3.42 *Approximation width.* Let $f_0, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ be given continuous functions. We consider the problem of approximating f_0 as a linear combination of $f_1, \ldots, f_n$. For $x \in \mathbf{R}^n$, we say that $f = x_1 f_1 + \cdots + x_n f_n$ approximates f_0 with tolerance $\epsilon > 0$ over the interval $[0, T]$ if $|f(t) - f_0(t)| \le \epsilon$ for $0 \le t \le T$. Now we choose a fixed tolerance $\epsilon > 0$ and define the *approximation width* as the largest T such that f approximates f_0 over the interval $[0, T]$:

another fn

$$W(x) = \sup\{T \mid |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon \text{ for } 0 \le t \le T\}.$$

Show that W is quasiconcave.

→ We take all T's for which the condition holds. Then we take the supremum of that set

so $W(x)$ could be T (or kinda upper bounded by T), so it is kinda $0 \le \alpha \le T$

$0 \dashrightarrow \alpha \dashrightarrow W(x) \dashrightarrow$

↳ could also be T

- W is quasiconcave if sets $\{x \mid W(x) \ge \alpha\}$ are convex

  $W(x) \ge \alpha$ if constraint $|x_1 f_1(t) + \cdots x_n f_n(t) - f_0(t)| \le \varepsilon \quad \forall t \in [0, \alpha]$

  which is the same as $-\varepsilon \le x_1 f_1(t) + \cdots x_n f_n(t) - f_0(t) \le \varepsilon \quad \forall t \in [0, \alpha]$

- If we fix t, $x_1 f_1(t) + \cdots x_n f_n(t) - f_0(t)$ is affine in x

  * so each inequality is a halfspace in x

    $f_0(t) - \varepsilon \le x_1 f_1(t) + \cdots x_n f_n(t) = f(t)^T x$

    $f_0(t) + \varepsilon \ge x_1 f_1(t) + \cdots x_n f_n(t) = f(t)^T x$

    $\{x \in \mathbb{R}^n \mid f(t)^T x \ge f_0(t) - \varepsilon\}$

    $\{x \in \mathbb{R}^n \mid f(t)^T x \le f_0(t) + \varepsilon\}$

  * 2 halfspaces per t

So $\{x \mid W(x) \ge \alpha\}$ is the set of x satisfying two inequalities per $t \in [0, \alpha]$, an intersection of infinite halfspaces, therefore $\{x \mid W(x) \ge \alpha\}$ is convex and W quasiconcave

3.49 Show that the following functions are log-concave.

(a) *Logistic function:* $f(x) = e^x/(1+e^x)$ with $\textbf{dom} f = \mathbf{R}$.

$$\log(f(x)) = \log(e^x) - \log(1+e^x) = x - \log(1+e^x)$$

$$\frac{d}{dx}\log f(x) = 1 - \frac{1}{1+e^x}\cdot e^x = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x}$$

$$(\log f)''(x) = -\frac{e^x}{(1+e^x)^2}, \quad e^x > \emptyset \text{ and } (1+e^x)^2 > \emptyset$$

$$\text{so } (\log f)''(x) < \emptyset \quad \forall x$$

(b) *Harmonic mean:*

$$f(x) = \frac{1}{1/x_1 + \cdots + 1/x_n}, \qquad \textbf{dom} f = \mathbf{R}^n_{++}. \qquad x \text{ is a vector}$$

$$h(x) = \log f(x) = \cancel{\log(1)} - \log\left(\sum_i \frac{1}{x_i}\right)$$

Gradient

$$\frac{\partial}{\partial x_i} h(x) = \frac{1}{x_i^2}\cdot\frac{1}{\sum_k \frac{1}{x_k}}$$

