IIIT Hyderabad

Signal Detection and Estimation Theory

TDOA Based Direct Positioning MLE and the CRLB

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Abstract

This report investigates the localization of a stationary emitter using spatially separated passive sensors. While the conventional two-step approach—estimating Time Differences of Arrival (TDOAs) followed by hyperbola intersection—is computationally simple, it is known to be suboptimal as it processes sensor pairs independently and estimates location indirectly. This study focuses on the Direct Position Determination (DPD) method, specifically the Maximum Likelihood Estimator (MLE), which estimates location directly from raw signal data. We reproduce the results of Vankayalapati et al., demonstrating that the MLE significantly outperforms the conventional method, particularly in low Signal-to-Noise Ratio (SNR) regimes. Furthermore, we extend the original 2D framework to a 3-dimensional space. This extension involves modifying the Jacobian matrix to include derivatives with respect to the z-coordinate. Simulation results confirm the validity of the 3D extension with error margins within $\pm 1\%$ at ASNR = -20 dB .

1 Introduction

Passive localization is a critical technology in surveillance, navigation, and sensor networks. The objective is to determine the coordinates of a stationary emitter using signals intercepted by a network of synchronized receivers.

1.1 The Conventional Approach (Two-Step)

Historically, localization has been treated as a two-step problem:

- Estimation: Compute the Time Differences of Arrival (TDOAs) between pairs of sensors (typically using cross-correlation).
- Localization: Use these TDOA estimates to define hyperbolas (in 2D) or hyperboloids (in 3D) and find their intersection point.

Limitations: This method is suboptimal because it ignores the constraint that all TDOA measurements must originate from a single, consistent source location. By processing pairs independently, information is lost, leading to poor performance at low SNRs (the "threshold effect").

1.2 The Proposed Solution: Direct Position Determination (DPD)

The Direct Position Determination (DPD) approach bypasses the intermediate TDOA estimation. Instead, it formulates a cost function based directly on the raw received signals. By solving for the position that best explains the observed data in a single step, DPD correctly handles nuisance parameters and approaches the theoretical performance limit defined by the Cramer-Rao Lower Bound (CRLB).

2 Problem Formulation

We consider a stationary emitter at an unknown location p transmitting a signal $s(t)$ over an observation interval $(0, T)$. The signal is intercepted by M spatially separated sensors.

2.1 Received Signal Model

The signal received at the i-th sensor is modeled as:

$$r_i(t) = A_i s(t - \tau_i) + w_i(t), \quad i = 0, 1, \ldots, M - 1 \tag{1}$$

Where:
A_i: Unknown attenuation factor (channel gain).
τ_i: Unknown Time of Arrival (TOA).
$w_i(t)$: Zero-mean Additive White Gaussian Noise (AWGN) with spectral density $N_0/2$.
The relationship between the TOA and the unknown position $p = [x_T, y_T, z_T]^T$ is given by:

$$\tau_i = \frac{\|p - p_i\|}{c} + t_0 \tag{2}$$

where p_i is the sensor location (km), c is the propagation speed (km/s), and t_0 is the unknown transmission time(s).

2.2 Signal Approximation

To make the Maximum Likelihood problem solvable, the continuous signal $s(t)$ is approximated using a Fourier Series with $2N - 1$ coefficients. This transforms the infinite-dimensional waveform estimation problem into a finite parameter estimation problem:

$$s(t) \approx h^T(t)\phi \tag{3}$$

where ϕ represents the vector of Fourier coefficients and $h(t)$ is the vector of trigonometric basis functions.

Rewriting the model in (1) as:

$$r_i(t) = A_i \mathbf{h}^T(t - \tau_i)\boldsymbol{\phi} + w_i(t), \quad 0 \le t \le T,\ i = 0, 1, ..., M - 1. \tag{4}$$

3 Case 1: Signal Unknown + Transmission Time Unknown

This section details the primary challenge addressed in the paper: estimating location when neither the signal waveform nor the transmission start time is known.

3.1 Singular FIM and Ambiguity

When both the signal waveform $s(t)$ and the transmission time t_0 are unknown, the Fisher Information Matrix (FIM) $\mathcal{I}_\theta$ becomes singular (rank deficient by at least two). This singularity arises because the unknowns cannot be uniquely determined due to inherent relationships:

- **Amplitude Ambiguity**: The observations $r_i(t)$ generated by parameters $(\bar{A}_i, \bar{s}(t))$ are identical to those generated by $(\bar{A}_i/\gamma, \gamma\bar{s}(t))$ for any non-zero constant γ.
- **Time Ambiguity**: The transmitted signal $s_T(t)$ generated by transmission time $\bar{t}_0$ and waveform $\bar{s}(t)$ is indistinguishable from one generated by $(\bar{t}_0 - \gamma)$ and waveform $\bar{s}(t-\gamma)$ for any constant γ.

