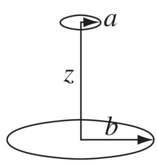
# problem set

## I — Mutual Inductance of Two Loops

*Griffiths 7.22*

A small loop of wire (radius $a$) is held a distance $z$ above a large loop (radius $b$), as shown in the figure below. The planes of the two loops are parallel, and perpendicular to the common axis.




**a.** Suppose current $I$ flows in the big loop. Find the flux through the little loop. Note the little loop is so small that you may consider the field of the big loop to be essentially constant.

***Solution***

The magnetic field a distance $z$ above a loop of radius $b$ is given by:

$$\boldsymbol{B} = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}} \hat{\boldsymbol{z}}$$

The flux through the small loop is:

$$\boldsymbol{\Phi}_B = \boldsymbol{B} \cdot A = \frac{\mu_0 I}{2} \frac{\pi a^2 b^2}{(b^2+z^2)^{3/2}}$$

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**b.** Suppose current $I$ flows in the little loop. Find the flux through the big loop. Note the little loop is so small that you may treat it as a magnetic dipole.

***Solution***

The field in this case is given by:

$$\boldsymbol{B} = \boldsymbol{\nabla} \cdot A = \frac{\mu_0}{4\pi} \frac{m}{r^3} (2\cos\theta \hat{\boldsymbol{r}} + \sin\theta \hat{\boldsymbol{\theta}})$$

where $m = I\pi a^2$. To find the flux through the big loop, integrate as follows:

$$\begin{aligned}\boldsymbol{\Phi}_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} &= \frac{\mu_0 I}{4\pi} \frac{\pi a^2}{r^3} \int 2\cos\theta r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\ &= \frac{\mu_0 I}{2} \frac{\pi a^2 b^2}{(b^2+z^2)^{3/2}}\end{aligned}$$

i.e. the flux is the same as in part a.

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**c.** Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

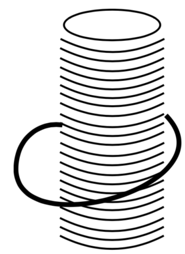
***Solution***

Since $\boldsymbol{\Phi}_1 = M_{12} I_2$ and $\boldsymbol{\Phi}_2 = M_{21} I_1$, it follows that:

$$M_{12} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}} = M_{21}$$

## II — Loop Around a Solenoid

Consider a single loop of wire wrapped around the outside of an infinite solenoid, as shown in the figure below. The solenoid is circular in cross section with radius $R$ and $n$ coils per unit length. The single loop is irregular in shape, but (of course) larger than the solenoid. Also, none of the answers require the loop to be oriented parallel to the solenoid. For simplicity, assume the loop is contained in one plane.



**a.** Find the mutual inductance $M$ between the loop and the solenoid.

***Solution***

Since the number of loops around the solenoid is 1, the mutual inductance between the loop and the solenoid is:

$$M = \mu_0 n A$$

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**b.** Suppose now that the loop goees around the solenoid twice. Again, find the mutual inductance $M$.

***Solution***

Since now the number of loops around the solenoid is 2, the mutual inductance is:

$$M = 2\mu_0 n A$$

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**c.** Find the self-inductance per unit length of the infinite solenoid, all by itself. Now check that the units of your answer work out correctly. Begin by expressing the units of $\mu_0$ in terms of Henries.

***Solution***

$$\varepsilon = -\frac{\mathrm{d}(N\mu_0 n I A)}{\mathrm{d}t} = -nl\mu_0 n A \frac{\mathrm{d}I}{\mathrm{d}t} = -\mathrm{L}\frac{\mathrm{d}I}{\mathrm{d}t}$$

Thus the self inductance $L$ is:

$$L = \mu_0 n^2 l A$$

And, the self inductance per unit length is:

$$\frac{L}{l} = \mu_0 n^2 A$$

The units on the LHS and RHS are both $\mathrm{kgm^2s^{-2}A^{-2}}$.

## III — Back EMF

*Griffiths 7.26*

An alternating current $I(t) = I_0 \cos \omega t$, which has amplitude 0.5 A and frequency 60 Hz, flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section, of inner radius 1 cm, outer radius 2cm, height 1 cm, and 1000 turns. The coil is connected to a 500 Ω resistor.

**a.** In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_R(t)$, in the resistor.

***Solution***

In the quasistatic approximation, the field is:

$$\boldsymbol{B} = \frac{\mu_0}{2\pi s}\hat{\boldsymbol{\phi}}$$

The flux through one turn of coil is:

$$\boldsymbol{\Phi}_B(\text{one turn}) = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{s} h \mathrm{d}s = \frac{\mu_0 I h}{2\pi}\ln\left(\tfrac{b}{a}\right)$$

The flux through $N$ turns of coil is thus:

$$\boldsymbol{\Phi}_B = \frac{\mu_0 N h}{2\pi}\ln\left(\tfrac{b}{a}\right) I_0 \cos\omega t$$

By Faraday's law, the induced emf is:

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mu_0 N h}{2\pi}\ln\left(\tfrac{b}{a}\right) I_0 \omega \sin\omega t$$

Substituting the values above:

$$\varepsilon = \frac{(4\pi\cdot 10^{-7})(10^3)(10^{-2})}{2\pi}\ln(2)(0.5)(2\pi\cdot 60)\sin\omega t$$

$$= 2.61\cdot 10^{-4}\sin\omega t \ \ \text{V}$$

The current in the resistor, by Ohm's law, is:

$$I_R = \frac{\varepsilon}{R} = \frac{2.61\cdot 10^{-4}}{500}\sin\omega t = 5.21\cdot 10^{-7}\sin\omega t \ \ \text{A}$$

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**b.** Calculate the back emf in the coil, due to the current $I_R(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in part a?

