ISSP tut

21st March

Spring 25

Concepts

↳ AR, MA, ARMA

↳ PSD, LTI

Some notes ....

While simulating:

ERGODICITY

we typically assume ergodicity!

Clearly, in most applications, determining whether or not a given process is ergodic is not practical. Therefore, whenever the solution to a problem requires knowledge of the mean, the autocorrelation, or some other ensemble average, the process is typically assumed to be ergodic and time averages are used to estimate these ensemble averages. Whether or not this assumption is appropriate will be determined by the performance of the algorithm that uses these estimates.

[3.3.6 Monson H Hayes]

White noise:

$$c_v(k) = \sigma^2 \delta(k)$$

Power spectrum

→ Random processes are ensembles of discrete time signals. cannot directly compute fourier transform.

→ Autocorr: time domain description of the 2nd order moment of a process/sequence.

→ $r_x(k)$ is deterministic

→ Fourier of $r_x(k)$ ⇒ ?

Wiener Khinchin thrm.

$$\boxed{P_x(e^{jw}) = \sum_{k=-\infty}^{\infty} r_x(k) e^{-jkw}}$$

: Power spectral density.

$$r_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{jw}) e^{jwk}\, dw$$

→ frequency domain description of 2nd order moment

$$P_x(z) = \sum_{k=-\infty}^{\infty} r_x(k) z^{-k}$$

autocorr is conjugate symmetric

Property 1—Symmetry. The power spectrum of a WSS random process $x(n)$ is real-valued, $P_x(e^{j\omega}) = P_x^*(e^{j\omega})$, and $P_x(z)$ satisfies the symmetry condition

$$P_x(z) = P_x^*(1/z^*)$$

In addition, if $x(n)$ is real then the power spectrum is *even*, $P_x(e^{j\omega}) = P_x(e^{-j\omega})$, which implies that

$$P_x(z) = P_x^*(z^*)$$

In addition to being real-valued, the power spectrum is nonnegative. Specifically,

Property 2—Positivity. The power spectrum of a WSS random process is nonnegative

$$P_x(e^{j\omega}) \geq 0$$

Property 3—Total power. The power in a zero mean WSS random process is proportional to the area under the power spectral density curve

$$E\{|x(n)|^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\omega})d\omega$$

This property follows from Eq. (3.70) with $k = 0$ and the fact that $r_x(0) = E\{|x(n)|^2\}$.

Intuition for PSD:

Autocorr: "how similar a signal is to the delayed version of itself" ⇒ captures patterns & structures in a rp.

Take its fourier transform? ⇒ "how much of each frequency contributes to this deterministic structure?"

↳ how does power distribute across frequencies?

Example 3.3.7 *The Power Spectrum*

The autocorrelation sequence of a zero mean white noise process is $r_v(k) = \sigma_v^2\delta(k)$ where σ_v^2 is the variance of the process. The power spectrum, therefore, is equal to a constant

$$P_v(e^{j\omega}) = \sigma_v^2$$

The random phase sinusoid, on the other hand, has an autocorrelation sequence that is sinusoidal,

$$r_x(k) = \frac{1}{2}A^2\cos(k\omega_0)$$

and has a power spectrum given by

$$P_x(e^{j\omega}) = \frac{1}{2}\pi A^2[u_0(\omega-\omega_0) + u_0(\omega+\omega_0)]$$

As a final example, consider the autocorrelation sequence

$$r_x(k) = \alpha^{|k|}$$

where $|\alpha| < 1$, which, as we will see in Section 3.6.2, corresponds to a first-order autoregressive process. The power spectrum is

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-jk\omega} = \sum_{k=0}^{\infty}\alpha^k e^{-jk\omega} + \sum_{k=0}^{\infty}\alpha^k e^{jk\omega} - 1$$

$$= \frac{1}{1-\alpha e^{-j\omega}} + \frac{1}{1-\alpha e^{j\omega}} - 1 = \frac{1-\alpha^2}{1-2\alpha\cos\omega+\alpha^2}$$

which is clearly real, even, and nonnegative.

