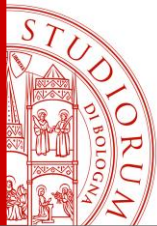




Graph Coloring

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The Graph Coloring Problem

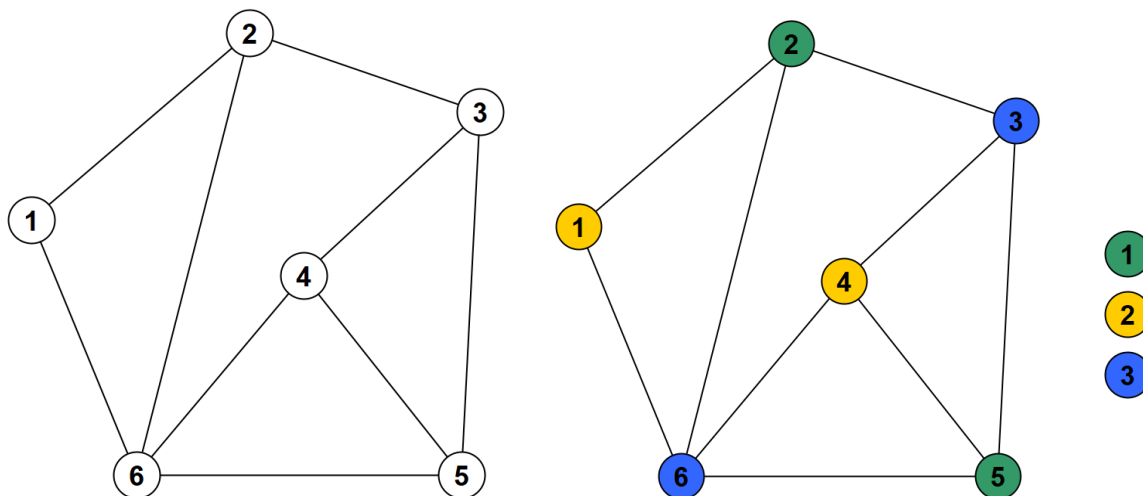
Colorazione dei grafi (Due nodi collegati da un arco devono avere colori diversi)

We are given an undirected graph G with node set V and edge set E . Edges in E are also called conflicts. The set C contains numbers that correspond to colors.

A **feasible coloring** of G is an assignment of its nodes to colors from C such that two adjacent nodes have different colors.

The **Graph Coloring Problem** asks for a feasible coloring of G using the minimum number of colors from C . * (Trovare una colorazione fattibile usando il minor numero possibile di colori)

*NP-hard (extremely difficult to optimize)



Applications:

- Frequency assignment
- Seating assignment
- Scheduling
- Experiment design
- Computer science, etc.

The Graph Coloring Problem

È il numero minimo di colori necessari per una colorazione fattibile del grafo G .

Definition: The **chromatic number** $\chi(G)$ of graph G is the minimum number of colors needed for a feasible coloring of G .

Definition: The **clique number** $\omega(G)$ of graph G is the size of a maximum clique.

È la dimensione della clique massima in G , ovvero il numero massimo di nodi che sono tutti collegati fra loro.

Proposition: $\chi(G) \geq \omega(G)$

perché per colorare una clique di dimensione k servono almeno k colori distinti.

Observation: Nodes of same color form an independent set.

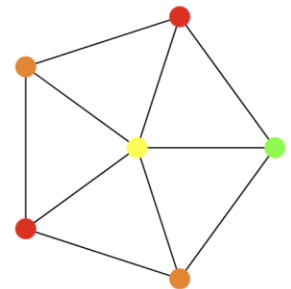
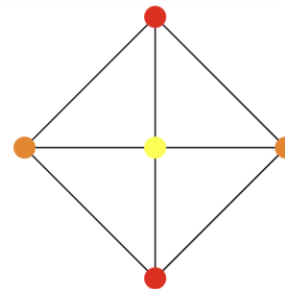
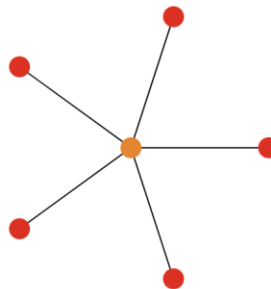
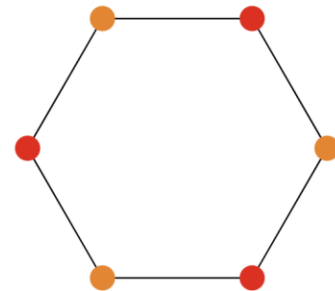
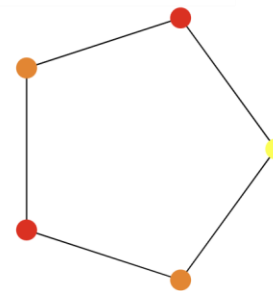
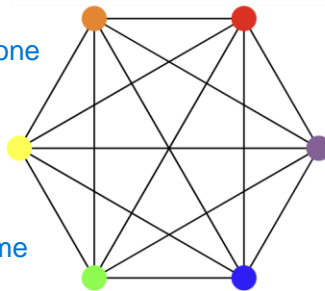
I nodi dello stesso colore formano un insieme indipendente (nessuno è adiacente).

