Liouville's Theorem
on integrability via elementary functions

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Introduction

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2}\,dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x}\,dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{\sinh x}{x}\,dx, \quad \int \log\log x\,dx$$

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These examples highlight the limitations of elementary functions in representing certain integrals.
In this presentation, we will:

- Introduce and prove a powerful tool: Liouville's theorem.
- Derive these integrals as a consequence.
- If time permits, discuss a special class of elliptic integrals, which also cannot be expressed in elementary terms:

$$\int \frac{dx}{\sqrt{P(x)}}$$

where $\deg P = 2, 3$ and P hasn't multiple roots.

Basic definitions

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Field F is differential if it's equipped with the unary function $'$ such that:

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Definition

Let F be a differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is an expression in elementary terms?

Definition

An extension E of F is called elementary if it can be written as $E = F(t_1, \dots, t_n)$, where for each i, the element t_i is either a logarithm, an exponent, or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense tells us that a function $f : \mathbb{C} \to \mathbb{C}$ is elementary if and only if it can be constructed using a finite number of radicals, sines, cosines, exponents, logarithms, and hyperbolic functions. One can see that this is consistent with our approach. Furthermore, our definition of elementarity is more general.

Liouville's Theorem (statement)

Theorem (Liouville, 1833–1841)

Let F be a differential field, and let K be its subfield of constants. If for some $\alpha \in F$, the equation $x' = \alpha$ has a solution in some elementary extension of F such that its subfield of constants remains K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots, c_m \in K$ and $u_1, \ldots, u_m, v \in F$.

The Main Lemma

Lemma

Let F be a differential field, and let t be transcendental over F. Suppose that t is either a logarithm or an exponent of some element from F. Let $f \in F[x]$ be a polynomial with $\deg f = k \geq 1$. Then:

- If t is a logarithm, then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and $k - 1$ if the leading coefficient is a constant.
- If t is an exponent, then the degree of $(f(t))'$ is k, and it is a multiple of f if and only if f is a monomial.

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Proof.

This is a straightforward technical exercise. $\square$

Liouville's Theorem (proof)

Let x be a solution of the differential equation mentioned above, and suppose that $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (without fixing the field F).
For brevity, we denote $t = t_1$.

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Using the inductive assumption, we obtain

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Here we use the fact that the subfield of constants of $F(t)$ is K.
Now we consider three cases:

- t is transcendental over F and is a logarithm;
- t is transcendental over F and is an exponent;
- t is algebraic over F.

t is a transcendental logarithm

First, let us consider the basic properties of logarithmic derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

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- If there is no u_i in the denominator of v, then there is no u_i in the denominator of v'. But then $\alpha \notin F$.

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- If there is u_i in the denominator of v, then the denominator of v' is divisible by u_i^2 and still cannot be reduced in the general sum, implying that $\alpha \notin F$.

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Therefore, $u_1, \ldots, u_m \in F$ and $v' \in F$.

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- If there is u_i in the denominator of v, then the denominator of v' is divisible by u_i^2 and still cannot be reduced in the general sum, implying that $\alpha \notin F$.

Therefore, $u_1, \dots, u_m \in F$ and $v' \in F$.
Then

$$v' = (ct + s)' = ct' + s' = c\frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c\frac{z'}{z} + s'$$

QED.

t is a transcendental exponent

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It is clear that $u_i = t$ is the only u that may not be in F, because t is the only irreducible monomial, and we would get the same contradiction otherwise.

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Then

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{m} \frac{u_i'}{u_i} + v' = \sum_{i=2}^{m} \frac{u_i'}{u_i} + (v + c_1 z)'$$

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$$\alpha = \frac{1}{k}\left(\sum_i \frac{\left(\prod_j u_i(\tau_j)\right)'}{\prod_j u_i(\tau_j)} + \left(\sum_j v(\tau_j)\right)'\right)$$

Since all rational symmetric functions of a polynomial's roots belong to the base field, the proof is complete.

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Remark

Here we used the fact that an algebraic extension can be equipped with a unique differentiation, but this is trivial in the case of meromorphic functions.

The main corollary

Corollary

Let $f, g \in \mathbb{C}(x), f \neq 0$ and g is not constant. If the integral of $f(x)\exp(g(x))$ can be expressed in elementary terms, then there exists $r \in \mathbb{C}(x)$ such that $f = r' + rg'$.

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Proof.

Denote $F = \mathbb{C}(x)$ and $t = \mathsf{exp}(g)$.

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Proof.

Denote $F = \mathbb{C}(x)$ and $t = \mathsf{exp}(g)$.
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One can use the main lemma and obtain:

$$c_i \frac{u_i'}{u_i} \in F, \quad v = \sum_{i \in I \subseteq \mathbb{Z}} b_i t^i$$

The rest of the proof is trivial. $\square$

Some special cases

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The following integrals are not elementary:

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Proof.

This case is slightly more complicated.
Denote $F = \mathbb{C}(x)$, $t = \exp(x)$, and using Liouville's theorem, we get

$$\frac{t^2-1}{tz} = \sum c_i \frac{u_i'}{u_i} + v'$$

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After some considerations, using the main lemma, we deduce an impossible equation:

On elliptic integrals

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Let P be a monic polynomial of degree $\geqslant 3$ with no repeated roots. Then

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Proof.

Here, we will use some more advanced concepts from the theory of Riemann surfaces.
Liouville's theorem allows us to write

$$\frac{1}{\sqrt{P(x)}} = \sum c_i \frac{u_i'}{u_i} + v'$$

with $c_1, \ldots, c_m \in \mathbb{C}$ and $u_1, \ldots, u_m, v \in \mathbb{C}(x, \sqrt{P(x)})$.

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with $c_1, \ldots, c_m \in \mathbb{C}$ and $u_1, \ldots, u_m, v \in \mathbb{C}(x, \sqrt{P(x)})$.
Now, we deduce an equation on the compact Riemann surface C associated with the equation $y^2 = P(x)$:

$$\frac{dx}{y} = \sum c_i \frac{du_i}{u_i} + dv$$

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Now, we deduce an equation on the compact Riemann surface C associated with the equation $y^2 = P(x)$:

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The left-hand side is a nonzero holomorphic 1-form on C.
However, such forms cannot be expressed as a linear combination of logarithmic meromorphic differentials $\frac{du}{u}$ and exact meromorphic differentials du.

Bibliography

Joseph Liouville, Sur la détermination des intégrales dont la valeur est algébrique, Journal de l'École Polytechnique, vol. 14, 1833, pp. 93–123.

Maxwell Rosenlicht, Integration in finite terms, American Mathematical Monthly, vol. 79, no. 9, 1972, pp. 963–972.

Keith Conrad, Impossibility theorems for elementary integration, American Mathematical Monthly, vol. 110, no. 5, 2003, pp. 459–462.