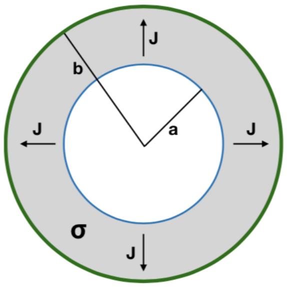
problem set

I — Spherical Resistor

Consider a resistor with conductivity σ filling the space between two concentric spherical shells with radii a and b.



a. If the shells are maintained at potential difference V, what current flows from one shell to the other?

Solution

If q is the charge on the inner shell, then the electric field in the space between the shells is:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Using Ohm's law, $\boldsymbol{J} = \sigma\boldsymbol{E}$, and the fact that current is related to current density by $I = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}$, the current flowing between the shells is:

$$I = \int \boldsymbol{J} \cdot \mathrm{da} = \sigma \int \boldsymbol{E} \cdot \mathrm{da} = \sigma\frac{q}{\epsilon_0}$$

Note the potential difference between the shells is:

$$\Delta V = V_a - V_b = -\int_b^a \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{1}{r^2}\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Thus the charge q can be expressed in terms of potential difference as:

$$q = \frac{4\pi\epsilon_0\Delta V}{\frac{1}{a} - \frac{1}{b}}$$

Substituting this into the equation above for current:

$$I = \frac{\sigma 4\pi\Delta V}{\frac{1}{a} - \frac{1}{b}}$$

where $\Delta V = V_a - V_b$.

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b. What is the resitance between the shells?

Solution

From Ohm's law:

$$R = \frac{\Delta V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

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c. Suppose we change the problem such that the conductivity is no longer uniform. Let's suppose $\sigma(r) = \frac{k}{r^2}$. Find the new resistance R between the shells. *Hint:* Since σ is not uniform, $\boldsymbol{\nabla} \cdot E \neq \frac{\boldsymbol{\nabla}\cdot J}{\sigma}$ and $\rho_e \neq 0$ in the resistive medium. This alters the behaviour of $\boldsymbol{E}$ in the medium. However, you can still use the fact that I is the same across any spherical surface.

Solution

Resistance is given by Ohm's law as $R = \frac{V}{I}$. To find resistance we must first find V and I.

Current is related to current density by $I = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}$, and by Ohm's law $\boldsymbol{J} = \sigma\boldsymbol{E}$, thus:

$$I = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} = \int \sigma\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = \int \frac{k}{r^2}\frac{q}{4\pi\epsilon_0 r^2}r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi$$
$$= \frac{kq}{4\pi\epsilon_0 r^2}\int_0^\pi \sin\theta\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi = \frac{kq}{\epsilon_0 r^2}$$

And, since $I = \frac{kq}{\epsilon_0 r^2}$, this means $q = \frac{I\epsilon_0 r^2}{k}$, and the electric field can thus be expressed:

$$\boldsymbol{E} = \frac{1}{4\pi}\frac{I}{k}$$

The potential is:

$$V = -\int_b^a \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\int_b^a \frac{1}{4\pi}\frac{I}{k}\mathrm{d}r = \frac{I}{4\pi k}(b - a)$$

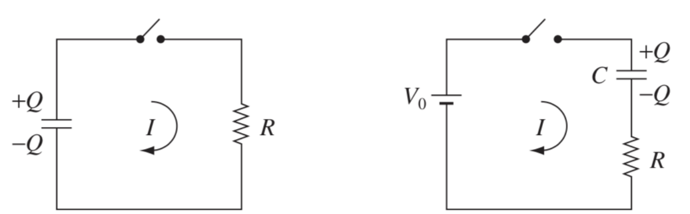
And, using Ohm's law, the resistance between the shells is thus:

$$R = \frac{V}{I} = \frac{b - a}{4\pi k}$$

II — RC Circuit

Griffiths 7.2

A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R, and begins to discharge.



