Liouville's Theorem
on integrability via elementary functions

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Introduction

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2}\,dx,\quad \int \frac{dx}{\ln x},\quad \int \frac{e^x}{x}\,dx,\quad \int \frac{\sin x}{x}\,dx,\quad \int \frac{\sinh x}{x}\,dx,\quad \int \log\log x\,dx$$

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These examples highlight the limitations of elementary functions in representing certain integrals.
In this presentation, we will:

- Introduce and prove a powerful tool: Liouville's theorem.
- Derive these integrals as a consequence.
- If time permits, discuss some special from of elliptic integrals, which also cannot be expressed in elementary terms.

  $$\int \frac{dx}{\sqrt{P(x)}}$$

  for $\deg P = 2, 3$ and P hasn't multiple roots.

Basic definitions

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Definition
Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is expression in elementary terms

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Common sense says us that some function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

Liouville's Theorem (statement)

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F$.

The Main Lemma

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynomial, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k-1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise. □

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (we don't fix the field F).
For short we denote $t = t_1$.

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Now we consider 3 cases

- t is trancendental over F and it is a logarithm;
- t is trancendental over F and it is an exponent;
- t is algebraic over F.

t is a trancendental logarithm

Firstly let us consider the basic properties of logarithmical derivatives:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

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Therefore $u_1, \ldots, u_m \in F$ and $v' \in F$

Then

$$v' = (ct + s)' = ct' + s' = c\frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c\frac{z'}{z} + s'$$

QED.

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It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

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By the main lemma $v' \in F$ iff $v \in F$.

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Then

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{m} \frac{u_i'}{u_i} + v' = \sum_{i=2}^{m} \frac{u_i'}{u_i} + (v + c_1 z)'$$

t is algebraic

Consider all congruent elements of t:

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

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$$\alpha = \frac{1}{k}\left(\sum_i \frac{(\prod_j u_i(\tau_j))'}{\prod_j u_i(\tau_j)} + (\sum_j v(\tau_j))'\right)$$

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The main corollary

TODO

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On elliptic integrals

TODO