Liouville's Theorem
on integrability via elementary functions

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Introduction

TODO

Basic definitions

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Through the all of presentation we will suppose that all fields have 0 characteristic.

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Field F is differential if it's equipped with the unary function $'$ such that:

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

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Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

What is integrability in elementary functions

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

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Remark

Common sense says us that some function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

Liouville's Theorem (statement)

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F$.

The Main Lemma

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k-1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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Proof.

It's a quite simple technical exercise. □

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (we don't fix the field F). And for short we will denote $t = t_1$.

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Now we consider 3 cases

- t is trancendental over F and it is a logarithm;
- t is trancendental over F and it is an exponent;
- t is algebraic over F.

t is a trancendental logarithm

TODO