TDOA Based Direct Positioning: MLE and CRLB

Extension of Vankayalapati et al.

Priya Mandot, Vaishnavi Shivkumar

IIIT Hyderabad

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Introduction: Passive Localization

- **Objective:** Locate a stationary emitter using spatially separated passive sensors.
- **Conventional Approach (Two-Step):**
  1. Estimate Time Differences of Arrival (TDOAs) between sensor pairs.
  2. Intersect hyperbolas defined by TDOAs to find the location.
- **The Problem:** The two-step method is suboptimal. It processes pairs independently and ignores the constraint that all measurements originate from a single point.
- **Proposed Solution:** Direct Position Determination (DPD). Estimate location directly from raw signal data in a single step.

Problem Setup: Received Signal Model

We observe the signal at M stationary sensors over interval $(0, T)$.

The Received Signal $r_i(t)$

$$r_i(t) = A_i s(t - \tau_i) + w_i(t), \quad i = 0, 1, \ldots, M - 1 \tag{1}$$

- $w_i(t)$: Zero-mean AWGN. Assumed independent and identical spectral density ($\frac{N_0}{2}$).
- A_i: Unknown attenuation factor.
- τ_i: Unknown Time of Arrival (TOA).

Goal: Estimate the emitter location $\mathbf{p} = [x_T, y_T]^T$ (2D) or $[x_T, y_T, z_T]^T$ (3D).

Relationship Between Parameters

The unknown TOA (τ_i) is linked to the unknown emitter position ($\mathbf{p}$) and unknown transmission time (t_0).

Time of Arrival τ_i

$$\tau_i = \frac{\sqrt{(x_T - x_i)^2 + (y_T - y_i)^2}}{c} + t_0 \tag{2}$$

- c: Signal propagation speed.
- The problem involves many unknown/nuisance parameters: $\mathbf{p}, t_0, A_i$, and the signal waveform $s(t)$.

The Signal Approximation

To solve the Maximum Likelihood problem, the continuous signal waveform $s(t)$ must be represented by a finite set of parameters. As we are assuming signal lies inside observation interval (0,T), we can assume signal is periodic with period T and can be written as a Fourier Series.

Fourier Series Approximation

$s(t)$ is approximated by $2N-1$ Fourier coefficients $\phi = [a_0 a_1 \ldots a_{N-1} b_1 \ldots b_{N-1}]^T$:

$$s(t) = \mathbf{h}^T(t)\phi \tag{3}$$

- $\mathbf{h}(t)$ is the vector of trigonometric basis functions (dimension $(2N-1) \times 1$).
- Rewriting the model in (1) as:

$$r_i(t) = A_i \mathbf{h}^T(t-\tau_i)\phi + w_i(t), \quad 0 \leq t \leq T, \; i = 0, 1, \ldots, M-1. \tag{4}$$

Case 1: Signal Unknown + Transmission Time Unknown $\rightarrow$ Singular FIM

When both the signal waveform $s(t)$ and the transmission time t_0 are unknown, the Fisher Information Matrix (FIM) $\mathcal{I}_\theta$ becomes **singular** (at least rank two deficient).

Ambiguity in Parameters

It is impossible to uniquely determine all unknowns due to inherent relationships.

- **Amplitude Ambiguity:** Observations $r_i(t)$ generated by $(\overline{A}_i, \overline{s}(t))$ are identical to those generated by $(\overline{A}_i/\gamma, \gamma\overline{s}(t))$ for any non-zero constant γ.
- **Time Ambiguity:** The transmitted signal $s_T(t)$ generated by $(\overline{t}_0, \overline{s}(t))$ is indistinguishable from one generated by $(\overline{t}_0 - \gamma, \overline{s}(t - \gamma))$ for any constant γ.

Transformation - Eliminating Ambiguity

To remove the singularity of the FIM, we transform the parameters into a set of identifiable quantities. We use sensor $i = 0$ as the reference.

