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**Specht Problem and Gelfand Conjecture**  
**Проблема Шпехта и гипотеза Гельфанда**

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## Abstract

Let  $F$  be a free pro- $p$  non-abelian group, and let  $\Delta$  represent a commutative Noetherian complete local ring with maximal ideal  $I$  such that  $\text{char}(\Delta/I) = p$ . We define the group

$$GL_2^1(\Delta) = \ker \left( GL_2(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_2(\Delta/I) \right)$$

A.N. Zubkov proved that  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$  for  $p \neq 2$ .

D. Ben-Ezra and E. Zelmanov further established that this embedding is not possible for  $p = 2$  and  $\text{char}(\Delta) = 2$ .

In this paper we aim to extend this result for  $p = 2$  and  $\text{char}(\Delta) = 4$ . As was shown in [15], the conjecture of non-linearity of free pro- $p$  groups is strongly connected with PI-theory.

In the second part we will investigate the connection between PI-theory and the old-standing Gelfand conjecture.

Thus, one can see that this paper is devoted to investigating combinatorial properties of substitutions.

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## 1. Introduction

We will explore the applications of PI-theory in two seemingly unrelated areas:

- In 2005, E. Zelmanov outlined the sketch of the proof regarding the non-linearity of certain free pro- $p$  groups. The proof predominantly relies on classical approaches of PI-theory (see [15]).
- Additionally, we will uncover a remarkable connection between Gelfand's conjecture and PI-theory, particularly with Grishin's methods ([8]).

Now, let's provide a brief historical overview of these problems and their formal statements.

### 1.1. On non-linearity of free non-abelian pro- $p$ groups

The problem of linearity of topological groups is a natural one and has been studied for many years. It is well known that discrete free groups are linear ([13]). Moreover, they can be embedded in  $GL_2(\mathbb{Z})$ .

So it's also quite natural to inquire whether a free pro- $p$  non-abelian group is linear.

**Definition 1.1.1.** *Commutative Noetherian complete local ring  $\Delta$  with a maximal ideal  $I$  is called pro- $p$  ring if  $\Delta/I$  is a finite field of characteristic  $p$ .*

Consider the congruence subgroup:

$$GL_2^1(\Delta) = \ker \left( GL_2(\Delta) \xrightarrow{\Delta \rightarrow \Delta/I} GL_2(\Delta/I) \right)$$

One can see that  $GL_2^1(\Delta)$  is a pro- $p$ -group. Thus, the main conjecture of this theory can be formulated as follows:

**Conjecture.** *Non-abelian free pro- $p$  group cannot be continuously embedded in  $GL_d^1(\Delta)$  for any pro- $p$  ring  $\Delta$ .*

There have been a lot of partial results for certain  $\Delta, d$  and  $p$ . Let us list them out.

- In 1987, A.N Zubkov ([17]) demonstrated that for  $d = 2, p \neq 2$  the conjecture holds true.
- In 1999, utilizing the profound results of Pink ([11]), Y. Barnea, M. Larsen ([2]) proved the conjecture for  $\Delta = (\mathbb{Z}/p\mathbb{Z})[[t]]$
- In 1991, J.D. Dixon, A. Mann, M.P.F. du Sautoy, D. Segal ([5]) established the conjecture for  $\Delta = \mathbb{Z}_p, GL_d^1(\mathbb{Z}_p) = \ker \left( GL_2(\mathbb{Z}_p) \xrightarrow{\mathbb{Z}_p \rightarrow \mathbb{F}_p} GL_2(\mathbb{F}_p) \right)$

- In 2020, D. Ben-Ezra, E. Zelmanov showed ([3]) that for  $d = 2, p = 2$  and  $\text{char}(\Delta) = 2$  the conjecture holds true.
- In 2005, E. Zelmanov ([15], [16]) announced that conjecture holds true for  $p \gg d$ .

One can observe that this subject has been extensively researched by many mathematicians. Therefore, first of all, we will provide a review of their methods.

We will primarily focus on Zubkov's approach, as well as Ben-Ezra and Zelmanov's methods. Zubkov's proof based on standard approaches of commutative algebra and the idea of generic matrices. Zelmanov and Ben-Ezra adopted Zubkov's method for the case  $p = 2$  using the trace identities which dates back to polynomial identities theory (PI-theory for short).

