ISSP tut 1

March 10th 2025

3.2

Let $x(n)$ be a stationary random process w $\mu = 0$

Autocorr: $r_x(k)$

$y(n) = x(n) + f(n)$

$m_y(n)$, $r_y(k_1, k_2)$

$E(y(n)) = E(x(n)) + E(f(n))$ deterministic

$E[y(n)] = f(n) \rightarrow m_y(n)$

$r_y(k_1, k_2) = E(y(k_1) y(k_2))$

$= E\{[x(k_1)+f(k_1)][x(k_2)+f(k_2)]\}$

$E[x(k_1)x(k_2)] + f(k_1)f(k_2)$

$r_x(k_1, k_2) + f(k_1)f(k_2)$

3.14 (i)

Determine whether or not random process is WSS

WSS: first & second order

$\hookrightarrow$ $\mu(n)$ is constant for any n.

$\hookrightarrow$ $r_{xx}(n_1, n_2) = r_{xx}(n_1+l, n_2+l) = r_{xx}(n_1 - n_2, 0)$

a) $x(n) = A$ $\quad$ pdf$(A) = f_A(\alpha)$

$m_x(n) = E\{x(n)\} = E\{A\} \rightarrow$ constant

$r_x(k, l) = E\{x(k)x(l)\} = E\{A \cdot A\} \rightarrow$ constant

WSS ✓✓

b) $x(n) = A\cos(n\omega_0)$ $\quad$ $A \rightarrow$ Gaussian rv m_A, σ_A^2

$m_x(n) = E\{A\cos(n\omega_0)\} = m_A \cos n\omega_0$

not WSS XX

[c] $x(n) = A\cos(n\omega_0 + \phi)$ $\quad$ $\phi \sim U[-\pi, \pi]$

$m_x(n) = E\{A\cos(n\omega_0 + \phi)\}$

$A\, E\{\cos(n\omega_0 + \phi)\}$

$\frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(n\omega_0 + \phi)\, d\phi \rightarrow \underline{0}$

$r_x(k, l)$

$= E\{A\cos(k\omega_0 + \phi) A\cos(l\omega_0 + \phi)\}$

$\frac{1}{2}A^2 E\{\cos[k-l]\omega_0 + \cos([k+l]\omega_0 + 2\phi)\}$

$\frac{1}{2}A^2\cos(k-l)\omega_0 \qquad 0$

$\hookrightarrow$ fn of $k-l$.

WSS ✓✓

[d] $x(n) = A\cos(n\omega_0) + B\sin(n\omega_0)$

A, B are uncorrelated, zero mean, σ^2

$m_x(n) = E[A\cos n\omega_0] + E[B\sin n\omega_0]$

$m_x(n) = 0$

$r_x(k, l) = E[[A\cos(k\omega_0) + B\sin(k\omega_0)][A\cos(l\omega_0) + B\sin(l\omega_0)]]$

solve -

e Bernoulli process

$Pr\{x(n) = 1\} \Rightarrow p$

$Pr\{x(n) = -1\} \Rightarrow 1-p$.

$E(x(n)) = p + p - 1 = 2p-1$ ✓✓ constant

$r_x(k, l) = \begin{cases} E(x^2(k)) & k = l \\ E(x(k)x(l)) & k \neq l \end{cases}$

$\begin{cases} p + 1 - p = 1 & k = l \\ (2p-1)(2p-1) & k \neq l \end{cases}$ uncorrelated since i.i.d.

$r_x(k, l)$ ✓✓

<u>WSS</u>

f) $y(n) = x(n) - x(n-1)$

$x(n)$ is bernoulli process

linear shift invarient op of $x(n)$.

So. WSS.

[calculate the other way too].

