Transcendental Numbers

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Definition: An **algebraic number** is $\alpha \in \mathbb{C}$, which is the root of a nonzero polynomial in $\mathbb{Q}[x]$ (or $\mathbb{Z}[x]$). We denote the set of algebraic numbers $\mathbb{A}$ or $\bar{\mathbb{Q}}$. Otherwise, it is **transcendental**.

$$\frac{p}{q} \in \mathbb{Q}$$

It is the root of $x - \frac{p}{q}$. Therefore, we can deduce that $\mathbb{Q} \subseteq \mathbb{A}$. i is the root of $x^2 + 1$.

Do transcendental numbers even exist? Are there numbers which are not algebraic?

Theorem: Yes.

Why? $\mathbb{C}$ is uncountable, $\mathbb{A}$ is countable. Therefore, there are elements in $\mathbb{C}$ not in $\mathbb{A}$. Likewise, since $\mathbb{R}$ is uncountable, there are real transcendental numbers as well.

$\mathbb{A}$ is countable. Let $p(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ and let the index of p be $|a_n| + \cdots + |a_0| + n$. For each $i > 0$, list all p with index i. Then, list their roots.

$$\text{index} = 1: \text{ None}$$

No new roots.

$$\text{index} = 2: \ \pm x$$

New roots: $\{0\}$

$$\text{index} = 3: \ \pm x^2, \pm x \pm 1, \pm 2x$$

New roots: $\{\pm 1\}$

$$\ldots$$

Eventually, every algebraic number will be listed because it is the root of a polynomial $\mathbb{Z}[x]$ with finite index. Thus, $\mathbb{A}$ is countable and has measure 0. However, it is dense in $\mathbb{C}$.

First Transcendental Number

Was the first transcendental number discovered e or π? No, in general, it is very hard to check if a number is transcendental. Therefore, the first transcendental number was constructed in 1844. This is called Liouville's constant:

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

$$= 0.\mathbf{11}000\mathbf{1}00000000000000000\mathbf{1}0\ldots$$

Liouville's Approximation Theorem: For any real $\alpha \in \mathbb{A}$ of degree n, there is $C(\alpha) > 0$ such that for rational approximations,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C(\alpha)}{q^n}$$

By MVT, $m(\alpha) - m\left(\frac{p}{q}\right) = \left(\alpha - \frac{p}{q}\right) m'(\xi)$. $m(\alpha) = 0, m\left(\frac{p}{q}\right)$ has denominator q^n. Thus, $\frac{C}{q^n} \leq \left|\alpha - \frac{p}{q}\right| \sup_{x \in (\alpha - 1, \alpha + 1)} |p'(x)|$.

Let $\frac{p_n}{q_n}$ be the sum of the first n terms of

$$L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

Then,

$$\begin{aligned}
\left| L - \frac{p_n}{q_n} \right| &= 10^{-(n+1)!} + 10^{-(n+2)!} + \dots \\
&< 2 \cdot 10^{-(n+1)!} \\
&< \left(10^{n!} \right)^{-n} \\
&= \frac{1}{(q_n)^n}
\end{aligned}$$

Pop Quiz!

Are the following numbers algebraic or transcendental? It is hard to determine whether a number we did not construct is transcendental.

Quiz

$$\log(2)$$

Transcendental, by the Lindemann-Weierstrass theorem.

$$\log_2(21441)$$

Transcendental, by the Gelfond-Schneider theorem.

$$\cos(1)$$

Transcendental by the Lindemann-Weierstrass theorem.

e, π

Hint: The Lindemann-Weierstrass theorem shows that if e is transcendental, then π is transcendental.

Both transcendental, e proven by Hermite in 1873. Using this fact, the Lindemann-Weierstrass theorem shows π is transcendental.

$$\sqrt{2}^{\sqrt{2}}$$

Transcendental, by the Gelfond-Schneider theorem.

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

Algebraic.

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2$$

$$i^i$$

Transcendental, by the Gelfond-Schneider theorem.

Transcendence is unknown for all of these except one; which one?

- $e + \pi$
- $e\pi$
- e^{π}
- π^{e}
- e^{e}
- $\pi^{\pi^{\pi^{\pi}}}$

Transcendence is unknown for all of these except one; which one?

