Introduction to Quantum Computing Complete Course - Quantum Soar

Introduction to the Qubit and Superposition

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Probability of measuring $|\psi\rangle$ as 0 is: $|\alpha|^2$

Probability of measuring $|\psi\rangle$ as 1 is: $|\beta|^2$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$|\psi_1\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \qquad |\psi_2\rangle = \begin{pmatrix} \frac{2}{3} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\left|\frac{\sqrt{3}}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = 1 \qquad \left|\frac{2}{3}\right|^2 + \left|\frac{\sqrt{2}}{2}\right|^2 \neq 1$$

Introduction to Dirac Notation

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underbrace{\alpha|0\rangle + \beta|1\rangle}_{\text{Dirac Notation}}$$

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{2\sqrt{3}}{4} \end{pmatrix} = \frac{1}{2}|0\rangle + \frac{2\sqrt{3}}{4}|1\rangle$$

Bloch sphere

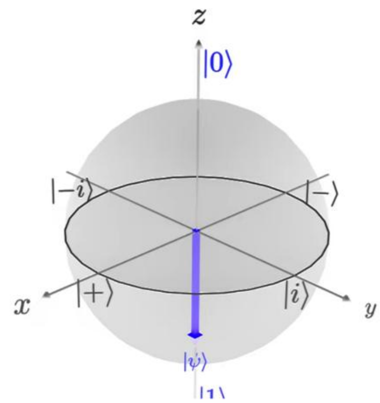
We can represent a qubit $|\psi\rangle$ as
a point on the surface of the Bloch Sphere

Higher vertically = Higher probability of measuring $|\psi\rangle$ as $|0\rangle$

$\implies$ if $|\psi\rangle$ is on the north pole, $|\psi\rangle = |0\rangle$

Lower vertically = Higher probability of measuring $|\psi\rangle$ as $|1\rangle$

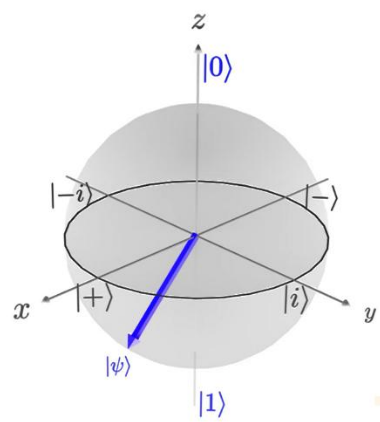
$\implies$ if $|\psi\rangle$ is on the south pole, $|\psi\rangle = |1\rangle$



We can represent a qubit $|\psi\rangle$ as
a point on the surface of the Bloch Sphere

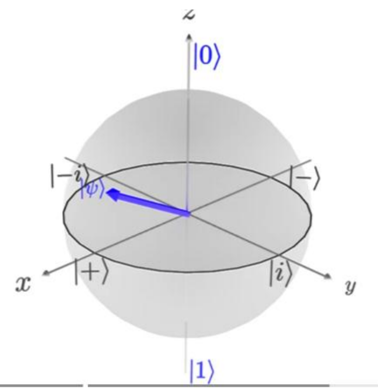
$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$\implies$ has $1/4$ chance of being measured as $|0\rangle$
and a $3/4$ chance of being measured as $|1\rangle$

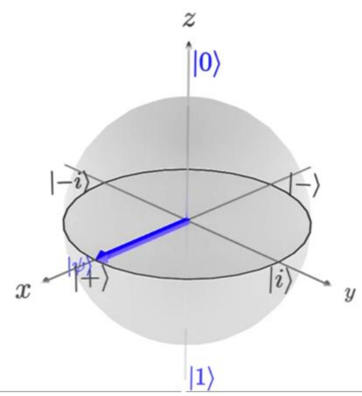


Manipulating a Qubit with Single Qubit Gates

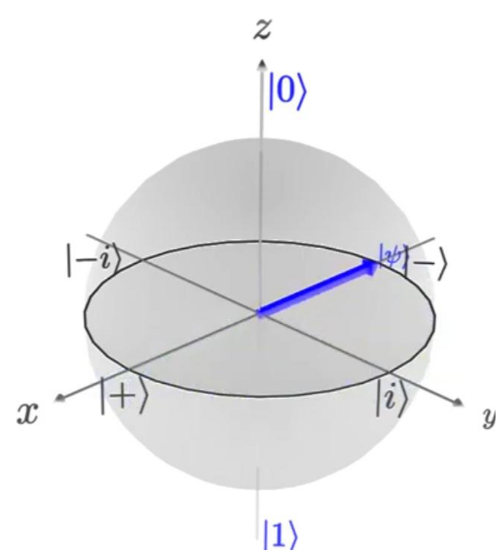
The X-gate flips the qubit π radians around the x-axis on the Bloch Sphere



