Modern Physics II - HW5 Solution - Winter 2016

☺*An old ant of 5 mg weight and very bad teeth is confined to an extremely delicate closed 3D box with uniform walls that are only 10 nm thick*. The walls represent a potential energy barrier of 7 10^{-9} Nm height. Since her teeth are worn down too much for any successful attempt to gnaw through the wall, she tries to bring the kinetic energy gained from running at 5 cm per second** to bear in attempts to "break through to the outside" by tunneling. What is this ant's probability of getting out of the box by applying this strategy? (☺ ☺ ☺ points and ☺ if you actually provide an approximate numerical value of the probability)*

** Extremely tiny holes in the walls prevent the ant from affixation and there are also caches of food and water that can sustain her for a year. The whole system is isolated so that no other ants or animal rights activist can come to her rescue. On the outside of the box, there is some anti-anteater repellent so that there is no danger from that side either. The necessary paperwork for doing this thought experiment has been filed with the proper authorities at the state and federal level.*

*** This kinetic energy corresponds of course to one of the possible discrete energy levels of an equally massive particle in an equally sized box.*

So in the sacred tradition of spherical cows let us assume that the ant is particle-like so this system can be approximated as a particle incident upon a finite boundary. Possibly questionable since the ant is huge(bunch of particles) compared to barrier thickness and we are blatantly ignoring the nature of the barrier/ant potential system. But even then we can still show that our spherical ant particle thingy isn't likely to escape. A bit of an aside, but we should note in a more realistic setting such as the ant being comprised of many particles that the probability of the ant being decapitated/dismembered/incrementally atomized is significantly higher than the probability of the ant making it through unscathed due to the nature of compound probabilities.

Solving first for the kinetic energy:

$$K_E = \frac{1}{2}mv^2 \approx \frac{1}{2}\left(5\text{E-}6_{kg}\right)\left(5\text{E-}2_{m/s}\right)^2 \approx 6.25\text{E-}9_J$$

Note that Newton meters and Joules are equivalent units by definition so we can see that our potential energy barrier is slightly higher than what the ant particle thingy is bringing to the table.

Calculating our decay constant gives:

$$\kappa = \frac{\sqrt{2m\left(V-K_E\right)}}{\hbar} \approx \frac{\sqrt{2\left(5\text{E-}6_{kg}\right)\left(\left(7\text{E-}9_J\right)-\left(6.25\text{E-}9_J\right)\right)}}{1.054\text{E-}34_{J\cdot s}} \approx 8.21\text{E}26_{m^{-1}}$$

This is a scary fast number, meaning the probability will very rapidly approach zero over very short distances. Showing our approximation criteria (should be obvious it is valid at this point):

$$1 \ll \kappa L \approx \left(8.21\text{E}26_{m^{-1}}\right)\left(10\text{E-}9_m\right) \approx 8.21\text{E}18$$

So we can use the approximate solution to get the transmission coefficient. We need to note something about the nature of calculators first though. The bit space for running calculations is finite, and the number we are trying to calculate for all effective purposes on about every machine we have in existence is going to round to zero. If at this point you say it is approximately zero I won't hold it against you since we are getting into the territory of infinitesimals. That said, it is possible to get a non-zero answer by asking in the right way but it will still be a rough representation which we can't write using standard exponential notation.

$$T \approx 16\frac{K_E}{V}\left(1-\frac{K_E}{V}\right)e^{-2\kappa L} \approx 1.53e^{-2(8.21\text{E}18)} \approx 1.53e^{-1.62\text{E}19} \approx 10^{-10^{18.84}} \approx 0$$

So yeah, assuming you were writing zeros every half second it would take ~8 ages of the universe to write down that number. This doesn't bode well for our ant escaping.

If she sustains attempts of getting out every single minute (by hitting the wall head on) for eight hours every day for five days each week, is there any chance at all that she will achieve her goal by tunneling before her food and water caches are gone (☺ point)

Well intuition says basically zero.. but yeah.

$$Attempts = 1_{\min^{-1}} \cdot 60_{\min/hr} \cdot 8_{hr/day} \cdot 5_{day/wk} \cdot 52_{wk} = 124800$$

Each attempt is equi-probable for escape so this many trials would give a probability of a single success on each trial. Figuring we didn't want to just cop out and say basically zero, we need to run the trials on the probability of not succeeding that many times and use that to calculate single success in the trials. While it is technically possible that our ant escapes on the first trial it is highly unlikely, so we can use binomial theorem from statistics to calculate it. The first thing to consider is what the outcomes are, either the ant escapes or it doesn't. The not escaping can be represented by the reflection coefficient, which can be calculated by taking unity less the transmission coefficient. Next we can calculate unity less the nth power of failures(reflections) which will give us the probability of not failing every single trial, or better said actually making it through the wall on at least once in the series of trials. The easiest way to think of this is to imaging you are rolling a die some number of times and you want to know what the probability of getting at least one 6 is within this number of rolls, the way you do this is to calculate the probability of getting 1 through 5 on n rolls in a row $\left(5/6\right)^n$. Since you either got a series of 1 through 5 or a 6 somewhere in there you can just calculate 1(all possibilities always add to 1) less the probability of getting a string of rolls for the values you didn't want to get your answer. Using this we can kind of calculate the probability of escape (numerically this is tricky due to precision like calculating a particular atom on a ball being on the very top of the ball after rolling it) but the concept is extendable to more reasonable problems so it is still worth noting:

$$R = 1 - T$$

$$P = 1 - R^{Attempts} = 1 - \left(1 - T\right)^{Attempts} \approx 1 - 1 \approx 0$$

Because the powers of numbers very close to one are still very close to one, and the closeness to one is much better(an understatement if I ever made one) than one part out of 124k we can assume for this case that it is basically one. Note that this is a bit hand wavy(if you want to verify feel free, even computers have significant problems with numbers like this), we should be a bit more careful even with extremely low probabilities if the trials are large enough to compensate, but for this case the ant will almost certainly not be going anywhere(particularly within its lifetime).

