Modern Physics II - HW4 Solution - Winter 2016

1. Suppose an electron is trapped in a quantum dot with a diameter of one nm with confining potential walls that are high enough for the infinite square well in one dimensions to be a reasonable approximation.

Taking the one dimensional approximation for the infinite square well we have our familiar wave function:

$$\Psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

In this case we have $L = 1_{nm}$.

(a) What are the energy, frequency, wavelength, speed , and momentum of the (circularly polarized) photon that gets emitted when the electron makes a transition from the second exited state to the ground state? (5 points)

This should be fairly procedural to most by now. Using the eigenvalue of the kinetic energy operator, since this problem does not depend on time:

$$\hat{K}\Psi = E\Psi$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = \frac{n^2\pi^2\hbar^2}{2mL^2}\Psi$$
$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

No potential energy to deal with so life is pretty easy we just need to calculate the change in E from the second excited state to the ground state and invoke conservation of energy to determine the properties of our photon. We know everything so this is just a plug and chug, but going to show it two different ways because of some confusion on the test particularly about the MeV/c^2 version of mass:

$$E_3 \approx \frac{(3)^2 \pi^2 \left(6.582\text{E-}16_{eV\cdot s}\right)^2}{2\left(5.110\text{E}5_{eV/c^2}\right)\left(1_{nm}\right)^2} \approx \frac{(3)^2 \pi^2 \left(6.582\text{E-}16_{eV\cdot s}\right)^2}{2\left(\left(\frac{5.110\text{E}5}{(2.998\text{E}8)^2}\right)_{eV\cdot s^2/m^2}\right)\left(1_{nm}\right)^2} \approx 3.387_{eV}$$

$$\approx \frac{(3)^2 \pi^2 \left(1.055\text{E-}34_{J\cdot s}\right)^2}{2\left(9.109\text{E-}31_{kg}\right)\left(1_{nm}\right)^2}\cdot 6.242\text{E}18_{eV/J} \approx 3.387_{eV}$$

Unit analysis is your friend, key thing to note c is a unit that is about $3E8_{m/s}$ and you can't ignore that if you are trying to get back to meters per second. A rule of thumb trick I use to figure out which $\hbar$ to use is by noting units of mass, generally if I see kilograms I think joules and do the conversion after if I want electron volts. In addition if you see some ridiculously large/small amount of electron volts when talking about electrons there is a good possibility of a calculator flub unless you have a really unusual situation. Same deal for ground state but $n = 1$ and then taking the delta of the energies:

$$E_1 \approx 0.376_{eV}$$
$$\Delta E_{3,1} \approx -3.011_{eV}$$

Negative delta just implies energy is leaving (photon) instead of coming in. Assuming conservation of energy is true (hope so) we can take that delta and assign it to the photon.

$$E_p = -\Delta E_{3,1} \approx 3.011_{eV}$$

The rest of the problem is super strait forward, run conversions on the energy to get the frequency, wavelength, and momentum (you already know the speed hopefully).

$$f = \frac{E_p}{h} \approx \frac{3.011_{eV}}{4.136\text{E-}15_{eV\cdot s}} \approx 7.277\text{E}14_{Hz}$$
$$\lambda = \frac{c}{f} \approx \frac{2.889\text{E}8_{m/s}}{7.277\text{E}14_{Hz}} \approx 412.0_{nm}$$
$$p = \frac{h}{\lambda} \approx \frac{6.626\text{E-}34_{J\cdot s}}{4.120\text{E-}7_{m}} \approx 1.608\text{E}-27_{N\cdot s}$$

And of course this photon is moving at the speed of light.

(b) Is that photon in the visible part of the electromagnetic spectrum? (2 points)

By the vague definition of visible, yes. Anything between 400_{nm} and 700_{nm} is generally considered visible. This is a rather blue photon.

(c) What is the angular momentum of that photon? You probably need to do some extra research on your own on this topic as it was only mentioned only once in class, but the correct answer is worth (3 points).

So this isn't evident in the 1D approximation(remember this thing is actually a sphere), but the excited state has an associated spin, where the ground state does not. Due to conservation of angular momentum we can't have any changes in the total system so the photon needs to carry that angular momentum with it. A bit beyond the scope of this homework to do a derivation so all that is needed is to note that the circularly polarized light carries with it $\pm\hbar$ (spin 1 particle)units of angular momentum with respect to the direction of propagation (the sign specifies right or left handed circular polarization).

