

Liouville's Theorem on integrability via elementary functions

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Through the all of presentation we will suppose that all fields have 0 characteristic.

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Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
- ▶ $(ab)' = a'b + ab'$

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

Definition

Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

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The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense says us that some function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constructed via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach.

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Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F$.

Lemma

Let F be a differential field, t is transcendental over F , and t is a logarithm or an exponent of some element from F . And let $f \in F[x]$ be a polynomial, $\deg f = k \geq 1$

- ▶ If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise. □

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Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$ We will use induction on n (we don't fix the field F).