↑ Good idea to take note of these ↑

LTI systems: some equations

[check notes / tb for proofs]

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) \qquad y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$\boxed{E\{y(n)\} = m_x H(e^{j0})}$$

$$\boxed{r_{yx}(k) = r_x(k) * h(k)}$$

$$r_y(k) = r_{yx}(k) * h^*(-k) = r_x(k) * h(k) * h^*(-k)$$

$$r_x(k) \rightarrow \boxed{h(k)} \rightarrow r_{yx}(k) \rightarrow \boxed{h^*(-k)} \rightarrow r_y(k)$$

$$\sigma_y^2 = r_y(0) = r_y(0) * h(0) * h^*(0)$$

$$P_y(e^{jw}) = \sum_{k=-\infty}^{\infty} r_x(k) * h(k) * h^*(k)\, e^{-jkw}$$

$$\boxed{P_y(e^{jw}) = P_x(e^{jw})|H(e^{jw})|^2}$$

Checkout pg 103 [figure 3.8 = eq 3.96] || Examples.

$$\text{WSS} \rightarrow \boxed{\text{LTI}} \rightarrow \text{WSS ??}$$

[generating specific structure in a random process]

$h[n]$

↓ $x[n]$

$r_x[k] = \sigma_x^2\delta[k]$

PSD ⇒ σ_x^2

↓ $y[n]$

$$|P_y(e^{jw}) = \sigma_x^2 |H(e^{jw})|^2$$

$$|y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \qquad | r_{yx}[k] = \sigma_x^2\delta[k]*h[k] = \sigma_x^2 h[k]$$

$$\boxed{\mu_y = \mu_x \sum_{m=-\infty}^{\infty} h[m]} \qquad | r_y[k] = (r_x * h * h^*)[k]$$

$$\sigma_x^2\delta[k] * (h * h^*)[k]$$

$$\sigma_x^2 (h * h^*)[k]$$

Moving Average process

$v(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$

$$y[n] = \sum_{l=0}^{q} h[k]\, v[n-k]$$

$$y[n] = h[0]v[n] + h[1]v[n-1] \cdots h[q]v[n-q]$$

window length $\rightarrow q+1$. If we use specific notation.

$$y[n] = b_0 v[n] + b_1 v[n-1] \cdots b_q v[n-q]$$

$$\frac{Y(z)}{V(z)} = b_0 + b_1 z^{-1} \cdots b_q z^{-q}$$

$$\frac{V(z)}{V(z)} = \sum_{i=0}^{q} b_i z^{-i} = B_q(z)$$

$$PSD = \sigma_w^2 \left| B_q(z=e^{jw}) \right|^2$$

Autoregressive process [IIR filter]

$$y[n] + a_1 y[n-1] + a_2 y[n-2] \cdots a_p y[n-p] = v[n]$$

output of previous time steps are reused ----

$$Y(z)\left[1 + a_1 z^{-1} \cdots a_p z^{-p}\right] = V(z)$$

$$\frac{Y(z)}{V(z)} = \frac{1}{\left[1 + a_1 z^{-1} \cdots a_p z^{-p}\right]} = \frac{1}{A_p(z)}$$

$$S_y(w) = \sigma_w^2 |H(w)|^2 = \frac{\sigma_w^2}{|A_p(e^{jw})|^2}$$

ARMA

$$y[n] + \sum_{i=1}^{p} a_i\, y[n-i] = \sum_{i=0}^{q} b_i\, v[n-i]$$

$\leftarrow$ AR $\rightarrow$ $\quad$ $\leftarrow$ MA $\rightarrow$

$$\frac{Y(z)}{V(z)} = H(z) = \frac{B_q(z)}{A_p(z)}$$

$$S_y(w) = \left| \frac{B_q(z=e^{jw})}{A_p(z=e^{jw})} \right|^2 \sigma_w^2$$

MA rp is always WSS $\Rightarrow$ FIR filter is stable

AR rp may not be WSS $\Rightarrow$ IIR filter not necessarily stable

$\Rightarrow$ AR parameter estimation

$$x[n] + a_1 x[n-1] \cdots a_p x[n-p] = v[n]$$

multiply by $x^*[n-l]$; take expectation

$$E\left[x[n]x^*[n-l] + a_1 x[n-1]x^*[n-l] \cdots a_p x[n-p]x^*[n-l]\right]$$
$$= E\left[v[n]x^*[n-l]\right]$$

$$r_x[l] + a_1 r_x[l-1] \cdots a_p r_x[l-p] = E[v[n]x^*[n-l]]$$

AR is a causal system. $x[n]$ depends on $v[n]$ & $x(n-1)$

$x[n-1]$ depends on $v[n-1]$ ... so on.