3.2 Transformation: Eliminating Ambiguity

To remove the singularity of the FIM, we transform the parameters into a set of identifiable quantities, using sensor $i = 0$ as the reference.

Transformed Parameters θ': The unknown transmission time t_0 is eliminated and is no longer a nuisance parameter.

$$\tau' = [\tau_1 - \tau_0, \tau_2 - \tau_0, \dots, \tau_{M-1} - \tau_0]^T \tag{5}$$

$$A' = (1/A_0)[A_1, A_2, \dots, A_{M-1}]^T \tag{6}$$

The vector of interest becomes $\alpha' = [\eta'^T, A'^T, \phi'^T]^T$, where $\eta' = [x_T, y_T]^T$ is the location vector.

3.3 CRLB Derivation Procedure

The Cramer-Rao Lower Bound (CRLB) is derived using the transformed parameters:

Joint Log-Likelihood (LL) Function: The cost function is defined relative to the reference sensor:

$$l(\theta') = -\frac{1}{N_0}\int_0^T (r_0(t) - h^T(t)\phi')^2 dt - \frac{1}{N_0}\int_0^T \sum_{i=1}^{M-1} (r_i(t) - A_i' h^T(t-\tau_i')\phi')^2 dt \tag{7}$$

Derive FIM $\mathcal{I}_{\theta'}$: The elements of the FIM are computed as the expectation of the negative second derivative of the log-likelihood:

$$[\mathcal{I}_{\theta'}]_{jj} = -E\left[\frac{\partial^2 l(\theta')}{\partial\theta_i'\partial\theta_j'}\right] \tag{8}$$

Transform FIM to $\mathcal{I}_{\alpha'}$: The final FIM for the location parameters is calculated using the Jacobian matrix:

$$\mathcal{I}_{\alpha'} = \left(\frac{\partial\theta'}{\partial\alpha'^T}\right)^T \mathcal{I}_{\theta'} \left(\frac{\partial\theta'}{\partial\alpha'^T}\right)$$

3.4 Maximum Likelihood Estimator (MLE)

The MLE solution involves correlating the received signals with basis vectors aligned by hypothesized TDOAs.

Correlation Vector y'_i: For each sensor i, we define a correlation vector:

$$y'_i = \int_0^T r_i(t)h(t - \tau'_i)dt$$

Cross-Correlation Matrix B': These vectors are concatenated into a matrix $Y' = [y'_0, y'_1, \ldots, y'_{M-1}]$, and the cross-correlation matrix is calculated as:

$$B' = Y'Y'^T$$

The MLE Solution: The maximum likelihood estimate of the location $(\hat{x}_T, \hat{y}_T)$ is found by maximizing the maximum eigenvalue of B' over all possible locations:

$$(\hat{x}_T, \hat{y}_T) = \arg \max_{(x_T, y_T)} \lambda_{\max}(B') \tag{9}$$

4 Case 2: Signal Known + Transmission Time Unknown

- Unknowns Reduced : In this case, the number of unknowns is significantly reduced to $2M$. The unknown parameter vector is simply $\zeta = [\tau^T, A^T]^T$. Since the signal parameters ϕ are known, they do not need to be estimated.

- Non-Singular FIM : Because the signal waveform is known, the unknown parameters can be uniquely determined without ambiguity. Therefore, the FIM $\mathcal{I}_\zeta$ is not singular, and no parameter transformation is required.

4.1 MLE Simplification

The MLE for this case simplifies to finding the location and transmission time that maximize the sum of squared correlation values across all sensors:

$$(\hat{x}_T, \hat{y}_T, \hat{t}_0) = \arg \max_{(x_T, y_T, t_0)} \sum_{i=0}^{M-1} \left(\int_0^T r_i(t)s(t - \tau_i)dt \right)^2 \tag{10}$$

5 Simulation Results on MATLAB

Code : https://github.com/v4ishnavi/sdet-source-estimation-project

5.1 2D - Reproducing the paper

- Low SNR Performance: At an Average SNR (ASNR) of -20 dB, the MLE variance was found to be 2 to 3 orders of magnitude lower than the conventional TDOA variance.

- Scatter Plot: The estimated locations were tightly concentrated around the true emitter position, falling within the 95% error ellipse derived from the CRLB.

- True Pos (X, Y): [130.00, 75.00] km. Mean Est (X, Y): [132.99, 76.11] km. Covariance Diagonal (Var X, Y): 0.3194380, 0.1273112 with 50 Monte Carlo Trials and Tolerance = 1e-4.

