Liouville's Theorem on integrability via elementary functions

Vanya Vorobiov

Sher

January 15, 2025

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

t is algebraic

Corollaries

The main corollary Some special cases

Appendix, on elliptic integrals

Introduction

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Appendix, on elliptic integrals

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Liouville's Theorem

Vanya Vorobiov

Introduction

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

elliptic integrals

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

t is aigebrai

Corollaries

The main corollary Some special cases Appendix, on elliptic integrals

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F, K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $E \supset F$ and there is the same differentiation ' on E.

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $E \supset F$ and there is the same differentiation ' on E.

Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{h}$

Theorem

Liouville's

Theorem (proof) t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

Some special cases Appendix, on elliptic integrals

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \ldots, t_{i-1}).$

Lemma Liouville's

Theorem (proof)

logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary Some special cases Appendix, on elliptic integrals

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \ldots, t_{i-1}).$

Remark

Common sense says us that some function $f: \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

Theorem (Liouville, 1833-1841)

What are elementary functions

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary Appendix, on elliptic integrals

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots, c_m \in K$, $u_1, \ldots, u_m, v \in F$.

Liouville's Theorem (statement)

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary Some special cases Appendix, on elliptic integrals

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $deg f = k \ge 1$

- If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k-1 if the leading coefficient is a constant.
- If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

Liouville's Theorem (statement)

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary Appendix, on elliptic integrals

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $\deg f = k \geqslant 1$

- ▶ If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k-1 if the leading coefficient is a constant.
- If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

Proof.

It's a quite simple technical exercise.

Liouville's Theorem (proof)

 $x \in F(t_1, \ldots, t_n).$

For short we denote $t = t_1$.

Let x be the solution of differential equation mentioned above. And

We will use induction on n (we don't fix the field F).

Theorem Vanya Vorobiov

Liouville's

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

> iouville's 'heorem (proof

t is a trancendental logarithm

t is a trancendental exponent t is algebraic

~---11---

Corollaries

The main corollary Some special cases

Appendix, on elliptic integrals

Liouville's Theorem (statement)

The Main Lemma

Theorem (proof

t is a trancendental logarithm

t is a trancendental exponent

exponent t is algebraic

Corollaries

The main corollary Some special cases Appendix, on elliptic integrals

4□ > 4♠ > 4 = > 4 = > 9 Q

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K, u_1, \ldots, u_m, v \in F(t)$. Here we use that the subfield of constants of F(t) is K.

elementary functions

Liouville's Theorem (statement)

Lemma

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary

Appendix, on

What are

The Main

elliptic integrals

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n).$

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F(t)$.

Here we use that the subfield of constants of F(t) is K.

Now we consider 3 cases

t is trancendental over F and it is a logarithm;

- ▶ t is trancendental over F and it is an exponent;
- t is algebraic over F.

t is a trancendental logarithm

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent t is algebraic

Corollaries

Corollaries

Some special cases
Appendix, on

t is a trancendental logarithm

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is algebraic

Corollaries

t is a trancendental logarithm

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

Corollaries

The main corollary

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

 ${f t}$ is a trancendental exponent

t is algebraic

Corollaries

oromaries

Some special cases Appendix, on

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

> Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

he main corollary

Some special cases

Appendix, on elliptic integrals

TODO

Appendix, on elliptic integrals

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What are elementary functions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries