Modern Physics II - HW1 Solution - Winter 2016

1. (a) Demonstrate the linearity of the Helmholtz equation on the example of the wave function of a free particle moving to the left. 4 points

The general form of the Helmholtz equation: $\nabla^2 B - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}B = 0$

The wave function for a free particle in one dimension: $\Psi(x,t) = Ae^{i(kx-\omega t)}$

Given the particle is moving left we should note that $k<0$ in this case due to convention since the wave vector k contains the information about momentum($k \propto p$).

Plugging the wave function into the Helmholtz equation:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi = 0$$

$$\nabla^2 Ae^{i(kx-\omega t)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}Ae^{i(kx-\omega t)} = 0$$

Since this is a one dimensional problem the Laplacian reduces to the second partial with respect to x:

$$\frac{\partial^2}{\partial x^2}Ae^{i(kx-\omega t)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}Ae^{i(kx-\omega t)} = 0$$

$$(ik)^2 Ae^{i(kx-\omega t)} - \frac{1}{c^2}(i\omega)^2 Ae^{i(kx-\omega t)} = 0$$

$$-k^2 Ae^{i(kx-\omega t)} + \frac{\omega^2}{c^2}Ae^{i(kx-\omega t)} = 0$$

$$\left(\frac{\omega^2}{c^2} - k^2\right)Ae^{i(kx-\omega t)} = 0$$

$$\left(\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{\lambda}\right)^2\right)Ae^{i(kx-\omega t)} = 0 \text{ , } \lambda = 2\pi\frac{c}{\omega}, k = \frac{2\pi}{\lambda}$$

Which is certainly valid and is important because it shows Ψ exists in the domain of the Helmholtz equation, so now we can use short hand to represent the wave function of a free particle for the rest of this problem. We can also note that the sign of k(direction) does not impact the validity of the statement.

So pressing on to show linearity we need to show two things(one for a and b, but both shown here):

$$f(\alpha\Psi) = \alpha f(\Psi) \qquad \text{and} \qquad f(\Psi_1 + \Psi_2) = f(\Psi_1) + f(\Psi_2)$$

, where α is an arbitrary scalar and Ψ_n are valid solutions.

In this case we define our function as the Helmholtz equation with its argument as the wave function:

$$f(\Psi) := \nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi$$
$$f(\Psi) = 0$$

First we take care of multiplication by a constant by utilizing commutativity and distributivity :

$$f(\alpha\Psi) = \nabla^2\alpha\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\alpha\Psi = \alpha\nabla^2\Psi - \alpha\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi = \alpha\left(\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi\right) = \alpha f(\Psi)$$

, we can factor α out of the Laplacian because it does not depend on the spatial coordinates and it can also be factored out of the time derivative because it does not depend on time.

Next we handle addition, noting the distributive property of the Laplacian stems from the properties of the gradient and divergence($\nabla^2 A = \nabla \cdot \nabla A$ for scalars), which in turn inherited this property from derivatives $\frac{d}{dx}\left(x_1 + x_2\right) = \frac{d}{dx}x_1 + \frac{d}{dx}x_2$:

$$\begin{aligned} f(\Psi_1 + \Psi_2) &= \nabla^2(\Psi_1 + \Psi_2) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\Psi_1 + \Psi_2) \\ &= \nabla^2\Psi_1 + \nabla^2\Psi_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_1 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_2 \\ &= \left(\nabla^2\Psi_1 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_1\right) + \left(\nabla^2\Psi_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_2\right) \\ &= f(\Psi_1) + f(\Psi_2) \end{aligned}$$

Which demonstrates the linearity of the Helmholtz equation.