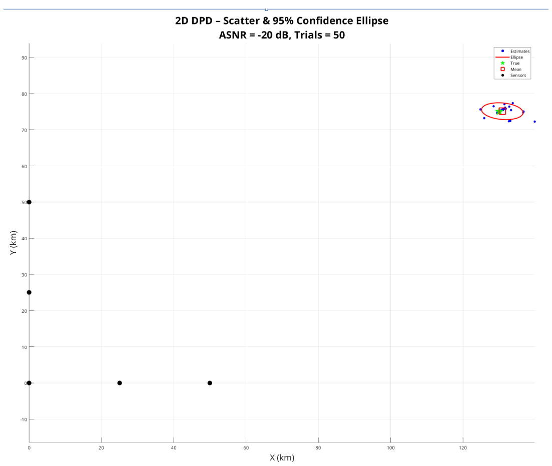

Figure 1: MLE Scatter and Sensor Locations

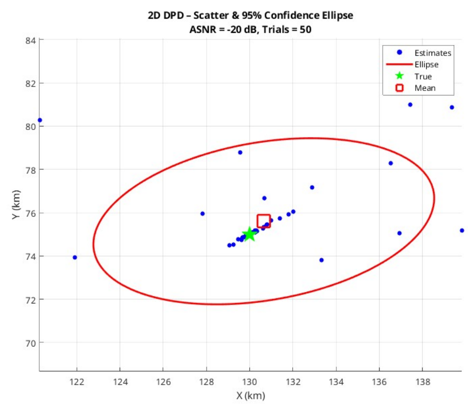

Figure 2: 95% confidence ellipse

5.2 3D - Further Exploration

As the primary contribution beyond the original paper, we extended the DPD framework to estimate 3D coordinates $[x_T, y_T, z_T]^T$.

5.2.1 Modifying the Jacobian

The core modification required lies in the Jacobian matrix that relates the TDOA vector τ' to the position vector η'. In the 2D case, this is an $(M-1) \times 2$ matrix. For 3D, we expanded this to an $(M-1) \times 3$ matrix by deriving the derivative of the relative time delays with respect to the z-coordinate.

The new derivative term is:

$$\frac{\partial \tau'_i}{\partial z} = \frac{1}{c} \left(\frac{z_T - z_i}{d_i} - \frac{z_T - z_0}{d_0} \right) \tag{11}$$

where d_i is the Euclidean distance from the emitter to sensor i.

Results True Pos: [35.00, 15.00, 12.00]. Mean Est: [35.55, 15.24, 12.15]. Covariance Diagonal (Var X, Y, Z): 0.9369315, 0.5124023, 0.3347965

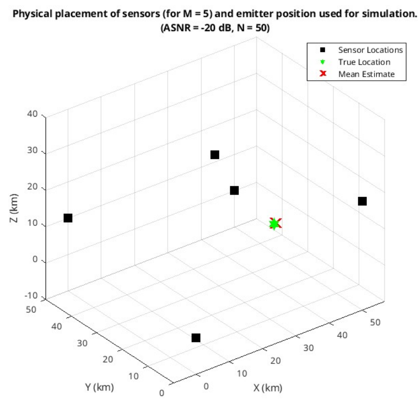

Figure 3: MLE Scatter and Sensor Locations

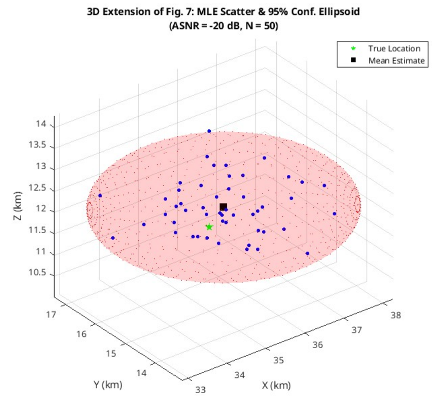

Figure 4: 95% confidence ellipse

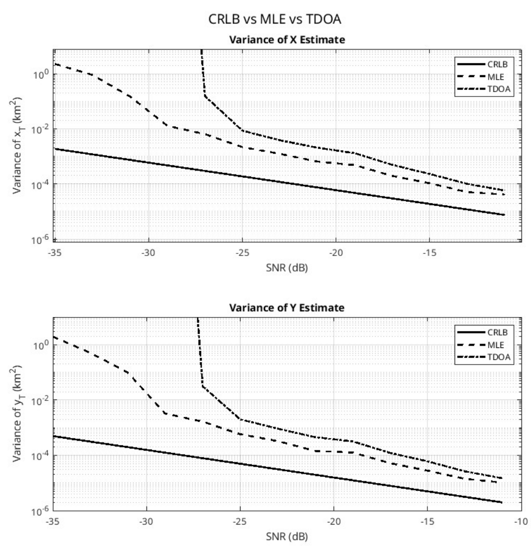

Figure 5: Comparison of variances for emitter location estimate using MLE and typical TDOA approach against CRLB for different ASNR values.r

References

[1] N. Vankayalapati, S. Kay and Q. Ding, "TDOA based direct positioning maximum likelihood estimator and the cramer-rao bound," in IEEE Transactions on Aerospace and Electronic Systems, vol. 50, no. 3, pp. 1616-1635, July 2014, doi: 10.1109/TAES.2013.110499.