Hessian

$$\frac{\partial^2}{\partial x_i^2} h(x) = \frac{1}{x_i^2}\cdot\frac{\frac{1}{x_i^2}}{\left(\sum_k \frac{1}{x_k}\right)^2} + \frac{1}{\sum_k \frac{1}{x_k}}\cdot\frac{-2x_i}{x_i^4} = \underbrace{\frac{1}{x_i^4}\cdot\frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2} - \frac{2}{x_i^3}\cdot\frac{1}{\sum_k \frac{1}{x_k}}}_{\text{diag-terms}}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} h(x) = \frac{1}{x_i^2}\cdot\frac{\frac{1}{x_j^2}}{\left(\sum_k \frac{1}{x_k}\right)^2} = \frac{1}{x_i^2 x_j^2}\cdot\frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2} \quad (i \neq j)$$

off-diag terms

Show NSD :

$$\overbrace{z^T \nabla^2 h(x)\, z}^{\text{scalar}} < \emptyset \quad \forall z \neq \emptyset$$

$$z^T \nabla^2 h(x)\, z = \sum_{i=1}^n z_i^2 \frac{\partial^2 h(x)}{\partial x_i^2} + \sum_{i\neq j} z_i z_j \frac{\partial^2 h(x)}{\partial x_i x_j}$$

$$\sum_{i=1}^n z_i^2 \frac{\partial^2 h(x)}{\partial x_i^2} = \sum_{i=1}^n z_i^2\left[\frac{1}{x_i^4}\cdot\frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2} - \frac{2}{x_i^3}\cdot\frac{1}{\sum_k \frac{1}{x_k}}\right] + \sum_{i\neq j} z_i z_j\left[\frac{1}{x_i^2 x_j^2}\cdot\frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2}\right]$$

$$= \frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2}\underbrace{\left[\sum_{i=1}^n \frac{z_i^2}{x_i^4} + \sum_{i\neq j}\frac{z_i z_j}{x_i^2 x_j^2}\right]}_{\text{off-diag + diag terms}} - \frac{2}{\sum_k \frac{1}{x_k}}\left[\sum_{i=1}^n \frac{z_i^2}{x_i^3}\right]$$

$$z^T \nabla^2 h(x)\, z = \frac{1}{\left(\sum_k \frac{1}{x_k}\right)^2}\underbrace{\left(\sum_{i=1}^n \frac{z_i}{x_i^2}\right)^2}_{\text{can't overpower for NSD to hold}} - \frac{2}{\sum_k \frac{1}{x_k}}\left[\sum_{i=1}^n \frac{z_i^2}{x_i^3}\right]$$

Cauchy-Schwarz inequality

For any vectors a, b: $(a^T b)^2 \leq (a^T a)(b^T b)$ or $(\sum_k a_k b_k)^2 \leq (\sum_k a_k^2)(\sum_k b_k^2)$

$$a_i = \frac{1}{\sqrt{x_i}}, \quad b_i = \frac{z_i}{x_i\sqrt{x_i}} = \frac{z_i}{x_i^{3/2}}$$

$$\left(\sum_{i=1}^n \frac{z_i}{x_i^2}\right)^2 = (a^T b)^2 \leq \|a\|_2^2 \|b\|_2^2 = \sum_{i=1}^n \frac{1}{x_i}\left[\sum_{i=1}^n \frac{z_i}{x_i^3}\right]$$

$$a^T b = \sum_{i=1}^n \frac{z_i}{x_i^2}$$

$$\|a\|_2^2 = \sum_{i=1}^n \frac{1}{x_i}$$

$$\|b\|_2^2 = \sum_{i=1}^n \frac{z_i^2}{x_i^3}$$

so

$$\frac{1}{(\sum_k \frac{1}{x_k})^2}(a^T b)^2 \leq 2\,\frac{1}{(\sum_k \frac{1}{x_k})^2}\|a\|_2^2\|b\|_2^2$$

$$\frac{1}{(\sum_k \frac{1}{x_k})^2}\left(\sum_{i=1}^n \frac{z_i}{x_i^2}\right)^2 \leq 2\,\frac{1}{(\sum_k \frac{1}{x_k})^{\cancel{2}}}\cancel{\sum_{i=1}^n \frac{1}{x_i}}\left[\sum_{i=1}^n \frac{z_i}{x_i^3}\right]$$

