

# Specht Problem and Gelfand Conjecture

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# Strucure of the work

There are two non-trivial applications of PI-theory those will be presented:

- Non-linearity of free pro- $p$  groups
- Gelfand's conjecture

Structure:

- 1 Preliminaries for pro- $p$  structures
- 2 A brief historical review
- 3 Problem statement (on non-linearity of free pro- $p$  group over 2-by-2 matrices)
- 4 Review of Zubkov's approach
- 5 Review of Zelmanov, Ben-Ezra's approach
- 6 Modification of Zelmanov, Ben-Ezra's approach for  $\text{char} \Delta = 4$
- 7 Gelfand conjecture, statement and reformulation: whether some module is Noetherian
- 8 Connection between Gelfand conjecture and PI-theory

## Definition

The inverse (projective) limit of the projective system of finite groups (rings) is called a profinite group (ring).

## Definition

The inverse limit of the projective system of  $p$ -groups is called pro- $p$  group.

## Definition

Commutative Noetherian complete local ring  $\Delta$  with a maximal ideal  $I$  is called pro- $p$  ring if  $\Delta/I$  is a finite field of characteristic  $p$ .

In that case:

$$\Delta = \varprojlim \Delta/I^n$$

## Definition

Let  $F$  be a free group generated by alphabet  $\mathcal{S}$ . Consider the completion  $\tilde{F}_p$  of  $F$  with respect to topology, defined by all normal subgroups of a finite index  $p^l$ ,  $\forall l \in \mathbb{N}$  which have almost all generators from  $\mathcal{S}$ . Then  $\tilde{F}_p$  is called a free pro- $p$ -group.

## Remark

*Here and later the completion with respect to normal subgroups means the inverse limit of the system of factorgroups.*

Let  $\Delta$  be pro- $p$  ring

$$GL_d^1(\Delta) = \ker \left( GL_d(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_d(\Delta/I) \right)$$

is a pro- $p$ -group.

# Main problem

## Conjecture

*A non-abelian free pro- $p$  group  $\tilde{F}_p$  cannot be continuously embedded in  $GL_d^1(\Delta)$  for any pro- $p$  ring  $\Delta$ .*

# Historical review

There are several partial results for certain  $\widetilde{F}_p, \Delta, p$ , that let us suppose that answer is positive for the general case:

- In 1987, A.N Zubkov ([?]) demonstrated that for  $d = 2, p \neq 2$  the conjecture holds true.
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([?]) established the conjecture for  $\Delta = \mathbb{Z}_p$ ,
$$GL_d^1(\mathbb{Z}_p) = \ker \left( GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \rightarrow \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$$
- In 1999, utilizing the profound results of Pink ([?]), Y. Barnea, M. Larsen ([?]) proved the conjecture for  $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 2005, E. Zelmanov ([?]) announced that conjecture holds true for  $p \gg d$ .
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([?]) that for  $d = 2, p = 2$  and  $\text{char}(\Delta) = 2$  the conjecture holds true.

# Zubkov's approach

## Theorem (Zubkov, 1987)

*Let  $F$  be a free non-abelian pro- $p$  group,  $\Delta$  is a pro- $p$  ring.  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $p > 2$ .*

The first non-trivial idea is to introduce the following definition:

## Definition

Let  $F$  be a free pro- $p$  group, and  $G$  be a pro- $p$  group. Then every  $1 \neq w \in F$  such that  $w \in \text{Ker}(\varphi)$  for all continuous homomorphisms  $\varphi : F \rightarrow G$  is called a pro- $p$  identity of  $G$ .

Zubkov consider the natural homomorphism to algebra of generic matrices: Let  $x, y \in \tilde{F}_p$  — generators,  $\pi : x \mapsto 1 + x^*, y \mapsto 1 + y^*$ , where  $x^*, y^*$  are the generic matrices over  $\mathbb{Z}_p$ . And one can continue  $\pi$  to the completion  $\langle\langle x, y \rangle\rangle$ , and it will map on closure of  $\langle 1 + x^*, 1 + y^* \rangle$ .