Feynman lectures has a bit more on this for anyone interested:

http://www.feynmanlectures.caltech.edu/III_18.html

2. An alpha particle is in a state given by the wave function $\Psi(x) = Ae^{-\left(\frac{x-b}{2\varepsilon}\right)^2}$

Ooh a scary looking thing that isn't scary at all if you know what it actually is. Anyone who has taken stats before should be able to see that this function looks very similar to a normal distribution. And if we were to square Ψ we would have the PDF for the normal distribution (this should make sense given how we have been calculating probability so far). Just by inspection then we would realize:

$$A = \sqrt{\frac{1}{\varepsilon\sqrt{2\pi}}} \qquad \langle x \rangle = b \qquad \Delta x = \varepsilon$$

All from the definition of the PDF for the normal distribution (standard deviation and uncertainty are equivalent based on our definition). Full points for being clever and noting function similarity to save time, but I'm guessing not everyone has taken stats so the longer form solutions are below.

(a) normalize this wave function (3 points)

Two ways to do this, u sub or note b is just an offset for the peak and set it to zero because it is obvious that an integral over all space is equivalent regardless of where the center is as long as the shape doesn't change. Will do the u sub because I'm sure someone will take issue with changing the function even though it has equivalence for the way we are using it.

We need the following to meet normalization:

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx = 1$$

So:

$$1 = \int_{-\infty}^{\infty} \left(Ae^{-\left(\frac{x-b}{2\varepsilon}\right)^2} \right) \left(Ae^{-\left(\frac{x-b}{2\varepsilon}\right)^2} \right) dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2\left(\frac{x-b}{2\varepsilon}\right)^2} dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-\left(\frac{x-b}{\sqrt{2}\varepsilon}\right)^2} dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-(u)^2} \sqrt{2}\varepsilon du \ , \ u = \frac{x-b}{\sqrt{2}\varepsilon}, \ du = \frac{1}{\sqrt{2}\varepsilon} dx$$

$$= A^2 \sqrt{\pi} \sqrt{2}\varepsilon \text{ , table solution for integral of that form}$$

$$= A^2 \sqrt{2\pi}\varepsilon$$

It follows then:

$$A^2 = \frac{1}{\varepsilon\sqrt{2\pi}}$$
$$A = \sqrt{\frac{1}{\varepsilon\sqrt{2\pi}}}$$

(b) calculate the expectation value of the position (4 points)

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} x e^{-(u)^2} du \ , \ u = \frac{x-b}{\sqrt{2}\varepsilon}, \ du = \frac{1}{\sqrt{2}\varepsilon} dx, \text{ from last part, reduce coefficient } A^2$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(u\sqrt{2}\varepsilon + b\right) e^{-(u)^2} du \ , x = u\sqrt{2}\varepsilon + b$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} b e^{-(u)^2} du \ , \text{ odd symmetry killed the first term}$$
$$= b$$

This reduces pretty quickly if you can recognize that odd functions add to zero (that converge specifically, this one does and infinity is weird for the ones that don't converge like sine because finite symmetrical regions do add to zero... I'm sure the mathematicians have an argument for this quirk).

But anyways our answer should be expected.. we have a nice symmetrical thing with the highest point at b, sure enough the average value will be in the center of this lump, right at b.

(c) calculate the uncertainty in position (4 points)

Need the spread of position, so we need expectation for the square of the position operator first:

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} x^2 e^{-(u)^2} du \ , \ \ u = \frac{x-b}{\sqrt{2\varepsilon}}, \ du = \frac{1}{\sqrt{2\varepsilon}} dx, \text{ from last part, reduce coefficient } A^2 \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(u\sqrt{2}\varepsilon + b \right)^2 e^{-(u)^2} du \ , x = u\sqrt{2}\varepsilon + b \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\left(u\sqrt{2}\varepsilon \right)^2 + 2u\sqrt{2}\varepsilon b + b^2 \right] e^{-(u)^2} du \ \text{, middle term dies from odd symmetry, last term reduces} \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(u\sqrt{2}\varepsilon \right)^2 e^{-(u)^2} du + b^2 \\
&= \frac{2\varepsilon^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-(u)^2} du + b^2 \\
&= \frac{2\varepsilon^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} + b^2 \ \text{, another table solution for the integral} \\
&= \varepsilon^2 + b^2
\end{aligned}
$$

Now we can figure out the uncertainty in position (read: spread/standard deviation):

$$
\begin{aligned}
\Delta x &= \sqrt{\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2} \\
&= \sqrt{\varepsilon^2 + b^2 - b^2} \\
&= \varepsilon
\end{aligned}
$$

Which matches the answer given by the definition of the PDF for a normal distribution.