problem set

I.i – Commutators (a)

Townsend 3.1
Verify for the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ that

$$\text{(a)} \qquad [\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$$

$$\text{(b)} \qquad [\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

Similarly, you can show that

$$\text{(c)} \qquad [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

Solution
The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined by the relationship:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Following this form, the commutator of the two operators $\hat{A}$ and $(\hat{B}+\hat{C})$ can be expanded as follows:

$$\begin{aligned}
[\hat{A}, \hat{B}+\hat{C}] &= \hat{A}(\hat{B}+\hat{C}) - (\hat{B}+\hat{C})\hat{A} \\
&= \hat{A}\hat{B} - \hat{B}\hat{A} + \hat{A}\hat{C} - \hat{C}\hat{A} \\
&= [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]
\end{aligned}$$

For the two operators $\hat{A}$ and $\hat{B}\hat{C}$, the commutator $[\hat{A}, \hat{B}\hat{C}]$ can be expressed as:

$$\begin{aligned}
[\hat{A}, \hat{B}\hat{C}] &= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} \\
&= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} \\
&= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]
\end{aligned}$$

where in the second step I added the terms $\pm\hat{B}\hat{A}\hat{C}$, and in the final step I factored out like terms.

Similarly, for the two operators $\hat{A}\hat{B}$ and $\hat{C}$:

$$\begin{aligned}
[\hat{A}\hat{B}, \hat{C}] &= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \\
&= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} \\
&= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}
\end{aligned}$$

I.ii – Commutators (b)

Townsend 3.8
Show that the operator $\hat{C}$ defined through $[\hat{A}, \hat{B}] = i\hat{C}$ is Hermitian, provided the operators $\hat{A}$ and $\hat{B}$ are Hermitian.

Solution
An operator $\hat{O}$ is said to be Hermitian if it satisfies the following property:

$$\hat{O}^\dagger = \hat{O}$$

i.e. if the matrix representing the operator is equal to its conjugate transpose.

In this example, the conjugate transpose of the commutator $[\hat{A}, \hat{B}]$ is:

$$\begin{aligned}
[\hat{A}, \hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger \\
&= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger = (i\hat{C})^\dagger
\end{aligned}$$

where in the final step I used the general result $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$. Since $\hat{A}$ and $\hat{B}$ are both Hermitian, it follows that $\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger = \hat{B}\hat{A} - \hat{A}\hat{B}$ and thus:

$$[\hat{A}, \hat{B}]^\dagger = -[\hat{A}, \hat{B}] = -i\hat{C}$$

But since $[\hat{A}, \hat{B}]^\dagger = (i\hat{C})^\dagger$, this means that:

$$\begin{aligned}
&(i\hat{C})^\dagger = -i\hat{C} \\
\implies &-i\hat{C}^\dagger = -i\hat{C}
\end{aligned}$$

Thus:

$$\hat{C}^\dagger = \hat{C}$$

Thus for an operator $\hat{C}$ defined through $[\hat{A}, \hat{B}] = i\hat{C}$, if $\hat{A}$ and $\hat{B}$ are Hermitian, $\hat{C}$ must also be Hermitian.

II – Schwarz Inequality

Townsend 3.7
Derive the Schwarz inequality

$$\langle\alpha|\alpha\rangle\,\langle\beta|\beta\rangle \geq |\,\langle\alpha|\beta\rangle\,|^2$$

and determine the value of λ that minimises the left-hand side of the equation.

Solution
Given the relation:

$$\big(\,\langle\alpha| + \lambda^*\,\langle\beta|\,\big)\big(\,|\alpha\rangle + \lambda\,|\beta\rangle\,\big) \geq 0$$

Choosing λ and λ^* as:

$$\lambda = \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} \qquad \lambda^* = -\frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}$$

Substituting:

$$\begin{aligned}
\big(\,\langle\alpha| + \lambda^*\,\langle\beta|\,\big)\big(\,|\alpha\rangle + \lambda\,|\beta\rangle\,\big) &= \left[\langle\alpha|\,\frac{-\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\,\langle\beta|\right]\left[|\alpha\rangle\,\frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\,|\beta\rangle\right] \\
&= \langle\alpha|\alpha\rangle\,\frac{-\langle\alpha|\beta\rangle^2}{\langle\beta|\beta\rangle^2}\,\langle\beta|\beta\rangle \\
&= \langle\alpha|\alpha\rangle\,\frac{-\langle\alpha|\beta\rangle^2}{\langle\beta|\beta\rangle} \geq 0
\end{aligned}$$

$$\therefore \;\; \langle\alpha|\alpha\rangle\,\langle\beta|\beta\rangle \geq \langle\alpha|\beta\rangle^2$$

