Transcendental Numbers (Rough Outline 9/30/23)

Vincent Lin

Fall 2023

Contents

Chapter 1

Preliminaries

1.1 Algebraic and Transcendental Numbers

Def An **algebraic number** is a complex number that is the root of a finite nonzero polynomial in one variable with rational coefficients. A **transcendental number** is a complex number that is not algebraic.

Note We will use $\mathbb{A}$ to denote the set of algebraic numbers and we will use $\mathbb{C} \setminus \mathbb{A}$ to denote the set transcendental numbers. We will use $\mathbb{R} \setminus \mathbb{A}$ to denote the set of real transcendental numbers.

To find some algebraic numbers, we can take a nonzero polynomial with rational coefficents and find its roots. By definition, these roots are algebraic numbers. For example, $\sqrt{2}$ is algebraic because it is a root of $x^2 - 2$. Also, i is algebraic because it is the root of $x^2 + 1$. All rational numbers are algebraic as well. Let $\frac{p}{q} \in \mathbb{Q}$ be rational, where $p, q \in \mathbb{Z}$ and q is nonzero. Then, it is the root of $x - \frac{p}{q}$.

What about transcendental numbers? Do they exist?

Theorem 1.1.1 *Yes, transcendental numbers exist.*

Proof Consider the set of algebraic numbers, which we will denote by $\mathbb{A}$. This set is countable. We will show this by forming a surjection

$$\phi : \left(\mathbb{N} \times \bigcup_{n \in \mathbb{N}} \mathbb{Q}^n \right) \to \mathbb{A} \,.$$

Note that $\mathbb{N} \times \bigcup\limits_{n \in \mathbb{N}} \mathbb{Q}^n$ is countable because it is the Cartesian product of a countable set with a countable union of countable sets. By the fundamental theorem of algebra, we know that a polynomial of degree k has k (not necessarily distinct) roots. Therefore, we can number the k roots from 1 to k for each polynomial. Thus, if we specify the nonzero rational coefficients $(a_0, \ldots, a_k)$ and an index i for $i \in [k]$ to be the i-th root of the polynomial $a_k x^k + \cdots + a_0$, we get an algebraic number. We define the map $\phi(i, (a_0, \ldots, a_k))$ to be the i-th root of $a_k x^k + \cdots + a_0$ if it exists. Otherwise, we map the input to zero. From the definition, we know that every algebraic number can be encoded this way since it is one of the roots of a nonzero rational polynomial. Thus, for all $a \in \mathbb{A}$, there must be an input x such that $\phi(x) = a$, making this a surjective map from a countable set to $\mathbb{A}$. This makes $\mathbb{A}$ countable. However, since $\mathbb{C}$ is uncountable, it cannot be the case that all complex numbers are algebraic. Thus, if a complex number is not algebraic, it must be transcendental. Furthermore, $\mathbb{R}$ is uncountable so there must be real transcendental numbers as well.

♦

If transcendental numbers exist, then can we find an example? To show that a number is algebraic, we just have to find a nonzero rational polynomial that it is a root of, evaluate that polynomial at that number, and verify that we get zero. However, checking if a number is transcendental is a very hard problem. The rest of this paper will discuss the transcendence of a large class of numbers and methods for determining the transcendence.

1.2 Fields

Before that, we will generalize the concept of transcendence to other fields. In order to do this, we will list several definitions. Let k, l be fields and let l/k be a field extension.

- $a \in l$ is **algebraic** over k if there is a nonzero polynomial $f \in k[x]$ such that $f(a) = 0$. Otherwise, a is **transcendental** over k. In other words, an algebraic number is a complex number that is algebraic over $\mathbb{Q}$ and a transcendental number is a complex number that is not algebraic over $\mathbb{Q}$.

- $X \subseteq L$ is **algebraically independent** over k if for all $a_1, \ldots, a_t$ distinct and all $f \in k[x_1, \ldots, x_t]$, $f(a_1, \ldots, a_t) = 0$ implies $f = 0$.

- If $a \in l$ is algebraic over k, the **minimal polynomial** of a over k, denoted m_a, is the unique monic irreducible polynomial which generates the kernel of the evaluation map $\phi(f) = f(a)$ for $f \in k[x]$. The **degree** of $\alpha \in k$ is the degree of the minimial polynomial m_a.

- l/k is an **algebraic extension** if every element in l is algebraic over k.

- l is **algebraically closed** if every nonconstant single variable polynomial in l has a root in l.

- An **algebraic closure** of k is an algebraic extension of k that is algebraically closed.

1.3 Liouville Numbers

Now we will start looking for our first transcendental number! First, we will start by proving a lemma.

Lemma 1.3.1 *For any nonzero polynomial $p(x)$ with a root at $x = \alpha$ of multiplicity $m > 0$, $p'(x)$ has the root α with multiplicity $m - 1$.*

Proof From the fundamental theorem of algebra, we can write $p(x)$ as $(x-\alpha)^m q(x)$ where $(x-\alpha) \nmid q(x)$. Now, taking the derivatives of both sides and using the product rule, we get

$$\begin{aligned}
p(x) &= (x-\alpha)^m q(x) \\
p'(x) &= ((x-\alpha)^m)' q(x) + (x-\alpha)^m q'(x) \\
&= m(x-\alpha)^{m-1} q(x) + (x-\alpha)^m q'(x) \\
&= (x-\alpha)^{m-1}\left(mq(x) + (x-\alpha)q'(x)\right)
\end{aligned}$$

Therefore, we get that $(x-\alpha)^{m-1} \mid p'(x)$ so $p'(x)$ has the root α of multiplicity at least $m-1$.