Transformed Parameters $\boldsymbol{\theta}'$

$$\boldsymbol{\tau}' = \begin{bmatrix}(\tau_1 - \tau_0) & (\tau_2 - \tau_0) & \cdots & (\tau_{M-1} - \tau_0)\end{bmatrix}^T$$

$$\mathbf{A}' = (1/A_0)\begin{bmatrix}A_1 & \cdots & A_{M-1}\end{bmatrix}^T$$

- The unknown transmission time t_0 is **eliminated** and is no longer a nuisance parameter.
- Let $\boldsymbol{\theta}' = [\boldsymbol{\tau}'^T \mathbf{A}'^T \phi'^T]^T$ be $(2M + 2N - 3) \times 1$ be the unknown transformed parameter vector.
- The vector of interest is $\boldsymbol{\alpha}' = [\boldsymbol{\eta}'^T \mathbf{A}'^T \phi'^T]^T$, where $\boldsymbol{\eta}' = [x_T, y_T]^T$ is the location vector.

CRLB Derivation Procedure: Signal Unknown

1. **Joint Log-Likelihood (LL) Function:**

$$l(\boldsymbol{\theta}') = -\frac{1}{N_0}\int_0^T (r_0(t) - \mathbf{h}^T(t)\phi')^2 dt \\ - \frac{1}{N_0}\int_0^T \sum_{i=1}^{M-1} (r_i(t) - A_i'\mathbf{h}^T(t-\tau_i')\phi')^2 dt \quad (5)$$

2. **Derive FIM $\mathcal{I}_{\boldsymbol{\theta}'}$:** The FIM elements are computed as:

$$[\mathcal{I}_{\boldsymbol{\theta}'}]_{ij} = -E\left[\frac{\partial^2 l(\boldsymbol{\theta}')}{\partial\theta_i'\partial\theta_j'}\right]$$

3. **Resulting FIM $\mathcal{I}_{\boldsymbol{\theta}'}$ is transformed to FIM $\mathcal{I}_{\boldsymbol{\alpha}'}$:** The FIM is calculated using the Jacobian matrix:

$$\mathcal{I}_{\boldsymbol{\alpha}'} = \left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right)^T \mathcal{I}_{\boldsymbol{\theta}'}\left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right) \quad (6)$$

Maximum Likelihood Estimator (MLE): Signal Unknown

- **Correlation Vector $\mathbf{y}_i'$:** The estimation procedure defines a correlation vector $\mathbf{y}_i'$ for each sensor i, correlating the received signal with the basis vector $\mathbf{h}(t)$ aligned by the hypothesized TDOA τ_i'.

$$\mathbf{y}_i' = \int_0^T r_i(t)\mathbf{h}(t - \tau_i')dt \tag{7}$$

- **Cross-Correlation Matrix $\mathbf{B}'$:** The matrix $\mathbf{Y}'$ concatenates these vectors, $\mathbf{Y}' = [\mathbf{y}_0' \ \mathbf{y}_1' \ \cdots \ \mathbf{y}_{M-1}']$. $\mathbf{B}'$ is calculated as $\mathbf{B}' = \mathbf{Y}'\mathbf{Y}'^T$.
- **The MLE Solution:** The maximum likelihood estimate of the location $(\hat{x}_T, \hat{y}_T)$ is found by maximizing the maximum eigenvalue of $\mathbf{B}'$ over all possible locations.

$$(\hat{x}_T, \hat{y}_T) = \arg \max_{(x_T, y_T)} \lambda_{max}(\mathbf{B}') \tag{8}$$