The Y-gate flips the qubit π radians around the y-axis on the Bloch Sphere



The Z-gate flips the qubit π radians around the z-axis on the Bloch Sphere



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|\psi\rangle = \frac{\sqrt{3}}{2} Y|0\rangle + \frac{1}{2} Y|1\rangle$$

$$Y|\psi\rangle = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 \\ i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$Y|\psi\rangle = \frac{\sqrt{3}}{2}\ i\ |1\rangle - \frac{1}{2}\ i\ |0\rangle$$

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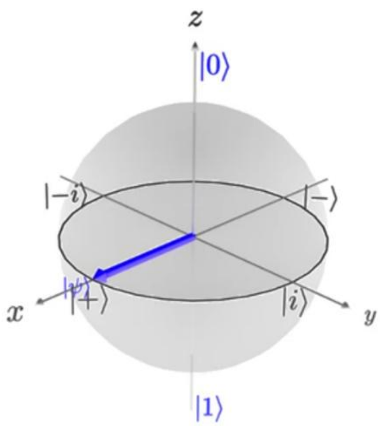
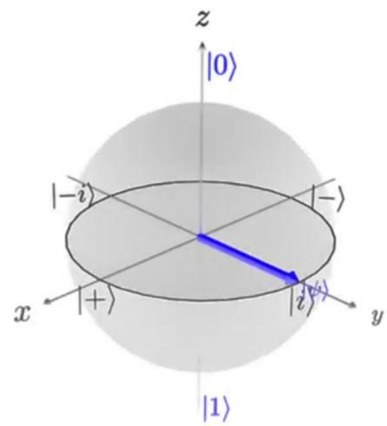
Introduction to Phase





$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad\qquad \frac{1}{\sqrt{2}}|0\rangle + i\frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

3y mulitplying the $|1\rangle$ by $e^{i\varphi}$, we rotate around the z-axis (on the Bloch Sphere) by φ radians

Global Phase

$$e^{i\varphi}(\alpha|0\rangle + \beta|1\rangle)$$
$$= e^{i\varphi}\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

Relative Phase

$$\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

Hardmard Gate:

On the equator of the Bloch sphere, there are four key states: the plus state, minus state, I state, and negative I state. Each of these has an equal chance of being measured as 0 or 1, but they differ in their relative phases. **The plus state has a phase of 0, the minus state has a phase of -1, the I state has a phase of i, and the negative I state has a phase of -i.**

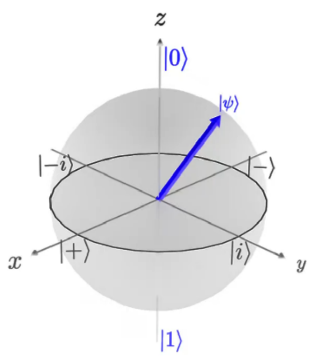
The Hadamard gate, represented by a specific matrix, transforms states on the Bloch sphere: it turns the **0 state into the plus state a**nd the **1 state into the minus state.** Applying the Hadamard gate **twice returns the original state**, making it its own inverse. The gate’s effect depends on the relative phase between the states, demonstrating the importance of phase in quantum computing. In upcoming sections, we'll explore quantum algorithms and why the Hadamard gate and phase are so powerful.

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$|-i\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard Gate

$$H(\alpha|0\rangle + e^{i\varphi}\beta|1\rangle)$$

$$= \alpha H|0\rangle + e^{i\varphi}\beta H|1\rangle$$

$$= \alpha|+\rangle + e^{i\varphi}\beta|-\rangle$$

$$|0\rangle \xrightarrow{H} |+\rangle \qquad |+\rangle \xrightarrow{H} |0\rangle$$

$$|1\rangle \xrightarrow{H} |-\rangle \qquad |-\rangle \xrightarrow{H} |1\rangle$$