In General: Given a clique Q and a feasible coloring with k colors. Then:

$$|Q| \leq \chi(G) \leq k$$

lower bound (LB)	upper bound (UB)
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massimo numero di colori usati in una colorazione fattibile



The Graph Coloring Problem

IP Model

Binary color assignment variable for each node and color:

$$x_{v,c} = \begin{cases} 1 & \text{if node } v \text{ will be colored with color } c, \\ 0 & \text{otherwise.} \end{cases}$$

Binary color usage variable for each color:

$$y_c = \begin{cases} 1 & \text{if color } c \text{ is used,} \\ 0 & \text{otherwise.} \end{cases}$$

Minimize $\sum_{c \in C} y_c$

Idea: Two connected nodes cannot be colored with the same color.

$$x_{u,c} + x_{v,c} \leq 1 \quad \forall \{u, v\} \in E, c \in C$$

Due nodi adiacenti (u e v) non possono essere colorati con lo stesso colore c . Questo vincolo garantisce che la colorazione sia fattibile.

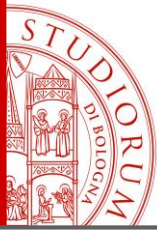
$$\frac{1}{|V|} \sum_{v \in V} x_{v,c} \leq y_c \quad \forall c \in C \quad \text{(Variable Linking)}$$

collega le variabili $x_{v,c}$ e y_c .

y_c indica se colore c è attivato; se non attivato, non ho nodi colorati da colore c ; se attivo, grazie a $1/|V|$ avrò ≤ 1

Ogni nodo deve essere colorato esattamente con un solo colore.

$$\sum_{c \in C} x_{v,c} = 1 \quad \forall v \in V \quad \text{(Color Assignment)}$$



The Graph Coloring Problem

Greedy Heuristic

Scelgo un nodo non colorato e lo coloro in modo corretto (coloro con un colore diverso dagli adiacenti (uso colore già usato ma se non posso, uso un colore nuovo))

Input: Undirected graph $G=(V,E)$; Colors $C=\{1,\dots,|V|\}$.

Nell'esempio, ho usato il metodo di scelta del nodo con indice più basso disponibile, per evitare di scegliere il prossimo nodo in modo random

1. Pick an uncolored node v in V .
→ Color v with a “feasible” used color, if possible, else use unused color.
2. If all nodes are colored return coloring, else go to 1.

The Graph Coloring Problem

DSatur Heuristic

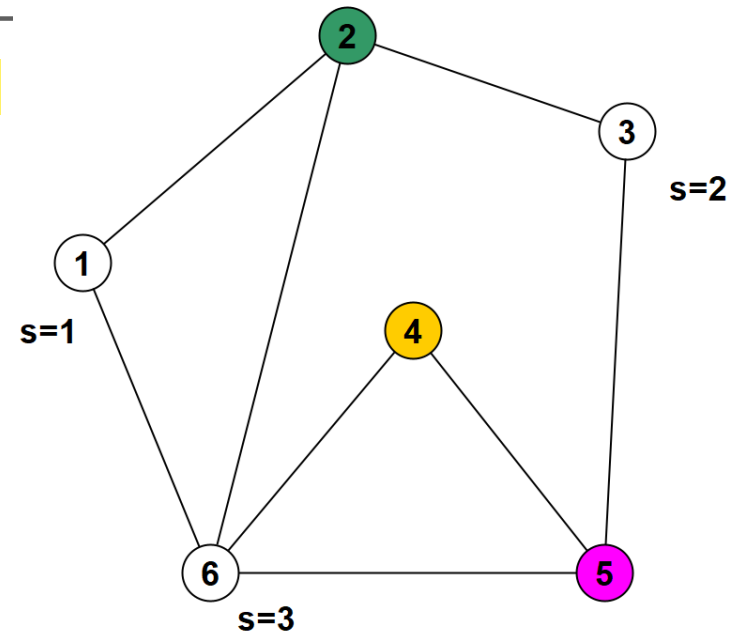
Definition: For a partial coloring of graph G , the saturation of an (uncolored) node is the number of distinct colors of its neighbors.

