CS10720 Problems and Solutions

Thomas Jansen

Today: Matrices

February 25$^{\text{th}}$

Plans for Today

Announcement

Tiny adjustment to submission deadline for weekly portfolio
new deadline each Friday, 8pm
(used to be Friday, 7pm)

My view on completing the portfolio

- find 30 minutes on Monday, Tuesday or Wednesday to write the summary for Monday
- attend the practicals on Tuesday/Wednesday and complete the practicals questions there (including writing up the answers as part of your blog entry)
- find 30 minutes on Thursday or Friday to write the summary for Thursday

Important Don't leave the practicals without having secured 50% of the portfolio mark!

Starting With Things You Already Know

You know

- numbers
- numbers can be added (and how to do that)
- numbers can be subtracted (and how to do that)
- numbers can be multiplied (and how to do that)
- it is useful to have numbers and to be able to add/subtract/multiply them

Today matrices
adding, subtracting and multiplying them

Definition

A matrix is a rectangular array of numbers (or other things) arranged in rows and columns.

Example $M = \begin{pmatrix} 3 & -7 & 12 \\ 1 & 0 & 42 \end{pmatrix}$

M is a 2×3 matrix (2 rows, 3 columns)

Matrices

Conventions

- when using variables as names for matrices use capital letters
- 1×1 matrices are okay (but not particular useful)
  e. g., (3)
- $1 \times n$ matrices are called row vectors
  e. g., $\begin{pmatrix} 7 & 18 & 4 & 11 \end{pmatrix}$
- $m \times 1$ matrices are called column vectors
  e. g., $\begin{pmatrix} 9 \\ 1 \end{pmatrix}$
- in $m \times n$ matrix A denote entry in row i and column j as $a_{i,j}$
  e. g., $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$
- $n \times n$ matrix with 1 on diagonal and 0 else is identity matrix
  e. g., $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Matrix Addition

We know matrices

We want be able to add matrices

Fact you can only add matrices of the same size
i. e., $m \times n$ matrix A and $o \times p$ matrix B
can only be added if $m = o$ and $n = p$

Definition

Two $m \times n$ matrices A and B are added entry-wise,

i. e., $$A + B = \begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,n} + b_{m,n} \end{pmatrix}$$

Again in different notation: three $m \times n$ matrices A, B, C
with $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$
$C = A + B$ means $c_{i,j} = a_{i,j} + b_{i,j}$

Example Matrix Addition

Remember three $m \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$
$C = A + B$ means $c_{i,j} = a_{i,j} + b_{i,j}$

$$\begin{pmatrix} 3 & 4 & -1 \\ 0 & 9 & 3 \end{pmatrix} + \begin{pmatrix} 6 & 3 & 2 \\ 7 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3+6 & 4+3 & -1+2 \\ 0+7 & 9+1 & 3+(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 7 & 1 \\ 7 & 10 & 2 \end{pmatrix}$$

Matrix Subtraction

We know matrices and how to add them

We want be able to subtract matrices

Fact you can only subtract matrices of the same size
i. e., $m \times n$ matrix A and $o \times p$ matrix B
can only be added if $m = o$ and $n = p$

Definition

Two $m \times n$ matrices A and B are subtracted entry-wise,

i. e., $$A - B = \begin{pmatrix} a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} & \cdots & a_{1,n} - b_{1,n} \\ a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} & \cdots & a_{2,n} - b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} - b_{m,1} & a_{m,2} - b_{m,2} & \cdots & a_{m,n} - b_{m,n} \end{pmatrix}$$

Again in different notation: three $m \times n$ matrices A, B, C
with $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$
$C = A - B$ means $c_{i,j} = a_{i,j} - b_{i,j}$

Example Matrix Subtraction

Remember three $m \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$
$C = A - B$ means $c_{i,j} = a_{i,j} - b_{i,j}$

$$\begin{pmatrix} 3 & 4 & -1 \\ 0 & 9 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 3 & 2 \\ 7 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3-6 & 4-3 & -1-2 \\ 0-7 & 9-1 & 3-(-1) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & -3 \\ -7 & 8 & 4 \end{pmatrix}$$

Matrix Multiplication (Introduction)

We know matrices and how to add/subtract them

We want be able to multiply matrices

What do we mean by this?

Observation more than one sensible option

- option 1: multiplication of a number with a matrix
- option 2: multiplication of a matrix with a matrix

Fact for 'simple number' $\cdot A$ (where A is a matrix)
the 'simple number' is called scalar

Scalar Matrix Multiplication

Definition

A scalar s and an $m \times n$ matrix A are multiplied entry-wise,

i. e., $s \cdot A = \begin{pmatrix} s \cdot a_{1,1} & s \cdot a_{1,2} & \cdots & s \cdot a_{1,n} \\ s \cdot a_{2,1} & s \cdot a_{2,2} & \cdots & s \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ s \cdot a_{m,1} & s \cdot a_{m,2} & \cdots & s \cdot a_{m,n} \end{pmatrix}$

Again in different notation: number s, two $m \times n$ matrices A, B
with $A = (a_{i,j})$, $B = (b_{i,j})$
$B = s \cdot A$ means $b_{i,j} = s \cdot a_{i,j}$