The Hadamard Gate is it’s own inverse

Representing Multiple Qubits

$$\Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\gamma|0\rangle + \delta|1\rangle\Big)$$

$$= \alpha|0\rangle \otimes \gamma|0\rangle + \alpha|0\rangle \otimes \delta|1\rangle + \beta|1\rangle \otimes \gamma|0\rangle + \beta|1\rangle \otimes \delta|1\rangle$$

$$= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$$\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$$\text{Prob(measuring } |00\rangle) = |\alpha\gamma|^2$$

$$\text{Prob(measuring } |01\rangle) = |\alpha\delta|^2$$

$$\text{Prob(measuring } |10\rangle) = |\beta\gamma|^2$$

$$\text{Prob(measuring } |11\rangle) = |\beta\delta|^2$$

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

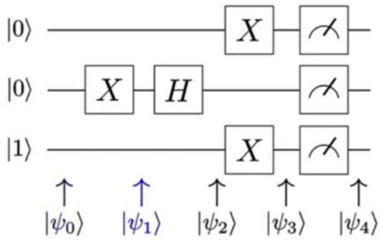
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}|10\rangle + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}|11\rangle$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

$$\underbrace{|0000\ldots0\rangle}_{n \text{ 0's}} = |0\rangle^{\otimes n}$$

$$|1\rangle^{\otimes 5} = |11111\rangle$$

Quantum Circuits



$$|\psi_1\rangle = |011\rangle$$

$$|\psi_2\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big) \otimes |1\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(\,|001\rangle - |011\rangle\,\Big)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\Big(\,|100\rangle - |110\rangle\Big)$$

$|\psi_4\rangle$ will be $|100\rangle$ $\frac{1}{2}$ of the time, and $|110\rangle$ $\frac{1}{2}$ of the time

Multi-Qubit Gates (CNOT, Toffoli, etc)

Nothing if control is 0, Reverse if control is 1

CNOT/Controlled X gate

$$CNOT\left(\frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{4}|11\rangle\right)$$

$$= \frac{\sqrt{3}}{4}CNOT|00\rangle + \frac{1}{2}CNOT|01\rangle + \frac{1}{\sqrt{2}}CNOT|10\rangle + \frac{1}{4}CNOT|11\rangle$$

$$= \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{4}|10\rangle$$

$$= \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{4}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

1st qubit is the control 2nd qubit is the target

Measuring Singular Qubits

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\text{Prob(measuring the first qubit as a 0)} = \text{Prob(measuring } |00\rangle) + \text{Prob(measuring } |01\rangle)$$

$$= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{2}$$

$$|\psi_0\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|001\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|101\rangle$$

Say we measure the middle qubit to be a 0, let $|\psi_1\rangle$ be the state after the measurement

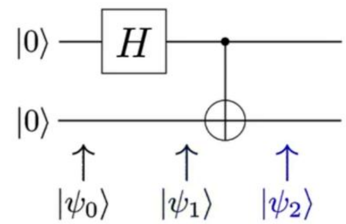
$$|\psi_1\rangle = A\left(\frac{1}{2}|000\rangle + \frac{1}{2}|001\rangle + \frac{1}{2}|101\rangle\right)$$

$$\left|\frac{A}{2}\right|^2 + \left|\frac{A}{2}\right|^2 + \left|\frac{A}{2}\right|^2 = 1 \Rightarrow \frac{A^2}{4} + \frac{A^2}{4} + \frac{A^2}{4} = 1 \Rightarrow \frac{3A^2}{4} = 1 \Rightarrow A^2 = \frac{4}{3} \Rightarrow A = \frac{2}{\sqrt{3}}$$

$$|\psi_1\rangle = \frac{2}{\sqrt{3}}\left(\frac{1}{2}|000\rangle + \frac{1}{2}|001\rangle + \frac{1}{2}|101\rangle\right) = \frac{1}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|101\rangle$$

Quantum Entanglement

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big)$$



$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + CNOT|10\rangle\Big)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

If we measure one of the qubits as a 0, $|\psi_2\rangle \rightarrow |00\rangle$

If we measure one of the qubits as a 1, $|\psi_2\rangle \rightarrow |11\rangle$

Bell States

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)$$

Hardmard Gate:

On the equator of the Bloch sphere, there are four key states: the plus state, minus state, I state, and negative I state. Each of these has an equal chance of being measured as 0 or 1, but they differ in their relative phases. **The plus state has a phase of 0, the minus state has a phase of -1, the I state has a phase of i, and the negative I state has a phase of -i.**

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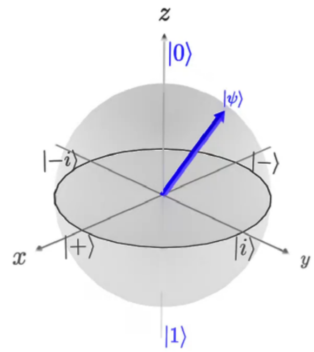