Now, we want to show that $p'(x)$ has root α with multiplicity at most $m-1$. In order to show this, we will shift the polynomial by α (all of the roots will therefore be shifted by α as well). Namely, we look at $p(x+\alpha) = x^m q(x+\alpha)$. Note that $p(x+\alpha)$ now has the root 0 with multiplicity m (it has m factors of x) and since $q(x)$ had no $x-\alpha$ factors, $q(x+\alpha)$ has no roots at zero. In other words, $x \nmid q(x+\alpha)$. Now, showing $p'(x)$ has root α with multiplicity $m-1$ is equivalent to showing $p'(x+\alpha)$ as root 0 with multiplicity $m-1$ (it has exactly $m-1$ factors of x). Thus, substituting $x+\alpha$ in $p'(x)$:

$$p'(x+\alpha) = x^{m-1}\left(mq(x+\alpha) + xq'(x+\alpha)\right)$$

We have $m-1$ factors of x from the x^{m-1} term. Therefore, we need to show the rest of the expression has no factors of x. In other words, $x \nmid (mq(x+\alpha) + xq'(x+\alpha))$, we can take the expression $(mq(x+\alpha) + xq'(x+\alpha))$ modulo x to get $mq(x+\alpha)$. Since $m > 0$ and $q(x+\alpha)$ has no factors of x, we know that x cannot divide the entire expression. Shifting back by α, we get that $x-\alpha$ cannot be a factor of $mq(x) + (x-\alpha)q'(x)$, so the multiplicity of $x-\alpha$ in $p(x)$ is at most $m-1$.

♦

Def A **Liouville number** is a real number x such that for all $n \in \mathbb{N}$, there exists integers p, q with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Essentially, Liouville numbers are real numbers that can be approximately really, really closely by a sequence of rationals of the form $\left\{\frac{p_i}{q_i}\right\}$, where $q_i > 1$, and the distance between x and $\frac{p_i}{q_i}$ is nonzero, but less than $\frac{1}{(q_i)^i}$.

Theorem 1.3.2 *(Liouville's Approximation Theorem) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an algebraic number of degree n. Then, for any rational approximation $\frac{p}{q}$ to α, we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^n}$$

Proof Let f be the minimal polynomial of α over $\mathbb{Q}$.

♦

1.4 Transcendence of e and π

Lemma 1.4.1 *Let $f(x)$ be a real polynomial with degree m. Let*

$$I(t) = \int_0^t e^{t-x} f(x) dx.$$

Then,

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t).$$

where $f^{(j)}(t)$ is the j-th derivative of f with respect to x evaluated at t.

Proof Probably induction and integration by parts. I will fill this in later.

Chapter 2

Lindemann-Weierstrass Theorem

Theorem 2.0.1 *(Lindemann-Weierstrass Theorem) Suppose*

$$\alpha_1, \ldots, \alpha_n$$

are algebraic numbers that are linearly independent over $\mathbb{Q}$*. Then,*

$$e^{\alpha_1}, \ldots, e^{\alpha_n}$$

are linearly independent over the algebraic numbers. In other words, the extension field $\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n})$ *has transcendence degree* n *over* $\mathbb{Q}$*.*

Proof

Chapter 3

Gelfond-Schneider Theorem

3.1 Useful Lemmas

In this chapter, we will prove the Gelfond-Schneider Theorem. We will introduce four lemmas to help prove this.

Lemma 3.1.1 *Let the functions*

$$a_1(t), \ldots, a_n(t)$$

be nonzero real polynomials $(\in \mathbb{R}[t])$ *of degree*

$$d_1, \ldots, d_n$$

respectively. Furthermore, let

$$w_1, \ldots, w_n$$

be (pairwise) distinct real numbers. Then, counting multiplicities, the function

$$f(t) = \sum_{j=1}^{n} a_j(t) e^{w_j t}$$

has at most $n - 1 + \sum\limits_{i=1}^{n} d_i$ *real roots.*

Proof Let $k = n + \sum\limits_{i=1}^{n} d_i$. We will prove this statement using strong induction on k.

Lemma 3.1.2 *Let $f(z)$ be an analytic function in the disk $D \subseteq \mathbb{C}$. Here, we define $D = \{z : |z| < d\}$ for some positive real d. Suppose f is continuous on the closure of D, $\overline{D} = \{z : |z| \leq d\}$. Furthermore, let $|f|_d = \max_{z \in \overline{D}, |z|=d} f(z)$. Then, for every $z \in \overline{D}$, $|f(z)| \leq |f|_d$.*

Proof Seems very difficult. Involves Maximum Modulus Principle, proof is omitted in most texts.

Lemma 3.1.3 *something from complex analysis*

Lemma 3.1.4 *something with matrices*

3.2 Main Theorem

Theorem 3.2.1 *(Gelfond-Schneider Theorem) Let α and β be algebraic numbers such that $\alpha \notin \{0, 1\}$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then, α^β is transcendental.*

Proof We will incorporate four lemmas in this proof

Chapter 4

Baker's Theorem

Theorem 4.0.1 *(Baker's Theorem) Let* $\mathbb{L} = \left\{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}\right\}$. *Then, if*

$$\lambda_1, \ldots, \lambda_n \in \mathbb{L}$$

are linearly independent over $\mathbb{Q}$,

Chapter 5

Schanuel's Conjecture

Conjecture 5.0.1 *(Schanuel's Conjecture) Suppose we have n complex numbers*

$$z_1, \ldots, z_n$$

that are linearly independent over $\mathbb{Q}$. Then, $\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$.

Index