Transcendental Numbers (Rough Outline 9/28/23)

Vincent Lin

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Chapter 1

Preliminaries

1.1 Algebraic and Transcendental Numbers

Def Let k, l be fields and l/k. Then, $a \in l$ is **algebraic** over k if there is a nonzero polynomial $f \in k[x]$ such that $f(a) = 0$. Otherwise, a is **transcendental** over k.

Def Let l/k be a field extension. We say that $X \subseteq L$ is **algebraically independent** over k if for all $a_1, \ldots, a_t$ distinct and all $f \in k[x_1, \ldots, x_t]$, $f(a_1, \ldots, a_t) = 0$ implies $f = 0$.

Now, we will consider the field extension $\mathbb{C} / \mathbb{Q}$.

Def An **algebraic number** is a complex number that is the root of a finite nonzero polynomial in one variable with rational coefficients. A **transcendental number** is a complex number that is not algebraic. In other words, an algebraic number is a complex number that is algebraic over $\mathbb{Q}$ and a transcendental number is a complex number that is not algebraic over $\mathbb{Q}$.

Note We will use $\mathbb{A}$ to denote the set of algebraic numbers and we will use $\mathbb{C} \setminus \mathbb{A}$ to denote the set transcendental numbers. We will use $\mathbb{R} \setminus \mathbb{A}$ to denote the set of real transcendental numbers.

To find some algebraic numbers, we can take a nonzero polynomial with rational coefficents and find its roots. By definition, these roots are algebraic numbers. For example, $\sqrt{2}$ is algebraic because it is a root of $x^2 - 2$. Also, i is algebraic because it is the root of $x^2 + 1$. All rational numbers are algebraic as well. Let $\frac{p}{q} \in \mathbb{Q}$ be rational, where $p, q \in \mathbb{Z}$ and q is nonzero. Then, it is the root of $x - \frac{p}{q}$.

What about transcendental numbers? Do they exist?

Theorem 1.1.1 *Yes, transcendental numbers exist.*

Proof Consider the set of algebraic numbers, which we will denote by $\mathbb{A}$. This set is countable. We will show this by forming a surjection

$$\phi : \left(\mathbb{N} \times \bigcup_{n \in \mathbb{N}} \mathbb{Q}^n\right) \to \mathbb{A}\,.$$

Note that $\mathbb{N} \times \bigcup_{n \in \mathbb{N}}$ is countable because it is the Cartesian product of a countable set with a countable union of countable sets. By the fundamental theorem of algebra, we know that a polynomial of degree k has k (not necessarily distinct) roots. Therefore, we can number the k roots from 1 to k for each polynomial. Thus, if we specify the nonzero rational coefficients $(a_0, \ldots, a_k)$ and an index i for $i \in [k]$ to be the i-th root of the polynomial $a_k x^k + \cdots + a_0$, we get an algebraic number. We define the map $\phi(i, (a_0, \ldots, a_k))$ to be the i-th root of $a_k x^k + \cdots + a_0$ if it exists. Otherwise, we return zero. From the definition, we know that every algebraic number can be encoded this way since it is one of the roots of a nonzero rational polynomial. Thus, for all $a \in \mathbb{A}$, there must be an input x such that $\phi(x) = a$, making this a surjective map from a countable set to $\mathbb{A}$, making $\mathbb{A}$ countable. However, since $\mathbb{C}$ is uncountable, it cannot be the case that all complex numbers are algebraic. Thus, if a complex number is not algebraic, it must be transcendental. Furthermore, $\mathbb{R}$ is uncountable so there must be real transcendental numbers as well.

♦

If transcendental numbers exist, then can we find an example? To show that a number is algebraic, we just have to find a nonzero rational polynomial that it is a root of, evaluate that polynomial at that number, and verify that we get zero. However, checking if a number is transcendental is a very hard problem. The rest of this paper will discuss the transcendence of a large class of numbers and methods for determining the transcendence of a large class of numbers.