$$E\left[v[n]x^*[n-l]\right] = \begin{cases} \sigma_v^2 & l=0 \rightarrow \text{correlated} \\ 0 & l=1 \cdots p \end{cases}$$

$x^*[n-l]$ would have not seen $v[n]$; which is in the future [causal system]

$$r_x[l] + \sum_{i=1}^{p} a_i r_x[l-i] = \begin{cases} \sigma_v^2 & l=0 \\ 0 & l=1 \cdots p \end{cases}$$

$$r_x[l] + \sum_{i=1}^{p} a_i r_x[l-i] = \sigma_v^2 \delta[l]$$

you can see the ARMA version of this in the textbook [page 109. -111]

further for AR, MA processes....

how do we calculate the coeffs?

Example:-

$$x[n] + a_1 x[n-1] + a_2 x[n-2] = v[n]$$

Given : $\{ r_x(0), r_x[1], r_x[2] \}$

find : $\{ a_1, a_2, \sigma_v^2 \}$ $\quad$ Intuition: use the autocorr's w/ diff lags!

Step 1: multiply the eqn w/ $x^*[n]$

$$r_x(0) + a_1 r_x(-1) + a_2 r_x(-2) = E[v[n]x^*(n)] = \sigma_v^2$$

$$\boxed{r_x(0) + a_1 r_x^*(1) + a_2 r_x^*(2) = \sigma_v^2}$$

Step 2: multiply w/ $x^*[n-1]$ $\quad$ $E[v[n]x^*[n-1]] = 0$

$$\boxed{r_x(1) + a_1 r_x(0) + a_2 r_x^*(1) = 0}$$

Step 3 $\rightarrow$ $x^*[n-2]$ $\quad$ taking these two equations..

$$\boxed{r_x(2) + a_1 r_x(1) + a_2 r_x(0) = 0}$$

$$\begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

Can calculate a_1, a_2 from here.

Yule walker equation:-

Generalized form

$$x(n) + \sum_{l=1}^{p} a_p(l)\, x(n-l) = \sum_{l=0}^{q} b_q(l)\, v(n-l)$$

multiplying $x^*(n-k)$ taking expectation.

$v(n) \rightarrow \boxed{h(n)} \rightarrow x(n)$

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \sum_{l=0}^{q} b_q(l)\, r_{vx}(k-l)$$

again --- check textbook :)

Problems : 3.3, 3.11, C 3.3, 3.5, 3.6 (a,b)

3.3

$$x(n) = \sum_{k=1}^{p} a(k)\, x(n-k) + w(n) \rightarrow \sigma_w^2$$

$$z(n) = x(n) + v(n) \rightarrow \sigma_v^2$$

$$\boxed{r_{vw}(k) = 0}$$

a] find PSD of $x(n)$

b] find PSD of $z(n)$

a] $$x(n) - \sum_{k=1}^{p} a(k)\, x(n-k) = w(n)$$

a way of looking at this is ... taking z transform of original signal

$$X(z) - a(1)X(z)z^{-1} - a(2)X(z)z^{-2} ---- - a(p)X(z)z^{-p} = W(z)$$

$$\frac{X(z)}{W(z)} = H(z) = \frac{1}{1-\sum_{k=1}^{p} a(k) z^{-k}}$$

PSD of $x(n)$ = $PSD_w \cdot |H(z=e^{jw})|^2$

$$\boxed{PSD_x = \frac{\sigma_w^2}{\left|1 - \sum_{k=1}^{p} a(k) e^{-jwk}\right|^2}}$$

b] $z(n) = x(n) + v(n)$

$r_{vw}(k) = 0.$

$r_{xv}(k) = E[x(n)\, v^*(n-k)]$

$$r_z(k) = E[x(n)x^*(n-k)] + \underbrace{E[x(n)v^*(n-k)]}_{r_{xv}(k)} + \underbrace{E[v(n)x^*(n-k)]}_{r_{xv}(-k)} + E[v(n)v^*(n-k)]$$

detour

Calculating $r_{xv}(k)$ ---

let $x(n) = \sum_{m=-\infty}^{\infty} h(m)\, w(n-m)$

where $h(m)$ is the unit impulse of AR process.

