Transcendental Numbers

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Chapter 1

Preliminaries

1.1 Transcendental Numbers

Def An **algebraic number** is a complex number that is the root of a finite nonzero polynomial in one variable with rational coefficients. A **transcendental number** is a complex number that is not algebraic.

To find some algebraic numbers, we can take a nonzero polynomial with rational coefficents and find its roots. By definition, these roots are algebraic numbers. For example, $\sqrt{2}$ is algebraic because it is a root of $x^2 - 2$. Also, i is algebraic because it is the root of $x^2 + 1$. All rational numbers are algebraic as well. Let $\frac{p}{q} \in \mathbb{Q}$ be rational, where $p, q \in \mathbb{Z}$ and q is nonzero. Then, it is the root of $x - \frac{p}{q}$.

What about transcendental numbers? Do they exist?

Theorem Yes, transcendental numbers exist.

Proof Consider the set of algebraic numbers. This set is countable.

Chapter 2

Lindemann-Weierstrass Theorem

Theorem (Lindemann-Weierstrass Theorem) If $\alpha_1, \ldots, \alpha_n$ algebraic numbers that are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over the algebraic numbers. In other words, the extension field $\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n})$ has transcendence degree n over $\mathbb{Q}$.

Proof

Chapter 3

Gelfond-Schneider Theorem

Theorem (Gelfond-Schneider Theorem) Let α and β be algebraic numbers such that $\alpha \notin \{0, 1\}$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then, α^β is transcendental.

Proof

Chapter 4

Weak Form of Baker's Theorem

Chapter 5

Schanuel's Conjecture

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