Example Scalar Matrix Multiplication

Remember number s, two $m \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$
$B = s \cdot A$ means $b_{i,j} = s \cdot a_{i,j}$

$$3 \cdot \begin{pmatrix} 3 & 4 & -1 \\ 0 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 3 & 3 \cdot 4 & 3 \cdot (-1) \\ 3 \cdot 0 & 3 \cdot 9 & 3 \cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 12 & -3 \\ 0 & 27 & 9 \end{pmatrix}$$

Matrix Multiplication

Definition

An $m \times n$ matrix A and an $n \times o$ matrix B are multiplied 'row · column', i. e., $A \cdot B =$

$$\begin{pmatrix} a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1} + \cdots + a_{1,n} \cdot b_{n,1} & a_{1,1} \cdot b_{1,2} + a_{1,2} \cdot b_{2,2} + \cdots + a_{1,n} \cdot b_{n,2} & \cdots & a_{1,1} \cdot b_{1,o} + a_{1,2} \cdot b_{2,o} + \cdots + a_{1,n} \cdot b_{n,o} \\ a_{2,1} \cdot b_{1,1} + a_{2,2} \cdot b_{2,1} + \cdots + a_{2,n} \cdot b_{n,1} & a_{2,1} \cdot b_{1,2} + a_{2,2} \cdot b_{2,2} + \cdots + a_{2,n} \cdot b_{n,2} & \cdots & a_{2,1} \cdot b_{1,m} + a_{2,2} \cdot b_{2,m} + \cdots + a_{2,n} \cdot b_{n,o} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} \cdot b_{1,1} + a_{m,2} \cdot b_{2,1} + \cdots + a_{m,n} \cdot b_{n,1} & a_{m,1} \cdot b_{1,2} + a_{m,2} \cdot b_{2,2} + \cdots + a_{m,n} \cdot b_{n,2} & \cdots & a_{m,1} \cdot b_{1,m} + a_{m,2} \cdot b_{2,m} + \cdots + a_{m,n} \cdot b_{n,o} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} a_{1,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{1,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{1,i} \cdot b_{i,o} \\ \sum_{i=1}^{n} a_{2,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{2,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{2,i} \cdot b_{i,o} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{m,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{m,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{m,i} \cdot b_{i,o} \end{pmatrix}$$

Matrix Multiplication

Definition

An $m \times n$ matrix A and an $n \times o$ matrix B are multiplied 'row · column', i. e.,

$$A \cdot B = \begin{pmatrix} \sum_{i=1}^{n} a_{1,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{1,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{1,i} \cdot b_{i,o} \\ \sum_{i=1}^{n} a_{2,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{2,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{2,i} \cdot b_{i,o} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{m,i} \cdot b_{i,1} & \sum_{i=1}^{n} a_{m,i} \cdot b_{i,2} & \cdots & \sum_{i=1}^{n} a_{m,i} \cdot b_{i,o} \end{pmatrix}$$

Again in different notation: three matrices A, B, C
$m \times n$ matrix $A = (a_{i,j})$, $n \times o$ matrix $B = (b_{i,j})$,
$m \times o$ matrix $C = (c_{i,j})$
$C = A \cdot B$ means $c_{i,j} = a_{i,1} \cdot b_{1,j} + a_{i,2} \cdot b_{2,j} + \cdots + a_{i,n} \cdot b_{n,j}$
Remember 'row · column'

Example Matrix Multiplication

Remember in different notation: three matrices A, B, C

$m \times n$ matrix $A = (a_{i,j})$, $n \times o$ matrix $B = (b_{i,j})$,

$m \times o$ matrix $C = (c_{i,j})$

$C = A \cdot B$ means $c_{i,j} = a_{i,1} \cdot b_{1,j} + a_{i,2} \cdot b_{2,j} + \cdots + a_{i,n} \cdot b_{n,j}$

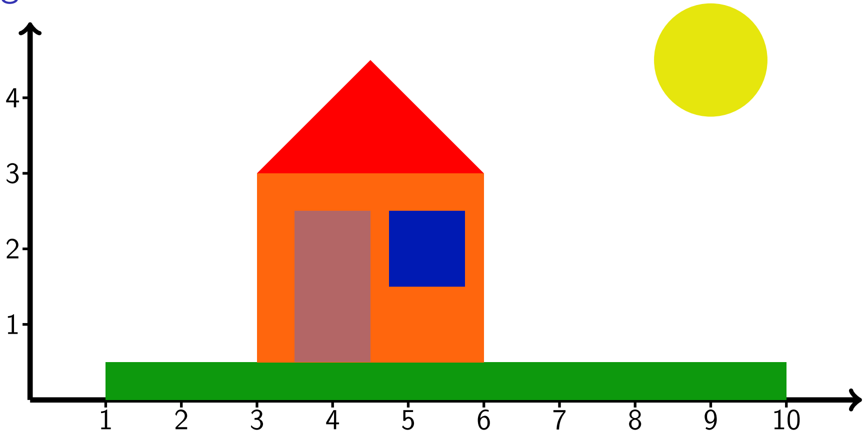
Remember 'row $\times$ column'