(b) Demonstrate the linearity of the time dependent Schrödinger equation on the example of the wave function of a free particle moving to the right. 4 points

The general form of the time dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

For this problem we have one dimension and a free particle, so we can ignore the other spatial derivatives and set the potential to 0, which gives us:

$$i\hbar\frac{\partial}{\partial t}\Psi + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = 0$$

Again we have the wave function for a free particle in one dimension and we note $k > 0$ since this particle is moving to the right for the same reason as last time $k \propto p$:

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$

As was done in the last part we should verify the solution exists in the domain of the equation:

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}Ae^{i(kx-\omega t)} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}Ae^{i(kx-\omega t)} &= 0 \\
i\hbar(-i\omega)Ae^{i(kx-\omega t)} + \frac{\hbar^2}{2m}(ik)^2 Ae^{i(kx-\omega t)} &= 0 \\
\hbar\omega Ae^{i(kx-\omega t)} - \frac{\hbar^2}{2m}k^2 Ae^{i(kx-\omega t)} &= 0 \\
\left(\hbar\omega - \frac{\hbar^2}{2m}k^2\right)Ae^{i(kx-\omega t)} &= 0
\end{aligned}$$

Realizing that the coefficient is a statement of conservation of energy and making the substitutions:

$$\begin{aligned}
\hbar\omega - \frac{\hbar^2}{2m}k^2 &= 0 \\
E - \frac{\hbar^2}{2m}\left(\frac{p}{\hbar}\right)^2 &= 0 \\
E - \frac{p^2}{2m} &= 0 \\
E - K_E &= 0
\end{aligned}$$

Which is what we should expect for a simple moving free particle(total energy is equal to kinetic energy) again noting the direction of the particle(sign of k)does not impact the validity of the statement. Being a little less general this time(only required to show it for a 1D free particle), but following the same procedure from part (a) we want to show linearity:

$$\begin{aligned}
&g(\Psi) := i\hbar\frac{\partial}{\partial t}\Psi + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi \\
&g(\Psi) = 0
\end{aligned}$$

As before the operations are strait forward since α does not depend on time or space.

$$g(\alpha\Psi) = i\hbar\frac{\partial}{\partial t}\alpha\Psi + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\alpha\Psi = \alpha i\hbar\frac{\partial}{\partial t}\Psi + \alpha\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = \alpha\left(i\hbar\frac{\partial}{\partial t}\Psi + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi\right) = \alpha g(\Psi)$$

Finally the summing property with the same arguments as before since derivatives are distributive:

$$\begin{aligned}
g(\Psi_1+\Psi_2) &= i\hbar\frac{\partial}{\partial t}\left(\Psi_1+\Psi_2\right)+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\Psi_1+\Psi_2\right) \\
&= i\hbar\frac{\partial}{\partial t}\Psi_1+i\hbar\frac{\partial}{\partial t}\Psi_2+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_1+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_2 \\
&= \left(i\hbar\frac{\partial}{\partial t}\Psi_1+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_1\right)+\left(i\hbar\frac{\partial}{\partial t}\Psi_2+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_2\right) \\
&= g(\Psi_1)+g(\Psi_2)
\end{aligned}$$

Which shows linearity. The general case is also linear which is easy to see because the Laplacian and multiplication by a scalar V is also linear but we were interested in a free particle in 1D so these were left out.

(c) Why is it very important for their usage in physics that these two equations are linear? 2 points

Numerous things can be mentioned here but superposition and conservation of energy are pretty key, additionally we would like to normalize the wave function so we can extract useful information about probability of state, which also requires linearity(the scaling property specifically). Managing these equations if 1+1 != 2 would make life painful.

2. (a) Determine the probability of finding a particle in the middle of an infinitely deep square well in one dimension. Use the wave function for the first exited state for this determination. 4 points

The first thing to note about this question it is a bit of a trick question because the probability of any continuous distribution at a point is zero. Starting with the normalized wave function for particle in an infinite potential well:

$$\Psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

The probability of it being somewhere inside the box is unity, but the probability of it being exactly in the center is zero:

$$P_{ab_n} = \int_a^b \Psi_n^*(x)\Psi_n(x)dx$$

Which is the probability of being between *a* and *b*. Now evaluating for the first excited state(n=2) at $L/2$:

$$\begin{aligned}
P_{\frac{L}{2}\frac{L}{2}} &= \frac{2}{L}\int_{\frac{L}{2}}^{\frac{L}{2}} \sin^2\left(\frac{2\pi x}{L}\right)dx \\
&= \frac{2}{L}\int_{\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2}\left(1-\cos\left(\frac{4\pi x}{L}\right)\right)dx \text{ , trig identity } \sin^2(\alpha x) = \frac{1}{2}\left(1-\cos(2\alpha x)\right) \\
&= \frac{1}{L}\left[x - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{\frac{L}{2}}^{\frac{L}{2}} \\
&= 0
\end{aligned}$$

Which is expected since this is an infinitely thin slice of a continuous probability and there is a null in the middle as well (zero by two accounts).