(a) (b)

a. Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

Solution

The potential of a capacitor is $V = \frac{Q}{C}$, and Ohm's law gives $V = IR$. This means $I = \frac{Q}{RC}$, and thus the charge as a function of time is:

$$Q(t) = \int -I\mathrm{d}t = \int -\frac{Q}{RC}\mathrm{d}t = Q_0 e^{-t/RC}$$

The initial charge at time $t = 0$ is $Q_0 = CV_0$, thus:

$$Q(t) = CV_0 e^{-t/RC}$$

From Ohm's law, the current through the resistor is $I = \frac{V}{R} = \frac{Q}{RC}$. As a function of time this is:

$$I(t) = \frac{Q(t)}{RC} = \frac{V_0}{R}e^{-t/RC}$$

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b. What was the original energy stored in the capacitor (eq. 2.55)? By integrating eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

Solution

From eq. 2.55, the original energy stored in the capacitor at time $t = 0$ was:

$$W = \frac{1}{2}CV_0^2$$

From eq. 7.7, power is given by $P = VI = I^2R$. The heat delivered to the resistor is:

$$\int_0^\infty P\mathrm{d}t = \int_0^\infty I^2R\mathrm{d}t = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}\mathrm{d}t$$
$$= \frac{V_0^2}{R}\Big(-\frac{1}{2}RCe^{-2t/RC}\Big)\Big|_0^\infty = \frac{1}{2}CV_0^2$$

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c. Next, imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (fig b). Again, determine $Q(t)$ and $I(t)$.

Solution

For a charging capacitor, the initial charge on the capacitor is zero, and the final charge is CV_0. From the answer in part (a), we can logically deduce that the charge as a function of time must be:

$$Q(t) = CV_0(1 - e^{-t/RC})$$

It also follows that the current as a function of time is simply:

$$I(t) = \frac{V_0}{R}e^{-t/RC}$$

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d. Find the total energy output of the battery, $\int V_0 I dt$. Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? Note that the answer is independent of R.

Solution

The total energy output of the battery is:

$$\int_0^\infty V_0I\mathrm{d}t = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}\mathrm{d}t = \frac{V_0^2}{R}\big(-RCe^{-t/RC}\big)\Big|_0^\infty = CV_0^2$$

The current as a function of time is:

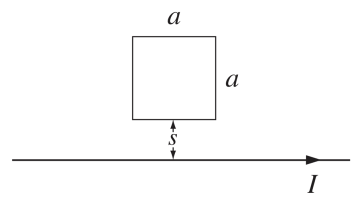
$$I(t) = \frac{V_0}{R}e^{-t/RC}$$

It follows that the heat delivered to the resistor is $\frac{1}{2}CV_0^2$ and the final energy stored by the capacitor is also $\frac{1}{2}CV_0^2$. In other words, *half* the work done by the battery shows up as energy in the capacitor.

III — Motional EMF

Griffiths 7.8

A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I.



a. Find the flux of $\boldsymbol{B}$ through the loop.

Solution

The field of the wire, given by Ampère's law, is:

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$

The flux of $\boldsymbol{B}$ through the loop is thus:

$$\Phi_B = \int \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{a} = \frac{\mu_0 I}{2\pi}\int_s^{s+a}\frac{a}{s}\mathrm{d}s$$
$$= \frac{\mu_0 Ia}{2\pi}\ln\left(\frac{s+a}{s}\right)$$

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b. If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or anticlockwise) does the current flow?

Solution

Faraday's law gives the emf $\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$. Using the answer from part (a):

$$\varepsilon = \frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\mu_0 Ia}{2\pi}\ln\left(\frac{s+a}{s}\right)\right] = -\frac{\mu_0 Ia}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ln(s+a) - \ln(s)\Big)$$

Using the fact that $\frac{\mathrm{d}s}{\mathrm{d}t} = v$, the expression above becomes:

$$\varepsilon = -\frac{\mu_0 Ia}{2\pi}\left(\frac{1}{s+a}\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{\mu_0 Ia^2 v}{2\pi s(s+a)}$$

Since the field points out of the page, the force on the loop acts *rightwards*, and Lenz's law gives the direction of the induced current as *anticlockwise*.

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c. What if the loop is pulled to the *right* at speed v?

Solution

If the loop is being pulled to the right, the flux through it is constant, and so the induced emf is zero.

$$\varepsilon = 0$$

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d. Place the loop under the wire with the centre of the loop a distance d from the wire. The wire is then released from rest, only to fall under the influence of gravity. Assume the loop has a resistance R. Find the current flowing in the wire as a function of time. Include the direction.