$$E[x(n)v^*(n-k)] = E\left[\sum_{m=-\infty}^{\infty} h(m)\, w(n-m)\, v^*(n-k)\right]$$

$$= \sum_{m=-\infty}^{\infty} h(m)\, \underbrace{E[w(n-m)v^*(n-k)]}_{\text{uncorrelated} = 0}$$

$$E[x(n)v^*(n-k)] = \sum_{m=-\infty}^{\infty} h(m).0 = \underline{\underline{0}}$$

$r_{xv}(k) = r_{xv}(-k) = \underline{\underline{0}}$

$\boxed{\text{So}}$

$r_z(k) = r_x(k) + r_v(k)$

PSD ---

$$P_z(e^{jw}) = \underbrace{P_x(e^{jw})}_{PSD_x} + \underbrace{P_v(e^{jw})}_{\sigma_v^2}$$

$$\boxed{P_z(e^{jw}) = \frac{\sigma_w^2}{|A(e^{jw})|^2} + \sigma_v^2}$$

3.5

Find the power spectrum for each of the following

$r_x(k) = 2\delta(k) + j\delta(k-1) - j\delta(k+1)$

z transform $z = e^{jw}$

$$S_x(z=e^{jw}) = 2 + je^{-jw} - je^{jw} = 2 + 2\sin w$$

[real, ≥ 0]

b] $r_x(k) = \delta(k) + 2(0.5)^{|k|}$

look at example in first page.

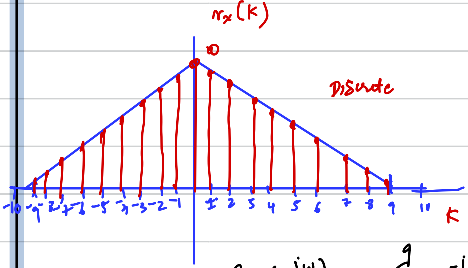
[c] $r_x(k) = 2\delta(k) + \cos\left(\frac{\pi k}{4}\right) = 2\delta(k) + \frac{e^{\frac{\pi k}{4}j}}{2} + \frac{e^{-\frac{\pi k}{4}j}}{2}$

looking at properties of DTFT

$$e^{jn w'} \longleftrightarrow 2\pi\delta(w-w')$$

$$P_x(e^{jw}) = 2 + \pi\delta\left(w - \frac{k}{4}\right) + \pi\delta\left(w + \frac{k}{4}\right)$$

d] $$r_x(k) = \begin{cases} 10-|k| & |k|<10 \\ 0 & \text{otherwise} \end{cases}$$



this can be written as a convolution of 2 rect.

$r_x(k) = \square(k) * \square(-k)$

$$\square(k) = \begin{cases} 1 & 0\leq k \leq 9 \\ 0 & \text{else} \end{cases}$$

$$S_\square(e^{jw}) = \sum_{k=0}^{9} e^{-jkw} = \frac{1-e^{-j10w}}{1-e^{-jw}}$$

$$\frac{e^{-j5w}\left[e^{j5w} - e^{-j5w}\right]}{e^{-j\frac{w}{2}}\left[e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right]}$$

$$= \underbrace{e^{-j\frac{9w}{2}}}_{\text{Angle}} \cdot \underbrace{\frac{\sin 5w}{\sin w/2}}_{\text{Amp}}$$

$$P_x(e^{jw}) = |S_\square(e^{jw})|^2 = \frac{\sin^2 5w}{\sin^2 w/2}$$

3.6 (a,b)

