

Liouville's Theorem on integrability via elementary functions

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Through the all of presentation we will suppose that all fields have 0 characteristic.

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Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
- ▶ $(ab)' = a'b + ab'$

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

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Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

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Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense says us that some function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constructed via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Furthermore our definition on elementarity is more general.

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Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F , such that its subfield of constants is still K , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F$.

Lemma

Let F be a differential field, t is transcendental over F , and t is a logarithm or an exponent of some element from F . And let $f \in F[x]$ be a polynomial, $\deg f = k \geq 1$

- ▶ If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise. □

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<1-> Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F). And for short we will denote $t = t_1$.

<2-> Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F(t)$.

Here we use that the subfield of constants of $F(t)$ is K .

<3-> Now we consider 3 cases

- ▶ t is transcendental over F and it is a logarithm;
- ▶ t is transcendental over F and it is an exponent;
- ▶ t is algebraic over F .

t is a transcendental logarithm

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