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard Gate

$$H(\alpha|0\rangle + e^{i\varphi}\beta|1\rangle)$$

$$= \alpha H|0\rangle + e^{i\varphi}\beta H|1\rangle$$

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$$|0\rangle \xrightarrow{H} |+\rangle \qquad |+\rangle \xrightarrow{H} |0\rangle$$

$$|1\rangle \xrightarrow{H} |-\rangle \qquad |-\rangle \xrightarrow{H} |1\rangle$$

The Hadamard Gate is it's own inverse

Quantum Computing Course: 2.1 Representing Multiple Qubits

$$\Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\gamma|0\rangle + \delta|1\rangle\Big)$$

$$= \alpha|0\rangle \otimes \gamma|0\rangle + \alpha|0\rangle \otimes \delta|1\rangle + \beta|1\rangle \otimes \gamma|0\rangle + \beta|1\rangle \otimes \delta|1\rangle$$

$$= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle$$

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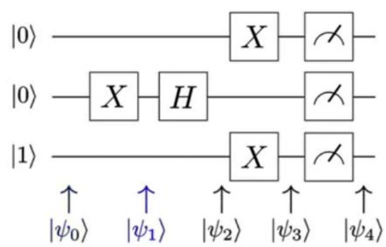
$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}|10\rangle + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}|11\rangle$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

$$\underbrace{|0000...0\rangle}_{n\ 0\text{'s}} = |0\rangle^{\otimes n} \qquad |1\rangle^{\otimes 5} = |11111\rangle$$

Quantum Computing Course: 2.2 Quantum Circuits



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$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\Big(|100\rangle - |110\rangle\Big)$$

$|\psi_4\rangle$ will be $|100\rangle$ $\frac{1}{2}$ of the time, and $|110\rangle$ $\frac{1}{2}$ of the time

Quantum Computing Course: 2.3 Multi-Qubit Gates (CNOT, Toffoli, etc)

Nothing if control is 0, Reverse if control is 1

CNOT/Controlled X gate

$$CNOT\left(\frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{4}|11\rangle\right)$$

$$= \frac{\sqrt{3}}{4}CNOT|00\rangle + \frac{1}{2}CNOT|01\rangle + \frac{1}{\sqrt{2}}CNOT|10\rangle + \frac{1}{4}CNOT|11\rangle$$

$$= \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{4}|10\rangle$$

$$= \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{4}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

1st qubit is the control 2nd qubit is the target

Quantum Computing Course: 2.4 Measuring Singular Qubits

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\text{Prob(measuring the first qubit as a 0)} = \text{Prob(measuring } |00\rangle) + \text{Prob(measuring } |01\rangle)$$

$$= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 \quad = \frac{1}{2}$$

$$|\psi_0\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|001\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|101\rangle$$

Say we measure the middle qubit to be a 0, let $|\psi_1\rangle$ be the state after the measurement

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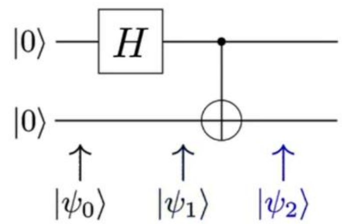
Quantum Entanglement

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big)$$



$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + CNOT|10\rangle\Big)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

If we measure one of the qubits as a 0, $|\psi_2\rangle \to |00\rangle$

If we measure one of the qubits as a 1, $|\psi_2\rangle \to |11\rangle$

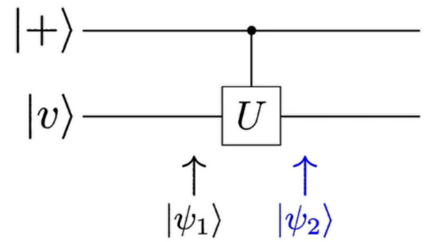
Bell States

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)$$

$|v\rangle$ is an eigenvector of U

$$U|v\rangle = e^{i\theta}|v\rangle$$



$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)|v\rangle$$

If we have a state $|v\rangle$ that is an eigenvector of a gate U, by apply a controlled-U gate with $|v\rangle$ as the target, we can 'kick' the phase onto the control qubit

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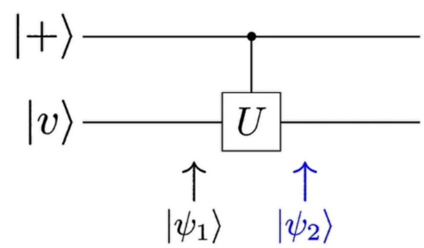
$$U|v\rangle = e^{i\theta}|v\rangle$$



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