Additionally, we intend to expand Zelmanov and Ben-Ezra's approach for the case where  $d = 2, p = 2$  and  $\text{char}(\Delta) = 4$ . Moreover, we anticipate that extending it to cases where  $\text{char}(\Delta) = 2^l$  and possibly even  $\text{char}(\Delta) = 0$  should be relatively straightforward.

## 1.2. Gelfand conjecture

In 2022 the remarkable connection between PI-theory (to be more precisely Grishin's methods) and Gelfand Conjecture stated at ICM'70 (see [7]) was found.

**Conjecture** (Gelfand). *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

This interesting connection was found during joint conversation between A.S. Khoroshkin, A.Ya. Kanel-Belov and with some assistance from the author. One can find Khoroshkin's sketch in [6], [4]. Additionally, we will incorporate findings from the author's coursework on finitely based  $T$ -spaces of commutative polynomials from the previous year.

## 2. Preliminaries

### 2.1. Profinite Objects

Let us provide a brief overview of the classical definitions in the theory of profinite groups.

**Definition 2.1.1.** *Inverse (projective) limit of finite groups is called a profinite group. In the case of finite  $p$ -groups we obtain pro- $p$  group.*

It is clear that profinite groups can be endowed with the topology induced by the Tikhonov's product topology.

**Definition 2.1.2.** *The free pro- $p$  group  $F_p(X)$  is the completion of the discrete free group  $F(X)$  with respect to a topology defined by all normal subgroups  $N \subseteq F(X)$  whose indices are equal to the order of  $p$  and which contain almost all generators of  $F(X)$ .*

One can also define the free pro- $p$  group in a classical manner using the universal property in the category of pro- $p$  groups.

Also note that if  $\Delta$  is a pro- $p$  ring as defined in introduction, and  $I$  is a maximal ideal, then

$$\Delta = \varprojlim \Delta/I^n$$

## 2.2. PI-theory

Now, let's introduce some basic definitions of PI-theory.

Let  $k$  be a field of characteristic zero and  $F = k\langle x_1, \dots, x_i, \dots \rangle$  be a free, countably generated, associative algebra over a field  $k$  and  $T$  be the endomorphism (substitution) semigroup of  $F$ .  $X = \{x_1, \dots, x_i, \dots\}$

Now let us give some classical definitions.

**Definition 2.2.1.** *An endomorphism  $\tau$  of  $F$  defined by the rule  $x_i \mapsto g_i, g_i \in F$ , is called a substitution of type  $(x_1, \dots, x_i, \dots) \mapsto (g_1, \dots, g_i, \dots)$ .*

**Definition 2.2.2.**  *$T$ -space in  $F$  is a vector subspace of  $F$ , that is closed under substitutions.*

**Definition 2.2.3.**  *$T$ -ideal in  $F$  is an ideal of  $F$  that is at the same time a  $T$ -space.*

**Definition 2.2.4.** *We say that a  $T$ -space  $M$  is finitely based if there exists a finite subset  $B \subset M$  such that  $T$ -space generated by  $B$  coincides with  $M$ .*

During the 1980s, A.R. Kemer's resolution of Specht problem was a significant breakthrough in the PI-theory ([10], see also simplified version of Kemer's proof in [1], [12]): A well-known reformulation of Kemer's theorem is:

**Theorem.** *Any  $T$ -ideal of the algebra  $F$  is finitely based.*

It's natural to ask the same question for  $T$ -spaces.

In 2001, V.V. Shchigolev combined Grishin ([8]) and Kanel-Belov's ([9]) methods. Then Shchigolev noticed that methods similar to Kemer's can be applied to localisation of Specht problem for the  $T$ -spaces. And finally he proved ([14]):

**Theorem** (V.V. Shchigolev, 2001). *Any  $T$ -space of the algebra  $F$  is finitely based.*

We will utilize one simple special case of Shchigolev's theorem, which was also proven in author's last year's coursework.

### 3. Main results

#### 3.1. Zubkov's approach

**Theorem** (Zubkov, 1987). *Let  $F$  be a free non-abelian pro- $p$  group,  $\Delta$  is a pro- $p$  ring.  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$ , when  $p > 2$ .*

The first non-trivial idea is to introduce the following definition:

**Definition 3.1.1.** *Let  $F$  be a free pro- $p$  group, and  $G$  be a pro- $p$  group. Then every  $1 \neq w \in F$  such that  $w \in \text{Ker}(\varphi)$  for all continuous homomorphisms  $\varphi : F \rightarrow G$  is called a pro- $p$  identity of  $G$ .*

Then Zubkov defines a ring of generic matrices over  $p$ -adic numbers  $\mathbb{Z}_p$ . The formal construction is quite lengthy, see [17]. Now the reader can suppose that it's some ring, that can be studied easier than a abstract pro- $p$  ring.

