

Liouville's Theorem on integrability via elementary functions

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Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{\sinh x}{x} dx, \quad \int \log \log x dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

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These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- ▶ Introduce and prove a powerful tool: Liouville's theorem.
- ▶ Derive these integrals as a consequence.
- ▶ If time permits, discuss some special form of elliptic integrals, which also cannot be expressed in elementary terms.

$$\int \frac{dx}{\sqrt{P(x)}}$$

for $\deg P = 2, 3$ and P has no multiple roots.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

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Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
- ▶ $(ab)' = a'b + ab'$

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Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Basic definitions

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Basic definitions

Remark

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

Definition

Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

What is expression in elementary terms

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Definition

The extension E of F is called elementary if it can be presented as

$E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

What is expression in elementary terms

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

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Remark

Common sense says us that some function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constructed via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Furthermore our definition on elementarity is more general.

Liouville's Theorem (statement)

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F , such that its subfield of constants is still K , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F$.

The Main Lemma

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Lemma

Let F be a differential field, t is transcendental over F , and t is a logarithm or an exponent of some element from F . And let $f \in F[x]$ be a polynomial, $\deg f = k \geq 1$

- ▶ If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of k if and only if f is a monomial.

The Main Lemma

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

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- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of k if and only if f is a monomial.

Proof.

It's a quite simple technical exercise. □

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Liouville's Theorem (proof)

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a transcendental logarithm

t is a transcendental exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F(t)$.

Here we use that the subfield of constants of $F(t)$ is K .

Liouville's Theorem (proof)

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a transcendental logarithm

t is a transcendental exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

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Here we use that the subfield of constants of $F(t)$ is K .

Now we consider 3 cases

- ▶ t is transcendental over F and it is a logarithm;
- ▶ t is transcendental over F and it is an exponent;
- ▶ t is algebraic over F .

t is a transcendental logarithm

Firstly let us consider the basic properties of logarithmical derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

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$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Then we can suppose that all of u_i are distinct monic irreducible polynomials.

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

t is a transcendental logarithm

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Then we can suppose that all of u_i are distinct monic irreducible polynomials.

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

t is a transcendental logarithm

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- If there's not u_i in the denominator of v , then there's not u_i in the denominator of v' . But then $\alpha \notin F$.

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- ▶ If there's not u_i in the denominator of v , then there's not u_i in the denominator of v' . But then $\alpha \notin F$.
- ▶ If there's u_i in the denominator of v , then the denominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

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Therefore $u_1, \dots, u_m \in F$ and $v' \in F$

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- ▶ If there's u_i in the denominator of v , then the denominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

Therefore $u_1, \dots, u_m \in F$ and $v' \in F$

Then

$$v' = (ct + s)' = ct' + s' = c \frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c \frac{z'}{z} + s'$$

QED.

t is a transcendental exponent

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

Let u_1, \dots, u_m be distinct monic irreducible again.

<4> Then

$$\alpha = c_1 \frac{t'_1}{t_1} + \sum_{i=2}^m \frac{u'_1}{u_i} + v' = \sum_{i=2}^m \frac{u'_1}{u_i} + (v + c_1 z)'$$

t is a transcendental exponent

Let u_1, \dots, u_m be distinct monic irreducible again. It's clear that $u_i = t$ is only u that can be not in F because only irreducible monomial, we would get the same contradiction otherwise.

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Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

t is a transcendental exponent

Let u_1, \dots, u_m be distinct monic irreducible again. It's clear that $u_i = t$ is only u that can be not in F because only irreducible monomial, we would get the same contradiction otherwise. By the main lemma $v' \in F$ iff $v \in F$. <4> Then

$$\alpha = c_1 \frac{t'_1}{t_1} + \sum_{i=2}^m \frac{u'_1}{u_i} + v' = \sum_{i=2}^m \frac{u'_1}{u_i} + (v + c_1 z)'$$

t is algebraic

TODO

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

The main corollary

TODO

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals

On elliptic integrals

TODO

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
expression in
elementary terms

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

t is a transcendental
exponent

t is algebraic

Corollaries

The main corollary

Some special cases

On elliptic integrals