Transcendental Numbers

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Introduction

Motivation, definitions, etc

Definition

An algebraic number is $\alpha \in \mathbb{C}$ which is a root of a nonzero polynomial in $\mathbb{Q}[x]$. $\alpha \in \mathbb{C}$ that are not algebraic numbers are called transcendental numbers.

Example

- $q \in \mathbb{Q}$ are algebraic numbers.

More Definitions

Can we construct a transcendental number?

Theorem

$\sum_{i=1}^{\infty} \frac{1}{10^{n!}}$ *is transcendental.*

Which of the following are transcendental?

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

$$\phi, \gamma, \lambda$$

Transcendental by the Lindemann-Weierstrass Theorem

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

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$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

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$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

$$\phi, \gamma, \lambda$$

Unknown. Schanuel's Conjecture implies transcendence.

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$\color{red}{e + \pi}, \color{red}{e\pi}, e^{\pi}, \color{red}{\pi^e}, \color{red}{e^e}$$

$$\phi, \gamma, \lambda$$

Algebraic. Minimal polynomial is $x^2 - x - 1$.

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

$$\phi, \gamma, \lambda$$

Both irrationality and transcendence are unknown.

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

$$\phi, \gamma, \lambda$$

Algebraic. Minimal polynomial has degree 71!

$$\log(2), \log_2(21441)$$

$$e, \pi, \sqrt[4]{\pi}$$

$$\cos 1$$

$$2^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}}, i^i$$

$$e + \pi, e\pi, e^\pi, \pi^e, e^e$$

$$\phi, \gamma, \lambda$$

Lindemann-Weierstrass Theorem

Theorem

If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers and linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

Corollary

If α is a nonzero algebraic number, then e^{α} is transcendental.

Gelfond-Schneider Theorem

Theorem

Let α be a nonzero algebraic not equal to 1. If β is not rational ($\in \mathbb{C} \setminus \mathbb{Q}$), then α^{β} is transcendental.

Baker's Theorem

Theorem

Schanuel's Conjecture

Theorem

References

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