# Zubkov's approach

Homomorphism  $\pi$  is called the universal representation, and that's why:

## Lemma

*Each  $1 \neq z \in \ker \pi$  is a pro- $p$  identity of  $GL_d^1(\Delta)$  for all pro- $p$  rings  $\Delta$ .*

This theorem can be proven with a standard commutative algebra approaches. And this implies that it would be enough to prove

## Theorem

*The universal representation of the degree 2 is not injective for  $p \neq 2$ .*

So we need to construct the pro- $p$  identity for generic matrices.



# Zubkov's approach

Zubkov investigated the lower central series of Lie algebra of generic matrices.

And he encountered a contradiction with the classical Witt's formula:

## Formula (Witt)

*Rank of  $r$ -th factor of the lower central series of  $\tilde{F}_p$  (as a  $\mathbb{Z}_p$  - module):*

$$\frac{1}{r} \sum_{m|r} \mu(m) \cdot 2^{\frac{r}{m}}$$

# Ben-Ezra, Zelmanov's approach

## Theorem (Ben-Ezra, Zelmanov, 2020)

*Let  $F$  be a free non-abelian pro-2 group,  $\Delta$  is a pro-2 ring.  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $\text{char}\Delta = 2$ .*

Ben-Ezra and Zelmanov modified Zubkov's universal representation for the case  $p = 2$ : analogous homomorphism to generic matrices over  $\mathbb{Z}/2\mathbb{Z}$  (instead of  $\mathbb{Z}_2$ ).

Then the following lemma still holds and has a pretty simple proof:

## Lemma

*Each  $1 \neq z \in \ker \pi$  is a pro-2 identity of  $GL_2^1(\Delta)$  for all pro-2 rings  $\Delta$  with  $\text{char}\Delta = 2$ .*

And the last theorem has a very hard proof: authors investigated Lie algebra using PI-theory approaches:

## Theorem

*The universal representation of the degree 2 is not injective.*

$$\text{char}\Delta = 4$$

## Conjecture

*Let  $F$  be a free non-abelian pro-2 group,  $\Delta$  is a pro-2 ring.  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $\text{char}\Delta = 4$ .*

We intend to prove it using the similar approaches, and believe that one can prove it even for the case  $\text{char}\Delta = 2^l$ .

Furthermore, maybe the case  $\text{char}\Delta = 0$  can be investigated if the above statement will be proved.

Let  $T$  be the endomorphism (substitution) semigroup of the free algebra  $F = k\langle x_1, \dots, x_i, \dots \rangle$ .

## Definition

An endomorphism  $\tau$  of  $F$  defined by the rule  $x_i \mapsto g_i, g_i \in F$ , is called a substitution of type  $(x_1, \dots, x_i, \dots) \mapsto (g_1, \dots, g_i, \dots)$ .

## Definition

$T$ -space in  $F$  is a vector subspace of  $F$ , that is closed under substitutions.

## Definition

$T$ -ideal in  $F$  is an ideal of  $F$  that is at the same time a  $T$ -space.

Following theorem (the special case of Shchigolev's [?]) is proved in author's last year coursework.

## Theorem

*Any  $T$ -space in algebra  $k[x_1, \dots, x_n]$  is finitely based.*

Furthermore, one can prohibit some of the substitutions and show that  $T$ -spaces are finitely based using some  $\tilde{T} \subset T$

The main idea is to use substitutions:

$$f(x_1, \dots, x_i, \dots, x_n) \mapsto f(x_1, \dots, 1 + \alpha_i P(x_i), \dots, x_n)$$

And then we linearize it on  $\alpha_i$ .

# Gelfand conjecture

## Conjecture (Gelfand, 1970, [?])

*The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

We denote by  $\mathcal{W}_n$  the Lie algebra of formal vector fields on an  $n$ -dimensional plane  $V$ .

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras  $\prod_{k=d}^{\infty} S^k V \otimes V^*$  of a finite codimension are denoted by  $L_d(n)$ .

# Gelfand conjecture

Using the classical considerations of homological algebra, one can reduce Gelfand conjecture to the following lemma:

## Lemma

*Any finitely generated  $L_d(n)$ -module is noetherian.*

Then we will observe how to use Grishin's methods to prove this lemma.

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