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 6 & 3 & 1 & 5 \\ 8 & 9 & 7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 6 + 4 \cdot 8 & 2 \cdot 3 + 4 \cdot 9 & 2 \cdot 1 + 4 \cdot 7 & 2 \cdot 5 + 4 \cdot 2 \\ 1 \cdot 6 + 3 \cdot 8 & 1 \cdot 3 + 3 \cdot 9 & 1 \cdot 1 + 3 \cdot 7 & 1 \cdot 5 + 3 \cdot 2 \\ 0 \cdot 6 + 5 \cdot 8 & 0 \cdot 3 + 5 \cdot 9 & 0 \cdot 1 + 5 \cdot 7 & 0 \cdot 5 + 5 \cdot 2 \end{pmatrix}$$

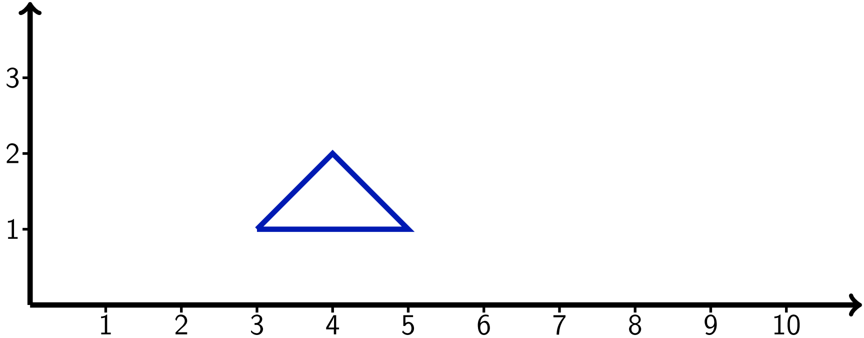
$$= \begin{pmatrix} 44 & 42 & 30 & 18 \\ 30 & 30 & 22 & 11 \\ 40 & 45 & 35 & 10 \end{pmatrix}$$

Images



Observation can describe image as list of points (in colours) and point by coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ (and colour)

Image Manipulation



Have image as list of points, points as coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$

Observations

- translation can be expressed as addition
- scaling can be expressed as multiplication

Image Manipulation as Matrix Computation

Have image as list of points, points as coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$

Know

- translation can be expressed as addition
  $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix}$
- scaling can be expressed as multiplication
  $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$

Fact

- rotation can be expressed as multiplication
  $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

Efficiency in Image Manipulation

Why would I care that I can express scaling and rotation as multiplication?

- scaling: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- rotation: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

Consider $1\,000\,000$ points that you want to

1. rotate by 45° and
2. scale by 1.75 in x direction and 1.25 in y direction.

Algorithm 1

1. Multiply $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ w. each point. $6\,000\,000$ ops.
2. Multiply $\begin{pmatrix} 7/4 & 0 \\ 0 & 5/4 \end{pmatrix}$ w. each point. $6\,000\,000$ ops.

total: $12\,000\,000$ operations

Algorithm 2

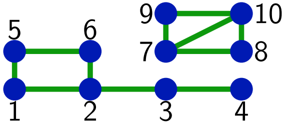
1. Multiply
$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 7/4 & 0 \\ 0 & 5/4 \end{pmatrix} = \begin{pmatrix} 7/8 & -5/\sqrt{3}/8 \\ 7\sqrt{3}/8 & 5/8 \end{pmatrix}$$
12 ops.
2. Multiply $\begin{pmatrix} 7/8 & -5/\sqrt{3}/8 \\ 7\sqrt{3}/8 & 5/8 \end{pmatrix}$ w. each point. $6\,000\,000$ ops.

total: $6\,000\,012$ operations

One Other Example

Have undirected graph
i. e., collection of n nodes and e edges between them

Example



Observation can be described as $n \times n$ matrix $M = (m_{i,j})$

with $m_{i,j} = \begin{cases} 1 & \text{if edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$

called adjacency matrix

Example

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

One Other Example: Reachability in Graphs

Have undirected graph
i. e., collection of n nodes and e edges between them
an be described as $n \times n$ matrix $M = (m_{i,j})$
with $m_{i,j} = \begin{cases} 1 & \text{if edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$
called adjacency matrix

Observations

- 0 in M shows 'cannot go from i to j via 1 edge'
- 0 in $M \cdot M$ shows 'cannot go from i to j via 2 edges'
- 0 in $M + M \cdot M$ shows 'cannot go from i to j via ≤ 2 edges'
- 0 in $M + M \cdot M + M \cdot M \cdot M = M + M^2 + M^3$ shows 'cannot go from i to j via ≤ 3 edges'
- ...
- 0 in $M + M^2 + M^3 + \cdots + M^{n-1} + M^n$ shows 'no path from i to j'

Summary & Take Home Message

Things to remember

- matrices
- vectors
- matrix addition
- scalar matrix multiplication
- matrix multiplication

Take Home Message

- Knowing about matrix multiplication can speed up things in image processing quite a bit.
- Matrices are very useful in many different areas.

Lecture feedback `http://onlineted.com`