

Liouville's Theorem on integrability via elementary functions

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From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{\sinh x}{x} dx, \quad \int \log \log x dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

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These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- ▶ Introduce and prove a powerful tool: Liouville's theorem.
- ▶ Derive these integrals as a consequence.
- ▶ If time permits, discuss some special form of elliptic integrals, which also cannot be expressed in elementary terms.

$$\int \frac{dx}{\sqrt{P(x)}}$$

for $\deg P = 2, 3$ and P has no multiple roots.

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Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
- ▶ $(ab)' = a'b + ab'$

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Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

Definition

Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

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Definition

The extension E of F is called elementary if it can be presented as

$E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense says us that some function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

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Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F , such that its subfield of constants is still K , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F$.

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Lemma

Let F be a differential field, t is transcendental over F , and t is a logarithm or an exponent of some element from F . And let $f \in F[x]$ be a polynomial, $\deg f = k \geq 1$

- ▶ If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of k if and only if f is a monomial.

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- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of k if and only if f is a monomial.

Proof.

It's a quite simple technical exercise.



Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

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We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F(t)$.

Here we use that the subfield of constants of $F(t)$ is K .

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Here we use that the subfield of constants of $F(t)$ is K .

Now we consider 3 cases

- ▶ t is transcendental over F and it is a logarithm;
- ▶ t is transcendental over F and it is an exponent;
- ▶ t is algebraic over F .

t is a transcendental logarithm

Firstly let us consider the basic properties of logarithmical derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

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Then we can suppose that all of u_i are distinct monic irreducible polynomials.

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Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

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Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

- If there's not u_i in the denominator of v , then there's not u_i in the denominator of v' . But then $\alpha \notin F$.

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- ▶ If there's not u_i in the denominator of v , then there's not u_i in the denominator of v' . But then $\alpha \notin F$.
- ▶ If there's u_i in the denominator of v , then the denominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

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Therefore $u_1, \dots, u_m \in F$ and $v' \in F$

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- ▶ If there's u_i in the denominator of v , then the denominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

Therefore $u_1, \dots, u_m \in F$ and $v' \in F$

Then

$$v' = (ct + s)' = ct' + s' = c \frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c \frac{z'}{z} + s'$$

QED.

t is a transcendental exponent

Let u_1, \dots, u_m be distinct monic irreducible again.

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Let u_1, \dots, u_m be distinct monic irreducible again.

It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

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It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

By the main lemma $v' \in F$ iff $v \in F$.

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It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

By the main lemma $v' \in F$ iff $v \in F$.

Then

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^m \frac{u_i'}{u_i} + v' = \sum_{i=2}^m \frac{u_i'}{u_i} + (v + c_1 z)'$$

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Consider all congruent elements of t :

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

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Consider all congruent elements of t :

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

Then it's clear that for all j :

$$\alpha = \sum_i \frac{u'_i(\tau_j)}{u_i(\tau_j)} + v'(\tau_j)$$

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Consider all congruent elements of t :

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

Then it's clear that for all j :

$$\alpha = \sum_i \frac{u_i'(\tau_j)}{u_i(\tau_j)} + v'(\tau_j)$$

$$\alpha = \frac{1}{k} \left(\sum_i \sum_j \frac{u_i'(\tau_j)}{u_i(\tau_j)} + \sum_j v'(\tau_j) \right)$$

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$$\alpha = \frac{1}{k} \left(\sum_i \sum_j \frac{u_i'(\tau_j)}{u_i(\tau_j)} + \sum_j v'(\tau_j) \right)$$

$$\alpha = \frac{1}{k} \left(\sum_i \frac{\left(\prod_j u_i(\tau_j) \right)'}{\prod_j u_i(\tau_j)} + \left(\sum_j v(\tau_j) \right)' \right)$$

Since all rational functions of polynomial's roots belongs to the basic field, proof is completed.

t is algebraic

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Since all rational functions of polynomial's roots belongs to the basic field, proof is completed.

Remark

Here we used that algebraic extension can be equipped with the unique differentiation, but it's trivial for the case of meromorphic functions.

The main corollary

Corollary

Let $f, g \in \mathbb{C}(x)$, $f \neq 0$ and g is not constant. If the integral of $f(x)\exp(g(x))$ can be expressed in elementary terms, then there's $r \in \mathbb{C}(x)$ such that $f = r' + rg'$.

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Proof.

One can use Liouville's theorem for the field $\mathbb{C}(x, \exp(g))$.
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Corollary

Functions

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \log \log x dx$$

are not elementary.

Corollary

Functions

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \log \log x dx$$

are not elementary.

Proof.

All of these cases can be simply proven with the main corollary and some basic asymptotical investigations. \square

Corollary

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Proof.

All of these cases can be simply proven with the main corollary and some basic asymptotical investigations. \square

Corollary

Functions

$$\int \frac{\sin x}{x} dx, \quad \int \frac{\sinh x}{x} dx$$

are not elementary.

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