(b) Determine the expectation value of x and the spread around this expectation value for the same state as in (a). 4 points

Starting with the definition of the expectation value of a normalized wave function and plugging in our case:

$$
\begin{aligned}
\langle x \rangle_2 &= \int_{-\infty}^{\infty} \Psi_2^*(x) x \Psi_2(x) dx \\
&= \int_{-\infty}^{\infty} x \Psi_2^*(x) \Psi_2(x) dx \\
&= \frac{2}{L} \int_0^L x \sin^2\left(\frac{2\pi x}{L}\right) dx \\
&= \frac{1}{L} \int_0^L x \left(1 - \cos\left(\frac{4\pi x}{L}\right)\right) dx \\
&= \frac{1}{L} \int_0^L x dx - \frac{1}{L} \int_0^L x \cos\left(\frac{4\pi x}{L}\right) dx \\
&= \frac{1}{L}\left(\frac{L^2}{2}\right) - \frac{1}{L}\left(\left[\frac{L}{4\pi} x \sin\left(\frac{4\pi x}{L}\right)\right]_0^L - \int_0^L \frac{L}{4\pi} \sin\left(\frac{4\pi x}{L}\right) dx\right) \text{, by parts} \\
&= \frac{L}{2} - \frac{1}{L}(0 - 0) \\
&= \frac{L}{2}
\end{aligned}
$$

Which makes sense that the average location is in the middle of the box because the absolute magnitude of the first excited state is symmetric about the middle.

Now we need the expectation value of the operator squared, which has roughly the same process as the expectation value above, but with slightly more involved calculus:

$$
\begin{aligned}
\left\langle x^2 \right\rangle_2 &= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{2\pi x}{L}\right)dx \\
&= \frac{1}{L}\int_0^L x^2\left(1-\cos\left(\frac{4\pi x}{L}\right)\right)dx \\
&= \frac{1}{L}\int_0^L x^2 dx - \frac{1}{L}\int_0^L x^2 \cos\left(\frac{4\pi x}{L}\right)dx \\
&= \frac{L^2}{3} - \frac{1}{L}\left(\left[\frac{L}{4\pi}x^2 \sin\left(\frac{4\pi x}{L}\right)\right]_0^L - \int_0^L \frac{L}{2\pi} x \sin\left(\frac{4\pi x}{L}\right)dx\right) \text{, by parts} \\
&= \frac{L^2}{3} - \frac{1}{L}\left((0) - \left(\left[-\frac{L^2}{8\pi^2} x\cos\left(\frac{4\pi x}{L}\right)\right]_0^L - \int_0^L -\frac{L^2}{8\pi^2}\cos\left(\frac{4\pi x}{L}\right)dx\right)\right)\text{, by parts again} \\
&= \frac{L^2}{3} - \frac{1}{L}\left((0) - \left(\left(-\frac{L^3}{8\pi^2}\right) - (0)\right)\right) \\
&= \frac{L^2}{3} - \frac{L^2}{8\pi^2} \approx 0.3207L^2
\end{aligned}
$$

The spread is given by:

$$
\begin{aligned}
\Delta x_n &= \sqrt{\left\langle x^2 \right\rangle_n - \left\langle x \right\rangle_n^2} \\
\Delta x_2 &= \sqrt{\left\langle x^2 \right\rangle_2 - \left\langle x \right\rangle_2^2} \\
&\approx \sqrt{0.3207L^2 - \left(\frac{L}{2}\right)^2} \\
&\approx 0.2658L
\end{aligned}
$$

(c) Why is the expectation value of x fuzzy? 2 points

Technical answer: Ψ is not an eigenfunction to the operator x (we would need a second differential operator for this to be the case).

Obviously the distribution is spread across the box, only the average over many samples would put the mean in the middle of the box. This implies that the variance is not zero, which further implies the value must be fuzzy. The physical interpretation then becomes that we are uncertain about the location of the particle in the box.