Liouville's Theorem on integrability via elementary functions

Vanya Vorobiov

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Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary
Some special cases
On alliptic integrals

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{\sinh x}{x} dx, \quad \int \ln(\ln x) dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- Derive these integrals as a consequence of a more general result.
- Introduce and prove a powerful tool for understanding this phenomenon: Liouville's theorem.
- If time permits, discuss elliptic integrals, which also cannot be expressed in elementary terms.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definition

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Lemma Liouville's

Theorem (proof)
t is a trancendental logarithm

t is a trancendental

exponent t is algebraic

t is algebraic

Corollaries

The main corollary Some special cases

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Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Lemma Liouville's

Theorem (proof)

logarithm
t is a trancendental

cponent

t is algebraic

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Corollaries

The main corollary Some special cases

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Subfield $K \subseteq F, K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Lemma Liouville's

Theorem (proof)

t is a trancendental logarithm t is a trancendental

xponent

t is algebraic

Corollaries

The main corollary

Some special cases

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation ' on E.

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Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{L}$

What is expression in elementary terms

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary
Some special cases

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, ..., t_{i-1})$.

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Common sense says us that some function $f: \mathbb{C} \to \mathbb{C}$ is elementary iff it

can be constucted via finite number of radicals, sines, cosines, exponents,

logarithms and hyperbolic functions. One can see that it's consistent with

our approach. Futhermore our definition on elementarity is more general.

Definition

Remark

 $F(t_1, \ldots, t_{i-1}).$

Liouville's

t is a trancendental

exponent

Corollaries

Basic definitions

Liouville's Theorem (statement)

Lemma

Theorem (proof)

logarithm t is a trancendental

t is algebraic

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

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Corollaries

The main corollary Some special cases

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

 $\text{for some } c_1, \ldots c_m \in K, \, u_1, \ldots, u_m, v \in F.$

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $deg f = k \ge 1$

- If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k-1 if the leading coefficient is a constant.
- If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

exponent t is algebraic

Corollaries

The main corollary Some special cases

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Proof.

It's a quite simple technical exercise.

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

iouville's Theorem (proof

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Liouville's Theorem (proof)

Theorem Vanya Vorobiov

Liouville's

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Theorem (proof

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases On elliptic integrals

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We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F(t)$. Here we use that the subfield of constants of F(t) is K. $x \in F(t_1, \ldots, t_n).$

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Now we consider 3 cases

t is algebraic over F.

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t is algebraic

Corollaries

The main corollary Some special cases

The Main

Lemma

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t is a trancendental logarithm

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Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendenta logarithm

t is a trancendental exponent t is algebraic

Corollaries

The main corollary Some special cases

t is a trancendental logarithm

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

> Liouville's Theorem (proof)

t is a trancendental logarithm

t is algebraic

Corollaries

t is a trancendental logarithm

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

Corollaries

The main corollary

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental

t is algebraic

Corollaries

The main and the second

The main corollary

ome special case

On elliptic integrals

TODO

On elliptic integrals

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

TODO