# Transcendental Numbers (Rough Outline 11/11/23)

Vincent Lin

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# Abstract

Research into transcendental numbers started with rational approximations. This paper covers preliminaries on the neccesary parts of field theory before going into the Lindemann-Weierstrass theorem, Gelfond-Schneider theorem, and Baker's theorem and concludes with a discussion of Schanuel's conjecture. This sequence of theorems are related to determining whether complex numbers are transcendental or not and is based off of the work by Baker.

# Chapter 1

# Preliminaries

## 1.1 Algebraic and Transcendental Numbers

**Def** An **algebraic number** is $\alpha \in \mathbb{C}$ which is the root of a nonzero polynomial in $\mathbb{Q}[x]$. A **transcendental number** is a complex number that is not algebraic.

**Note** We will use $\mathbb{A}$ to denote the set of algebraic numbers and we will use $\mathbb{C} \setminus \mathbb{A}$ to denote the set transcendental numbers. We will use $\mathbb{R} \setminus \mathbb{A}$ to denote the set of real transcendental numbers.

To find examples of algebraic numbers, we can take any nonzero polynomial in $\mathbb{Q}[x]$ find its roots. By definition, these roots are algebraic numbers. For example, $\sqrt{2}$ is algebraic because it is a root of $x^2 - 2$. Also, $i$ is algebraic because it is the root of $x^2 + 1$. All rational numbers are algebraic as well. Let $\frac{p}{q} \in \mathbb{Q}$ be rational, where $p, q \in \mathbb{Z}$ and $q$ is nonzero. Then, it is the root of $x - \frac{p}{q}$. Therefore, $\mathbb{Q} \subseteq \mathbb{A}$.

What about transcendental numbers? Do they exist?

**Theorem 1.1.1** *Yes, transcendental numbers exist.*

**Proof** Consider the set of algebraic numbers, which we will denote by $\mathbb{A}$. This set is countable. We will show this by forming a surjection

$$\phi : \left( \mathbb{N} \times \bigcup_{n \in \mathbb{N}} \mathbb{Q}^n \right) \to \mathbb{A}\,.$$

Note that $\mathbb{N} \times \bigcup\limits_{n \in \mathbb{N}} \mathbb{Q}^n$ is countable because it is the Cartesian product of a countable set with a countable union of countable sets. By the fundamental theorem of algebra, we know that a polynomial of degree $k$ has $k$ (not necessarily distinct) roots. Therefore, we can number the $k$ roots from 1 to $k$ for each polynomial. Thus, if we specify the nonzero rational coefficients $(a_0, \ldots, a_k)$ and an index $i$ for $i \in [k]$ to be the $i$-th root of the polynomial $a_k x^k + \cdots + a_0$, we get an algebraic number. We define the map $\phi(i, (a_0, \ldots, a_k))$ to be the $i$-th root of $a_k x^k + \cdots + a_0$ if it exists. Otherwise, we map the input to zero. From the definition, we know that every algebraic number can be encoded this way since it is one of the roots of a nonzero rational polynomial. Thus, for all $a \in \mathbb{A}$, there must be an input $x$ such that $\phi(x) = a$, making this a surjective map from a countable set to $\mathbb{A}$. This makes $\mathbb{A}$ countable. However, since $\mathbb{C}$ is uncountable, it cannot be the case that all complex numbers are algebraic. Thus, if a complex number is not algebraic, it must be transcendental. Furthermore, $\mathbb{R}$ is uncountable so there must be real transcendental numbers as well.

♦

If transcendental numbers exist, then can we find an example? To show that a number is algebraic, we just have to find a nonzero rational polynomial that it is a root of, evaluate that polynomial at that number, and verify that we get zero. However, checking if a number is transcendental is a very hard problem. The rest of this paper will discuss the transcendence of a large class of numbers and methods for determining the transcendence.

## 1.2 Fields

Before that, we will generalize the concept of transcendence to other fields. In order to do this, we will list several definitions. Let $k, l$ be fields and let $l/k$ be a field extension.

- $a \in l$ is **algebraic** over $k$ if there is a nonzero polynomial $f \in k[x]$ such that $f(a) = 0$. Otherwise, $a$ is **transcendental** over $k$. In other words, an algebraic number is a complex number that is algebraic over $\mathbb{Q}$ and a transcendental number is a complex number that is not algebraic over $\mathbb{Q}$.

- $X \subseteq L$ is **algebraically independent** over $k$ if for all $a_1, \ldots, a_t$ distinct and all $f \in k[x_1, \ldots, x_t]$, $f(a_1, \ldots, a_t) = 0$ implies $f = 0$.

- If $a \in l$ is algebraic over $k$, the **minimal polynomial** of $a$ over $k$, denoted $m_a$, is the unique monic irreducible polynomial which generates the kernel of the evaluation map $\phi(f) = f(a)$ for $f \in k[x]$. The **degree** of $\alpha \in k$ is the degree of the minimial polynomial $m_a$.

- $l/k$ is an **algebraic extension** if every element in $l$ is algebraic over $k$.