Case 2: Signal Known + Transmission Time Unknown

- **Unknowns Reduced:** The number of unknowns is reduced to $2M$. The unknown parameter vector is $\boldsymbol{\zeta} = [\boldsymbol{\tau}^T \mathbf{A}^T]^T$. The signal parameters ϕ are known.
- **FIM Not Singular:** Since the signal waveform is known, the unknown parameters can be uniquely determined. Therefore, the FIM $\mathcal{I}_{\zeta}$ is **not singular** $\implies$ No transformation
- **MLE Simplifies:** The location and transmission time are estimated by maximizing the sum of squared correlation values across all sensors.

$$(\hat{x}_T, \hat{y}_T, \hat{t}_0) = \arg \max_{(x_T, y_T, t_0)} \sum_{i=0}^{M-1} \left(\int_0^T r_i(t) s(t - \tau_i) dt \right)^2 \quad (9)$$

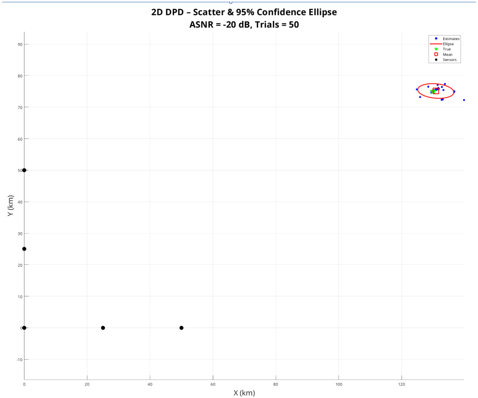
2D Simulation Setup

Configuration:

- 5 Sensors in a 2D L-shape.
- Emitter Location: (130, 75) km.
- Signal: Gaussian Chirp.
- ASNR Range: -35 dB to -10 dB.

Results:

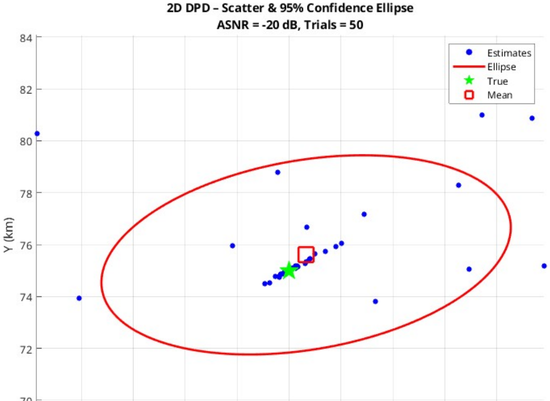
1. True Pos (X, Y): [130.00, 75.00] km
2. Mean Est (X, Y): [132.99, 76.11] km
3. Covariance Diagonal (Var X, Y): 0.3194380, 0.1273112



MLE Scatter and 95% confidence ellipse

2D Results: Scatter Plot

- Estimates are highly concentrated near the true location even at low ASNR (e.g., -20 dB).
- The 95% error ellipse confirms that the estimates are tightly bound, confirming the low variance.
- The estimator operates reliably far below the SNR threshold where the conventional TDOA method breaks down (≈ -17 dB).



2D Results: Performance Comparison

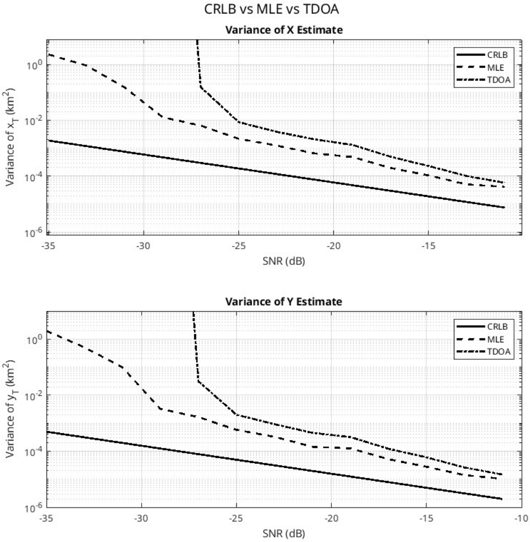


Figure 1: Variance Comparison: MLE vs. TDOA vs. CRLB

Extension to 3D: Modifying the Jacobian

Extending the DPD model to 3D requires modifying the geometry-dependent terms.

