# Liouville's Theorem
# on integrability via elementary functions


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January 15, 2025

# Introduction

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2}\,dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x}\,dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{\sinh x}{x}\,dx, \quad \int \ln(\ln x)\,dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.
In this presentation, we will:

- Derive these integrals as a consequence of a more general result.
- Introduce and prove a powerful tool for understanding this phenomenon: Liouville's theorem.
- If time permits, discuss elliptic integrals, which also cannot be expressed in elementary terms.

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## Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{b}$

# What is expression in elementary terms

### Definition

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## Remark

Common sense says us that some function $f : \mathbb{C} \to \mathbb{C}$ is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Futhermore our definition on elementarity is more general.

# Liouville's Theorem (statement)

## Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots c_m \in K$, $u_1, \dots, u_m, v \in F$.

# The Main Lemma

## Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynom, $\deg f = k \geqslant 1$

- If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of is not a constant, and it has degree $k-1$ if the leading coefficient is a constant.
- If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

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## Proof.

It's a quite simple technical exercise. □

# Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \ldots, t_n)$.
We will use induction on n (we don't fix the field F).
For short we denote $t = t_1$.

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Here we use that the subfield of constants of $F(t)$ is $K$.
Now we consider 3 cases

- $t$ is trancendental over $F$ and it is a logarithm;
- $t$ is trancendental over $F$ and it is an exponent;
- $t$ is algebraic over $F$.

# $t$ is a trancendental logarithm

TODO

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TODO

# The main corollary

TODO

TODO

# On elliptic integrals

TODO