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Specht Problem and Gelfand Conjecture

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Abstract

Let F be a free pro-p non-abelian group, and let Δ represent a commutative Noetherian complete local ring with maximal ideal I such that $\mathrm{char}(\Delta/I) = p$.
We define the group

$$GL_2^1(\Delta) = \ker\left(GL_2(\Delta) \xrightarrow{\Delta\to\Delta/I} GL_2(\Delta/I)\right)$$

A.N. Zubkov proved that F cannot be continuously embedded in $GL_2^1(\Delta)$ for $p \neq 2$.
D. Ben-Ezra and E. Zelmanov further established that this embedding is not possible for $p = 2$ and $\mathrm{char}(\Delta) = 2$.
In this paper we aim to extend this result for $\mathrm{char}(\Delta) = 4$.
In the second part we will investigate the connection between PI-theory and the old-standing Gelfand conjecture.

1. Introduction

1.1. On non-linearity of free non-abelian pro-p groups

The problem of linearity of topological groups is a natural one and has studied for many years. It is well known that discrete free groups are linear ([13]). Moreover, they can be embedded in $GL_2(\mathbb{Z})$.
So it's also quite natural to inquire whether a free pro-p non-abelian group is linear.

Definition 1.1.1. *Commutative Noetherian complete local ring Δ with a maximal ideal I is called pro-p ring if Δ/I is a finite field of characteristic p.*

Consider the congruence subgroup:

$$GL_2^1(\Delta) = \ker\left(GL_2(\Delta) \xrightarrow{\Delta\to\Delta/I} GL_2(\Delta/I)\right)$$

One can see that $GL_2^1(\Delta)$ is a pro-p-group. Thus, the main conjecture of this theory can be formulated as follows:

Conjecture. *Non-abelian free pro-p group cannot be continuously embedded in $GL_d^1(\Delta)$ for any pro-p ring Δ.*

There have been a lot of partial results for certain Δ, d and p. Let us list them out.

- In 1987, A.N Zubkov ([17]) demonstrated that for $d = 2, p \neq 2$ the conjecture holds true.
- In 1999, utilizing the profound results of Pink ([11]), Y. Barnea, M. Larsen ([2]) proved the conjecture for $\Delta = (\mathbb{Z}/p\mathbb{Z})\,[[t]]$
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([5]) established the conjecture for $\Delta = \mathbb{Z}_p$, $GL_d^1(\mathbb{Z}_p) = \ker\left(GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \to \mathbb{F}_p} GL_2(\mathbb{F}_p)\right)$
- In 2020, D. Ben-Ezra, E. Zelmanov showed ([3]) that for $d = 2, p = 2$ and $\mathrm{char}(\Delta) = 2$ the conjecture holds true.
- In 2005, E. Zelmanov ([15], [16]) announced that conjecture holds true for $p \gg d$.

One can see that this subject has be researched by many mathematicians. So first of all we are going to give a review of their methods.
We will focus mostly on Zubkov's and Ben-Ezra, Zelmanov's methods.
Zubkov's proof based on standard approaches of commutative algebra and the idea of generic matrices. Zelmanov and Ben-Ezra adopted Zubkov's method for the case $p = 2$ using the trace identities which dates back to polynomial identities theory (PI-theory for short).
Additionally, we intend to extend Zelmanov and Ben-Ezra's approach for $d = 2, p = 2$ and $\mathrm{char}(\Delta) = 4$. We also hope that it will be quite easy to extend it for $\mathrm{char}(\Delta) = 2^l$, and maybe even for $\mathrm{char}(\Delta) = 0$.

1.2. Gelfand conjecture

In 2022 the remarkable connection between PI-theory (to be more precisely Grishin's methods) and Gelfand Conjecture stated at ICM'70 (see [7]) was found.

Conjecture (Gelfand)**.** *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

This interesting connection was found during joint conversation between A.S. Khoroshkin, A.Ya. Kanel-Belov and with a little help of author. One can find Khoroshkin's sketch in [6], [4]. Also we will use results from author's last year's coursework about finitely based T-spaces of commutative polynomials.

2. Preliminaries

2.1. Profinite objects

Let us remind the reader of the classical definitions of the profinite groups theory.

Definition 2.1.1. *Inverse (projective) limit of finite groups is called a profinite group. In the case of finite p-groups we get pro-p group.*

It is clear that profinite groups can be equipped with the topology induced by the Tikhonov's product topology.

Definition 2.1.2. *The free pro-p group $F_p(X)$ is the completion of the discrete free group $F(X)$ with respect to a topology defined by all normal subgroups $N \subseteq F(X)$ whose indices are equal to the order of p and which contain almost all generators of $F(X)$.*

One can define the free pro-p group in a classical way using universal property in the category of pro-p groups.
Also note that if Δ is a pro-p ring as defined in introduction, and I is a maximal ideal. Then

$$\Delta = \varprojlim \Delta/I^n$$

2.2. PI-theory

Now let us give some basic definitions of PI-theory.
Let k be a field of characteristic zero and $F = k\langle x_1, \ldots, x_i, \ldots\rangle$ be a free, countably generated, associative algebra over a field k and T be the endomorphism (substitution) semigroup of F. $X = \{x_1, \ldots, x_i, \ldots\}$
Now let us give some classical definitions.