$E[y(n)] = E[x(n) - x(n-1)]$

$2(2p-1) \Rightarrow$ ✓✓

$r_x(k, l) = E[[x(k) - x(k-1)][x(l) - x(l-1)]]$

$E[x(k)x(l)] - E[x(k-1)x(l)]$

$- E[x(k)x(l-1)] + E[x(k-1)x(l-1)]$

Calc from before :)

3.8

remember: noise is uncorrelated -

$x(n) = A\cos(n\omega_0 + \phi) + w(n)$

$w(n) \rightarrow \{0, \sigma_w^2\}$

find autocorrelation seq when.

a) A is Gaussian $\{0, \sigma_A^2\}$

$r_x(k, l) = E[(A\cos(k\omega_0 + \phi) + w(k))(A\cos(l\omega_0 + \phi) + w(l))]$

$E[A^2\cos(k\omega_0 + \phi)\cos(l\omega_0 + \phi)] + \cancel{E[w(k)]E[\text{term}]}$

$\cancel{E(w(l))E[\text{term}]} + E(w(k)w(l))$.

$\cos(k\omega_0 + \phi)\cos(l\omega_0 + \phi)\sigma_A^2 + \sigma_w^2\delta(k-l)$

3.8 b) $\phi \sim U[-\pi, \pi]$.

← from earlier →

$$r_x(k,l) = E\left[A^2 \cos(k\omega_0+\phi)\cos(l\omega_0+\phi)\right]$$
$$+ E(w(k))\cancel{E[\text{term}_1]} + E[w(l)]\cancel{E[\text{term}_2]}$$
$$+ E[w(k)w(l)]$$
$$\frac{A^2}{2}\cos((k-l)\omega_0) + \sigma_w^2\delta(k-l)$$

3.8 c) ω_0 is a random var $\omega_0-\Delta$, $\omega_0+\Delta$

$$r_x(k,l) = E\left[A^2\cos(k\omega_0+\phi)\cos(l\omega_0+\phi)\right] + \sigma_w^2\delta(k-l)$$
$$\frac{A^2}{2}\int_{\omega_0-\Delta}^{\omega_0+\Delta}\left[\cos((k-l)\omega_0) - \cos((k+l)\omega_0+2\phi)\right]d\omega_0$$
$$\left[\frac{\sin(k-l)\omega_0}{k-l}\right]_{\omega_0-\Delta}^{\omega+\Delta} - \left[\frac{\sin((k+l)\omega_0+2\phi)}{k+l}\right]_{\omega_0-\Delta}^{\omega_0+\Delta}$$

solve.

3.15

Which of the following is valid autocorr for rp

what makes a valid autocorrelation?

1) $r_x(k) = r_x(-k)$
2) PSD non negative : $P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-jk\omega}$ $\forall\ \omega$
3) "sanity check": $r_x(0) \geq |r_x(k)|$

[a] $r_x[k] = \delta(k-1) + \delta(k+1)$ ✗ $r_x[0] < r_x[l]$

[b] $r_x[k] = 8\delta(k) + 2\delta(k-1) + 2\delta(k+1)$ ✗ PSD must be non neg

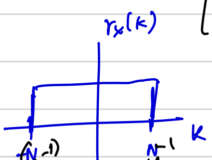
[c] $r_x(k) = \exp\left(\frac{jk\pi}{4}\right) \longrightarrow \sin\frac{k\pi}{4} - j\cos\left(\frac{k\pi}{4}\right)$

✓✓

1) $r_x(k) = r_x(-k)$

2) PSD: $\sum_{k=-\infty}^{\infty} e^{j(\frac{k\pi}{4}-k\omega)}$ [always non neg].

d] $r_x(k) = \begin{cases} 1 ; & |k| < N \\ 0 ; & \text{else}\end{cases}$



1) $r_x(k) = r_x(-k)$ ✓

2) $PSD = \sum_{-(N-1)}^{(N-1)} 1 \cdot e^{-jk\omega}$

let $M = N-1$

$$PSD = \sum_{k=-M}^{M} e^{-jk\omega} = e^{jM\omega}\sum_{m=0}^{2M} e^{-jm\omega}$$
$$PSD = e^{jM\omega}\left[\frac{1-e^{-j(2M-1)\omega}}{1-e^{-j\omega}}\right]$$
$$PSD = e^{jM\omega}\,\frac{2j e^{-j(M+\frac{1}{2})\omega}\sin\left((M+\frac{1}{2})\omega\right)}{2j e^{-j\frac{\omega}{2}}\sin\left(\frac{\omega}{2}\right)}$$ ✗.
$$PSD = \frac{\sin\left((M+\frac{1}{2})\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \Rightarrow \frac{\sin\left((N-\frac{1}{2})\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$ not non neg everywhere

