CS10720 Problems and Solutions

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Today: Induction

March 17th

Organisational Issues

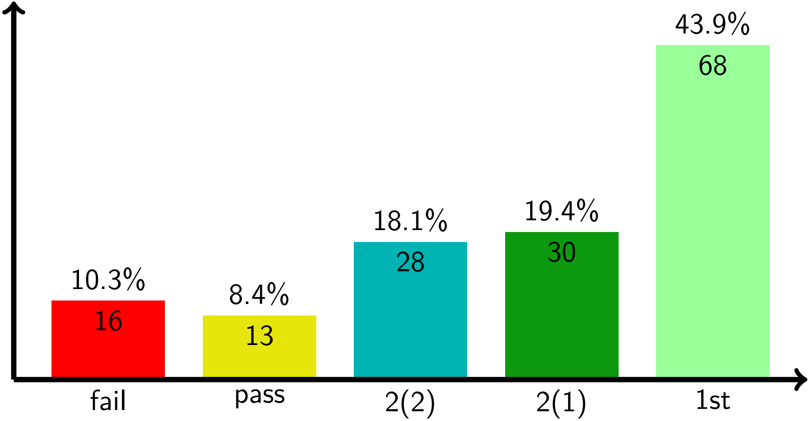
Reminder today last lecture before the Easter break
resuming lectures on the 11th April 2016

Reminder portfolio submission deadline tomorrow, 8pm
summary for Monday's and today's lecture
plus answer to practical questions from Tuesday/Wednesday

Announcement portfolio feedback will be delayed by one day
(i. e., feedback sent on Tuesday instead of Monday)
because I can send feedback only when in my office
and I'll be in Birmingham on Monday
Sorry!

In-Class Test

Announcement results (unmoderated!) later today (before 5pm)



Basic statistics 155 participants
minimum 3 $\widehat{=}$ 10%; maximum 30 $\widehat{=}$ 100%
average 19.59 $\widehat{=}$ 65%; median 20 $\widehat{=}$ 66%

Feedback for the In-Class Test

Reminder in-class test including solutions
posted on Blackboard (general discussion forum)

Announcement you will receive
marks breakdown for all parts you have answered
(and if you have answered more than one question
an indication which of them determines the mark:
the one where you did better)
you will not receive
additional written feedback

What if I want more feedback?

Simple! Send me an email to let me know
and we arrange a time to meet in my office
to look at your in-class test together

Plans for Today

1. Induction: Motivation
    - Reminder: Analysing Insertion Sort
    - Induction
2. Another Example
    - Tower of Hanoi
    - Analysis and Result
3. Correctness of Algorithms
    - Insertion Sort
    - Merge Sort
4. Summary
    - Summary & Take Home Message

Remember: Analysing Insertion Sort

Insertion Sort

```
for ( i=1; i<n; i++ ) {
  pos=search(keys, keys[i], i-1); /* find position of keys[i] in keys[0..i-1] */
  for ( j=i; j>pos; j--)
    keys[j]=keys[j-1]; /* shift keys[pos..i-1] one to the right */
  keys[pos]=i;
}
```

Analysis number of items shifted in the worst case $s(n) = \Theta(n^2)$ with a very simple proof

What if someone claimed $s(n) = n(n-1)/2$?

Observation knowing more is always better
$s(n) = \frac{n(n-1)}{2}$ much more informative than $s(n) = \Theta(n^2)$

How can we know if it is true?

How can we prove this to be true?

What is a Proof?

A Different Example

Consider someone claims 17399 and 17401 are both prime

Prove 17399 not prime because $17399/127 = 137$ ✓

Prove 17401 is prime because $2 \nmid 17401$, $3 \nmid 17401$, $5 \nmid 17401$, $7 \nmid 17401$, $11 \nmid 17401$, $13 \nmid 17401$, $17 \nmid 17401$, $19 \nmid 17401$, $23 \nmid 17401$, $29 \nmid 17401$, $31 \nmid 17401$, $37 \nmid 17401$, $41 \nmid 17401$, $43 \nmid 17401$, $47 \nmid 17401$, $53 \nmid 17401$, $59 \nmid 17401$, $61 \nmid 17401$, $67 \nmid 17401$, $71 \nmid 17401$, $73 \nmid 17401$, $79 \nmid 17401$, $83 \nmid 17401$, $89 \nmid 17401$, $97 \nmid 17401$, $101 \nmid 17401$, $103 \nmid 17401$, $107 \nmid 17401$, $109 \nmid 17401$, $113 \nmid 17401$, $127 \nmid 17401$, $131 \nmid 17401$ ✓

Why do these proofs work?