The Jacobian Matrix $\frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\eta}'^T}$ (Equation 40)

The Jacobian relates changes in the location vector $\boldsymbol{\eta}' = [x_T, y_T, z_T]^T$ to changes in the TDOA vector $\boldsymbol{\tau}'$.

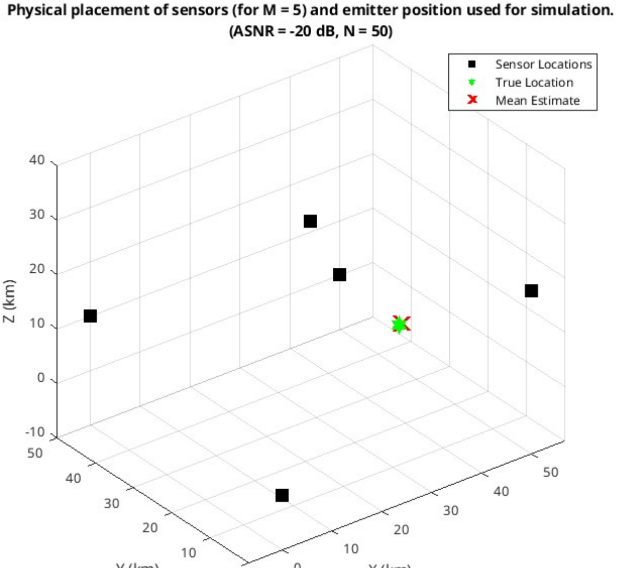
- **Change:** The Jacobian matrix ($\frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\eta}'^T}$) must be expanded from $(M-1) \times 2$ to $(\mathbf{M}-\mathbf{1}) \times \mathbf{3}$.
- **New Term:** Each row (TDOA) gains a column representing the derivative with respect to z:

$$\frac{\partial \tau_i'}{\partial z} = \frac{1}{c}\left(\frac{z_T - z_i}{d_i} - \frac{z_T - z_0}{d_0}\right)$$

- This modified Jacobian ensures the FIM and subsequent CRLB calculations are valid for the 3D position vector.

3D Simulation Setup and Results

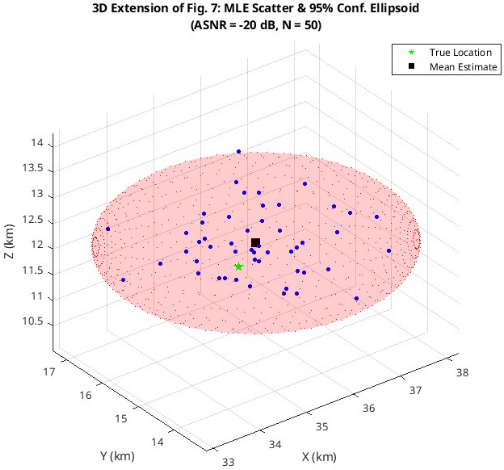
- **Setup:** 5 non-coplanar sensors are used to guarantee a non-singular FIM for the 3D problem.



3D Simulation - 95% error ellipse for MLE

Results:

- True Pos: [35.00, 15.00, 12.00]
- Mean Est: [35.55, 15.24, 12.15]
- Covariance Diagonal (Var X, Y, Z): 0.9369315, 0.5124023, 0.3347965



Conclusion

- **Methodology:** DPD-MLE is the optimal single-step localization technique, correctly handling nuisance parameters.
- **Key Insight:** Ambiguities in the unknown signal case necessitate parameter transformation (using relative measurements) to ensure a non-singular FIM and valid CRLB.
- **Extension:** The DPD framework extends seamlessly to 3D by expanding the Jacobian to include derivatives w.r.t. the z-coordinate.