He defines a natural homomorphism  $\pi$  from a free pro- $p$  group  $F$  generated by  $X, Y$  to the pro- $p$  group generated by  $1+x_*, 1+y_*$ , where  $x_*, y_*$  are generic matrices.

This homomorphism is called a universal representation:

**Theorem 3.1.1.** *Each  $1 \neq w(X, Y) \in \ker \pi$  is a pro- $p$  identity of  $GL_d^1(\Delta)$  for all pro- $p$  rings  $\Delta$ .*

This theorem has pretty simple proof using the standard commutative algebra approaches. Surprisingly, it remains to prove that

**Theorem 3.1.2.** *The universal representation is not injective for  $p \neq 2$ .*

Zubkov proved this by investigating the Lie algebra of generic  $2 \times 2$  matrices through rather lengthy (but not very complicated) calculations.

### 3.2. Ben-Ezra and Zelmanov's approach

**Theorem** (Ben-Ezra, Zelmanov, 2020). *Let  $F$  be a free non-abelian pro-2 group,  $\Delta$  be a pro-2 ring with  $\text{char}(\Delta) = 2$ . Then  $F$  cannot be continuously embedded in  $GL_2^1(\Delta)$ .*

The main ideas of Ben-Ezra and Zelmanov's proof are quite similar to Zubkov's. But they use a slightly different way to define the universal representation  $\pi$ . They define generic matrices over  $\mathbb{Z}/2\mathbb{Z}$  instead of  $\mathbb{Z}_2$  as Zubkov did. Nevertheless, the analogous theorem about the kernel of universal representation still holds true at least in the case  $\text{char}(\Delta) = 2$ :

**Theorem 3.2.1.** *Each  $1 \neq w(X, Y) \in \ker \pi$  is a pro-2 identity of  $GL_2^1(\Delta)$  for all pro-2 rings  $\Delta$  with  $\text{char}(\Delta) = 2$ .*

Thus, the main result boils down to

**Theorem 3.2.2.** *The universal representation is not injective.*

This was proven using lengthy calculations spanning 20 pages of fairly complex methods. The main idea is to investigate Lie algebra of generic matrices using PI-theory methods.

### 3.3. An extension of Ben-Ezra and Zelmanov's approach

We intend to establish a slightly different version of Theorem 3.2.1 for the case where  $\text{char}(\Delta) = 4$ . Moreover, we aim to explore the possibility of extending this result to the cases where  $\text{char}(\Delta) = 2^l$ . We anticipate that similar methodologies and calculations as those used in the original proof will suffice.

Additionally, we will discuss the case  $\text{char}(\Delta) = 0$  and how it can be investigated using the aforementioned approaches.

### 3.4. Gelfand's conjecture

We will begin by considering a specific case of Shchigolev's result ([14]):

**Theorem 3.4.1.** *Any  $T$ -space in algebra of commutative polynomials  $k[x_1, \dots, x_n]$  is finitely based.*

Then we will discuss Gelfand's conjecture.

**Conjecture.** *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

We denote by  $\mathcal{W}_n$  the Lie algebra of formal vector fields on an  $n$ -dimensional plane  $V$ .

Well known that

$$\mathcal{W}_n \simeq \prod_{k=0}^{\infty} S^k V \otimes V^*$$

The subalgebras  $\prod_{k=d}^{\infty} S^k V \otimes V^*$  of a finite codimension are denoted by  $L_d(n)$ .

Utilizing classical considerations of homological algebra (which will be omitted), one can reduce Gelfand's conjecture to the following lemma:

**Lemma 3.4.1.** *Any finitely generated  $L_d(n)$ -module is Noetherian.*

Finally, it is worth noting that the methods from Theorem 3.4.1 can be applied to prove this lemma. The action of linearized substitutions is similar with action of differentiations.

Because of this, one can view that Noetherity of  $L_d(n)$ -module can be considered as some finite basability problem.

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