Liouville's Theorem on integrability via elementary functions

Vanya Vorobiov

Sher

January 15, 2025

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary
Some special cases
On alliptic integrals

Theorem (proof) t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

Some special cases

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} \, dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} \, dx, \quad \int \frac{\sin x}{x} \, dx, \quad \int \frac{\sinh x}{x} \, dx, \quad \int \log \log x \, dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

The Main Lemma Liouville's

Theorem (proof) t is a trancendental

logarithm t is a trancendental

t is algebraic

Corollaries

From high school, we are familiar with the idea that some integrals cannot be expressed in terms of elementary functions. For instance:

$$\int e^{\pm x^2} \, dx, \quad \int \frac{dx}{\ln x}, \quad \int \frac{e^x}{x} \, dx, \quad \int \frac{\sin x}{x} \, dx, \quad \int \frac{\sinh x}{x} \, dx, \quad \int \log \log x \, dx$$

These examples highlight the limitations of elementary functions in representing certain integrals.

In this presentation, we will:

- Introduce and prove a powerful tool: Liouville's theorem.
- ▶ Derive these integrals as a consequence.
- If time permits, discuss some special from of elliptic integrals, which also cannot be expressed in elementary terms.

$$\int \frac{\mathrm{dx}}{\sqrt{\mathrm{P(x)}}}$$

for deg P = 2, 3 and P hasn't multiple roots.

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definition

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Theorem (proof)
t is a trancendental logarithm

t is a trancendental

exponent t is algebraic

t is algebraic

Corollaries

The main corollary Some special cases

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Theorem (proof)

logarithm
t is a trancendental

cponent

t is algebraic

angebraic

Corollaries

The main corollary Some special cases

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F, K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Theorem (proof)

t is a trancendental logarithm t is a trancendental

xponent

t is algebraic

Corollaries

The main corollary

Some special cases

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation ' on E.

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Definition

Field F is differential if it's equipped with the unary function ' such that:

- (a + b)' = a' + b'
- (ab)' = a'b + ab'

Definition

Subfield $K \subseteq F$, $K = \{a \in F \mid a' = 0\}$ is called subfield of constants.

Definition

Differential extension of the differential field F is field E such that $E \supset F$ and there is the same differentiation ' on E.

Definition

Let F be the differential field. Then

- b is called the logarithm of a if $b' = \frac{a'}{a}$
- b is called the exponent of a if $a' = \frac{b'}{L}$

What is expression in elementary terms

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, ..., t_{i-1})$.

The extension E of F is called elementary if it can be presented as

 $E = F(t_1, ..., t_n)$ and for all i t_i is logarithm or exponent or algebraic over

Common sense says us that some function $f: \mathbb{C} \to \mathbb{C}$ is elementary iff it

can be constucted via finite number of radicals, sines, cosines, exponents,

logarithms and hyperbolic functions. One can see that it's consistent with

our approach. Futhermore our definition on elementarity is more general.

Definition

Remark

 $F(t_1, \ldots, t_{i-1}).$

Liouville's

t is a trancendental

exponent

Corollaries

Basic definitions

Liouville's Theorem (statement)

Lemma

Theorem (proof)

logarithm t is a trancendental

t is algebraic

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental exponent

t is algebraic

t is aigebraic

Corollaries

The main corollary Some special cases

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F, such that its subfield of constants is still K, then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

 $\text{for some } c_1, \ldots c_m \in K, \, u_1, \ldots, u_m, v \in F.$

Liouville's Theorem (proof)

t is a trancendental logarithm t is a trancendental

is a trancenden kponent

t is algebraic

Corollaries

The main corollary
Some special cases

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynomial, $\deg f = k \geqslant 1$

- ▶ If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k - 1 if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary
Some special cases

Lemma

Let F be a differential field, t is trancendental over F, and t is a logarithm or an exponent of some element from F. And let $f \in F[x]$ be a polynomial, $\deg f = k \geqslant 1$

- ▶ If t is a logarithm then the degree of (f(t))' is k if the leading coefficient of is not a constant, and it has degree k - 1 if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of (f(t))' is k and it's multiple of f if and only if f is a monomial.

Proof.

It's a quite simple technical exercise.

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

iouville's heorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

Liouville's Theorem (proof)

Theorem Vanya Vorobiov

Liouville's

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases On elliptic integrals

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, ..., t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^{m} c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K, u_1, \ldots, u_m, v \in F(t)$. Here we use that the subfield of constants of F(t) is K.

Liouville's Theorem (statement)

The Main Lemma

t is a trancendental

logarithm

t is a trancendental

t is algebraic

Corollaries

The main corollary Some special cases

Let x be the solution of differential equation mentioned above. And

 $x \in F(t_1, \ldots, t_n).$

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \ldots c_m \in K$, $u_1, \ldots, u_m, v \in F(t)$.

Here we use that the subfield of constants of F(t) is K.

Now we consider 3 cases

- t is trancendental over F and it is a logarithm;
- ▶ t is trancendental over F and it is an exponent;
- t is algebraic over F.

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent t is algebraic

Corollaries

The main coro

Some special cases On elliptic integral:

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

logarithm

t is a trancendental

exponent t is algebraic

Corollaries

Corollaries
The main cor

Some special cases
On elliptic integral

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

> Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendenta logarithm

t is a trancendental exponent t is algebraic

Corollaries

Corollaries
The main core

Some special cases

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

▶ If there's not u_i in the denominator of v, then there's not u_i in the denominator of v'. But then $\alpha \notin F$.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

logarithm

t is a trancendental exponent t is algebraic

Corollaries

The main corollary Some special cases On elliptic integrals

Liouville's Theorem (statement)

The Main Lemma Liouville's

Theorem (proof)

t is a trancendental

t is algebraic

Corollaries

The main corollary Some special cases On elliptic integrals

 $\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$

Firstly let us consider the basic properties of logarithmical derivatives:

Then we can suppose that all of u_i are distinct monic irreducible polynomials.

