Modern Physics II - HW8 Solution - Winter 2016

1. Assume that 5 identical particles have to share 7 units of energy (E) and that the lowest energy is zero. Assume further that (a) the particles are either classical, i.e. distinguishable by their paths, (b) in-distinguishable particles that are bosons, and (c) in-distinguishable particles that are fermions.

Calculate the probabilities of finding a particle with 0E, 1E, 2E, 3E, 4E, 5E, 6E, and 7E for each of the cases (a), (b), (c). Present your results in a table. (Hint: the Serway Moses Moyer book on line) [10 points]

Ok lets enumerate some states. We have 7 bits of energy share between 5 particles. Let's build a table to represent the number of ways to spread this without worrying about microstates. Stars and stripes stats doesn't work for this since our "bins" the particles don't have explicit identity yet e.g. particle 1 having 7 units of energy is the same as particle 2 having 7 units of energy, the microstates express those differences. So we just go through and carefully enumerate by using a descending cascade to prevent enumerating duplicate energy patterns. Once that is done we can calculate the microstates by using:

$$N_{MB} = \frac{N!}{n_1!n_2!n_3!...}$$

Which is the number of microstates given by the quotient of the factorial of the number of particles over the product of factorials for the number of particles in a given energy state. That probably sounds more confusing that it needs to be, example calc for the [3,2,1,1,0] column below:

$$N_{MB} = \frac{5!}{1_0!2_1!1_2!1_3!0_4!0_5!0_6!0_7!} = 60$$

With the number of particles in a particular energy level (indicated by subscript).

$N_{particle}$	**Energy, all columns add up to 7**												
1	7	6	5	5	4	4	4	3	3	3	3	2	2
2	0	1	2	1	3	2	1	3	2	2	1	2	2
3	0	0	0	1	0	1	1	1	2	1	1	2	1
4	0	0	0	0	0	0	1	0	0	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	1	0	1
Microstates	5	20	20	30	20	60	20	30	30	60	5	20	10

Noting first that we have 13 indistinguishable states (will need this for the other stats) and then summing up the microstates for these configurations we see that we have 330 total(which is the same number you get if you run a multiset calculation $(5+7-1)!/7!(5-1)! = 330$. Now we can run calculations for our particular statistics.

a) For Maxwell-Boltzmann on our distinguishable particles we take each microstate as being equiprobable so to figure out the average number of particles in a given energy state and then use that to figure out the probability of particular energy showing up in the set.

$$p(E_n) = \frac{\overline{n}_n}{N}$$

So as a demo calc lets figure out the probability of 4E showing up in the set:

$$p(E_4) = \frac{\overline{n}_4}{5} = \frac{\left[(1)\frac{20}{330}\right] + \left[(1)\frac{60}{330}\right] + \left[(1)\frac{20}{330}\right]}{5} \approx 0.0606 \approx 6.06\%$$

Making another table:

E_n	0	1	2	3	4	5	6	7
MB - $p(E_n)$, %	36.36%	25.45%	16.97%	10.61%	6.06%	3.03%	1.21%	0.30%

Which gives us the probability of occurrence of a particular energy level for this set.

b) Now for the bosons we lose our microstates since the particles are indistinguishable Bose-Einstein, so our probability calculation is a bit simpler since we only have 13 states to deal with. Similarly for 4E again we end up with:

$$p(E_4) = \frac{\overline{n}_4}{5} = \frac{\left[(1)\frac{1}{13}\right] + \left[(1)\frac{1}{13}\right] + \left[(1)\frac{1}{13}\right]}{5} \approx 0.0462 \approx 4.62\%$$

E_n	0	1	2	3	4	5	6	7
BE - $p(E_n)$, %	36.92%	27.69%	15.38%	9.23%	4.62%	3.08%	1.54%	1.54%

Which is reasonably similar to the Maxwell-Boltzmann probabilities but we notice a bit of an increase in probability on the higher energy end of things.

c) Finally we need to deal with the fermions. The key thing to keep in mind is that we can have at max two particles in the same energy state due to exclusion. Looking at our energy table we can just toss out anything that has three or more occupancies. Everything else will be legit to keep:

$N_{particle}$	Energy, all columns add up to 7												
1	7	6	5	5	4	4	4	3	3	3	3	2	2
2	0	1	2	1	3	2	1	3	2	2	1	2	2
3	0	0	0	1	0	1	1	1	2	1	1	2	1
4	0	0	0	0	0	0	1	0	0	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	1	0	1

So that leaves us with 5 states to play with for our Fermi-Dirac distribution. Doing a demo calculation with 4E yet again we have:

$$p(E_4) = \frac{\overline{n}_4}{5} = \frac{\left[(1)\frac{1}{5}\right]}{5} \approx 0.0400 \approx 4.00\%$$

E_n	0	1	2	3	4	5	6	7
FD - $p(E_n)$, %	36.00%	24.00%	16.00%	16.00%	4.00%	4.00%	0.00%	0.00%

A bit more granular obviously since we have fewer states to throw things into, but this should be expected since the exclusion principal is a fairly stringent requirement to meet.

Now just for comparison sake we can put all the tables together:

E_n	0	1	2	3	4	5	6	7
MB - $p(E_n)$, %	36.36%	25.45%	16.97%	10.61%	6.06%	3.03%	1.21%	0.30%
BE - $p(E_n)$, %	36.92%	27.69%	15.38%	9.23%	4.62%	3.08%	1.54%	1.54%
FD - $p(E_n)$, %	36.00%	24.00%	16.00%	16.00%	4.00%	4.00%	0.00%	0.00%

And now we can all "enjoy" the fruits of this moderately tedious calculation by readily being able to appreciate the variations in probability due to the differing forms of statistics.

2. Turn the Maxwell's speed distribution function into the kinetic energy distribution function. (hint: basic steps are somewhere in the lecture slides ...) [5 points]

Relatively strait forward, starting with the Maxwell speed distribution:

$$F(v)dv = 4\pi C e^{-\frac{1}{2}\beta m v^2} v^2 dv$$

We want this function in terms of energy instead of speed(not quite velocity since we are ignoring the direction). We know that:

$$E = \frac{1}{2}mv^2$$

We also know that:

$$dE = mvdv$$
$$dv = \frac{1}{mv}dE$$

We want everything in terms of energy though so let's substitute that *v*:

$$mv = \sqrt{2m\frac{1}{2}mv^2} = \sqrt{2mE}$$

Plugging in everything we have:

$$F(E)dE = 4\pi C e^{-\beta E}\left(\frac{2E}{m}\right)\frac{1}{\sqrt{2mE}}dE$$
$$= \frac{8\pi C}{\sqrt{2}m^{3/2}}e^{-\beta E}\sqrt{E}dE$$

Which is what we were looking for.