Solution

The field at distance d from the wire is given by Ampère's law as:

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r}$$

The total flux through the loop, when at a distance d from the wire, is:

$$\Phi = \frac{\mu_0 I a}{2\pi} \int_{d-a/s}^{d+a/2} \frac{1}{r} \mathrm{d}r = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{2d+a}{2d-a}\right)$$

Faraday's law gives the induced emf as:

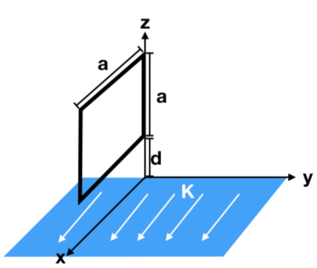
$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mu_0 a}{2\pi} \ln\left(\frac{2d+a}{2d-a}\right) \frac{\mathrm{d}I}{\mathrm{d}t}$$

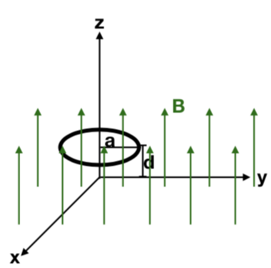
And, thus, the current in the loop is:

$$I(t) = -\frac{\mu_0 a}{2\pi R} \ln\left(\frac{2d+a}{2d-a}\right) \frac{\mathrm{d}I}{\mathrm{d}t}$$

where $\frac{\mathrm{d}I}{\mathrm{d}t}$ is the rate of change of current in the *straight* wire.

IV — Faraday Induced EMF





a. A time-dependent areal current density $K = \alpha \sin(\beta t)$ flows in the x-direction on a sheet in the xy plane. A square loop of wire with resistance R and side length a is in the xz plane, with its bottom side a distance d above the current sheet. Find the time-dependent current (magnitude and direction) in the loop.

Solution

For $z > 0$, the field due to the current sheet is:

$$\boldsymbol{B} = \frac{\mu_0 K}{2}(-\hat{\boldsymbol{y}})$$

The flux through the loop is thus:

$$\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 K a^2}{2}$$

Faraday's law gives the induced emf as:

$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mu_0 a^2}{2} \alpha \frac{\mathrm{d}}{\mathrm{d}t} \sin\beta t = -\frac{\mu_0 a^2}{2} \alpha\beta \cos\beta t$$

Ohm's law gives the current as $I = \frac{V}{R}$, in this case:

$$I(t) = \frac{\mu_0 a^2}{2R} \alpha\beta \cos\beta t$$

Lenz's law gives that the direction of the induced current as that which opposes the change in flux (which in this case oscillates).

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b. A circular loop of radius a, a distance d above the origin and the xy plane, is in a magnetic field $\boldsymbol{B} = ks^3 z \sin\frac{\phi}{3} \cos\omega t \hat{\boldsymbol{z}}$ (cylindrical coordinates), where k is a constant. Find the emf in the loop.

Solution

The field at $z = d$ is:

$$\boldsymbol{B} = ks^3 d \sin\frac{\phi}{3} \cos\omega t \hat{\boldsymbol{z}}$$

The flux through the loop is:

$$\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = kd\cos\omega t \int s^3 \sin\frac{\phi}{3} s\mathrm{d}s\mathrm{d}\phi$$

$$= kd\cos\omega t \left[\frac{s^5}{5}\right]_0^a \left[-3\cos\frac{\phi}{3}\right]_0^{2\pi} = -\frac{9}{10} kda^5 \cos\omega t$$

Faraday's law gives the induced emf in the loop as:

$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \frac{9}{10} kda^5 \omega \sin\omega t$$

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c. A circular loop of radius a and resistance R in the xy plane is in a uniform, constant magnetic field pointing in the z-direction. Suddenly the magnetic field turns off. How much charge passes through the loop during the current flow?

