Def Let $z \in \mathbb{C}$. The **Lambert W function** is a multivalued function denoted $W(z)$ and is equal to all $w \in \mathbb{C}$ where $we^w = z$.

Let the function $\Gamma : \{z : \Re(z) > 0\} \to \mathbb{C}$ via

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Def The meromorphic **gamma function** is the analytic continuation of Γ defined above and its poles are zero and the negative integers.

Theorem 0.0.1 *For $z \in \mathbb{C}$ with $\Re(z) > 0$, $\Gamma(z+1) = z\Gamma(z)$.*

Proof Using integration by parts, we get that

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z e^{-t} dt \\
&= \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt \\
&= \lim_{t\to\infty} (-t^z e^{-t}) + z \int_0^\infty t^{z-1} e^{-t} dt \\
&= 0 + z \int_0^\infty t^{z-1} e^{-t} dt \\
&= z\Gamma(z)
\end{aligned}$$

♦

Note Note that at $z = 1$, the function evaluates to

$$\int_0^\infty e^{-t} dt = 1.$$

Using this and the recurrence relation from before, we know

$$\Gamma(n) = (n-1)!$$

for positive integers n. Also, note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This is because using u substitution, we get

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt \\
&= \int_0^\infty u^{-1} e^{-u^2} 2u du \\
&= 2\int_0^\infty e^{-u^2} du \\
&= \int_{-\infty}^\infty e^{-u^2} du.
\end{aligned}$$

To evaluate the last integral, we will change to polar coordinates:

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-u^2} du\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} rdrd\theta \\
&= 2\pi \int_{0}^{\infty} re^{-r^2} dr \\
&= 2\pi \int_{-\infty}^{0} \frac{1}{2} e^{s} ds \\
&= \lim_{x\to-\infty} \pi(e^0 - e^x) \\
&= \pi.
\end{aligned}$$

Therefore, our original integral evaluates to $\sqrt{\pi}$.

Theorem 0.0.2 *(Cauchy's Residue Theorem) Let $U \subseteq \mathbb{C}$ be a simply connected open set containing a finite number of points $\{a_1, \ldots, a_n\}$. Then, let U_0 be $U \setminus \{a_1, \ldots, a_n\}$. Let $f : U_0 \to \mathbb{C}$ be holomorphic on U_0. Furthermore, let γ be a closed rectifiable curve in U_0, $\operatorname{Res}(f, a_i)$ denote the residue of f at each a_k, and $I(\gamma, a_k)$ be the winding number of γ around a_k. Then,*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k) \operatorname{Res}(f, a_k).$$