Liouville's Theorem
on integrability via elementary functions

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Introduction

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2}\,dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x}\,dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{\sinh x}{x}\,dx, \quad \int \log\log x\,dx$$

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These examples highlight the limitations of elementary functions in representing certain integrals.
In this presentation, we will:

- Introduce and prove a powerful tool: Liouville's theorem.
- Derive these integrals as a consequence.
- If time permits, discuss some special from of elliptic integrals, which also cannot be expressed in elementary terms.

  $$\int \frac{dx}{\sqrt{P(x)}}$$

  for $\deg P = 2, 3$ and P hasn't multiple roots.

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Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is expression in elementary terms

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense says us that some function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

Liouville's Theorem (statement)

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F$.

The Main Lemma

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynomial, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k-1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise. □

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (we don't fix the field F).
For short we denote $t = t_1$.

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Using the inductive assumption, we get

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Now we consider 3 cases

- t is trancendental over F and it is a logarithm;
- t is trancendental over F and it is an exponent;
- t is algebraic over F.

t is a trancendental logarithm

Firstly let us consider the basic properties of logarithmical derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

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- If there's u_i in the denominator of v, then the demominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

Therefore $u_1, \ldots, u_m \in F$ and $v' \in F$
Then

$$v' = (ct + s)' = ct' + s' = c\frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c\frac{z'}{z} + s'$$

QED.

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It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

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Then

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{m} \frac{u_i'}{u_i} + v' = \sum_{i=2}^{m} \frac{u_i'}{u_i} + (v + c_1 z)'$$

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Remark

Here we used that algebraic extension can be equipped with the unique differentiation, but it's trivial for the case of meromorphic functions.

The main corollary

Corollary

Let $f, g \in \mathbb{C}(x), f \neq 0$ and g is not constant. If the integral of $f(x)\exp(g(x))$ can be expressed in elementary terms, then there's $r \in \mathbb{C}(x)$ such that $f = r' + rg'$.

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Denoting $F = \mathbb{C}(x)$, $t = \exp(g)$.

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Using Liouville's theorem

$$ft = \sum c_i \frac{u_i'}{u_i} + v'$$

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One can use the main lemma and get:

$$c_i \frac{u_i'}{u_i} \in F, \quad v = \sum_{i \in I\mathbb{Z}} b_i t^i$$

The rest of the proof is trivial. □

Corollary

Functions

$$\int e^{\pm x^2}\,dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x}\,dx, \quad \int \log\log x\,dx$$

are not elementary.

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This case is slightly more complicated.
Denoting $F = \mathbb{C}(x)$, $t = \exp(x)$ and using Liouville's theorem

$$\frac{t^2 - 1}{tz} = \sum c_i \frac{u_i'}{u_i} + v'$$

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After some considirations using the main lemma we will deduce an impossible equation

$$\frac{1}{x} = a' + a$$

On elliptic integrals

Corollary

Let P be a monic polynomial with degree $\geqslant 3$ and no repeated roots. Then $\int \frac{dx}{\sqrt{P(x)}}$ is not elementary.

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Proof.

Here we will use a bit more advanced stuff on Riemann surfaces. Liouville's theorem enable us to write

$$\frac{1}{\sqrt{P(x)}} = \sum c_i \frac{u_i'}{u_i} + v'$$

with $c_1, \ldots, c_m \in \mathbb{C}$ and $u_1, \ldots, u_m, v \in \mathbb{C}(x, \sqrt{P(x)})$.

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Now we deduce an equation on the compact Riemann surface C associated to the equation $y^2 = P(x)$

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The left hand side is a nonzero holomorphic 1-form on C.
But such type of the forms cannot be expressed as a linear combination of a logarithmic meromorphic differentials du/u and exact meromorphic differentials dv. $\square$

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