Probability and Statistics. Week 3

1. Distribution of random variable ξ is given by $\xi \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0.1 & 0.3 & 0.2 & p & 0.2 \end{pmatrix}$. Determine the probability p. Draw the graph of a cumulative distribution function $F_\xi(t)$.

   Answer: $p = 0.2$.

2. Let us consider the circle of radius R centered at O. Point M is chosen at random inside this circle. Random variable ξ is equal to the length of OM. Find the cumulative distribution and probability density functions.

   Answer: $F_\xi(r) = \begin{cases} 0, \ r < 0 \\ \frac{r^2}{R^2}, \ 0 \le r < R \\ 1, r \ge R \end{cases}$, $f_\xi(r) = \begin{cases} \frac{2r}{R^2}, \ 0 \le r < R \\ 0, \text{ otherwise} \end{cases}$.

3. Let us consider the ball of radius R centered at O. Point M is chosen at random inside this ball. Random variable ξ is equal to the distance from point M to the sphere. Find the cumulative distribution and probability density functions.

   Answer: $F_\xi(x) = \begin{cases} 0, \ x < 0 \\ 1 - \frac{(R-x)^3}{R^3}, \ 0 \le x < R \\ 1, \ x \ge R \end{cases}$; $f_\xi(x) = \begin{cases} 3\frac{(R-x)^2}{R^3}, \ 0 < x \le R \\ 0, \text{ otherwise} \end{cases}$

4. Find all values of C such that function $F(x) = \begin{cases} 0, \ x < 1 \\ 1 - \frac{C}{x}, \ x \ge 1 \end{cases}$ can be a cumulative distribution function for some random variable ζ. Find the probability density function.

   Answer: $C = 1$, $f(x) = \begin{cases} \frac{1}{x^2}, \ x > 1 \\ 0, \text{otherwise} \end{cases}$

5. Is it possible that for some values of C the functions below are probability density functions of random variables? If it is so, find the values of C.

   (a) $f(x) = \begin{cases} Ce^{-2x}, \ x > 0 \\ 0, \ x \le 0 \end{cases}$

   (b) $f(x) = Ce^{-|x|}$, $x \in \mathbb{R}$.

   Answer: (a) $C = 2$ (b) $C = 0.5$.

6. Find such value of C that function $f(x) = \frac{C}{1+x^2}$ is probability density function of a random variable.

   Answer: $C = \frac{1}{\pi}$

7. (Walpole 3.5) Determine the value c so that each of the following functions can serve as a probability distribution of the discrete random variable X.

   (a) $f(x) = c(x^2 + 4)$, $x \in \{0, 1, 2, 3\}$;

   (b) $f(x) = c\binom{2}{x}\binom{3}{3-x}$, $x \in \{0, 1, 2\}$.

Answer: (a) $\frac{1}{30}$, (b) $\frac{1}{10}$.

8. (Walpole 3.9) The proportion of people who respond to a certain mail-order solicitation is a continuous random variable X that has the density function $f(x) = \begin{cases} \frac{2(x+2)}{5}, 0 < x < 1 \\ 0, \text{elsewhere} \end{cases}$.
   (a) Show that $P(0 < X < 1) = 1$;
   (b) Find the probability that more than $\frac{1}{4}$ but fewer than $\frac{1}{2}$ of the people contacted will respond to this type of silicitation.

   Answer: (b) $\frac{19}{80}$.

9. (Walpole 3.35) Suppose it is known from large amounts of historical data that X, the number of cars that arrive at a specific intersection during a 20-second time period, is characterized by the following discrete probability function: $f(x) = e^{-6}\frac{6^x}{x!}$, for $x = 0, 1, 2, \ldots$.
   (a) Find the probability that in a specific 20-second time period, more than 8 cars arrive at the intersection.
   (b) Find the probability that only 2 cars arrive.

   Hint: $e^6 \approx 400$.

   Answer: (a) 0.1528 (b) 0.0446

10. (Walpole 3.42) Let X and Y denote the lengths of life, in years, of two components in an electronic system. If the joint density function of these variables is $f(x, y) = \begin{cases} e^{-(x+y)}, x > 0, y > 0 \\ 0, \text{elsewhere} \end{cases}$

    find $P(0 < X < 1|Y = 2)$.

    Answer: $1 - \frac{1}{e} \approx 0.6321$

11. (Walpole 3.68) Consider the following joint probability density function of the random variables X and Y:

$$f(x, y) = \begin{cases} \frac{3x-y}{9}, 1 < x < 3, 1 < y < 2 \\ 0, \text{elsewhere} \end{cases}$$

    (a) Find the marginal density functions of X and Y;
    (b) Are X and Y independent?
    (c) Find $P(X > 2)$.

    Answer: (a) $g(x) = \frac{x}{3} - \frac{1}{6}$ for $1 < x < 3$ and $h(y) = \frac{4}{3} - \frac{2}{9}y$; (b) No; (c) $\frac{2}{3}$.

12. Consider a system of components in which there are 5 independent components, each of which possesses an operational probability of 0.92. The system does have a redundancy built in such that it does not fail if 3 out of the 5 components are operational.
    (a) Write down the probability mass function;
    (b) What is the probability that the total system is operational?

    Answer: (a) $f(x) = \begin{cases} \binom{5}{x} \cdot 0.92^x \cdot (1 - 0.92)^{5-x}, x \in \{0, 1, 2, 3, 4, 5\} \\ 0, \text{otherwise} \end{cases}$; (b) 0.9955