Definition 2.2.1. *An endomorphism τ of F defined by the rule $x_i \mapsto g_i, g_i \in F$, is called a substitution of type $(x_1, \ldots, x_i, \ldots) \mapsto (g_1, \ldots, g_i, \ldots)$.*

Definition 2.2.2. *T-space in F is a vector subspace of F, that is closed under substitutions.*

Definition 2.2.3. *T-ideal in F is an ideal of F that is at the same time a T-space.*

Definition 2.2.4. *We say that T-space M is finitely based if there is a finite subset $B \subset M$ such that T-space generated by B coincides with M.*

During the 1980s, A.R. Kemer's resolution of Specht problem was a significant breakthrough in the PI-theory ([10], see also simplified version of Kemer's proof in [1], [12]):
Here is a well known reformulation of Kemer's theorem:

Theorem. *Any T-ideal of the algebra F is finitely based.*

It's natural to ask the same question for T-spaces.
In 2001, V.V. Shchigolev combined Grishin ([8]) and Kanel-Belov's ([9]) methods. Then Shchigolev noticed that methods similar to Kemer's can be applied to localisation of Specht problem for the T-spaces. And finally he proved ([14]):

Theorem (V.V. Shchigolev, 2001)**.** *Any T-space of the algebra F is finitely based.*

We will use one simple special case of Shchigolev's theorem, which was also proven in author's last year's coursework.

3. Main results

3.1. Zubkov's approach

Theorem (Zubkov, 1987)**.** *Let F be a free non-abelian pro-p group, Δ is a pro-p ring. F cannot be continuously embedded in $GL_2^1(\Delta)$, when $p > 2$.*

The first non-trivial idea is to give the following definition:

Definition 3.1.1. *Let F be a free pro-p group, and G be a pro-p group. Then every $1 \neq w \in F$ such that $w \in Ker(\varphi)$ for all continuous homomorphisms $\varphi : F \to G$ is called a pro-p identity of G.*

Then Zubkov defines a ring of generic matrices over p-adic numbers $\mathbb{Z}_p$ (the formal way to do it is quite long, see [17])
He defines a natural homomorphism π from a free pro-p group F generated by X, Y to the pro-p group generated by $1+x_*, 1+y_*$, where x_*, y_* are generic matrices.
This homomorphism is called a universal representation:

Theorem 3.1.1. *Each $1 \neq w(X,Y) \in \ker \pi$ is a pro-p identity of $GL_d^1(\Delta)$ for all pro-p rings Δ.*

This theorem has pretty simple proof using the standard commutative algebra approachs. Surprisingly, it remains to prove that

Theorem 3.1.2. *The universal representation is not injective for $p \neq 2$.*

Zubkov proved it by investigating the Lie algebra of generic 2-by-2 matrices by fairly long calculations.

3.2. Ben-Ezra and Zelmanov's approach

Theorem (Ben-Ezra, Zelmanov, 2020)**.** *Let F be a free non-abelian pro-2 group, Δ is a pro-2 ring, such that* $\mathrm{char}(\Delta) = 2$*. F cannot be continuously embedded in $GL_2^1(\Delta)$.*

The main ideas of Ben-Ezra and Zelmanov's proof are quite similar to Zubkov's ones. But they use a slightly different way to define the universal representation π. They define generic matrices over $\mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{Z}_2$ as Zubkov. Then the analogous theorem about the kernel of universal representation is still true (at least in the case $\mathrm{char}(\Delta) = 2$):

Theorem 3.2.1. *Each $1 \neq w(X, Y) \in \ker \pi$ is a pro-2 identity of $GL_2^1(\Delta)$ for all pro-2 rings Δ, such that* $\mathrm{char}(\Delta) = 2$*.*

Thus, the main result boils down to

Theorem 3.2.2. *The universal representation is not injective.*

This was proven using the 20 pages fairly difficult calculations.

3.3. An extension of Ben-Ezra and Zelmanov's approach

We are going to prove a slightly different version of the theorem 3.2.1 for the case $\mathrm{char}(\Delta) = 4$ and perhaps we will be able to do the similar thing for $\mathrm{char}(\Delta) = 2^l$. Then we hope that the same calculations will complete the proof.
Also, we will discuss the case $\mathrm{char}(\Delta) = 0$ and how it can be investigated with aforementioned approaches.

3.4. Gelfand's conjecture

We are going to consider one special case of Shchigolev's result ([?]:

Theorem 3.4.1. *Any T-space in algebra $k[x_1, \ldots, x_n]$ is finitely based.*

Then we will discuss Gelfand's conjecture.

Conjecture. *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

We denote by $\mathcal{W}_n$ the Lie algebra of formal vector fields on an n-dimensional plane V.
Well known that

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras $\prod\limits_{k=d}^{\infty} S^k V \otimes V^*$ of a finite codimension are denoted by $L_d(n)$.
Using the classical considerations of homological algebra (which will be omitted), one can reduce Gelfand's conjecture to the following lemma

Lemma 3.4.1. *Any finitely generated $L_d(n)$-module is Noetherian.*

Finally, we will notice that the methods from the theorem 3.4.1 can be applied to prove this lemma.

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