Types of Proofs

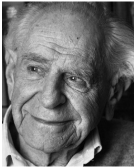
Consider someone claims 17399 and 17401 are both prime
Prove 17399 not prime because $17399/127 = 137$ ✔
Prove 17401 is prime because $n \not| \ 17401$
for all $n \in \{2, 3, 4, \ldots, 131\}$ (and $132 \cdot 132 > 17401$)
(or for all prime numbers in $\{2, 3, 4, \ldots, 131\}$)

Why do these proofs work?

Observation for 17399 a single counterexample suffices

Observation for primality testing
sufficient to test a finite number of statements
but $s(n) = n(n-1)/2$ (for all $n \in \mathbb{N}$) is statement
about infinite many values of n

Compare Popper's critical rationalism
a single counterexample tells us
a general statement is false
but there is no proof by example



Karl Raimund Popper 1902–1994

Proving Statements For Infinitely Many Numbers

How can we prove statement holds for infinitely many numbers?

Method to prove statement holds for (almost) all natural numbers called induction

1. Prove the statement holds for some number $b \in \mathbb{N}$.
2. Prove if the statements holds for some $n \in \mathbb{N}$ $(n \geq b)$ then it holds for $n + 1$, too.

Consequence Statement is true for all $n \in \mathbb{N}$ with $n \geq b$.

Example $s(n) = 1 + 2 + 3 + \cdots + n - 1$

Statement 'For all $n \in \mathbb{N}$, $s(n) = n(n-1)/2$ holds.

Remark this is the Insertion Sort example
because number of items shifted in the worst case
is 'all that are already sorted in each round'
is 1 in the 1st round, 2 in the 2nd round,
3 in the 3rd round, ..., $n - 1$ in the last round
together $1 + 2 + 3 + \cdots + n - 1$

Proving $s(n) = n(n-1)/2$ by Induction

Example $s(n) = 1 + 2 + 3 + \cdots + n - 1$

Statement 'For all $n \in \mathbb{N}$, $s(n) = n(n-1)/2$ holds.

Base Case $b = 1$: $s(b) = s(1) = 0$

$b(b-1)/2 = 1(1-1)/2 = 0$ ✓

Inductive step Assume $s(n) = n(n-1)/2$

$$s(n+1) = 1 + 2 + 3 + \cdots + (n+1) - 1$$
$$= 1 + 2 + 3 + \cdots + (n-1) + n = s(n) + n$$
$$= \frac{n(n-1)}{2} + n = \frac{n^2-n}{2} + \frac{2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$
$$= \frac{(n+1)((n+1)-1)}{2}$$ ✓

Consequence for all $n \in \mathbb{N}$, $s(n) = n(n-1)/2$ □

Another 'Number Example'

Claim For all $n \in \mathbb{N}$, $11^n - 6$ can be divided by 5 without a remainder.

Base Case $b = 1$: $11^b - 6 = 11^1 - 6 = 11 - 6 = 5$ ✓

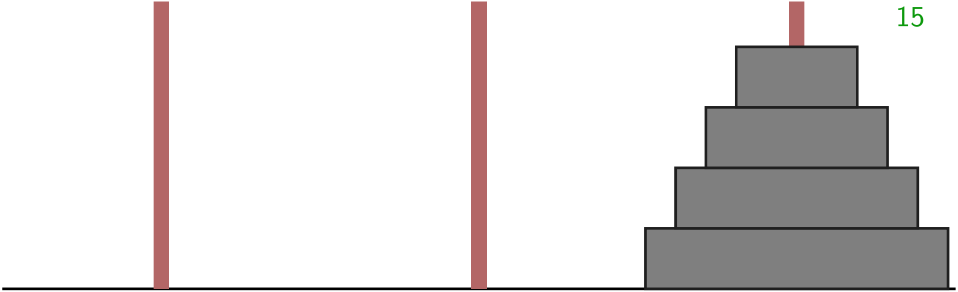
Inductive step Assume $11^n - 6$ can be divided by 5
i. e., $11^n - 6 = 5 \cdot k$ for some $k \in \mathbb{N}$

$$11^{n+1} - 6 = 11 \cdot 11^n - 6 = 11 \cdot (11^n - 6 + 6) - 6$$
$$= 11 \cdot (5k + 6) - 6 = 55k + 66 - 6 = 55k + 60$$
$$= 5 \cdot 11k + 5 \cdot 12 = 5 \cdot (11k + 12)$$ ✓

