Probability and Statistics. Week 6

1. (Walpole 4.71) The length of time Y, in minutes, required to generate a human reflex to tear gas has the density function $f(y) = \begin{cases} \frac{1}{4}e^{-\frac{y}{4}}, \ 0 \le y < \infty \\ 0, \text{ elsewhere} \end{cases}$
   (a) What is the mean time to reflex?
   (b) Find $\mathrm{E}\, Y^2$ and $\mathrm{Var}\, Y$.

2. (Walpole 4.73) Let random variable Y have a probability density function $f(y) = \begin{cases} 1, \ 7 \le y \le 8 \\ 0, \text{ elsewhere} \end{cases}$. Find the expectation of e^Y in two ways:
   (a) $\mathrm{E}\, e^Y = \int_7^8 e^y f(y) dy$,
   (b) Using the second order approximation.

3. (Walpole 4.74) Consider the random variable Y as in the previous problem. Find $\mathrm{Var}\, e^Y$ using the exact approach and the approximation of the variance.

4. A fair die is rolled until a four is obtained. Find the expected value of a sum obtained in all of the rolls.
   Answer: $\mathrm{E}\, S = 21$.

5. Find a correlation coefficient between the number of sixes and the number of fives obtained in K rolls of a fair die.
   Answer: $\rho(\xi, \eta) = -\frac{1}{5}$,

6. ζ is the number of threes and η is the number of odd digits obtained when rolling a fair die K times. Find the correlation coefficient between η and ζ.
   Answer: $\rho(\zeta, \eta) = \frac{1}{\sqrt{5}}$.

7. Let ξ be a random variable with a finite variance. Prove the inequality: $P(|\xi - \mathrm{E}\,\xi| \le 3\sqrt{\mathrm{Var}\,\xi}) \ge \frac{8}{9}$.

8. Let $\epsilon > 0$ and $0 < p < \frac{1}{2}$ and $\xi \sim \begin{pmatrix} -\epsilon & 0 & \epsilon \\ p & 1-2p & p \end{pmatrix}$. Show that Chebyschev's inequality turns into equality for this distribution: $P(|\xi - \mathrm{E}\,\xi| \ge \epsilon) = \frac{\mathrm{Var}\,\xi}{\epsilon^2}$.

9. Prove that if random variable ξ is non-negative, integer, and it's expected value $\mathrm{E}\,\xi$ is finite, then $\mathrm{E}\,\xi = \sum_{k=1}^{\infty} P(\xi \ge k)$.

10. Calculate the expected value of $\xi \sim G(p)$ using the result from the previous task.
    Answer: $\mathrm{E}\,\xi = \frac{1}{p}$.

11. N fair dice are rolled. Random variable ξ is the smallest digit obtained.
    (a) Calculate expected value of ξ if $N = 6$;
    (b) What happens to this expected value as $N \to \infty$?
    Answer: (a) $\sum_{k=1}^{6} (\frac{k}{6})^6 \approx 1.4397$; (b) $\sum_{k=1}^{6} (\frac{k}{6})^N \to 1$.

12. A marksman hits the target with probability 0.8. The marksman is shooting until he misses the target at least once and hits the target at least once. Find the expected value of the number of shots.

    Answer: $\mathrm{E}\,\xi = 5.25$.

13. Random variable ζ is uniformly distributed on set $\{-1, 0, 1\}$. Let us consider the random variables $\xi = 1 - \zeta^{1000}$ and $\eta = 1 - \zeta^{1001}$.

    (a) Determine if ξ and η are independent;

    (b) Determine if ξ and η are correlated.

14. Two independent random variables η and ξ have geometric distribution with parameter p. Prove that $P(\xi = k | \xi + \eta = n) = \frac{1}{n-1}$.

15. Two independent random variables ξ_1, ξ_2 have geometric distribution with parameters p_1 and p_2. Find the distribution law of random variable $\xi = \min(\xi_1, \xi_2)$.

    Answer: $\xi \sim \begin{pmatrix} 1 & 2 & \dots & k & \dots \\ 1 - q_1 q_2 & q_1 q_2 (1 - q_1 q_2) & \dots & q_1^{k-1} q_2^{k-1} (1 - q_1 q_2) & \dots \end{pmatrix}$.