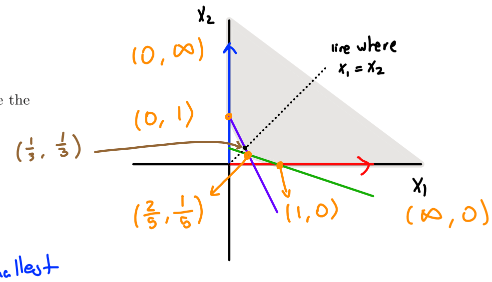
$$z^T \nabla^2 h(x)\, z = \frac{1}{(\sum_k \frac{1}{x_k})^2}\left(\sum_{i=1}^n \frac{z_i}{x_i^2}\right)^2 - \frac{2}{\sum_k \frac{1}{x_k}}\sum_{i=1}^n \frac{z_i}{x_i^3} \leq \emptyset$$

4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \ge 1 \\ & x_1 + 3x_2 \ge 1 \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.
(b) $f_0(x_1, x_2) = -x_1 - x_2$.
(c) $f_0(x_1, x_2) = x_1$.
(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.
(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.



$x_2 \ge 1 - 2x_1$

$x_2 \ge \frac{1}{3} - \frac{1}{3}x_1$

$1 - 2x_1 = \frac{1}{3} - \frac{1}{3}x_1$

$\frac{2}{3} = \frac{5}{3}x_1 \therefore x_1 = \frac{2}{5}$

$x_2 = \frac{1}{5}$

a) $x^* = (\frac{2}{5}, \frac{1}{5})$, $f_0(x^*) = \frac{3}{5}$ smallest sum

b) unbounded below

c) $X_{opt} = \{(0, x_2) \mid x_2 \ge 1\}$ smallest x_1 is 0, but x_2 can be any

$f(x_1^*) = 0$

d) $x_2 = x_1 \to$ dotted line

$x_2 \ge 1 - 2x_1 \therefore x_2 \ge \frac{1}{3}$,

for the other line, $x_1 = x_2 = \frac{1}{4}$, but no longer in the set

$x^* = (\frac{1}{3}, \frac{1}{3})$, $f_0(x^*) = \frac{1}{3}$

e) 1 constraint $\rightarrow$ $f_0(x_1, x_2) = (\frac{1}{2} - \frac{1}{2}x_2)^2 + 9x_2^2 = \frac{1}{4} - \frac{1}{2}x_2 + \frac{1}{4}x_2^2 + 9x_2^2 = \frac{37}{4}x_2^2 - \frac{1}{2}x_2 + \frac{1}{4}$

$2x_1 + x_2 \ge 1$

$x_1 = \frac{1}{2}(1 - x_2)$

$\frac{d}{dx_2} f_0 = \frac{37}{2}x_2 - \frac{1}{2} = 0$

$x_2 = \frac{1}{37}$, $x_1 = \frac{18}{37}$ $\leftarrow$ Not in the set

$x_1 + 3x_2 = \frac{18}{37} + \frac{3}{37} = \frac{21}{37} < 1$

2nd constraint $\rightarrow$ $f_0(x_1, x_2) = (1 - 3x_2)^2 + 9x_2^2 = 1 - 6x_2 + 9x_2^2 + 9x_2^2 = 18x_2^2 - 6x_2 + 1$

$x_1 + 3x_2 \ge 1$

$x_1 = 1 - 3x_2$

$\frac{d}{dx_2} f_0 = 36x_2 - 6 = 0$

$x_2 = \frac{1}{6}$, $x_1 = \frac{1}{2}$

$x^* = (\frac{1}{2}, \frac{1}{6})$

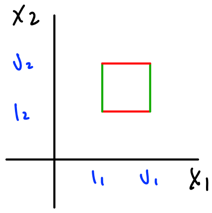
$f(x^*) = \frac{1}{2}$

4.8 *Some simple LPs.* Give an explicit solution of each of the following LPs.

(c) *Minimizing a linear function over a rectangle.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \preceq x \preceq u, \end{array}$$

where l and u satisfy $l \preceq u$.



if $c_i > \emptyset$, $x_i^* = l_i$

if $c_i < \emptyset$, $x_i^* = u_i$

$\underset{x_i \in [l_i, u_i]}{\arg\min} \left(-|c_i| x_i\right) = \underset{x}{\arg\max} (\cdots)$ $|c_i| x_i$

$$\boxed{x_i^* = \begin{cases} l_i & c_i > \emptyset \\ u_i & c_i < \emptyset \end{cases}}$$

a) $c^T x = c_1 x_1 + \cdots + c_n x_n$

$l \preceq x \preceq u$: rectangle

$c_1 l_1 \le c_1 x_1 \le c_1 u_1$

$c_2 l_2 \le c_2 x_2 \le c_2 u_2$

(d) *Minimizing a linear function over the probability simplex.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0. \end{array}$$

What happens if the equality constraint is replaced by an inequality $\mathbf{1}^T x \le 1$?

