

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\text{char}\Delta = 4$.

We intend to prove it using the similar approaches, and believe that one can prove it even for the case $\text{char}\Delta = 2^l$.

Furthermore, maybe the case $\text{char}\Delta = 0$ can be investigated if the above statement will be proved.

Let T be the endomorphism (substitution) semigroup of the free algebra $F = k\langle x_1, \dots, x_i, \dots \rangle$.

Definition

An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \dots, x_i, \dots) \mapsto (g_1, \dots, g_i, \dots)$.

Definition

T -space in F is a vector subspace of F , that is closed under substitutions.

Definition

T -ideal in F is an ideal of F that is at the same time a T -space.

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

Theorem

Any T -space in algebra $k[x_1, \dots, x_n]$ is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T -spaces are finitely based using some $\tilde{T} \subset T$

The main idea is to use substitutions:

$$f(x_1, \dots, x_i, \dots, x_n) \mapsto f(x_1, \dots, 1 + \alpha_i P(x_i), \dots, x_n)$$

And then we linearize it on α_i .

Gelfand conjecture

Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by \mathcal{W}_n the Lie algebra of formal vector fields on an n -dimensional plane V .

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.

Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

Lemma

Any finitely generated $L_d(n)$ -module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.

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