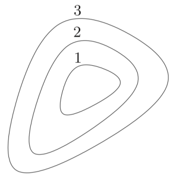
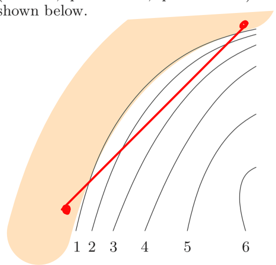
3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.

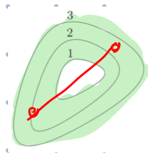


Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



① Seems to be quasiconvex.
↳ Sublevel sets seem to be convex

* Superlevel sets are not convex.
$\{x \mid f(x) \geq \alpha\}$



$\alpha = 1$

epi f not convex

② Not convex. $\{x \mid f(x) \leq \alpha\}$
Sublevel sets are not convex, then are not quasiconvex, but could be concave (then not quasilinear).

3.5 [RV73, page 22] *Running average of a convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. Show that its *running average* F, defined as

nonnegative reals

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad \mathbf{dom}\, F = \mathbf{R}_{++},$$

↳ strictly positive reals

is convex. *Hint.* For each s, $f(sx)$ is convex in x, so $\int_0^1 f(sx)\, ds$ is convex.

$\mathbb{R}_+$ means: $f(x)$ is defined for every $x \geq 0$
$\mathbb{R}_+ \subseteq \operatorname{dom} f$: the nonnegative real numbers $x \in \mathbb{R} : x \geq 0$ is a subset/contained in the domain of f (set of inputs where f is defined)

$t = sx$

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt = \frac{1}{x}\int_0^1 f(sx)\, x\, ds = \int_0^1 f(sx)\, ds$$

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

l'hopital rule

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{\frac{d}{d\alpha}(x^\alpha - 1)}{\frac{d}{d\alpha}\alpha} = \lim_{\alpha \to 0} \frac{x^\alpha \log(x) \cdot 1}{1} = \log(x)$$

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation.*

$$U_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = \frac{1-1}{\alpha} = \frac{0}{\alpha} = 0, \quad U_0(1) = \log(1) = 0$$

$$U_\alpha'(x) = \alpha \cdot (\alpha x^{\alpha-1} - 0) / \alpha^2 = x^{\alpha-1}$$

MONOTONE INCREASE

* For $x > 0$, then $x^{\alpha-1} > 0$. Hence $U_\alpha'(x) > 0 \; \forall x > 0$, strictly increasing on $(0, \infty)$
↳ $\operatorname{dom} U_0 = \mathbb{R}_{++}$

* At $x = 0$, $U_\alpha(0) = -\frac{1}{\alpha}$, and $U_\alpha(0) \leq U_\alpha(x) \;\; \forall\, x > 0$:

$$U_\alpha(X) - U_\alpha(\emptyset) = \frac{x^\alpha - 1}{\alpha} + \frac{1}{\alpha} = \frac{x^\alpha}{\alpha}$$

Since $x \geq \emptyset$, $x^\alpha \geq \emptyset$ and $\alpha > \emptyset$, so $\frac{x^\alpha}{\alpha} \geq \emptyset$

↳ $\text{dom } U_\alpha = \mathbb{R}_+$

Then, $U_\alpha(x) - U_\alpha(\emptyset) \geq \emptyset$, so $U_\alpha(x) \geq U_\alpha(\emptyset)$.

So U_α is monotone increasing on $[\emptyset, \infty)$

CONCAVITY

$$U''_\alpha(x) = (\alpha - 1)\, x^{\alpha-2}$$

Since $\emptyset < \alpha \leq 1$, then $\alpha - 1 \leq \emptyset$, so

$U''_\alpha(x) \leq \emptyset \quad \forall\ x > \emptyset$ Hence $U_\alpha(x)$ is concave

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

a) $f''(x) = e^x$, $f''(x) > \emptyset$. Convex

b) Quasiconcave

$$\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad z^T(\nabla^2 f)z = [z_1\ z_2]^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1z_2$$

If $z = (1,1)$, then $z^T(\nabla^2 f)z = 2 > \emptyset$

If $z = (1,-1)$, then $z^T(\nabla^2 f)z = -2 < \emptyset$

$\nabla^2 f$ is not PSD nor NSD. Neither convex nor concave

Quasiconvex if $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \leq \alpha\}$