Saturazione di un nodo non colorato = numero di colori distinti dei nodi adiacenti al nodo non colorato

Cerco il/i nodo/i con alta saturazione e se ne ho più di uno, controllo il degree e se ne ho più di uno, scelgo quello con indice più basso

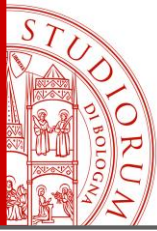
Se ho nodi con stessa saturazione e degree, scelgo il nodo con indice più basso

Scelgo nodo non colorato con saturazione più alta e degree più alto (in caso di nodi non colorati con simil saturazione). Coloro con un colore diverso da adiacenti (uso colore già usato, ma se non posso, uso un nuovo colore)



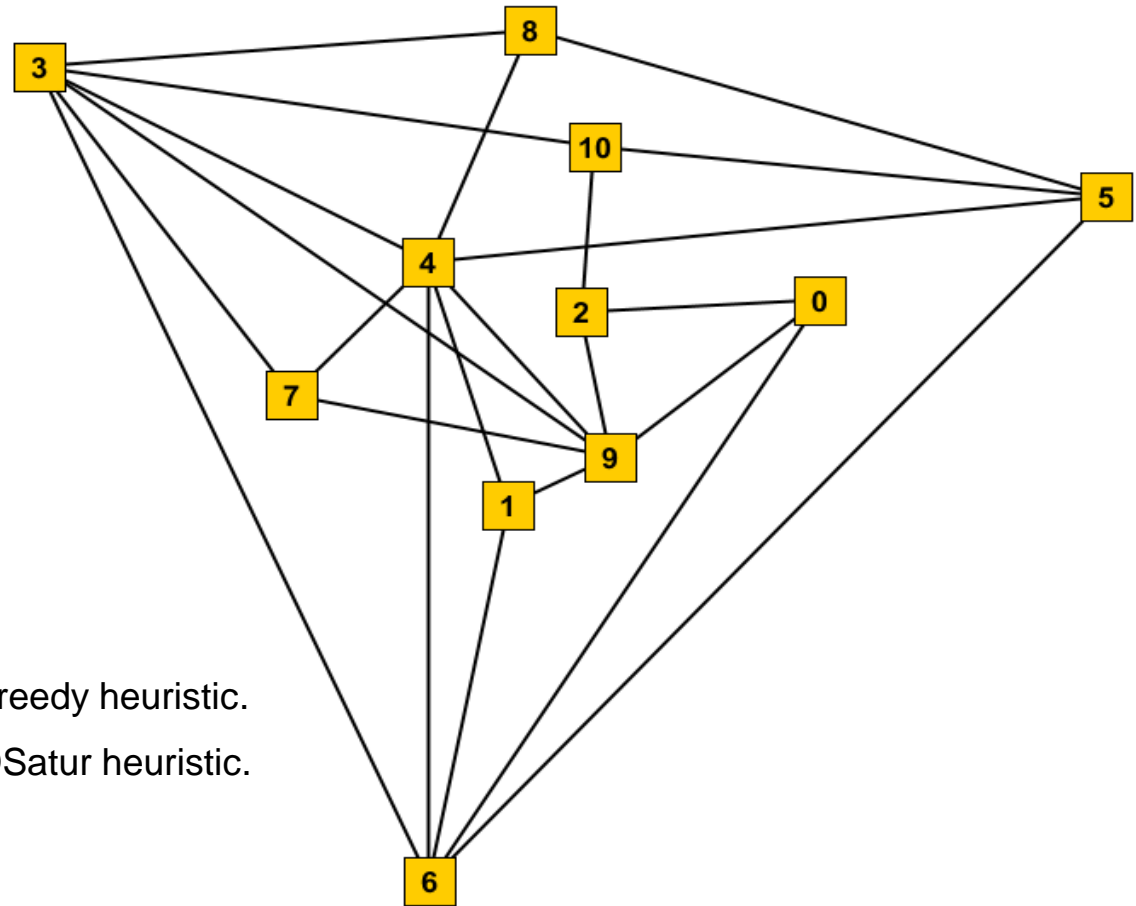
Input: Undirected graph $G=(V,E)$; Colors $C=\{1,\dots,|V|\}$.