Consequence for all $n \in \mathbb{N}$, 5 divides $11^n - 6$ □

http://onlineted.com

Tower of Hanoi



- move all disk from left-most rod to right-most rod
- move only one disk at a time
- place all disks only on rods at all times
- never place a bigger disk on a smaller one
- generalises obviously to arbitrary number of disks n

In how many moves can you finish?

Analysing the Tower of Hanoi

In how many moves can you finish?

Let $M(n)$ denote the required number of moves needed for n disks

Lower bound
largest disk needs to be moved to right (1 move)
can only work if $n-1$ smaller disks
are on other rod ($M(n-1)$ moves)
$n-1$ smaller disks need to be moved to right rod
($M(n-1)$ moves)
Consequence $M(n) \geq 2M(n-1)+1$

Upper bound
move $n-1$ smaller disks to middle rod ($M(n-1)$ moves)
move largest disk to right rod (1 move)
move $n-1$ smaller disks to right rod ($M(n-1)$ moves)
Consequence $M(n) \leq 2M(n-1)+1$

We have $M(n) = 2M(n-1)+1$

Can we have a closed form?

Closed Form for Number of Moves

We have $M(n) = 2M(n-1) + 1$ (and $M(1) = 1$)

We want closed form for $M(n)$

Observe $M(n) + 1 = 2M(n-1) + 2 = 2(M(n-1) + 1)$

Define $D(n) := M(n) + 1$ (so, $D(1) = 2$)
implying $D(n) = 2D(n-1) = 4D(n-2) = 8D(n-3)$
$= \cdots = 2^{n-1}D(n-(n-1)) = 2^{n-1} \cdot 2 = 2^n$

Prove $M(n) = 2^n - 1$ by induction

Induction base $b = 1\colon\ 2^b - 1 = 2^1 - 1 = 2 - 1 = 1 = M(1)$ ✓

Induction step Assume $M(n) = 2^n - 1$
$M(n+1) = 2M(n) + 1 = 2\,(2^n - 1) + 1$
$= 2^{n+1} - 2 + 1 = 2^{n+1} - 1$ ✓

Consequence Tower of Hanoi with n disks can be solved in $2^n - 1$ moves (and no less)

Correctness of Insertion Sort

```
Insertion Sort
for ( i=1; i<n; i++ ) {
  pos=search(keys, keys[i], i-1); /* find position of keys[i] in keys[0..i-1] */
  for ( j=i; j>pos; j--)
    keys[j]=keys[j-1]; /* shift keys[pos..i-1] one to the right */
  keys[pos]=i;
}
```

Claim For all $n \in \mathbb{N}$, insertion sort sorts n items correctly

Induction base $b = 1$: 1 item sorted; insertion sort does nothing ✔

Induction step Assume insertion sort works for n items
for size $n + 1$, after round with $i = n - 1$
first n items sorted correctly by assumption
Observe item at position n is correctly sorted in
Consequence $n + 1$ items sorted correctly ✔

Correctness of Merge Sort

```
Merge Sort for n = 2^k items (k ∈ ℕ)
if ( n > 1) {
  Use merge sort to sort first n/2 = 2^(k−1) items.
  Use merge sort to sort after n/2 = 2^(k−1) items.
  Merge the two halves.
}
```

Claim For all $k \in \mathbb{N}$, merge sort sorts $n = 2^k$ items correctly

Induction base $b = 0$: $2^0 = 1$ item sorted; merge sort does nothing ✓

Induction step Assume insertion sort works for $n = 2^k$ items
for size $2n = 2^{k+1}$, both halves sorted correctly

Observe after merging all items sorted correctly ✓

Note proof was over k (not over n)

Summary & Take Home Message

Things to remember

- proof methods
  - enumeration
  - counterexample
  - induction
- Tower of Hanoi
- proof 5 divides $11^n - 6$
- correctness: insertion sort; merge sort

Take Home Message

- Induction allows to prove statements for all integers.
- Induction is useful to prove correctness of recursive algorithms.
- Induction is useful to prove correctness of closed forms for recursive formulas.


Happy
Easter