- $l$ is **algebraically closed** if every nonconstant single variable polynomial in $l$ has a root in $l$.

- An **algebraic closure** of $k$ is an algebraic extension of $k$ that is algebraically closed.

## 1.3 Liouville Numbers

Now we will start looking for our first transcendental number! First, we will start by proving a lemma.

**Lemma 1.3.1** *For any nonzero polynomial $p(x)$ with a root at $x = \alpha$ of multiplicity $m > 0$, $p'(x)$ has the root $\alpha$ with multiplicity $m - 1$.*

**Proof** From the fundamental theorem of algebra, we can write $p(x)$ as $(x-\alpha)^m q(x)$ where $(x-\alpha) \nmid q(x)$. Now, taking the derivatives of both sides and using the product rule, we get

$$\begin{aligned} p(x) &= (x-\alpha)^m q(x) \\ p'(x) &= ((x-\alpha)^m)' q(x) + (x-\alpha)^m q'(x) \\ &= m(x-\alpha)^{m-1} q(x) + (x-\alpha)^m q'(x) \\ &= (x-\alpha)^{m-1} \big(mq(x) + (x-\alpha)q'(x)\big) \end{aligned}$$

Therefore, we get that $(x-\alpha)^{m-1} \mid p'(x)$ so $p'(x)$ has the root $\alpha$ of multiplicity at least $m-1$.

Now, we want to show that $p'(x)$ has root $\alpha$ with multiplicity at most $m-1$. In order to show this, we will shift the polynomial by $\alpha$ (all of the roots will therefore be shifted by $\alpha$ as well). Namely, we look at $p(x+\alpha) = x^m q(x+\alpha)$. Note that $p(x+\alpha)$ now has the root 0 with multiplicity $m$ (it has $m$ factors of $x$) and since $q(x)$ had no $x-\alpha$ factors, $q(x+\alpha)$ has no roots at zero. In other words, $x \nmid q(x+\alpha)$. Now, showing $p'(x)$ has root $\alpha$ with multiplicity $m-1$ is equivalent to showing $p'(x+\alpha)$ as root 0 with multiplicity $m-1$ (it has exactly $m-1$ factors of $x$). Thus, substituting $x+\alpha$ in $p'(x)$:

$$p'(x+\alpha) = x^{m-1} \big(mq(x+\alpha) + xq'(x+\alpha)\big)$$

We have $m-1$ factors of $x$ from the $x^{m-1}$ term. Therefore, we need to show the rest of the expression has no factors of $x$. In other words, $x \nmid (mq(x+\alpha) + xq'(x+\alpha))$, we can take the expression $(mq(x+\alpha) + xq'(x+\alpha))$ modulo $x$ to get $mq(x+\alpha)$. Since $m > 0$ and $q(x+\alpha)$ has no factors of $x$, we know that $x$ cannot divide the entire expression. Shifting back by $\alpha$, we get that $x-\alpha$ cannot be a factor of $mq(x) + (x-\alpha)q'(x)$, so the multiplicity of $x-\alpha$ in $p(x)$ is at most $m-1$.

♦

**Def** A **Liouville number** is a real number $x$ such that for all $n \in \mathbb{N}$, there exists integers $p, q$ with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Essentially, Liouville numbers are real numbers that can be approximately really, really closely by a sequence of rationals of the form $\left\{\frac{p_i}{q_i}\right\}$, where $q_i > 1$, and the distance between $x$ and $\frac{p_i}{q_i}$ is nonzero, but less than $\frac{1}{(q_i)^i}$.

**Theorem 1.3.2** *(Liouville's Approximation Theorem) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an algebraic number of degree $n$. Then, for any rational approximation $\frac{p}{q}$ to $\alpha$, we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^n}$$

**Proof** Let $f$ be the minimal polynomial of $\alpha$ over $\mathbb{Q}$. Note that $f'(\alpha) \neq 0$. This is because $\deg(f') < \deg(f)$ and $f$ is the minimal polynomial. Furthermore, $f$ is irreducible in $\mathbb{R}[x]$, so it cannot be the case that $(x - r) \mid f$ for a rational $r$. In other words, $f$ has no rational roots. Now let $\frac{p}{q}$ be a rational number which we will use to approximate $\alpha$ and plug it into $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

$$\begin{aligned} f\left(\frac{p}{q}\right) &= a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_0 \\ &= \frac{C}{q^n} \end{aligned}$$

for some constant $C \neq 0$ as $\frac{p}{q}$ is not a root of $f$. Also, note that since we want a good approximation of $\alpha$ that is close to $\frac{1}{q^n}$ away, this rational must be at most distance 1 away from $\alpha$.

