

# Liouville's Theorem on integrability via elementary functions

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January 15, 2025

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From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{\sinh x}{x} dx, \quad \int \log \log x dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

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These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- ▶ Introduce and prove a powerful tool: Liouville's theorem.
- ▶ Derive these integrals as a consequence.
- ▶ If time permits, discuss some special form of elliptic integrals, which also cannot be expressed in elementary terms.

$$\int \frac{dx}{\sqrt{P(x)}}$$

for  $\deg P = 2, 3$  and  $P$  has no multiple roots.

# Basic definitions

## Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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## Definition

Field  $F$  is differential if it's equipped with the unary function  $'$  such that:

- ▶  $(a + b)' = a' + b'$
- ▶  $(ab)' = a'b + ab'$

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Differential extension of the differential field  $F$  is field  $E$  such that  $E \supseteq F$  and there is the same differentiation  $'$  on  $E$ .

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## Definition

Let  $F$  be the differential field. Then

- ▶  $b$  is called the logarithm of  $a$  if  $b' = \frac{a'}{a}$
- ▶  $b$  is called the exponent of  $a$  if  $a' = \frac{b'}{b}$



# What is expression in elementary terms

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## Definition

The extension  $E$  of  $F$  is called elementary if it can be presented as

$E = F(t_1, \dots, t_n)$  and for all  $i$   $t_i$  is logarithm or exponent or algebraic over  $F(t_1, \dots, t_{i-1})$ .

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## Remark

Common sense says us that some function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is elementary iff it can be constructed via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Furthermore our definition on elementarity is more general.

# Liouville's Theorem (statement)

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## Theorem (Liouville, 1833-1841)

Let  $F$  be a differential field, and  $K$  is its subfield of constants. If for  $\alpha \in F$  equation  $x' = \alpha$  has the solution in some elementary extension of  $F$ , such that its subfield of constants is still  $K$ , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some  $c_1, \dots, c_m \in K$ ,  $u_1, \dots, u_m, v \in F$ .

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## Lemma

Let  $F$  be a differential field,  $t$  is transcendental over  $F$ , and  $t$  is a logarithm or an exponent of some element from  $F$ . And let  $f \in F[x]$  be a polynomial,  $\deg f = k \geq 1$

- ▶ If  $t$  is a logarithm then the degree of  $(f(t))'$  is  $k$  if the leading coefficient of  $f$  is not a constant, and it has degree  $k - 1$  if the leading coefficient is a constant.
- ▶ If  $t$  is an exponent then the degree of  $(f(t))'$  is  $k$  and it's multiple of  $k$  if and only if  $f$  is a monomial.

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- ▶ If  $t$  is an exponent then the degree of  $(f(t))'$  is  $k$  and it's multiple of  $k$  if and only if  $f$  is a monomial.

## Proof.

It's a quite simple technical exercise.



# Liouville's Theorem (proof)

Let  $x$  be the solution of differential equation mentioned above. And  $x \in F(t_1, \dots, t_n)$ .

We will use induction on  $n$  (we don't fix the field  $F$ ).

For short we denote  $t = t_1$ .

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For short we denote  $t = t_1$ .

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some  $c_1, \dots, c_m \in K$ ,  $u_1, \dots, u_m, v \in F(t)$ .

Here we use that the subfield of constants of  $F(t)$  is  $K$ .

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Here we use that the subfield of constants of  $F(t)$  is  $K$ .

Now we consider 3 cases

- ▶  $t$  is transcendental over  $F$  and it is a logarithm;
- ▶  $t$  is transcendental over  $F$  and it is an exponent;
- ▶  $t$  is algebraic over  $F$ .



# t is a transcendental logarithm

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Firstly let us consider the basic properties of logarithmical derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

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Then we can suppose that all of  $u_i$  are distinct monic irreducible polynomials.

Now suppose that some  $u_i \notin F$ . It's clear that  $\frac{u_i'}{u_i}$  is already in lowest terms (because  $\deg u_i > \deg u_i'$ )

# $t$ is a transcendental exponent

TODO

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