***Solution***

The back emf, $\varepsilon_b$, is given by $\varepsilon_b = -L\frac{\mathrm{d}I_R}{\mathrm{d}t}$, where in this case the self-inductance, $L$, is:

$$L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\tfrac{b}{a}\right) = 1.4\cdot 10^{-3} \ \ \text{H}$$

Thus the back emf is:

$$\varepsilon_b = -(1.4\cdot 10^{-3})(5.21\cdot 10^{-7}\omega\cos\omega t) = -2.78\cdot 10^{-7}\cos\omega t \ \ \text{V}$$

The ratio of amplitudes of the back emf and the "direct" emf is:

$$\frac{2.78\cdot 10^{-7}}{2.61\cdot 10^{-4}} = 1.07\cdot 10^{-3}$$

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## IV — Energy Stored in a Rotating Cylinder

*Griffiths 7.33*

An infinite cylinder of radius $R$ carries a uniform surface charge $\sigma$. We propose to set it spinning about its axis, at a final angular velocity $\omega_f$. How much work will this take, per unit length? Do it two ways, and compare your answers:

**a.** Find the magnetic field and the induced electric field (in the quasistatic approximation), inside and outside the cylinder, in terms of $\omega, \dot{\omega}$, and $s$ (the distance from the axis). Calculate the torque you must exert, and from that obtain the work done per unit length ($W = \int N \mathrm{d}\phi$).

***Solution***

The magnetic field for a solenoid is:

$$\boldsymbol{B} = \begin{cases} \mu_0 \sigma\omega R\hat{\boldsymbol{z}} & s < R \\ 0 & s > R \end{cases}$$

The electric field is:

$$\boldsymbol{E} = \begin{cases} -\frac{Rs}{2}\mu_0\sigma\dot{\omega}\hat{\boldsymbol{\phi}} & s < R \\ \\ -\frac{R^3}{2s}\mu_0\sigma\dot{\omega}\hat{\boldsymbol{\phi}} & s > R \end{cases}$$

When $s = R$, the eleectric field is:

$$\boldsymbol{E} = -\frac{1}{2}\mu_0 R^2\sigma\dot{\omega}\hat{\boldsymbol{\phi}} \qquad\qquad s = R$$

Thus the torque on a length of cylinder $l$ is:

$$\boldsymbol{\tau} = -R\sigma 2\pi R l\cdot \tfrac{1}{2}\mu_0 R^2\sigma\dot{\omega}\hat{\boldsymbol{z}} = -\pi\mu_0 R^4\sigma^2\dot{\omega} l\hat{\boldsymbol{z}}$$

The work done per unit length is:

$$\frac{W}{l} = -\pi\mu_0\sigma^2 R^4\int\frac{\mathrm{d}\omega}{\mathrm{d}t}\mathrm{d}\phi = -\frac{\mu_0\pi}{2}(\sigma\omega_f R^2)^2$$

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**b.** Use eq. 7.35 to determine the energy stored in the resulting magnetic field.

***Solution***

Equation 7.35:

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 \mathrm{d}V$$

Since the magnetic field is uniform inside the solenoid, and zero outside the solenoid:

$$W = \frac{1}{2\mu_0}B^2\pi R^2 l$$

Thus the work per unit length is:

$$\frac{W}{l} = \frac{1}{2\mu_0}(\mu_0\sigma\omega_f R)^2\pi R^2$$

$$= \frac{\mu_0\pi}{2}(\sigma\omega_f R^2)^2$$

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# V — Magnetic Field in an Alternating Capacitor

Consider a parallel-plate capacitor in an RLC circuit such that it experiences alternating current. The plates are circles of radius $R$ spaced a distance $d$ apart. Assume the surface charge on the plates is uniform and given as a function of time as $\sigma(t) = \sigma_0 \sin\omega t$.

**a.** Find the electric fieeld beetween the plates as a function of time. Assume the spacing $d$ between the plates is much smaller than the dimensions of the plates $d \ll R$ such that when computing the electric field you can assume the limit where the plates are infinitely large.

***Solution***

The electric field as a function of time is simply:

$$\boldsymbol{E}(t) = \frac{\sigma(t)}{\epsilon_0} = \frac{\sigma_0}{\epsilon_0}\sin\omega t$$

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**b.** Find the magnetic field between the plates as a function of time. Approximate the electric field as being nonzero only between the plates, i.e. $\boldsymbol{E}(s > R) = 0$.