The big take away is that when things get big we get back to the classical limit (which is a good thing) of not being able to pass through potential energy barriers with energy less than the barrier height. This tells us why we don't fall through floors and walls when we come in contact with them since the probability of doing so is so vanishingly small that it is never observed.

2. (a) An electron is confined by a 0.2 nm finite square-well of strength 1 keV. What is the approximate ground state energy in J? 2.5 points each

Iterative math problems... this is where programming is your friend unless you are masochistic and like calculating the same thing multiple times. I'll show the first iteration as a first order approximation but then will compare with a script to see how far off the convergence is. Luckily the well is fairly deep compared to the energy of the electron so it should converge pretty quickly so even a single iteration approximation should be within a percent. Starting with the approximate form for the energy:

$$E_n \approx \frac{n^2\pi^2\hbar^2}{2m\left(L+2\delta\right)^2} \qquad \delta = \frac{\hbar}{\sqrt{2m\left(V-E\right)}}$$

Where $L+2\delta$ is the effective well length. We are interested in ground state so n=1. Making a quick guess to give us a starting point we assume that E<<V:

$$E_{1_0} \approx \frac{\pi^2\hbar^2}{2m\left(L+2\delta\right)^2} \approx \frac{\pi^2\hbar^2}{2m\left(L+\frac{2\hbar}{\sqrt{2m\left(V-E\right)}}\right)^2} \approx \frac{\pi^2\hbar^2}{2m\left(L+\frac{2\hbar}{\sqrt{2mV}}\right)^2}$$

$$\approx \frac{\pi^2\left(1.054\text{E-}34_{J\cdot s}\right)^2}{2\left(9.109\text{E-}31_{kg}\right)\left(0.2\text{E-}9_m+\frac{2\left(1.054\text{E-}34_{J\cdot s}\right)}{\sqrt{2\left(9.109\text{E-}31_{kg}\right)\left(1\text{E}3_{eV}\cdot 1.602\text{E-}19_{J/eV}\right)}}\right)^2}$$

$$\approx 1.3361\text{E-}18_J$$

Taking our initial guess we can plug it back in to get a more accurate result:

$$E_{1_1} \approx \frac{\pi^2 \left(1.054\text{E-}34_{J\cdot s}\right)^2}{2\left(9.109\text{E-}31_{kg}\right)\left(0.2\text{E-}9_m + \dfrac{2\left(1.054\text{E-}34_{J\cdot s}\right)}{\sqrt{2\left(9.109\text{E-}31_{kg}\right)\left(1\text{E}3_{eV}\cdot 1.602\text{E-}19_{J/eV} - 1.336\text{E-}18_J\right)}}\right)^2}$$

$$\approx 1.3354793\text{E-}18_J$$

Yes woah sigfigs (justified because I'm using a lot more than shown here in my calculations the equation is cumbersome enough as is). Just wanted to have something to compare for a few iterations down. Taking it to the 10th iteration we have:

$$E_{1_{10}} \approx 1.3354796\text{E-}18_J$$

Which is in extremely good agreement with our first iteration.

(b) An alpha particle is confined by a 3 fm finite square-well of strength 40 MeV. What is the approximate ground state energy in Nm?

Same deal with a significantly smaller well and a much more massive particle:

$$m_\alpha \approx 6.644657\text{E-}27_{kg}$$

$$E_{1_0} \approx \frac{\pi^2\hbar^2}{2m\left(L+2\delta\right)^2} \approx \frac{\pi^2\hbar^2}{2m\left(L+\dfrac{2\hbar}{\sqrt{2m\left(V-E\right)}}\right)^2} \approx \frac{\pi^2\hbar^2}{2m\left(L+\dfrac{2\hbar}{\sqrt{2mV}}\right)^2}$$

$$\approx \frac{\pi^2\left(1.054\text{E-}34_{J\cdot s}\right)^2}{2\left(6.644\text{E-}27_{kg}\right)\left(3.0\text{E-}15_m + \dfrac{2\left(1.054\text{E-}34_{J\cdot s}\right)}{\sqrt{2\left(6.644\text{E-}27_{kg}\right)\left(40\text{E}6_{eV}\cdot 1.602\text{E-}19_{J/eV}\right)}}\right)^2}$$

$$\approx 5.95976\text{E-}13_J$$

$$\approx 5.95976\text{E-}13_{N\cdot m}$$

Basic unit analysis will tell you that a joule is a newton meter by definition. The significantly higher energy should not be surprising since this well is about 5 orders of magnitude smaller (squeezed things get angry). Taking the next iteration the same as last time we have:

$$E_{1_1} \approx 5.845691\text{E-}13_{N\bullet m}$$

Not nearly as small of a step as last time, this is because the well energy barrier compared to the ground state energy isn't as excessive compared the last problem(infinite well gives ground energy around in the range of 9E-13J vs. well barrier at about 6.4E-12J), so we should see it walk a bit more before converging. Comparing to the 10th iteration again:

$$E_{1_{10}} \approx 5.847959\text{E-}13_{N \cdot m}$$

Which shows good agreement even though it is converging slowly in comparison to the first problem.