Task 1

$N = 116$ samples total

$X = 17$ – with cysts.

θ – pr. of a sample to contain cysts

1. $X|\theta \sim Bin(N, \theta)$

2. $E\theta = 0.2$, $\sigma = 0.16$

$$E\theta = \frac{\alpha}{\alpha+\beta}$$

$$Var\,\theta = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\beta E\theta}{(\alpha+\beta)(\alpha+\beta+1)} =$$

$$= \frac{\beta (E\theta)^2}{\alpha(\alpha+\beta+1)}$$

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = 0.2 \\ \frac{\beta \cdot 6 \cdot (0.2)^2}{\alpha(\alpha+\beta+1)} = 0.16^2 \end{cases} \begin{cases} \beta = 4\alpha \\ \frac{4\alpha}{\alpha(5\alpha+1)} = \frac{16}{25} \end{cases}$$

$$\begin{cases} \beta = 4\alpha \\ 100\alpha = 80\alpha^2 + 16\alpha \end{cases} \begin{cases} \beta = 4\alpha \\ 80\alpha^2 - 84\alpha = 0 \end{cases}$$

$$\begin{cases} \alpha = \beta = 0 - \text{not a sln.} \\ \alpha = \frac{21}{20}, \ \beta = \frac{84}{20} \end{cases}$$

$$\text{Var}\,\theta = \frac{\frac{84}{20} \cdot 0.04}{\frac{21}{20} \cdot \left(\frac{125}{20}\right)} = \frac{84 \cdot 0.04 \cdot 20}{21 \cdot 125} =$$

$$= \frac{4 \cdot 0.04}{25} = \frac{4}{100^2} = \sigma^2 \Rightarrow \sigma = \frac{4^2}{100} = 0.16$$

Hence, $\alpha = \frac{21}{20}, \beta = \frac{84}{20}$

$\alpha = 1$, $\beta = 4$ (rounding)

3. $$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} = \frac{P(X|\theta)P(\theta)}{\int P(X|\theta)P(\theta)d\theta}$$

Since beta is conjugate prior distribution for a binomial one, resulting posterior is also beta distribution.

$$P(\theta|X) \sim B(\alpha + X, \beta + N - X)$$

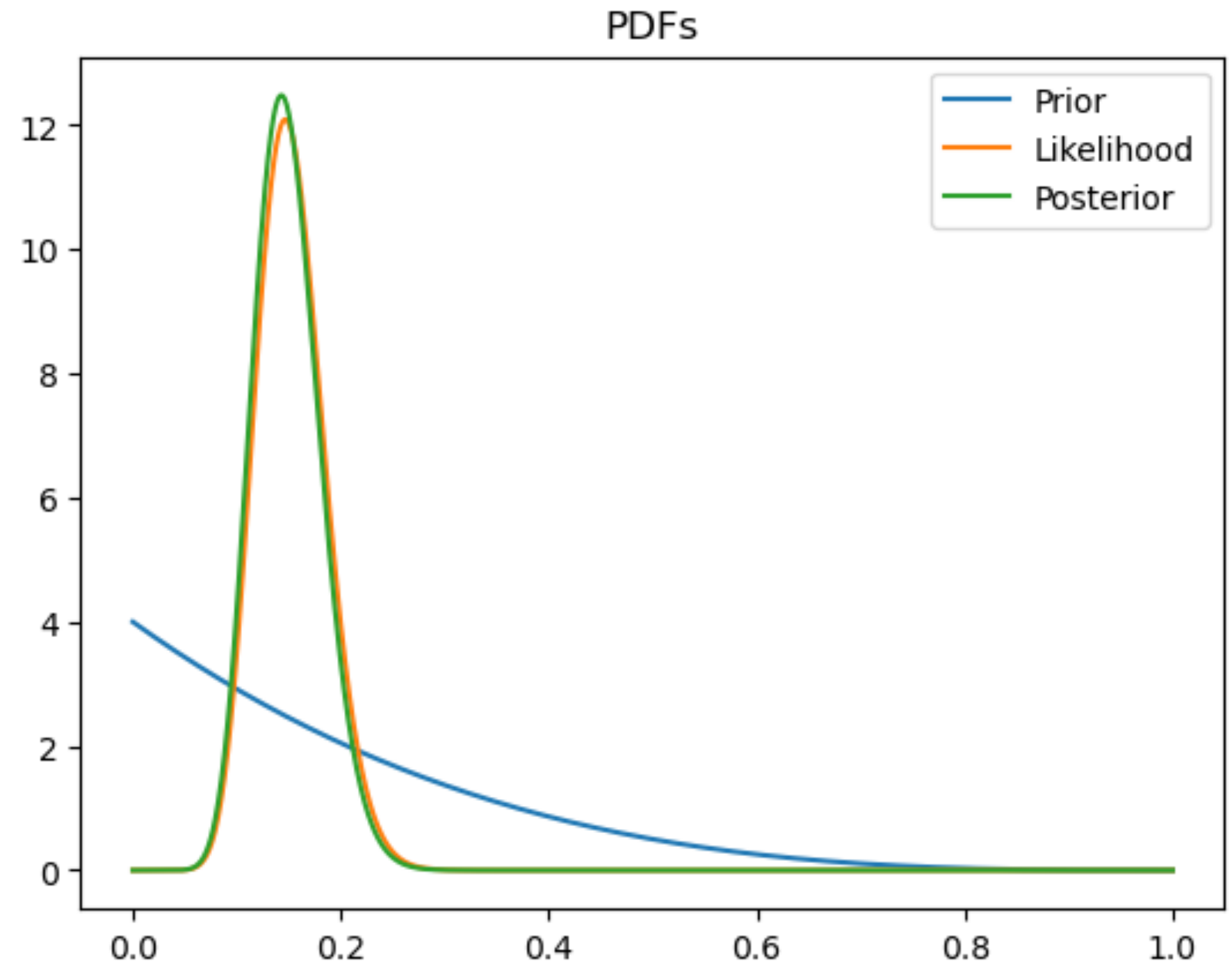
$1 + 17 = 18$ $\quad 9 + 116 - 17 = 103$

$$E(\theta|X) = \frac{18}{18+103} = \frac{18}{121}$$

$$Var(\theta|X) = \frac{18 \cdot 103}{(18+103)^2 \cdot (18-103+1)} =$$

$$= \frac{18 \cdot 103}{121^2 \cdot 122} = \frac{9 \cdot 103}{121^2 \cdot 61} \approx 10^{-3}$$

4.



5. $p(\theta < 0.1) \approx 0.053$

6. Central 95% confidence interval.

$0.0913 \leq \theta \leq 0.2171$

All calculations related to 5&6 are available in the .ipynb attached.