The "inverse" of $f(x) = xe^x$ is **"W"**eird to compute

An introduction to the Lambert W function

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Outline

Outline for section 1

Background

Definition

Let $W(x)$ be a function such that it satisfies $W(x)e^{W(x)} = x$

- In other words, $W(x)$ is the inverse of xe^x
- Technically, $W : \mathbb{C} \to \mathbb{C}$, but there are cases where $W : \mathbb{R} \to \mathbb{R}$ holds

Background

We have a problem!

- More than one value $W(x)$ exist for the interval $x \in (-\frac{1}{e}, 0)$ "fails vertical line test!"
- So $W(x)$ contradicts the definition of a function?

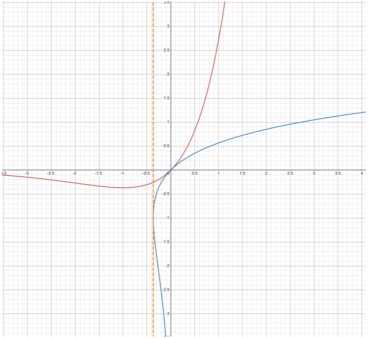

Figure: Something doesn't feel right

Background

The fix: Using branches

- Set $W_0(x)$ (principal branch) for $W(x) > -1$
- Set $W_{-1}(x)$ for $W(x) \leq -1$
- For complex solutions, there are infinitely many branches

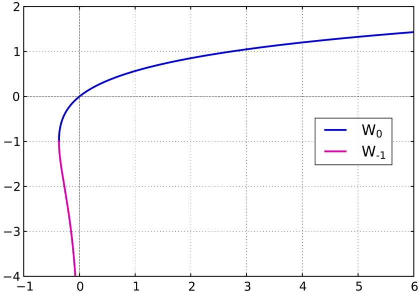

Figure: W_0 and W_{-1} branch shown (source: Wikipedia)

Background

Why did I choose this topic?

- Its a very weird function (no pun intended!)
- Can give an exact solution to certain types of equations
- Does show up in engineering, physics, disease-modelling, and analysis

Interesting things

- $W(x)$ is not an elementary function
- $W(x)$ is tricky to be represented as a combination of elementary equations

Outline for section 2

Why do we need Lambert W?

Solving nasty equations

- There are some equations where its very difficult to find an exact solution, but can be found with Lambert W.
- If we have $xe^x = z$, then $x = W(z)$.

An exponential + polynomial

$2^x + x = 5$

x^x

$x^x = 3$

Iterated exponents

$h(x) = x^{x^{x^{\cdots}}}$

Why do we need Lambert W?

Solving nasty equations

W(x) can also be used to find exact equations of certain functions with complex solutions

Real-Life Applications

Solving nasty equations

W(x) can also be used to find exact equations of certain functions with complex solutions

Interesting Note!

The derivative and integral of $W(x)$ is solvable with first year calculus!

Real-Life Applications

- There are many applications that make use of Lambert W function in engineering (chemical, electrical), fluids, disease spreading, physics, etc.
- We delve into two of them!

Current Diode Equation

Finding exact solution for non-linear circuits

Time Delay ODEs

Solution to $x'(t) = ax(t - \tau)$

Current Diode Equation

Definition

- V_s = voltage of source/battery
- V_d = voltage of diode
- I_d = diode current

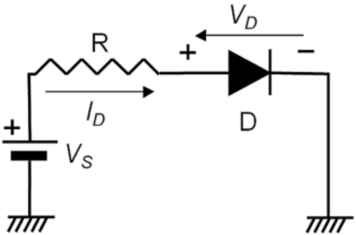

Figure: Current diode circuit (source: J.A. Gazquez et al.)

Current Diode Equation

Current Diode Equation

$$I_d = \frac{1}{R}[V_s - V_d f(I_d)] \tag{1}$$

$$I_d = I_s(e^{\frac{V_d}{\eta}} - 1) \tag{2}$$

$$V_s = V_d + RI_s e^{\frac{V_d}{V_t}} \tag{3}$$

Setting $a = \frac{1}{V_T}, b = RI_s, x = V_D, y = V_S$

$$y = f(x) = x + be^{ax} \tag{4}$$

Time Delay ODEs

Solving it

We want to find the general solution to $x'(t) = ax(t - \tau)$
If $x(t) = Ae^{\lambda t}$ is a solution iff

$$A\lambda e^{\lambda t} = Aae^{\lambda(t-\tau)}$$
$$\lambda\tau e^{\lambda t} = a\tau$$

