Datenstrukturen und Algorithmen, WS2024, Übungsblatt

Konstantin Krasser

October 17, 2024

1. **Aufgabe 1 (2 Points).** Prove that $\sum_{i=1}^{n} i \in \mathcal{O}\left(n^2\right)$.

   Lösung:

   The sum $\sum_{i=1}^{n} i$ is known to be equal to $\frac{n(n+1)}{2}$. Simplifying, we have:

   $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \approx \frac{n^2}{2}$$

   Since $\frac{n^2}{2} \leq C \cdot n^2$ for some constant $C > 0$ and large n, it follows that:

   $$\sum_{i=1}^{n} i \in \mathcal{O}(n^2)$$

2. **Aufgabe 2 (2 Points).** Prove or disprove that $2^{2n} \in \mathcal{O}\left(2^n\right)$.

   Lösung:

   We consider the limit:

   $$\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^n = \infty$$

   Since the limit goes to infinity, 2^{2n} grows faster than 2^n. Therefore, $2^{2n} \notin \mathcal{O}(2^n)$.

3. **Aufgabe 3 (2 Points).** Prove or disprove using the limit criterion: $\sqrt{n} = \mathcal{O}(\log n)$.

   Lösung:

   We evaluate the limit:

   $$\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$$

   As $n \to \infty$, $\sqrt{n}$ grows much faster than $\log n$. Thus:

   $$\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$$

   Since the limit is infinite, $\sqrt{n} \notin \mathcal{O}(\log n)$.

4. **Aufgabe 4 (2 Points).** Let $f, f', g, g' : N \to R^+$ such that $f \in \mathcal{O}(g)$ and $f' \in \mathcal{O}\left(g'\right)$. Show that:

   $$ff' \in \mathcal{O}\left(gg'\right)$$

Does this statement also hold analogously for asymptotically tight bounds?

Lösung:

By definition, there exist constants $C_1, C_2 > 0$ and $n_0, m_0 \in N$ such that:

$$f(n) \leq C_1 g(n) \quad \text{and} \quad f'(n) \leq C_2 g'(n) \quad \text{for all } n \geq \max(n_0, m_0)$$

Multiplying these inequalities:

$$f(n) \cdot f'(n) \leq C_1 \cdot C_2 \cdot g(n) \cdot g'(n)$$

Thus, $ff' \in \mathcal{O}(gg')$.

For asymptotically tight bounds (Θ), if $f \in \Theta(g)$ and $f' \in \Theta(g')$, then:

$$ff' \in \Theta(gg')$$

This holds analogously.

5. **Aufgabe 5 (2 Points).** Prove or disprove that $1 + \sum_{k=2}^{\frac{n}{2}} \log(2k) \in \mathcal{O}(n \log n)$.

   Lösung:

   We analyze the sum:

   $$\sum_{k=2}^{\frac{n}{2}} \log(2k)$$

   This can be approximated by $\sum_{k=2}^{\frac{n}{2}} \log n$, as $\log(2k) \leq \log n$. Therefore:

   $$\sum_{k=2}^{\frac{n}{2}} \log(2k) \leq \frac{n}{2} \log n = \mathcal{O}(n \log n)$$

   Thus, $1 + \sum_{k=2}^{\frac{n}{2}} \log(2k) \in \mathcal{O}(n \log n)$.

6. **Aufgabe 6 (2 Points).** Show that $\sum_{i=0}^{\log_2(n)-1} 8^i \in O\left(n^3\right)$.

   Lösung:

   The sum is a geometric series:

   $$\sum_{i=0}^{\log_2(n)-1} 8^i = \frac{8^{\log_2(n)} - 1}{8 - 1} \approx \frac{8^{\log_2(n)}}{7}$$

   Since $8^{\log_2(n)} = n^{\log_2(8)} = n^3$, we have:

   $$\sum_{i=0}^{\log_2(n)-1} 8^i \in O(n^3)$$