We can interpret this LP as a simple portfolio optimization problem. The vector x represents the allocation of our total budget over different assets, with x_i the fraction invested in asset i. The return of each investment is fixed and given by $-c_i$, so our total return (which we want to maximize) is $-c^T x$. If we replace the budget constraint $\mathbf{1}^T x = 1$ with an inequality $\mathbf{1}^T x \le 1$, we have the option of not investing a portion of the total budget.

$c_1 = c_2 = \cdots = c_k \le c_{k+1} \le \cdots \le c_n$

$c_{min} = c_1 (\mathbf{1}^T x) \le c^T x$

If $x_1 + \cdots + x_k = 1$, $x_{k+1} = \cdots = x_n = \emptyset$

$\boxed{p^* = c_{min}}$

4.15 *Relaxation of Boolean LP.* In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n. \end{array} \qquad (4.67)$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \le x_i \le 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \quad i = 1, \ldots, n. \end{array} \qquad (4.68)$$

We refer to this problem as the *LP relaxation* of the Boolean LP (4.67). The LP relaxation is far easier to solve than the original Boolean LP.

(a) Show that the optimal value of the LP relaxation (4.68) is a lower bound on the optimal value of the Boolean LP (4.67). What can you say about the Boolean LP if the LP relaxation is infeasible?

(b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0,1\}$. What can you say in this case?

a) $C = [0, 1]^n$ $\qquad \hat{C} \supseteq C$

$\hat{C} = \{0, 1\}^n$

$p^*_{\hat{C}} = \min_{x \in \hat{C}} c^T x \le p^*_C = \min_{x \in C} c^T x$

If the relaxation is infeasible, the Boolean LP is infeasible

b) The optimal solution for the relaxation is also optimal for the Boolean LP

3.30 *A general vector composition rule.* Suppose

$$f(x) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $g_i : \mathbf{R}^n \to \mathbf{R}$. Suppose that for each i, one of the following holds:

- h is nondecreasing in the ith argument, and g_i is convex
- h is nonincreasing in the ith argument, and g_i is concave
- g_i is affine.

Show that f is convex. This composition rule subsumes all the ones given in the book, and is the one used in software systems that are based on disciplined convex programming (DCP) such as CVX*. You can assume that $\mathbf{dom}\, h = \mathbf{R}^k$; the result also holds in the general case when the monotonicity conditions listed above are imposed on $\tilde{h}$, the extended-valued extension of h.

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

$$f(x) = h(g_1(z), \cdots g_k(z))$$

Let $z = \theta x_1 + (1-\theta)x_2$, with $x_1, x_m, \theta \in [0,1]$.

Re-arrange indexes so that g_i is affine for $i = 1, \cdots, k$, then g_i is convex for $i = k+1, \cdots, m$, and g_i concave for $i = m+1, \cdots, p$

Then
$$g_i(z) = \theta g_i(x_1) + (1-\theta) g_i(x_2), \quad i = 1, \cdots, k$$
$$g_i(z) \leq \theta g_i(x_1) + (1-\theta) g_i(x_2), \quad i = k+1, \cdots, m$$
$$g_i(z) \geq \theta g_i(x_1) + (1-\theta) g_i(x_2) \quad i = m+1, \cdots, p$$

And
$$\begin{aligned} f(z) &= h(g_1(z), \cdots, g_p(z)) \\ &\leq h(\theta g_1(x_1) + (1-\theta) g_1(x_2), \cdots, \theta g_p(x_1) + (1-\theta) g_p(x_2)) \\ &\leq \theta h(g_1(x_1), \cdots, g_p(x_1)) + (1-\theta) h(g_1(x_2), \cdots, g_p(x_2)) \\ &= \theta f(x_1) + (1-\theta) f(x_2) \end{aligned}$$

Second inequality follows from the monotonicity of h in each coordinate together with the convexity/concavity of functions g_i

For $i = 1, \cdots, k$, g_i is convex, so

$$g_i(z) \leq \theta g_i(x_1) + (1-\theta) g_i(x_2)$$

As h is nondecreasing in its ith arg, replacing $g_i(z)$ by the larger value $\theta g_i(x_1) + (1-\theta) g_i(x_2)$ can only increase the value of h.

