Probability and Statistics. Week 7

1. Expected value μ and covariance matrix $\mathcal{K}$ of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ are provided: $\mu = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$, $\mathcal{K} = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 35 \end{pmatrix}$.

   Calculate the expected value and variance of:

   (a) $\eta = \xi_1 - \xi_3$;

   (b) $\eta = 2\xi_1 - \xi_2 + 3\xi_3$;

   (c) $\eta = -2\xi_1 + 3\xi_2 - \xi_3$.

   Answer: (a) $\mathrm{E}\,\eta = -1$, $\mathrm{Var}\,\eta = 42$; (b) $\mathrm{E}\,\eta = 6$, $\mathrm{Var}\,\eta = 314$; (c) $\mathrm{E}\,\eta = -10$, $\mathrm{Var}\,\eta = 66$.

2. Covariance matrix $\mathcal{K} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & \lambda \\ 1 & \lambda & 2 \end{pmatrix}$ of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ depends on a parameter λ. Find the value of λ such that variance of random variable $\zeta = \lambda\xi_1 + 2\xi_2 - \xi_3$ reaches its minimum.

   Answer: $\lambda = 5 \pm \sqrt{11}$

3. A book of 600 pages has on average one typo per page. Evaluate the following probabilities using Poisson approximation for binomial distribution.

   (a) Page 13 does not have a single typo;

   (b) There are exactly 2 typos on page 13;

   (c) There are no more than three typos on page 13.

   Answer: (a) $\frac{1}{e}$, (b) $\frac{1}{2e}$, (c) $\frac{8}{3e}$.

4. When baking raisin cupcakes, it sometimes (with probability 0.003) happens that a cupcake does not contain any raisins at all. Estimate the probability than in a batch of 1000 cupcakes

   (a) There are no cupcakes withous raisins;

   (b) There are exactly three cupcakes without raisins;

   (c) There are at least three cupcakes without raisins.

   Answer: (a) $\frac{1}{e^3}$, (b) $\frac{9}{2e^3}$, (c) $1 - \frac{17}{2e^3}$

5. How many raisins do the cupcakes described in the previous problem have to contain on average so that the probability that a cupcake does not have a single raisin does not exceed 0.01?

   Answer: $\lambda > \ln 100$.

6. (Walpole 5.15) It is known that 60% of mice inoculated with a serum are protected from a certain disease. If 5 mice are inoculated, find the probability that

   (a) none contracts the disease;

   (b) fewer than 2 contract the disease;

Answer: (a) 0.0778, (b) 0.337.

7. (Walpole 5.18) In testing a certain kind of truck tire over rugged terrain, it is found that 25% of the trucks fail to complete the test run without a blowout.
   (a) How many of the 15 trucks would you expect to have blowouts?
   (b) What is the variance of the number of blowouts experienced by the 15 trucks?
   Answer: (a) $\mathrm{E}\, X = 3.75$, (b) $\mathrm{Var}\, X = 2.8125$.

8. (Walpole 5.58 and 5.63) A certain area of the eastern US is, on average, hit by 6 hurricanes a year.
   (a) Find the probability that in a given year that area will be hit by fewere than 4 hurricanes;
   (b) What is the mean and variance of the number of hurricanes per year?
   Answer: (a) 0.1512, (b) $\mathrm{E}\, X = \mathrm{Var}\, X = 6$.

9. (Walpole 5.44) An urn contains 3 green balls, 2 blue balls and 4 red balls. In a random sample of 5 balls, find the probability that both blue balls and at least 1 red ball are selected.
   Answer: $\frac{17}{63}$.

10. Random variable ζ has a distribution $\zeta \sim \begin{pmatrix} 0 & \frac{\pi}{2} & \pi & \frac{3\pi}{2} \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$. Consider the random variables $\eta = \cos\zeta$ and $\xi = \sin\zeta$.
    (a) Determine if η and ξ are independent;
    (b) Determine if η and ξ are correlated.

11. The card is being drawn from a shuffled deck of 52 cards. The person checks if the card is red or it is a picture (jack, queen or king), returns the card to the deck and shuffles it again. The experiment is repeated K times. Random variable η is the number of the red cards observed and random variable ξ is the number of the pictures observed. Find the correlation coefficient between η and ξ.

12. You are rolling a biased die until you get 1. Probability that you get 1 is $\frac{1}{7}$ and probabilities to get 2, 3, 4, 5 or 6 are equal. Find the expected sum of the numbers observed (including 1).
    Answer: $\mathrm{E}\, S = 25$.