[e] $r_x(k) = \begin{cases} \frac{N-|k|}{N} & |k| < N \\ 0 & \text{else}\end{cases}$

1] $r_x(k) = r_x(-k)$ ✓

2) $PSD: \sum_{k=-\infty}^{\infty} r_x(k)e^{-j\omega w}$

$$PSD = \sum_{k=-(N-1)}^{(N-1)} e^{-jk\omega} - \sum_{0}^{N-1}\frac{k}{N}e^{-j\omega k} + \sum_{-(N-1)}^{0}\frac{k}{N}e^{-j\omega k}$$
$$\downarrow$$
$$\frac{\sin\left(N\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

way too long of a calc.
they dont know PSD yet.

[f] $r_x(k) = 2^{-k^2}$

$r_x(k) = r_x(-k)$ ✓✓

$$\sum_{k=-\infty}^{\infty} 2^{-k^2}e^{-j\omega_0 k}$$
$$\sum_{k=-\infty}^{\infty} e^{(\ln 2)-k^2}e^{-j\omega_0 k}$$

Again, long of a calc, they dont know PSD yet.

3.25

$x(n) \rightarrow$ WSS

one value $x(n_0)$ is missing

$$x = [x(0), x(1), \cdots x(n_0-1), x(n_0+1) \ldots x(N)]^T$$

a] $R_x = E\{xx^H\}$

which is true?

1) R_x is toeplitz. → toeplitz: Diagonal constant matrix
2) R_x is Hermitian → $A = A^H$ square matrix = its conjugate transpose
3) R_x is positive semidefinite → non negative eigenvalues.

$$xx^H = \begin{bmatrix} x[0]x^*[0] & x[0]x^*[1] & \cdots & x[0]x^*[N] \\ x[1]x^*[0] & - & \cdots & x[1]x^*[N] \\ x[2]x^*[0] & - & \cdots & \vdots \\ \vdots & & & \vdots \\ x[n_0-1]x^*[0] & - & \cdots & \vdots \\ x[n_0+1]x^*[0] & & & \vdots \\ \vdots & & & \vdots \\ x[N]x^*[0] & - & \cdots & x[N]x^*[N] \end{bmatrix}$$

$$E[xx^H] = \begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-N) \\ r_x(1) & - & \cdots & \\ \vdots & & \ddots & \\ r_x(n_0-1) & - & \cdots & \\ r_x(n_0+1) & - & \cdots & \\ r_x(N) & r_x(N-1) & \cdots & r_x(0) \end{bmatrix}$$

we know that

$$r_x[k] = r_x^*[-k]$$

$$r_x[-k] = r_x^*[k]$$

$$R_x = \begin{bmatrix} r_x[0] & r_x^*[1] & \cdots & r_x^*[N] \\ \vdots & r_x[0] & \cdots & \vdots \\ r_x(n_0-1) & r_x(n_0-2) & \cdots & \vdots \\ r_x(n_0+1) & r_x(n_0) & \cdots & \vdots \\ \vdots & & & \\ r_x(N) & \cdots & \cdots & r_x[0] \end{bmatrix}$$

$\Rightarrow$ All diagonals are not constant. Hence not Toeplitz.