Solution

Turning off the magnetic field causes a change in flux:

$$\Delta\Phi_B = \Phi_B\ (\text{final}) - \Phi_B\ (\text{initial}) = -B\pi a^2$$

Faraday's law gives the induced emf as:

$$-\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} B\pi a^2$$

The charge that passes through the loop during the current flow is:

$$\mathrm{d}q = I\mathrm{d}t = \frac{V}{R}\mathrm{d}t = \frac{\frac{\mathrm{d}}{\mathrm{d}t} B\pi a^2}{R}\mathrm{d}t = \frac{B\pi a^2}{R}$$

V — Induced Electric Fields

Griffiths 7.15

A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\boldsymbol{\phi}}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

Solution

In the quasistatic approximation, the field in the solenoid is:

$$\boldsymbol{B} = \begin{cases} \mu_0 n I \hat{\boldsymbol{z}} & s < a \\ 0 & s > a \end{cases}$$

Inside the loop, Ampère's law gives:

$$\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = B\pi s^2 = \mu_0 n I \pi s^2$$

Faraday's law gives:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

where, in this case

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = E 2\pi s$$

$$-\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_0 n I \pi s^2\right) = -\mu_0 n \pi s^2 \frac{\mathrm{d}I}{\mathrm{d}t}$$

Thus:

$$E 2\pi s = -\mu_0 n \pi s^2 \frac{\mathrm{d}I}{\mathrm{d}t}$$

and the electric field *inside* the solenoid is:

$$\boldsymbol{E} = -\frac{1}{2}\mu_0 n s \frac{\mathrm{d}I}{\mathrm{d}t} \hat{\boldsymbol{\phi}}$$

Outside the loop, Ampère's law gives:

$$\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = B\pi a^2 = \mu_0 n I \pi a^2$$

And, in this case:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = E 2\pi s$$

$$-\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\mu_0 n I \pi a^2 = -\mu_0 n \pi a^2 \frac{\mathrm{d}I}{\mathrm{d}t}$$

Using Faraday's law:

$$E 2\pi s = -\mu_0 n \pi a^2 \frac{\mathrm{d}I}{\mathrm{d}t}$$

and the electric field *outside* the solenoid is:

$$\boldsymbol{E} = -\frac{1}{2s}\mu_0 n a^2 \frac{\mathrm{d}I}{\mathrm{d}t} \boldsymbol{\phi}$$

Thus the electric field inside and outside the solenoid is:

$$\boldsymbol{E} = \begin{cases} -\frac{1}{2}\mu_0 n s \frac{\mathrm{d}I}{\mathrm{d}t} \hat{\boldsymbol{\phi}} & s < a \\ \\ -\frac{1}{2s}\mu_0 n a^2 \frac{\mathrm{d}I}{\mathrm{d}t} \hat{\boldsymbol{\phi}} & s > a \end{cases}$$

VI — Variable Toroid

Griffiths 7.19

A toroidal coil has a rectangular cross section, with inner radius a, outer radius $a+w$, and height h. It carries a total of N tightly wound turns, and the current is increasing at a constant rate, i.e. $\frac{\mathrm{d}I}{\mathrm{d}t} = k$. If w and h are both much less than a, find the electric field at a point z above the centre of the toroid. *Hint:* exploit the analogy between Faraday fields and magnetostatic fields, and refer to ex. 5.6.

Solution

The magnetic field inside and outside a toroid is given by Ampère's law as:

$$\boldsymbol{B} = \begin{cases} \frac{\mu_0 N I}{2\pi s} \hat{\boldsymbol{\phi}} & \text{inside} \\ \\ 0 & \text{outside} \end{cases}$$

The flux around the toroid is:

$$\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 N I}{2\pi} \int_a^{a+w} \frac{h}{s} \mathrm{d}s = \frac{\mu_0 N I h}{2\pi} \ln\left(1 + \frac{w}{a}\right)$$

$$\approx \frac{\mu_0 N I h w}{2\pi a}$$

Given the shape of the toroid, the expression for the electric field is equivalent to that for the magnetic field of a circular current:

$$\boldsymbol{E} = \frac{1}{2}\mu_0 I \frac{a^2}{(a^2 + z^2)^{3/2}} \hat{\boldsymbol{z}}$$

Using Faraday's law, we can see that the current I can be expressed:

$$I = -\frac{1}{\mu_0}\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{N h w k}{2\pi a}$$

Substituting this into the expression for electric field above gives:

$$\boldsymbol{E} = -\frac{\mu_0}{4\pi}\frac{N h w k a}{(a^2 + z^2)^{3/2}} \hat{\boldsymbol{z}}$$