- $e + \pi$
- $e\pi$
- e^{π} Gelfond's constant
- π^{e}
- e^{e}
- $\pi^{\pi^{\pi^{\pi}}}$ Open problem: is this an integer?

Schanuel's conjecture, if true, would show these are transcendental:

- $e + \pi \leftarrow$
- $e\pi \leftarrow$
- e^{π}
- $\pi^{e} \leftarrow$
- $e^{e} \leftarrow$
- $\pi^{\pi^{\pi^{\pi}}} \leftarrow$

$$\phi,$$

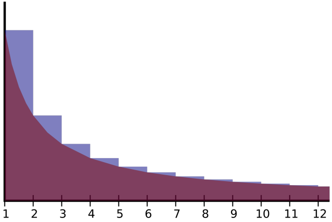
the golden ratio or $\frac{1+\sqrt{5}}{2}$

Algebraic. Minimal polynomial is $x^2 - x - 1$.

$$\gamma,$$

Euler-Mascheroni constant or $0.5772156649\ldots$

$$\lim_{n\to\infty}\left(-\log(n)+\sum_{k=1}^{n}\frac{1}{k}\right)$$



Both transcendence and irrationality are unknown.

λ,

Conway's constant or $1.3035772690\ldots$

Look and say sequence

$$(a_n) = 1, 11, 21, 1211, 111221, 312211, \ldots$$

$$\lambda = \lim_{n \to \infty} \frac{\text{length}(a_{n+1})}{\text{length}(a_n)}$$

Quiz

Algebraic. Minimal polynomial has degree 71!

								$+1x^{71}$	
$-1x^{69}$	$-2x^{68}$	$-x^{67}$	$+2x^{66}$	$+2x^{65}$	$+1x^{64}$	$-1x^{63}$	$-1x^{62}$	$-1x^{61}$	$-1x^{60}$
$-1x^{59}$	$+2x^{58}$	$+5x^{57}$	$+3x^{56}$	$-2x^{55}$	$-10x^{54}$	$-3x^{53}$	$-2x^{52}$	$+6x^{51}$	$+6x^{50}$
$+1x^{49}$	$+9x^{48}$	$-3x^{47}$	$-7x^{46}$	$-8x^{45}$	$-8x^{44}$	$+10x^{43}$	$+6x^{42}$	$+8x^{41}$	$-5x^{40}$
$-12x^{39}$	$+7x^{38}$	$-7x^{37}$	$+7x^{36}$	$+x^{35}$	$-3x^{34}$	$+10x^{33}$	$+1x^{32}$	$-6x^{31}$	$-2x^{30}$
$-10x^{29}$	$-3x^{28}$	$+2x^{27}$	$+9x^{26}$	$-3x^{25}$	$+14x^{24}$	$-8x^{23}$		$-7x^{21}$	$+9x^{20}$
$+3x^{19}$	$-4x^{18}$	$-10x^{17}$	$-7x^{16}$	$+12x^{15}$	$+7x^{14}$	$+2x^{13}$	$-12x^{12}$	$-4x^{11}$	$-2x^{10}$
$+5x^{9}$		$+x^{7}$	$-7x^{6}$	$+7x^{5}$	$-4x^{4}$	$+12x^{3}$	$-6x^{2}$	$+3x^{1}$	$-6x^{0}$

Definition: Let L/K be a field extension. Then $S = \{s_1, \ldots, s_t\} \subseteq L$ is **algebraically independent** over K if for all $f \in K[x_1, \ldots, x_t]$, $f(s_1, \ldots, s_t) = 0 \implies f = 0$.

Stronger than linear independence. $\{\pi\}$ is algebraically independent over $\mathbb{Q}$.

Definition: Let L/K be a field extension. The maximal algebraically independent subset of L is called a **transcendence basis** of L/K. It's cardinality is called the **transcendence degree** denoted $\operatorname{trdeg}(L/K)$.

All transcendence bases have the same size.

$$\operatorname{trdeg}(M/K) = \operatorname{trdeg}(M/L) + \operatorname{trdeg}(L/K).$$

$$\operatorname{trdeg}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = 0$$

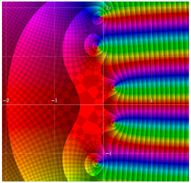
$$\operatorname{trdeg}(\mathbb{Q}(\pi)/\mathbb{Q}) = 1$$

$$\operatorname{trdeg}(\mathbb{R}/\mathbb{Q}) = 2^{\aleph_0}$$

Lindemann-Weierstrass Theorem: If $\alpha_1, \ldots, \alpha_n \in \mathbb{A}$, then $\{e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ is $\mathbb{A}$-linearly independent.

equivalently, $\operatorname{trdeg}(\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n})/\,\mathbb{Q}) = n$.