♦

## 1.4 Transcendence of $e$ and $\pi$

**Lemma 1.4.1** *Let $f(x)$ be a real polynomial with degree $m$. Let*

$$I(t) = \int_0^t e^{t-x} f(x)dx.$$

*Then,*

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t).$$

*where $f^{(j)}(t)$ is the $j$-th derivative of $f$ with respect to $x$ evaluated at $t$.*

**Proof** Probably induction and integration by parts. I will fill this in later.

# Chapter 2

# Lindemann-Weierstrass Theorem

**Theorem 2.0.1** *(Lindemann-Weierstrass Theorem) Suppose*

$$\alpha_1, \ldots, \alpha_n$$

*are algebraic numbers that are linearly independent over $\mathbb{Q}$. Then,*

$$e^{\alpha_1}, \ldots, e^{\alpha_n}$$

*are linearly independent over the algebraic numbers. In other words, the extension field $\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n})$ has transcendence degree $n$ over $\mathbb{Q}$.*

**Proof**

# Chapter 3

# Gelfond-Schneider Theorem

## 3.1 Useful Lemmas

In this chapter, we will prove the Gelfond-Schneider Theorem. We will introduce a few definitions and lemmas to help prove this.

**Def** The square **Vandermonde matrix** of size $n$ is the matrix $V = V(x_1, \ldots, x_n)$ is the $n \times n$ matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

with nonzero element $x_i^j$ in the $i$-th row and $j+1$-th column column for $i \in \{1, \ldots, n\}$ where $0 \leq j \leq n-1$ and $1 \leq i \leq n$.

**Lemma 3.1.1** *Let $V = V(x_1, \ldots, x_n)$ be the square Vandermonde matrix of size $n$. Then, the determinant of the square Vandermonde matrix is*

$$\prod_{1 \leq i < j \leq n} (x_j - x_i).$$

*Therefore the determinant vanishes if and only if $x_i = x_j$ for any $i \neq j$.*

**Proof**

**Lemma 3.1.2** *Let the functions*

$$a_1(t), \ldots, a_n(t)$$

*be nonzero real polynomials* ($\in \mathbb{R}[t]$) *of degree*

$$d_1, \ldots, d_n$$

*respectively. Furthermore, let*

$$w_1, \ldots, w_n$$

*be (pairwise) distinct real numbers. Then, counting multiplicities, the function*

$$f(t) = \sum_{j=1}^{n} a_j(t) e^{w_j t}$$

*has at most* $n - 1 + \sum_{i=1}^{n} d_i$ *real roots.*

**Proof** Note that we can multiply the function $f(t)$ by $e^{-w_n t}$ and the number of roots stays the same. This is because this function is nonzero

Let $k = n + \sum_{i=1}^{n} d_i$. This means that we want to show $f(t)$ has at most $k-1$ roots. We will prove this statement using strong induction on $k$. For our base case $k = 1$. In this case, we know that

$$1 = n + \sum_{i=1}^{n} d_i$$

so it must be the case that $n = 1$, making $d_1 = 0$. Therefore, $a_1(t)$ is a nonzero constant polynomial so $f(t) = a_1(t)e^{w_j t} = ce^{w_j t}$ for some nonzero constant $c$. This is an exponential function which is either always greater than zero or always less than zero. Therefore, it has $k - 1 = 0$ roots. This completes the base case.

Now assume that $k > 1$ and that the proposition holds true for all $k'$, where $1 \le k' < k$. Now, consider $f'(t)$. We will try to bound the number of roots in $f'(t)$. Firstly, note that $f(t)$ is differentiable over the reals. Therefore, between any two different roots, i.e. $f(i) = f(j) = 0$, where $i \neq j$ there is a point with derivative zero from Rolle's theorem. Therefore, we know that if $f(t)$ has $N$ roots.

**Lemma 3.1.3** *Let $f(z)$ be an analytic function in the disk $D \subseteq \mathbb{C}$. Here, we define $D = \{z : |z| < d\}$ for some positive real $d$. Suppose $f$ is continuous on the closure of $D$, $\overline{D} = \{z : |z| \le d\}$. Furthermore, let $|f|_d = \max_{z \in \overline{D}, |z| = d} f(z)$. Then, for every $z \in \overline{D}$, $|f(z)| \le |f|_d$.*

**Proof** Seems very difficult. Involves Maximum Modulus Principle, proof is omitted in most texts.

**Lemma 3.1.4** *something from complex analysis*

**Lemma 3.1.5** *something with matrices*

## 3.2 Main Theorem

**Theorem 3.2.1** *(Gelfond-Schneider Theorem) Let $\alpha$ and $\beta$ be algebraic numbers such that $\alpha \notin \{0, 1\}$ and $\beta \in \mathbb{C} \setminus \mathbb{Q}$. Then, $\alpha^\beta$ is transcendental.*

## 3.3 Consequences

**Proof** We will incorporate four lemmas in this proof

# Chapter 4

# Baker's Theorem

**Theorem 4.0.1** *(Baker's Theorem) Let* $\mathbb{L} = \left\{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}\right\}$. *Then, if*

$$\lambda_1, \ldots, \lambda_n \in \mathbb{L}$$

*are linearly independent over* $\mathbb{Q}$*,*

# Chapter 5

# Schanuel's Conjecture

**Conjecture 5.0.1** *(Schanuel's Conjecture) Suppose we have $n$ complex numbers*

$$z_1, \ldots, z_n$$

*that are linearly independent over $\mathbb{Q}$. Then, $\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ has transcendence degree at least $n$ over $\mathbb{Q}$.*

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