Specht Problem and Gelfand Conjecture

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Strucure of the work

There are two non-trivial applications of PI-theory those will be presented:

- Non-linearity of free pro-p groups
- Gelfand's conjecture

Structure:

1. Preliminaries for pro-p structures
2. A brief historical review
3. Problem statement (on non-linearity of free pro-p group over 2-by-2 matrices)
4. Review of Zubkov's approach
5. Review of Zelmanov, Ben-Ezra's approach
6. Modification of Zelmanov, Ben-Ezra's approach for $\mathrm{char}\Delta = 4$
7. Gelfand conjecture, statement and reformulation: whether some module is Noetherian
8. Connection between Gelfand conjecture and PI-theory

Preliminaries

Definition

The inverse (projective) limit of the projective system of finite groups (rings) is called a profinite group (ring).

Definition

The inverse limit of the projective system of p-groups is called pro-p group.

Definition

Commutative Noetherian complete local ring Δ with a maximal ideal I is called pro-p ring if Δ/I is a finite field of characteristic p.

In that case:

$$\Delta = \varprojlim \Delta/I^n$$

Preliminaries

Definition

Let F be a free group generated by alphabet $\mathcal{S}$. Consider the completion $\widetilde{F}_p$ of F with respect to topology, defined by all normal subgroups of a finite index p^l, $\forall l \in \mathbb{N}$ which have almost all generators from $\mathcal{S}$. Then $\widetilde{F}_p$ is called a free pro-p-group.

Remark

Here and later the completion with respect to normal subgroups means the inverse limit of the system of factorgroups.

Let Δ be pro-p ring

$$GL_d^1(\Delta) = \ker\left(GL_d(\Delta) \xrightarrow{\Delta \to \Delta/I} GL_d(\Delta/I)\right)$$

is a pro-p-group.

Main problem

Conjecture

A non-abelian free pro-p group $\widetilde{F}_p$ cannot be continuously embedded in $GL_d^1(\Delta)$ for any pro-p ring Δ.

Historical review

There are several partial results for certain $\widetilde{F}_p, \Delta, p$, that let us suppose that answer is positive for the general case:

- In 1987, A.N Zubkov ([?]) demonstrated that for $d = 2, p \neq 2$ the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([?]) established the conjecture for $\Delta = \mathbb{Z}_p$,
$$GL_d^1(\mathbb{Z}_p) = \ker\left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p)\right)$$
- In 1999, utilizing the profound results of Pink ([?]), Y. Barnea, M. Larsen ([?]) proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})\,[[t]]$
- In 2005, E. Zelmanov ([?]) announced that conjecture holds true for $p \gg d$.
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([?]) that for $d = 2, p = 2$ and $\mathrm{char}(\Delta) = 2$ the conjecture holds true.

Zubkov's approach

Theorem (Zubkov, 1987)

Let F be a free non-abelian pro-p group, Δ is a pro-p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.

The first non-trivial idea is to introduce the following definition:

Definition

Let F be a free pro-p group, and G be a pro-p group. Then every $1 \neq w \in F$ such that $w \in Ker(\varphi)$ for all continuous homomorphisms $\varphi : F \to G$ is called a pro-p identity of G.

Zubkov consider the natural homomorphism to algebra of generic matrices: Let $x, y \in \widetilde{F}_p$ — generators, $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$, where x^*, y^* are the generic matrices over $\mathbb{Z}_p$. And one can continue π to the completion $\langle\langle x, y \rangle\rangle$, and it will map on closure of $\langle 1 + x^*, 1 + y^* \rangle$.

Zubkov's approach

Homomorphism π is called the universal representation, and that's why:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-p identity of $GL_d^1(\Delta)$ for all pro-p rings Δ.

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

Theorem

The universal representation of the degree 2 is not injective for $p \neq 2$.

So we need to construct the pro-p identity for generic matrices.

Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.
And he encountered a contradiction with the classical Witt's formula:

Formula (Witt)

Rank of r-th factor of the lower central series of $\widetilde{F}_p$ (as a $\mathbb{Z}_p$ − module):

$$\frac{1}{r}\sum_{m|r}\mu(m)\cdot 2^{\frac{r}{m}}$$

Ben-Ezra, Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 2$.

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case $p = 2$: analogous homomorphism to generic matrices over $\mathbb{Z}/2\mathbb{Z}$ (instead of $\mathbb{Z}_2$).
Then the following lemma still holds and has a pretty simple proof:

Lemma

Each $1 \neq z \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ with $\mathrm{char}\Delta = 2$.

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

Theorem

The universal representation of the degree 2 is not injective.

$\mathrm{char}\Delta = 4$

Conjecture

Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $\mathrm{char}\Delta = 4$.

We intend to prove it using the similar approaches, and believe that one can prove it even for the case $\mathrm{char}\Delta = 2^l$.
Furthermore, maybe the case $\mathrm{char}\Delta = 0$ can be investigated if the above statement will be proved.

PI-theory, preliminaries

Let T be the endomorphism (substitution) semigroup of the free algebra $F = k\langle x_1, \ldots, x_i, \ldots \rangle$.

Definition

An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$.

Definition

T-space in F is a vector subspace of F, that is closed under substitutions.

Definition

T-ideal in F is an ideal of F that is at the same time a T-space.

PI-theory, preliminaries

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

Theorem

Any T-space in algebra $k[x_1, \ldots, x_n]$ is finitely based.

Furthermore, one can prohibit some of the substitutions and show that T-spaces are finitely based using some $\widetilde{T} \subset T$
The main idea is to use substitutions:

$$f(x_1, \ldots, x_i, \ldots, x_n) \mapsto f(x_1, \ldots, 1 + \alpha_i P(x_i), \ldots, x_n)$$

And then we linearize it on α_i.

Gelfand conjecture

Conjecture (Gelfand, 1970, [?])

The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.

We denote by $\mathcal{W}_n$ the Lie algebra of formal vector fields on an n-dimensional plane V.

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.

Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

Lemma

Any finitely generated $L_d(n)$-module is noetherian.

Then we will observe how to use Grishin's methods to prove this lemma.

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