

Liouville's Theorem on integrability via elementary functions

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Introduction

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
 t is a transcendental
logarithm

TODO

Basic definitions

Remark

Through the all of presentation we will suppose that all fields have 0 characteristic.

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
 t is a transcendental
logarithm

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Field F is differential if it's equipped with the unary function $'$ such that:

- ▶ $(a + b)' = a' + b'$
- ▶ $(ab)' = a'b + ab'$

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Differential extension of the differential field F is field E such that $E \supseteq F$ and there is the same differentiation $'$ on E .

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Let F be the differential field. Then

- ▶ b is called the logarithm of a if $b' = \frac{a'}{a}$
- ▶ b is called the exponent of a if $a' = \frac{b'}{b}$

What is integrability in elementary functions

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
 t is a transcendental
logarithm

Definition

The extension E of F is called elementary if it can be presented as $E = F(t_1, \dots, t_n)$ and for all i t_i is logarithm or exponent or algebraic over $F(t_1, \dots, t_{i-1})$.

What is integrability in elementary functions

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
 t is a transcendental
logarithm

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Remark

Common sense says us that some function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elementary iff it can be constructed via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach. Furthermore our definition on elementarity is more general.

Liouville's Theorem (statement)

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
t is a transcendental
logarithm

Theorem (Liouville, 1833-1841)

Let F be a differential field, and K is its subfield of constants. If for $\alpha \in F$ equation $x' = \alpha$ has the solution in some elementary extension of F , such that its subfield of constants is still K , then

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F$.

Lemma

Let F be a differential field, t is transcendental over F , and t is a logarithm or an exponent of some element from F . And let $f \in F[x]$ be a polynomial, $\deg f = k \geq 1$

- ▶ If t is a logarithm then the degree of $(f(t))'$ is k if the leading coefficient of f is not a constant, and it has degree $k - 1$ if the leading coefficient is a constant.
- ▶ If t is an exponent then the degree of $(f(t))'$ is k and it's multiple of f if and only if f is a monomial.

The Main Lemma

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)
 t is a transcendental
logarithm

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Proof.

It's a quite simple technical exercise. □

Liouville's Theorem (proof)

Let x be the solution of differential equation mentioned above. And $x \in F(t_1, \dots, t_n)$.

We will use induction on n (we don't fix the field F).

For short we denote $t = t_1$.

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

Liouville's Theorem (proof)

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

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Using the inductive assumption, we get

$$\alpha = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some $c_1, \dots, c_m \in K$, $u_1, \dots, u_m, v \in F(t)$.

Here we use that the subfield of constants of $F(t)$ is K .

Liouville's Theorem (proof)

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm

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Now we consider 3 cases

- ▶ t is transcendental over F and it is a logarithm;
- ▶ t is transcendental over F and it is an exponent;
- ▶ t is algebraic over F .

t is a transcendental logarithm

TODO

Liouville's
Theorem

Vanya Vorobiov

Introduction

Basic definitions

What is
integrability in
elementary
functions

Liouville's
Theorem
(statement)

The Main
Lemma

Liouville's
Theorem (proof)

t is a transcendental
logarithm