Liouville's Theorem on integrability via elementary functions

Vanya Vorobiov

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Liouville's Theorem

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From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{-x^2} dx, \quad \int \frac{\sin(x)}{x} dx$$

These examples highlight the limitations of elementary functions in representing certain integrals. In this presentation, we will:

- Explore additional examples of integrals that defy expression in elementary terms.
- Introduce and prove a powerful tool for understanding this phenomenon: Liouville's theorem.

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Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

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Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

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Differential extension of the differential field F is field E such that $E \supset F$ and there is the same differentiation ' on E.

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Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{L}$

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Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, ..., t_{i-1})$.

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 $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over

Common sense says us that some function $f: \mathbb{C} \to \mathbb{C}$ is elementary iff it

can be constucted via finite number of radicals, sines, cosines, exponents,

logarithms and hyperbolic functions. One can see that it's consistent with

our approach. Futhermore our definition on elementarity is more general.

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 $F(t_1, \ldots, t_{i-1}).$

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Theorem (Liouville, 1833-1841)

that its subfield of constants is still K, then

for some $c_1, \ldots, c_m \in K$, $u_1, \ldots, u_m, v \in F$.

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such

 $\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$

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Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $deg f = k \ge 1$

- If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k-1 if the leading coefficient is a constant.
- If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise.

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n).$

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

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For short we denote $t = t_1$.

Using the inductive assumption, we get

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F(t)$.

Let x be the solution of differential equation mentioned above. And

 $\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$

We will use induction on n (we don't fix the field F).

Here we use that the subfield of constants of F(t) is K.

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Some special cases On elliptic integrals $x \in F(t_1, \ldots, t_n).$

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Now we consider 3 cases

t is algebraic over F.

Using the inductive assumption, we get

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Let x be the solution of differential equation mentioned above. And

 $\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$

We will use induction on n (we don't fix the field F).

Here we use that the subfield of constants of F(t) is K.

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▶ t is trancendental over F and it is an exponent;

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