Liouville's Theorem (Differential algebra)

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Introduction

TODO

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

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Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is integrability in elementary functions

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \ldots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \ldots, t_{i-1})$.

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Common sense say us that any function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach.

Liouville's Theorem (statement)

Theorem (Liouville (1833-1841))

Let F be the differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some differential extension of F, then

$$a = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F$.