

# Liouville's Theorem (Differential algebra)

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January 14, 2025

# Introduction

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# Basic definitions

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Field  $F$  is differential if it's equipped with the unary function  $'$  such that:

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- ▶  $(ab)' = a'b + ab'$

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Differential extension of the differential field  $F$  is field  $E$  such that  $E \supseteq F$  and there is the same differentiation  $'$  on  $E$ .

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## Definition

Let  $F$  be the differential field. Then

- ▶  $b$  is called the logarithm of  $a$  if  $b' = \frac{a'}{a}$
- ▶  $b$  is called the exponent of  $a$  if  $a' = \frac{b'}{b}$

# What is integrability in elementary functions

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## Definition

The extension  $E$  of  $F$  is called elementary if it can be presented as  $E = F(t_1, \dots, t_n)$  and for all  $i$   $t_i$  is logarithm or exponent or algebraic over  $F(t_1, \dots, t_{i-1})$ .



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## Remark

Common sense say us that any function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is elementary iff it can be constucted via finite number of radicals, sines, cosines, exponents, logarithms and hyperbolic functions. One can see that it's consistent with our approach.

# Liouville's Theorem (statement)

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## Theorem (Liouville, 1833-1841)

Let  $F$  be the differential field, and  $K$  is its subfield of constants. If for  $\alpha \in F$  equation  $x' = \alpha$  has the solution in some differential extension of  $F$ , then

$$a = \sum_{i=1}^m c_i \frac{u_i'}{u_i} + v'$$

for some  $c_1, \dots, c_m \in K$ ,  $u_1, \dots, u_m, v \in F$ .

# The Main Lemma

Lemma  
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