Then, we know $\lambda\tau = W(a\tau)$

$$x(t) = A_w e^{W(a\tau)t/\tau}$$

- Used in certain ways of dealing with control systems (e.g. PID control)

Outline for section 3

Newton's Method

- Recall from first year calculus as a way to find roots of an equation
- $x_{n+1} = x_n + \dfrac{f(x_n)}{f'(x_n)}$

Using Newton's Method

$$f(x) = xe^x - z$$

$$f'(x) = (x+1)e^x$$

$$x_{n+1} = x_n - \frac{xe^x - z}{(x+1)e^x}$$

With $x_1 = 0$

Halley's Method

- Used in MATLAB / Python function
- Much computationally faster way than Newton's method
- Converges cubically (Newton's converges quadratically)
- Requires second derivative as well

Using Halley's Method

$$f''(x) = (x+2)e^x$$

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}$$

With $x_1 = 0$

Outline for section 4

Inverse Langrange Theorem

Definition

Let $f(x)$ be an analytic function. If the function is analytic at $x = a$ and $f'(a) \neq 0$. Then we can express the inverse, centered at a as

$$f^{-1}(x) = a + \sum_{n=1}^{\infty} \frac{c_n(x - f(a)^n}{n!}$$

Where $c_n = \lim_{x \to a} \frac{d^{n-1}}{dy^{n-1}} \left[\left(\frac{x-a}{f(x) - f(a)} \right)^n \right]$

Inverse Langrange Theorem

Applying it to Lambert W

- Let $x = ye^y$. We want to find $W(x)$ centered at $x = 0$.
- If $x = 0$, then we know $y = 0$
- We find the coefficient c_n

$$
\begin{aligned}
c_n &= \lim_{y \to 0} \frac{d^{n-1}}{dy^{n-1}} \left[\left(\frac{y-0}{ye^y - 0} \right)^n \right] \\
&= \lim_{y \to 0} \frac{d^{n-1}}{dy^{n-1}} \left(e^{-ny} \right) \\
&= \lim_{y \to 0} \frac{d^{n-1}}{dy^{n-1}} (-n)^{n-1} e^{-ny} \\
&= (-n)^{n-1}
\end{aligned}
$$

Applying it to Lambert W

- With c_n found, we then have

$$W(x) = \sum_{n=1}^{\infty} \frac{c_n(x-0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

- Using ratio test we get that the radius of converge is $|x| < \frac{1}{e}$

A weird definite integral

- There are ways to represent branches of Lambert W using definite integrals (albeit with complex analysis involved)

Steljles explicit representation (Kalugin et al., 2011)

For $z > -\frac{1}{e}$, we can express $W(x)$ as

$$W(x) = \frac{z}{2\pi} \int_0^{\pi} \frac{v^2 + (1 - v \cot v)^2}{z + v \csc(v) e^{-v \cot v}} dv$$

- The proof involves messing with knowing where $W(x)$ is holomorphic and using Cauchy's integral formula.
- See `https://math.stackexchange.com/questions/3347447/proof-for-integral-representation-of-lambert-w-function`

THEOREM 2.2 *The following representation of function $W(z)/z$ holds [28].*

$$\frac{W(z)}{z} = \frac{1}{\pi}\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{z + v\csc(v)e^{-v\cot v}}\,dv\ , \quad (|\arg z| < \pi)\ . \tag{4}$$

Proof From [15], we take

$$\frac{W(z)}{z} = \frac{1}{\pi}\int_{1/e}^{\infty} \frac{1}{z+t}\frac{\Im W(-t)}{t}\,dt\ , \tag{5}$$

and change to the variable $v = \Im W(t)$. From [15, Eq.1.10], this implies

$$t = t(v) = -v\csc(v)e^{-v\cot v}\ . \tag{6}$$

The integral becomes

$$\frac{W(z)}{z} = \frac{1}{\pi}\int_0^{\pi} \frac{v}{t(z-t)}\frac{dv}{v'(t)}\ , \tag{7}$$

Further simplification gives (4). ∎

Remark 1 Since the integrand in (4) is an even function in v, the integral admits the symmetric form

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v^2 + (1 - v\cot v)^2}{z + v\csc(v)e^{-v\cot v}}\,dv\ , \quad (|\arg z| < \pi)\ .$$

Figure: FYI (source: Kalugin et al., 2011)

References

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