Liouville's Theorem
on integrability via elementary functions

Vanya Vorobiov

Sber

January 15, 2025

Introduction

TODO

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- $(a+b)' = a' + b'$
- $(ab)' = a'b + ab'$

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- $(\mathrm{a}+\mathrm{b})' = \mathrm{a}' + \mathrm{b}'$
- $(\mathrm{ab})' = \mathrm{a}'\mathrm{b} + \mathrm{ab}'$

Definition

Subfield $\mathrm{K} \subseteq \mathrm{F}$, $\mathrm{K} = \{\mathrm{a} \in \mathrm{F} \mid \mathrm{a}' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $\mathrm{E} \supseteq \mathrm{F}$ and there is the same differentiation $'$ on E.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function $'$ such that:

- $(a+b)' = a' + b'$
- $(ab)' = a'b + ab'$

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E.

Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is integrability in elementary functions

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

What is integrability in elementary functions

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \ldots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \ldots, t_{i-1})$.

Remark

Common sense says us that some function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach.

Liouville's Theorem (statement)

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F$.

The Main Lemma

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

The Main Lemma

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k-1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

Proof.

It's a quite simple technical exercise. $\square$

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (we don't fix the field F). And for short we will denote $t = t_1$.
Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K_1$, $u_1, \ldots, u_m, v \in F(t_1)$.