1.2 Liouville Numbers

Def A **Liouville number** is a real number x such that for all $n \in \mathbb{N}$, there exists integers p, q with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Essentially, Liouville numbers are real numbers that can be approximately really, really closely by a rational $\frac{p}{q}$, where $q > 1$.

Theorem 1.2.1 *(Liouville's Approximation Theorem) For any algebraic number α of degree $n > 2$, a rational approximation $\frac{p}{q}$ to α,*

Proof We will show that no algebraic number has this property. Namely, if $\alpha \in \mathbb{A}$, then for any approximation $\frac{p}{q}$ with $q > 1$, there will be some natural n where $|\alpha - \frac{p}{q}| > \frac{1}{q^n}$.

blah blah blah

1.3 Transcendence of e and π

Lemma 1.3.1 *Let $f(x)$ be a real polynomial with degree m. Let*

$$I(t) = \int_0^t e^{t-x} f(x) dx.$$

Then,

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t).$$

where $f^{(j)}(t)$ is the j-th derivative of f with respect to x evaluated at t.

Proof Probably induction and integration by parts. I will fill this in later.

Chapter 2

Lindemann-Weierstrass Theorem

Theorem 2.0.1 *(Lindemann-Weierstrass Theorem) Suppose*

$$\alpha_1, \ldots, \alpha_n$$

are algebraic numbers that are linearly independent over $\mathbb{Q}$*. Then,*

$$e^{\alpha_1}, \ldots, e^{\alpha_n}$$

are linearly independent over the algebraic numbers. In other words, the extension field $\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n})$ *has transcendence degree* n *over* $\mathbb{Q}$*.*

Proof

Chapter 3

Gelfond-Schneider Theorem

3.1 Useful Lemmas

In this chapter, we will prove the Gelfond-Schneider Theorem. We will introduce four lemmas to help prove this.

Lemma 3.1.1 *Let the functions*

$$a_1(t), \ldots, a_n(t)$$

be nonzero real polynomials $(\in \mathbb{R}[t])$ *of degree*

$$d_1, \ldots, d_n$$

respectively. Furthermore, let

$$w_1, \ldots, w_n$$

be (pairwise) distinct real numbers. Then, counting multiplicities, the function

$$f(t) = \sum_{j=1}^{n} a_j(t) e^{w_j t}$$

has at most $n - 1 + \sum_{i=1}^{n} d_i$ *real roots.*

Proof Let $k = n + \sum_{i=1}^{n} d_i$. We will prove this statement using strong induction on k.

Lemma 3.1.2 *Let $f(z)$ be an analytic function in the disk $D \subseteq \mathbb{C}$. Here, we define $D = \{z : |z| < d\}$ for some positive real d. Suppose f is continuous on the closure of D, $\overline{D} = \{z : |z| \leq d\}$. Furthermore, let $|f|_d = \max\limits_{z \in \overline{D}, |z|=d} f(z)$. Then, for every $z \in \overline{D}$, $|f(z)| \leq |f|_d$.*

Proof Seems very difficult. Involves Maximum Modulus Principle, proof is omitted in most texts.

Lemma 3.1.3 *something from complex analysis*

Lemma 3.1.4 *something with matrices*

3.2 Main Theorem

Theorem 3.2.1 *(Gelfond-Schneider Theorem) Let α and β be algebraic numbers such that $\alpha \notin \{0, 1\}$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then, α^β is transcendental.*

Proof We will incorporate four lemmas in this proof

Chapter 4

Baker's Theorem

Theorem 4.0.1 *(Baker's Theorem) Let* $\mathbb{L} = \left\{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}\right\}$*. Then, if*

$$\lambda_1, \ldots, \lambda_n \in \mathbb{L}$$

are linearly independent over $\mathbb{Q}$*,*

Chapter 5

Schanuel's Conjecture

Conjecture 5.0.1 *(Schanuel's Conjecture) Suppose we have n complex numbers*

$$z_1, \ldots, z_n$$

that are linearly independent over $\mathbb{Q}$. Then, $\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$.

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