$\Rightarrow$ Hermitian means $R_x^H = R_x$.

$$R_x^H \begin{bmatrix} r_x[0] & r_x^*[1] & \cdots & r_x^*[N] \\ r_x[1] & r_x(0) & \cdots & \\ \vdots & \vdots & & \\ r_x(n_0-1) & r_x(n_0-2) & \cdots & \\ r_x(n_0+1) & r_x(n_0) & \cdots & \\ \vdots & \vdots & & \\ r_x(N) & & & r_x[0] \end{bmatrix}$$

$R_x^H = R_x$

hence, the matrix is hermitian

$\Rightarrow$ Positive semi definite means:

$a^H R_x a$ for any vector $a \in \mathbb{C}^{N+1}$ $a \neq 0$.
if $a^H R_x a \geq 0$.

$a^H \mathbb{E}[xx^H] a^H$

$\mathbb{E}[(a^H x) x^H a^H] = \mathbb{E}[(a^H x)(a^H x)^*]$

$\mathbb{E}[|a^H x|^2]$ always non negative.

So, it is positive semi definite.

[b] Our process is a WSS. hence, our autocorrelation matrix without missing data is supposed to be toeplitz. We can use this to construct our missing data.

Our process is stationary, so the expected value of x' (including data at n_0) will not change from the the expected value of x [not including n_0 values].

Now for checking $R_{x'}$

$$x' = \begin{cases} x[t] & t \neq n_0 \\ x[n_0] & t = n_0 \end{cases}$$

$$R_{x'} = \begin{bmatrix} \mathbb{E}[x[0]x^*[0]] & \cdots & \mathbb{E}[x[0]x^*[N]] \\ \mathbb{E}[x[1]x^*[0]] & \cdots & \mathbb{E}[x[1]x^*[N]] \\ \vdots & & \vdots \\ \mathbb{E}[x[n_0-1]x^*[0]] & & \mathbb{E}[x[n_0-1]x^*[N]] \\ \mathbb{E}[x[n_0]x^*[0]] & & \mathbb{E}[x[n_0]x^*[N]] \\ \mathbb{E}[x[n_0+1]x^*[0]] & & \mathbb{E}[x[n_0+1]x^*[N]] \\ \vdots & & \vdots \\ \mathbb{E}[x[N]x^*[0]] & & \mathbb{E}[x[N]x^*[N]] \end{bmatrix}$$

$$R_{x'} = \begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-N) \\ \vdots & & & \vdots \\ r_x(n_0-1) & & & \\ r_x(n_0) & & & \\ r_x[n_0+1] & r_x(n_0) & \cdots & \\ \vdots & & & \\ r_x(N) & & \cdots & r_x(0) \end{bmatrix}$$

from R_x and $R_{x'}$ we can see that all extra autocorrelation values ~~from~~ in $R_{x'}$ are present in R_x.

Hence, if we keep a memoization mechanism to store our $r_x(k)$ values, we can retrieve $R_{x'}$ from it.

Also, through interpolation and autoregressive methods on $x[n]$, we can get $x'[n]$.

(Again, this is because autocorrelation matrix stores all time difference correlation b/w 0 to N).

Coding Questions.

C.3.3

[a] we segment our 1000 white noise into 10 different sequences. The autocorrelations are calculated for each one of these segments and averaged out. This brings the estimate closer to true autocorrelation as compared to [b]; but there are still variations

Calculation for tutorial Demo

$x[n] = A\sin(w_0 n + \phi)$ $\quad \phi \sim U[-\pi, \pi]$

$E[x[n]] = A\int_{-\pi}^{\pi} \sin(w_0 n + \phi)\, d\phi \cdot \frac{1}{2\pi}$

$- [\cos(w_0 n + \phi)]_{-\pi}^{\pi} \cdot \frac{A}{2\pi}$

$\underline{0}$

$r_x(k,l) = E[A^2 \sin(w_0 k + \phi)\sin(w_0 l + \phi)]$

$- \frac{A^2}{2} E[\cos(w_0(k+l) + 2\phi) - \cos(w_0(k-l))]$

from earlier calc here

$-\frac{A^2}{2} \cdot (-\cos w_0(k-l)) = \frac{A^2}{2}\cos(w_0(k-l))$

THATS IT :)