For $i = k+1, \cdots, m$, g_i is concave, so

$$g_i(z) \geq \theta g_i(x_1) + (1-\theta) g_i(x_2)$$

As h is nonincreasing in its ith arg, replacing $g_i(z)$ by the smaller value $\theta g_i(x_1) + (1-\theta) g_i(x_2)$ can only increase the value of h.

For $i = m+1, \cdots, p$, g_i is affine and equality holds

Therefore
$$h(g_1(z), \cdots, g_p(z)) \leq h(\theta g_1(x_1) + (1-\theta) g_1(x_2), \cdots, \theta g_p(x_1) + (1-\theta) g_p(x_2))$$

The third inequality follows directly from the convexity of h

3.49 *DCP representation of inverse product.* The function $f(x) = 1/(xy)$, with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$, is convex. CVXPY includes an atom for it, called `inv_prod()`. Here we ask you to implement or express this function using other atoms and of course the DCP rules. The atoms you can use are

```
square, inv_pos, sqrt, quad_over_lin, geo_mean, norm,
pos, max, min, log, exp, log_sum_exp, power,
```

as well as affine operations like sum, difference, matrix multiply, slicing, and stacking. You can assume sign-dependent monotonicity, *e.g.*, `square` is known to be decreasing if its argument is nonpositive.

power (geo_mean (hstack ([x, y])), -2)

3.54 *Conjugate of pinball loss function.* The *pinball loss function* $f : \mathbf{R} \to \mathbf{R}$ has the form

$$f(x) = \begin{cases} -ax & x \le 0 \\ (1-a)x & x > 0, \end{cases}$$

where $a \in [0,1]$ is a parameter. (The pinball loss is used for quantile regression, but that's not relevant for this problem.)

What is the conjugate of the pinball loss? That is, what is $f^*(y)$? Be sure to specify its domain if it is not all of $\mathbf{R}$.

$\mathrm{dom}\, f = \mathbb{R}$

$$f(x) = (1-a)x^+ - ax^- \qquad f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^T x - f(x) \right)$$

For $x \le \emptyset$

$$yx + ax = (y+a)x$$

$$f^*(y) = \sup_{x \le \emptyset} (y+a)x = \begin{cases} \infty & \text{if } y+a < \emptyset \\ \emptyset & \text{if } y+a \ge \emptyset \end{cases}$$

For $x > \emptyset$

$$yx - (1-a)x = (y-1+a)x$$

$$f^*(y) = \sup_{x > \emptyset} (y-1+a)x = \begin{cases} \infty & \text{if } y+a-1 > \emptyset \\ \emptyset & \text{if } y+a-1 \le \emptyset \end{cases}$$

We need:

$y \ge -a$, $y \le 1-a$ $\rightarrow$ $-a \le y \le 1-a$ for finiteness

$$\text{So} \quad f^*(y) = \sup_{x \in \mathrm{dom} f} xy - f(x) = \begin{cases} \emptyset & \text{if } -a \le y \le 1-a \\ \infty & \text{otherwise} \end{cases}$$

3.66 *Fractional or relative error.* The fractional or relative error between two positive numbers u, v is defined as

$$E(u,v) = \frac{|u-v|}{\min\{u,v\}}.$$

Are the statements below true or false?

(a) E is a convex function of (u, v).

(b) E is a quasiconvex function of (u, v).