1. Pick an uncolored node with (1) highest saturation and (2) highest degree.
→ Color node with a “feasible” used color, if possible, else use unused color.
2. If all nodes are colored return coloring, else go to 1.

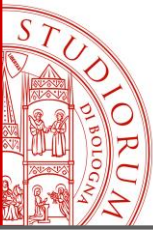


The Graph Coloring Problem

Example:



- a) Find a feasible coloring using the greedy heuristic.
- b) Find a feasible coloring using the DSatur heuristic.
- c) Find an optimal coloring using IP.



The Graph Coloring Problem

1) Exercises:

1. Consider the graph with 8 nodes.

2. Find an optimal coloring using IP.

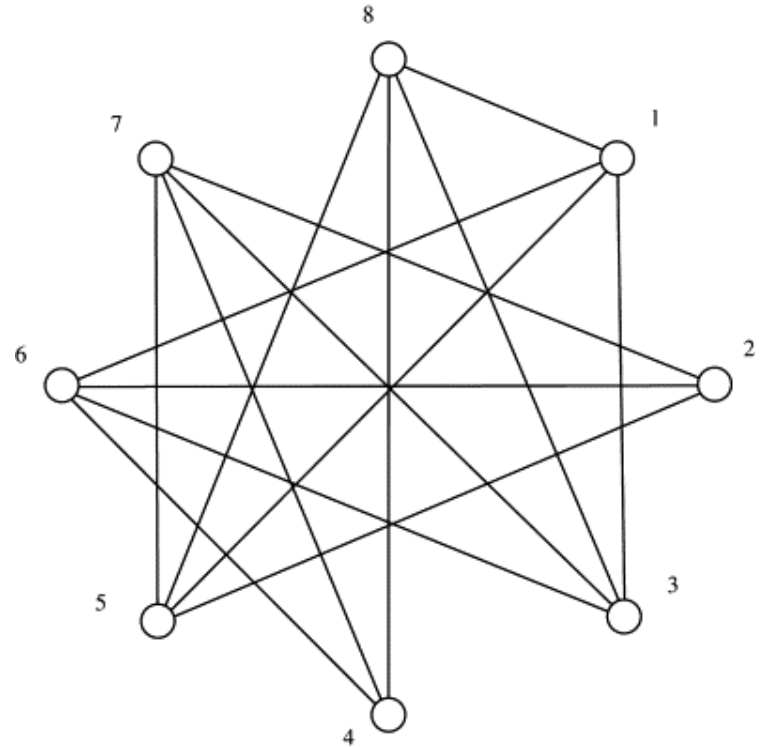
Verify your results visually.

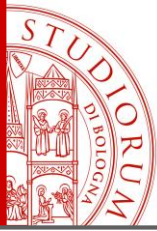
3. Find a coloring using the DSatur heuristic.

Verify your results visually.

4. Add a minimal number of edges to increase the number of needed colors by one.

Verify your results using the IP model.





The Graph Coloring Problem

2) Exercises:

1. Create a random graph with 100 nodes and an arbitrary number of edges in yEd.

Use a convenient layout for your visualization.

Format as necessary.

You can export the edge list using the TGF format.

2. Find an optimal coloring using IP.

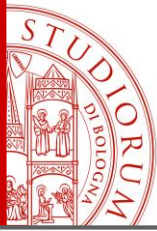
Verify your results visually.

3. Find an optimal coloring using MiniZinc.

Verify your results visually.

4. For what larger/denser graphs can you still answer the questions above?

230	9	80
231	9	81
232	10	21
233	10	38
234	10	60
235	10	82
236	10	83
237	10	104
238	10	140
239	10	68
240	10	80
241	10	81
242	11	12
243	11	13
244	11	14
245	11	15
246	11	54
247	11	84
248	11	85
249	12	13
250	12	14
251	12	15
252	12	53
253	12	54
254	12	55
255	12	84
256	12	85
257	12	41
258	12	144
259	12	143
260	13	14
261	13	15
262	13	53
263	13	54
264	13	55
265	13	84
266	13	85
267	13	144
268	13	143
269	14	15
270	14	53
271	14	55
272	14	84
273	14	85
274	14	143
275	14	144
276	15	53
277	15	54
278	15	52
279	15	55



The Graph Coloring Problem

3) Exercises:

1. Use the class social network data (Virtuale).
2. Can you find a partition of minimum size such that people in the partition set do not know each other?
Solve the graph coloring problem.
Then nodes of the same color correspond to individuals that do not know each other.
3. Can you visualize your results from 2?