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u'_i$).

- If there's not u; in the denominator of v, then there's not u; in the denominator of v'. But then $\alpha \notin F$.
- If there's u_i in the denominator of v, then the demominator of v' is divisible by u_i² and it still cannot be reduced in the general sum and $\alpha \notin F$.

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

- ▶ If there's not u_i in the denominator of v, then there's not u_i in the denominator of v'. But then $\alpha \notin F$.
- ▶ If there's u_i in the denominator of v, then the demominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

Therefore $u_1, \ldots, u_m \in F$ and $v' \in F$

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases On elliptic integrals

$$\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}, \quad \frac{(1/a)'}{1/a} = -\frac{a'}{a}$$

Now suppose that some $u_i \notin F$. It's clear that $\frac{u_i'}{u_i}$ is already in lowest terms (because $\deg u_i > \deg u_i'$).

- ▶ If there's not u_i in the denominator of v, then there's not u_i in the denominator of v'. But then $\alpha \notin F$.
- ▶ If there's u_i in the denominator of v, then the demominator of v' is divisible by u_i^2 and it still cannot be reduced in the general sum and $\alpha \notin F$.

Therefore $u_1, \dots, u_m \in F$ and $v' \in F$ Then

$$v' = (ct + s)' = ct' + s' = c\frac{z'}{z} + s'$$

$$\alpha = \sum c_i \frac{u_i'}{u_i} + c\frac{z'}{z} + s'$$

QED.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental

exponent t is algebraic

Corollaries

The main corollary Some special cases

On elliptic integra

t is a trancendental exponent

Let u_1, \ldots, u_m be distinct monic irreducible again.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

Corollaries

t is a trancendental exponent

Let u_1, \ldots, u_m be distinct monic irreducible again. It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is algebraic

Corollaries

Some special cases

t is a trancendental exponent

Let u_1, \ldots, u_m be distinct monic irreducible again. It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise. By the main lemma $v' \in F$ iff $v \in F$.

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof) t is a trancendental

logarithm

t is algebraic

Corollaries

Some special cases

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

> Liouville's Theorem (proof)

t is a trancendental

t is algebraic

Corollaries

Some special cases

Let u_1, \ldots, u_m be distinct monic irreducible again.

It's clear that $u_i = t$ is only u that can be not in F because t is an only irreducible monomial, and we would get the same contradiction otherwise. By the main lemma $v' \in F$ iff $v \in F$.

Then

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{m} \frac{u_i'}{u_i} + v' = \sum_{i=2}^{m} \frac{u_i'}{u_i} + (v + c_1 z)'$$

Consider all congurent elements of t:

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

Corollaries

Since all rational functions of polynomial's roots are belongs to the basic field, proof is completed.

 $\tau_1 = t, \tau_2, \ldots, \tau_k$

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

Corollaries

Some special cases

Consider all congurent elements of t:

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

Then it's clear that for all j:

$$\alpha = \sum_{i} \frac{u_i'(\tau_j)}{u_i(\tau_j)} + v'(\tau_j)$$

Since all rational functions of polynomial's roots are belongs to the basic field, proof is completed.

Consider all congurent elements of t:

Then it's clear that for all i:

Liouville's Theorem (statement)

The Main Lemma

> Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

Corollaries

The main corollary Some special cases

On elliptic integrals

Since all rational functions of polynomial's roots are belongs to the basic field, proof is completed.

 $\tau_1 = t, \tau_2, \dots, \tau_k$

 $\alpha = \sum_{\cdot} \frac{\mathrm{u}_{\mathrm{i}}'(\tau_{\mathrm{j}})}{\mathrm{u}_{\mathrm{i}}(\tau_{\mathrm{i}})} + \mathrm{v}'(\tau_{\mathrm{j}})$

 $\alpha = \frac{1}{k} \left(\sum_{i} \sum_{j} \frac{u_{i}'(\tau_{j})}{u_{i}(\tau_{j})} + \sum_{j} v'(\tau_{j}) \right)$

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

Corollaries

The main corollary

Some special cases On elliptic integrals

Consider all congurent elements of t:

$$\tau_1 = t, \tau_2, \dots, \tau_k$$

Then it's clear that for all i:

$$\alpha = \sum_{i} \frac{u_i'(\tau_j)}{u_i(\tau_i)} + v'(\tau_j)$$

$$lpha = rac{1}{k} \left(\sum_{i} \sum_{j} rac{u_i'(au_j)}{u_i(au_j)} + \sum_{j} v'(au_j)
ight)$$

$$\alpha = \frac{1}{k} \left(\sum_{i} \frac{(\prod_{j} u_{i}(\tau_{j}))'}{\prod_{j} u_{i}(\tau_{j})} + (\sum_{j} v(\tau_{j}))' \right)$$

Since all rational functions of polynomial's roots are belongs to the basic field, proof is completed.

The main corollary

TODO

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental

t is algebraic

Corollaries

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental

t is a trancendental

t is algebraic

Corollaries

The main and the second

The main corollary

ome special case

On elliptic integrals

TODO

On elliptic integrals

Liouville's Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is expression in elementary terms

Liouville's Theorem (statement)

The Main Lemma

Liouville's Theorem (proof)

t is a trancendental logarithm

t is a trancendental exponent

t is algebraic

Corollaries

The main corollary Some special cases

TODO