(c) E is a convex function of u, for fixed v.

$$E(u,v) = \frac{|u-v|}{\min\{u,v\}} = \begin{cases} \frac{u-v}{v} = \frac{u}{v} - 1 & \text{if } u \geq v \\ \frac{v-u}{u} = \frac{v}{u} - 1 & \text{if } u < v \end{cases}$$

$$E(u,v) = \max\left(\frac{u}{v}, \frac{v}{u}\right) - 1$$

a) $x_1 = (\emptyset, \emptyset)$ $\quad$ $x_2 = (2, 3)$

$E(x_1) = \emptyset$ $\quad$ $E(x_2) = 0.5$

$$E\left(\tfrac{1}{2}(\emptyset,\emptyset) + \tfrac{1}{2}(2,3)\right) = E(1, 1.5) \leq \tfrac{1}{2}E(\emptyset,\emptyset) + \tfrac{1}{2}E(2,3)$$

$$0.5 \leq 0.25$$

× not convex

b) $\{(u,v) \mid E(u,v) \leq \alpha\}$

$$E(u,v) = \max\left(\frac{u}{v}, \frac{v}{u}\right) - 1 \leq \alpha$$

- $\frac{u}{v} \leq \alpha + 1 \quad \therefore \quad u \leq v(\alpha+1)$
- $\frac{v}{u} \leq \alpha + 1 \quad \therefore \quad \frac{v}{(\alpha+1)} \leq u$

So $\left\{(u,v) \mid \frac{v}{(\alpha+1)} \leq u \leq v(\alpha+1),\ u > \emptyset,\ v > \emptyset\right\}$

Intersection of convex halfspaces, so quasiconvex

c) convex

4.2 *'Hello World' in CVXPY.* Use CVXPY to verify the optimal values you obtained (analytically) for exercise 4.1 in *Convex Optimization.*

```
import cvxpy as cp

x1 = cp.Variable()
x2 = cp.Variable()

constraints = [
    2*x1 + x2 >= 1,
    x1 + 3*x2 >= 1,
    x1 >= 0,
    x2 >= 0
]

# f0 = x1 + x2
# f0 = -x1 - x2
# f0 = x1
# f0 = cp.maximum(x1, x2)
f0 = cp.square(x1) + 9*cp.square(x2)

prob = cp.Problem(cp.Minimize(f0), constraints)
val = prob.solve()

print("status:", prob.status)
print("optimal value:", val)
print("x1*, x2* =", x1.value, x2.value)
```

```
(convex) saveasmtz@pacman:~/Documents/EE364A$ python cvx.py
status: optimal
optimal value: 0.6000000001640435
x1*, x2* = 0.4000000000142378 0.20000000014980568
(convex) saveasmtz@pacman:~/Documents/EE364A$ python cvx.py
status: unbounded
optimal value: -inf
x1*, x2* = None None
(convex) saveasmtz@pacman:~/Documents/EE364A$ python cvx.py
status: optimal
optimal value: -1.95729336465049e-11
x1*, x2* = -1.95729336465049e-11 1.6915974374433624
(convex) saveasmtz@pacman:~/Documents/EE364A$ python cvx.py
status: optimal
optimal value: 0.3333333337083394
x1*, x2* = 0.3333333337083394 0.3333333316865374
(convex) saveasmtz@pacman:~/Documents/EE364A$ python cvx.py
status: optimal
optimal value: 0.5000000000000002
x1*, x2* = 0.5000000000000001 0.1666666666666667
```

Results are similar to 4.1

20.1 *Battery energy.* The state of a chemical battery (such as a Lithium ion battery) is characterized by its terminal voltage $v \geq 0$ (in V, volts) and its charge $q \geq 0$ (in C, coulombs). These are related by $v = \phi(q)$, where $\phi : \mathbf{R} \to \mathbf{R}$ is an increasing differentiable function with $\phi(0) = 0$.

The energy (in J, Joules) stored in the battery is a function of its charge q, given by

$$E(q) = \int_0^q \phi(r)\, dr.$$

What can you say about the function E, with no further information? Is E convex, concave, or neither? Briefly justify your answer.

Remarks.

- You don't need to know any electrical or chemical engineering to answer this question.
- In EE dialect, the battery model above is a nonlinear capacitor.

$E'(q) = \phi(r)$. As $\phi(r)$ increases, so does $E'(q)$

$E''(q) = \phi'(r) = v' \geq 0$, so E is convex.