# Liouville's Theorem on integrability via elementary functions

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#### Liouville's Theorem

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From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

 $\int e^{\pm x^2} \, dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} \, dx, \quad \int \frac{\sin x}{x} \, dx, \quad \int \frac{\sinh x}{x} \, dx, \quad \int \log \log x \, dx$ 

These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- Derive these integrals as a consequence of a more general result.
- Introduce and prove a powerful tool for understanding this phenomenon: Liouville's theorem.
- If time permits, discuss elliptic integrals, which also cannot be expressed in elementary terms.

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### Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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### Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

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Subfield  $K \subseteq F, K = \{a \in F \mid a' = 0\}$  is called subfield of constants.

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### Definition

Let F be the differential field. Then

- b is called the logarithm of a if  $b' = \frac{a'}{a}$
- b is called the exponent of a if  $a' = \frac{b'}{L}$

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### Definition

The extension E of F is called elementary if it can be presented as  $E = F(t_1, ..., t_n)$  and for all i  $t_i$  is logarithm or exponent or algebraic over  $F(t_1, ..., t_{i-1})$ .

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Common sense says us that some function  $f: \mathbb{C} \to \mathbb{C}$  is elementary iff it

can be constucted via finite number of radicals, sines, cosines, exponents,

logarithms and hyperbolic functions. One can see that it's consistent with

our approach. Futhermore our definition on elementarity is more general.

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 $F(t_1, \ldots, t_{i-1}).$ 

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### Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for  $\alpha \in F$  equation  $x' = \alpha$  has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

 $\text{for some } c_1, \ldots c_m \in K, \, u_1, \ldots, u_m, v \in F.$ 

### Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let  $f \in F[x]$  be a polynom,  $deg f = k \ge 1$ 

- If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k-1 if the leading coefficient is a constant.
- If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

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### Proof.

It's a quite simple technical exercise.

### Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And  $x \in F(t_1, \dots, t_n)$ .

We will use induction on n (we don't fix the field F).

For short we denote  $t = t_1$ .

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We will use induction on n (we don't fix the field F).

For short we denote  $t = t_1$ .

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some  $c_1, \ldots c_m \in K$ ,  $u_1, \ldots, u_m, v \in F(t)$ . Here we use that the subfield of constants of F(t) is K.  $x \in F(t_1, \ldots, t_n).$ 

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Now we consider 3 cases

t is algebraic over F.

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