Not quasiconvex as $x_1 x_2 \in \mathbb{R}^2_{++}$ and could be $> \alpha$

Quasiconcave if $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$ if $\alpha = \emptyset$, then holds (book def.)

c) Convex and quasiconvex

$$\nabla^2 f(x) = \frac{1}{x_1 x_2}\begin{bmatrix} 2/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & 2/x_2^2 \end{bmatrix}$$, since $x_1 x_2 \in \mathbb{R}^2_{++}$, then $\nabla^2 f(x) \succeq \emptyset$

d) Quasilinear

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & 2\frac{x_1}{x_2^3} \end{bmatrix}$$

Not PSD nor NSD.
Not convex nor concave

$\{(x_1, x_2) \mid \frac{x_1}{x_2} \leq \alpha\}$ $x_1, x_2 > 0$. Halfspace, convex

$\{(x_1, x_2) \mid \frac{x_1}{x_2} \geq \alpha\}$ $x_1, x_2 > 0$. Halfspace, convex.

e)

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{x} & -2\frac{x_1}{x_2^2} \\ -2\frac{x_1}{x_2^2} & 2\frac{x_1}{x_2^3} \end{bmatrix} = \frac{2}{x_2}\begin{bmatrix} 1 \\ -\frac{x_1}{x_2} \end{bmatrix}\begin{bmatrix} 1 & -\frac{x_1}{x_2} \end{bmatrix} \rightarrow$$

$x_2 > 0$, and matrix is an outer product, so $\nabla^2 f(x) \succeq 0$

so only convex and quasiconvex

3.17 *Weighted log-sum-exp.* Consider the function

$$f(w,x) = \log(w_1 \exp x_1 + \cdots + w_n \exp x_n), \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++} \times \mathbf{R}^n.$$

(a) For fixed $w \in \mathbf{R}^n_{++}$, what is the curvature of $g(x) = f(w,x)$? Is it convex, concave, both (*i.e.*, affine), or neither?

(b) For fixed $x \in \mathbf{R}^n$, what is the curvature of $h(w) = f(w,x)$? Is it convex, concave, both (*i.e.*, affine), or neither?

a) $g(x) = \log\left(\sum_{i=1}^{n} w_i \exp(x_i)\right) = \log\left(\sum_{i=1}^{n} \exp(x_i + \log w_i)\right)$ CONVEX

$x + \log w$ is an affine transformation that preserves convexity

b) $a_i = \exp(x_i) > 0$

$h(w) = \log(a^T w)$. log is concave
$a^T w$ affine in w

CONCAVE

3.21 *Some functions of the values of a probability distribution.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $p \succeq 0$, $\mathbf{1}^T p = 1$ is the given vector of probabilities. Below we give several functions of $a \in \mathbf{R}^n$, the values that the random variable takes. Is each of these functions convex, concave, affine, or neither? For each function, choose one of these. (If you select affine, this means you think the function is both convex and concave.)

Note. In this problem we consider p as given, and the quantities below as functions of a. In some homework problems, you considered the opposite, where a was given and we considered p as the variable.

(a) *Mean.* $\mathbf{E}\, x$.

(b) *Second moment.* $\mathbf{E}\, x^2$.

(c) *Third moment.* $\mathbf{E}\, x^3$.

(d) *Variance.* $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}\, x)^2$.

a) Affine

b) $\mathbb{E}x^2 = \sum_{i=1}^{n} p_i a_i^2$ Weighted sum of cvx fns Convex

c) $\mathbb{E}x^3 = \sum_{i=1}^{n} p_i a_i^3 = y$ $y'' = 6 \sum_{i=1}^{n} p_i a_i$. change of sign. Not convex nor concave on all $\mathbb{R}$

Neither

d) Convex

$$\mathrm{var}(X) = \mathbb{E}x^2 - (\mathbb{E}x)^2 = \sum_{i=1}^{n} p_i a_i^2 - (p^T a)^2$$

$$\sum_{i=1}^{n} p_i a_i^2 = a^T B a, \quad B = \mathrm{diag}(p)$$

$$(p^T a)^2 = a^T (p p^T) a$$

$$\mathrm{var}(x) = a^T B a - a^T (p p^T) a = a^T (B - p p^T) a$$

For any z,

$$z^T (B - p p^T) z = \sum_i p_i z_i^2 - \left(\sum_i p_i z_i\right)^2 \geq 0$$

$B - pp^T$ is PSD, so $\mathrm{var}(x)$ is convex