Autocorr corresponding to the sequences

a) $P_x(e^{jw}) = 3 + 2\cos w$

b) $P_x(e^{jw}) = \frac{1}{5+3\cos w}$

a] $P_x(e^{jw}) = 3 + e^{jw} + e^{-jw}$

$r_x(0) = 3,\ r_x(1) = 1,\ r_x(-1) = 1.$

b) $$P_x(e^{jw}) = \frac{1}{5+\frac{3}{2}(e^{jw}+e^{-jw})}$$

$$P_x(e^{jw}) = \frac{2}{10+3(e^{jw}+e^{-jw})}$$

Converting it to z form

$$P_x(z) = \frac{2}{10+3z+3z^{-1}}$$

$$P_x(z) = \frac{2z}{3z^2+10z+3}$$

partial fraction expansion

$g(k) = h(-k)$ $\quad a(n) * b(n)$

$a(m) b(z-m)$

$$\frac{2z}{(3z+1)(z+3)} = \frac{A}{(3z+1)} + \frac{B}{(z+3)}$$

$$2z = A(z+3) + B(3z+1)$$

$A + 3B = 2$ $\quad 3B - \frac{B}{3} = 2$

$3A + B = 0$ $\quad \frac{8B}{3} = 2$

$A = -B/3$ $\quad B = \frac{3}{4}, \; A = -1/4$

we know that $\frac{1}{z-a} \xleftrightarrow{z^{-1}} a^n u(n)$

$$\frac{-1}{4} \times \frac{1}{3}\left(\frac{1}{z+1/3}\right) = \left(\frac{-1}{3}\right)^n u(n) \cdot \frac{-1}{12}$$

$$\frac{3}{4} \cdot \frac{1}{z+3} = (-3)^n u(n) \cdot \frac{3}{4}$$

$$x_x[n] = \frac{-1}{12}\left(\frac{-1}{3}\right)^n u[n] + \frac{3}{4}(-3)^n u[n]$$

3.11

Consider 1st order AR process..

$y(n) = a y(n-1) + w(n)$

where $|\alpha| < 1$, $w(n) \sim \mathcal{N}(0, \sigma_w^2)$

a] Unit sample response of filter that generate $y(n)$ from $w(n)$

b] find autocorr of $y[n]$

c] find power spectrum of $y[n]$

a] take z transform of the signals

$Y(z) = a Y(z) z^{-1} + W(z)$

$$\frac{Y(z)}{W(z)} = \frac{1}{1 - a z^{-1}}$$

taking z inverse

$h[n] = a^n u[n]$

b] $y(n) - a y(n-1) = w(n)$

multiply by $y^*(n-l)$ take exp

$E[y(n) y^*(n-l)] - a E[y(n-1) y^*(n-l)] = E[w(n) y^*(n-l)]$

$r_y(l) - a r_y(l-1) = E[w(n) y^*(n-l)]$

$\sigma_w^2 \; l=0$, else 0

[causality]

$$r_y(l) = \begin{cases} a r_y(l-1) + \sigma_w^2 & \text{when } l=0 \\ a r_y(l-1) & \text{otherwise} \end{cases}$$

hmm... doesnt answer our question..

$y(n) = h(n) * w(n)$

$h(n) = a^n u[n] \rightarrow \square$

$r_y(k) = r_x(k) * h(k) * h^*(-k)$

$\sigma_w^2 \cdot \Delta = \sigma_w^2 \cdot \left(a^n u[n] * a^{-n} u[-n]\right)$

$$\sigma_w^2 \sum_{m=-\infty}^{\infty} h(m)\, g(k-m)$$

$$\sigma_w^2 \sum_{m=-\infty}^{\infty} h(m)\, h(-(k-m))$$

$$\sigma_w^2 \sum_{m=-\infty}^{\infty} a^m a^{m-k} u[m]\, u[-(k-m)]$$

$$\sigma_w^2 a^{-k} \sum_{m=-\infty}^{\infty} a^{2m} u[m]\, u[-(k-m)]$$

$\leftarrow m \geq 0 \rightarrow \leftarrow m \geq k \rightarrow$

$-(k-m) \geq 0$

$k - m \leq 0$

$-m \leq -k$

$m \geq k$

m

$m - k \geq 0$

$m \geq k$

$$\sigma_w^2 a^{-k} \sum_{K=0}^{\infty} a^{2m}$$

$$\boxed{r_y(k) = \sigma_w^2 \frac{1}{1-a^2} \alpha^{-k}}$$

[verify this...]

c] power spectrum

$$S_y(w) = S_x(w) \left| H(z = e^{jw}) \right|^2$$

$$\sigma_w^2 \left| \frac{1}{1 - a e^{-jw}} \right|^2$$

$$= \frac{\sigma_w^2}{1 + a^2 - 2a\cos w}$$

[calculate this on your own!]

C.3.3

check file.