***Solution***

The magnetic field as a function of time is:

$$\boldsymbol{B}(t) = \tfrac{1}{2}\mu_0\sigma\omega r\cos\omega t$$

where $r$ is the distance from the center of the plates.

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# VI — Magnetic Monopoles

**a.** *Griffiths 7.38:* Assuming that "Coulomb's law" for magnetic charges ($q_m$) reads:

$$\boldsymbol{F} = \frac{\mu_0}{4\pi}\frac{q_{m_1}q_{m_2}}{r^2}\hat{\boldsymbol{r}}$$

work out the force law for a monopole $q_m$ moving with velocity $\boldsymbol{v}$ through electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

***Solution***

For a monopole, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = \mu_0\rho_m$, and thus its field would be :

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\boldsymbol{r}}$$

This gives the following force law:

$$\boldsymbol{F} = q_m\big(\boldsymbol{B} - \tfrac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\big)$$

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**b.** *Griffiths 7.64:*

**i.** Show that Maxwell's equatoins with magnetic charge (eq 7.44) are invariant under the duality transformation:

$$\boldsymbol{E}' = \boldsymbol{E}\cos\alpha + c\boldsymbol{B}\sin\alpha$$
$$c\boldsymbol{B}' = c\boldsymbol{B}\cos\alpha - \boldsymbol{E}\sin\alpha$$
$$cq_e' = cq_e\cos\alpha + q_m\sin\alpha$$
$$q_m' = q_m\cos\alpha - cq_e\sin\alpha$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ and $\alpha$ is an arbitrary rotation angle in "$\boldsymbol{E}/\boldsymbol{B}$-space". Charge and current densities transform in the same way as $q_e$ and $q_m$. This means, in particular, that if you know the fields produced by a configuration of electric charge, you can immediately (using $\alpha = 90^o$) write down the fields produced by the corresponding arrangement of magnetic charge.

***Solution***

Checking each of the Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}' = (\boldsymbol{\nabla}\cdot\boldsymbol{E})\cos\alpha + c(\boldsymbol{\nabla}\cdot\boldsymbol{B})\sin\alpha = \frac{1}{\epsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha$$

$$= \frac{1}{\epsilon_0}\left(\rho_e\cos\alpha + \frac{1}{c}\rho_m\sin\alpha\right) = \frac{1}{\epsilon_0}\rho_e'$$

q.e.d.

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}' = (\boldsymbol{\nabla}\cdot\boldsymbol{B})\cos\alpha - \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\sin\alpha = \mu_0\rho_m\cos\alpha - \frac{1}{c\epsilon_0}\rho_e\sin\alpha$$

$$= \mu_0(\rho_m\cos\alpha - c\rho_e\sin\alpha) = \mu_0\rho_m'$$

q.e.d.

$$\boldsymbol{\nabla}\times\boldsymbol{E}' = (\boldsymbol{\nabla}\times\boldsymbol{E})\cos\alpha + c(\boldsymbol{\nabla}\times\boldsymbol{B})\sin\alpha$$

$$= \left(-\mu_0\boldsymbol{J}_m - \frac{\partial\boldsymbol{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\boldsymbol{J}_e + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\sin\alpha$$

$$= \mu_0\boldsymbol{J}_m' - \frac{\partial\boldsymbol{B}'}{\partial t}$$

q.e.d.

$$\boldsymbol{\nabla}\times\boldsymbol{B}' = (\boldsymbol{\nabla}\times\boldsymbol{B})\cos\alpha - \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E})\sin\alpha$$

$$= \left(\mu_0 J_e + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\cos\alpha - \frac{1}{c}\left(-\mu_0\boldsymbol{J}_m - \frac{\partial\boldsymbol{B}}{\partial t}\right)\sin\alpha$$

$$= \mu_0\boldsymbol{J}_e' + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}'}{\partial t}$$

q.e.d.

Thus, we can conclude that Maxwell's equations with magnetic charges are invariant under the duality transformation.

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**ii.** Show that the force law (prob 7.38):

$$\boldsymbol{F} = q_e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + q_m\big(\boldsymbol{B} - \tfrac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\big)$$

is also invariant under the duality transformation.

***Solution***

From problem 7.38:

$$\boldsymbol{F}' = q_e'(\boldsymbol{E}' + \boldsymbol{v}\times\boldsymbol{B}') + q_m'(\boldsymbol{B}' - \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}')$$

$$= \left(q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha\right)\left[(\boldsymbol{E}\cos\alpha + c\boldsymbol{B}\sin\alpha) + \boldsymbol{v}\times\left(\boldsymbol{B}\cos\alpha - \frac{1}{c}\boldsymbol{E}\sin\alpha\right)\right]$$

$$+\left(q_m\cos\alpha - cq_e\sin\alpha\right)\left[\left(\boldsymbol{B}\cos\alpha - \frac{1}{c}\boldsymbol{E}\sin\alpha\right) - \frac{1}{c^2}\boldsymbol{v}\times(\boldsymbol{E}\cos\alpha + c\boldsymbol{B}\sin\alpha)\right]$$

$$= q_e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + q_m\left(\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\right) = \boldsymbol{F}$$

Thus, the force law for a monopole is also invariant under the duality transformation.