Proof uses a cleverly constructed auxiliary function with nice properties (many zeroes or zeroes with high multiplicity) to form a contradiction. In Liouville's Approximation theorem, the auxiliary function was the minimal polynomial.



Gelfond's Auxiliary Function

Let the function

$$P(x,y) = \sum_{m=0}^{D-1} \sum_{n=1}^{D-1} a_{m,n} x^m y^n \in \mathbb{Z}[x,y]$$

Lindemann-Weierstrass theorem uses the auxiliary function

$$F(z) = P(e^z, e^{iz})$$

Lindemann-Weierstrass Theorem: If $\alpha_1, \ldots, \alpha_n \in \mathbb{A}$, then $\{e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ is $\mathbb{A}$-linearly independent.

Set $\alpha_1 = 0, \alpha_2 = 1$. Shows e is transcendental.
Set $\alpha_1 = 0, \alpha_2 = \alpha_2$. Shows e^{α_2} is transcendental.
AFSOC, set $\alpha_1 = 0, \alpha_2 = \pi i$. Shows πi is transcendental; we know $i \in \mathbb{A}$ so $\pi \notin \mathbb{A}$.

Gelfond-Schneider Theorem: If $\alpha, \beta \in \mathbb{A}$, $\alpha \neq 0, 1$, $\beta \notin \mathbb{Q}$, then any value of α^β is transcendental.

Answers Hilbert's Seventh Problem.

$$2^{\log_2(21441)} = 21441$$

so $\log_2(21441)$ is either transcendental or rational.

$$e^\pi = \left(e^{\pi i}\right)^{-i} = (-1)^{-i} \text{ is transcendental.}$$

Let the function

$$P(x,y)=\sum_{m=0}^{D-1}\sum_{n=1}^{D-1}a_{m,n}x^m y^n\in\mathbb{Z}[x,y]$$

Gelfond used Taylor series of $F(z)=P(e^z,e^{\beta z})$. Has zeroes of high multiplicity.

Schneider used $F(z)=P(e^z,e^{\log(\alpha)z})$. Has many zeroes of multiplicity one.

Baker's Theorem: Let nonzero $\alpha_1, \ldots, \alpha_n \in \mathbb{A}$. Then, if $2\pi i, \log(\alpha_1), \ldots, \log(\alpha_n)$ are $\mathbb{Q}$-linearly independent, they are also $\mathbb{A}$-linearly independent.

Alan Baker later showed it the statement is still true without $2\pi i$. Auxiliary function was

$$F(z_1, \ldots, z_n) =$$

$$\sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \cdots \sum_{\lambda_n=0}^{L} p(\lambda_1, \lambda_2, \ldots, \lambda_n) \alpha_1^{(\lambda_1+\lambda_n\beta)z_1} \cdots \alpha_{n-1}^{(\lambda_{n-1}+\lambda_n\beta)z_{n-1}}$$

Baker's theorem generalizes the Lindemann-Weierstrass and Gelfond-Schneider theorems. For $a_1, \ldots, a_n \in \mathbb{A}$ not 0 or 1, and $\beta_1, \ldots, \beta_n \in \mathbb{A} \setminus \mathbb{Q}$ and $\mathbb{Q}$-linearly independent, ${a_1}^{\beta_1} \cdots {a_n}^{\beta_n}$ is transcendental.

Schanuel's Conjecture: If $z_1, \ldots, z_n$ be $\mathbb{Q}$-linearly independent, $\operatorname{trdeg}(\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})/\,\mathbb{Q}) \geq n$.

No progress for 60 years. Setting $z_1 = 1, z_2 = \pi i$ shows algebraic independence of e and π. Setting $z_1 = 1$, $z_2 = e$ shows transcendence of e^e. Setting $z_1 = \log\log\pi, z_2 = 1 + \log\log\pi, z_3 = \log\pi, z_4 = e\log\pi, z_5 = i\pi$, one can show π^e is transcendental.

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Thank You!

Questions?