

Liouville's Theorem (Differential algebra)

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Sber

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Introduction

Liouville's
Theorem

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Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

TODO

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

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Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

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The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense say us that any function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach.

Liouville's Theorem (statement)

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Theorem

Liouville (1833-1841) Let F be the differential field, and K is its subfield of constants. If equation $a' = x$ has the solution in some differential extension of F , then

$$a =$$