III.i – Spin $\frac{3}{2}$ (a)

Townsend 3.22
Arsenic atoms in the ground state are spin-$\frac{3}{2}$ particles. A beam of arsenic atoms enters an SGx device, a Stern-Gerlach device with its inhomogeneous magnetic field oriented in the x direction. Atoms with $S_x = \frac{1}{2}\hbar$ then enter an SGz device. Determine the fraction of the atoms that exit the SGz device with $S_z = \frac{3}{2}\hbar$, $S_z = \frac{1}{2}\hbar$, $S_z = -\frac{1}{2}\hbar$, $S_z = -\frac{3}{2}\hbar$.

Solution
Our goal here is to compute the probability amplitude of finding a particle with $S_x = \frac{1}{2}\hbar$ in one of the S_z states. To do this we can use the representation of $\hat{S}_x$ in the S_z basis:

$$\hat{S}_x \longrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

The eigenvalue equation:

$$\hat{S}_x \left|\frac{3}{2},\mu\right\rangle_x = \mu\hbar\left|\frac{3}{2},\mu\right\rangle_x$$

where μ represents one of the spin states, has the following matrix representation:

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mu\hbar\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

which can be rearranged as follows:

$$\begin{bmatrix} -\mu & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & -\mu & 1 & 0 \\ 0 & 1 & -\mu & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & -\mu \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Setting $\mu = \frac{1}{2}$ (since in this case atoms entering the SGz device are in the state $S_x = \frac{1}{2}\hbar$, we get the following eigenvector in the S_z basis representing the eigenstate of $\hat{S}_x$ with eigenvalue $\frac{1}{2}$:

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle_x \xrightarrow[S_z]{} \begin{bmatrix} -1 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}$$

Requiring that the state be normalised:

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle_x \xrightarrow[S_z]{} \sqrt{\frac{3}{8}}\begin{bmatrix} -1 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}$$

Expressing this in terms of kets:

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle_x = -\sqrt{\frac{3}{8}}\left|\frac{3}{2},\frac{3}{2}\right\rangle - \frac{1}{2\sqrt{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle + \frac{1}{2\sqrt{2}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{3}{8}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

The probability of particles leaving the SGz with $S_z = \frac{3}{2}\hbar$ is:

$$\left|\left\langle\frac{3}{2},\frac{3}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\right|^2 = \left|-\sqrt{\frac{3}{8}}\right|^2 = \frac{3}{8}$$

Similarly the probability of particles leaving with $S_z = \frac{1}{2}\hbar$ is:

$$\left|\left\langle\frac{3}{2},\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\right|^2 = \left|-\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

The probaility of particles leaving with $S_z = -\frac{1}{2}\hbar$ is:

$$\left|\left\langle\frac{3}{2},-\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\right|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

The probability of particles leaving with $S_z = -\frac{3}{2}\hbar$ is:

$$\left|\left\langle\frac{3}{2},-\frac{3}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\right|^2 = \left|\sqrt{\frac{3}{8}}\right|^2 = \frac{3}{8}$$

III.ii – Spin $\frac{3}{2}$ (b)

Townsend 3.23 (modified)
For a spin-$\frac{3}{2}$ particle the matrix representation of the operator $\hat{S}_x$ in the S_z basis is given by

$$\hat{S}_x \longrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

Check that all four of the following states are eigenvectors with the correct eigenvalues. Check the states are orthonormal.

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \qquad \left|\frac{3}{2},\frac{1}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix} \qquad \left|\frac{3}{2},-\frac{3}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix}$$

Solution
The eigenstates and eigenvalues of $\hat{S}_x$ are given by the eigenvalue equation:

$$\hat{S}_x\left|s,m\right\rangle = m\hbar\left|s,m\right\rangle$$

where $|s,m\rangle$ represents a spin state of a spin-s particle. For a spin $\frac{3}{2}$ particle, there are four possible m values: $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$, which correspond to the four possible spin states. From the equation above, the eigenvalues for these spin states are: $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, $-\frac{3}{2}\hbar$. The proposed eigenvectors in the question can be verified by substituting them into the eigenvalue equation above.

For the state $\left|\frac{3}{2},\frac{3}{2}\right\rangle$:

$$\hat{S}_x\left|\frac{3}{2},\frac{3}{2}\right\rangle = \frac{3}{2}\hbar\left|\frac{3}{2},\frac{3}{2}\right\rangle$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3\hbar}{4\sqrt{2}} \\ \frac{\sqrt{3}\hbar}{4\sqrt{2}} + \frac{2\sqrt{3}\hbar}{4\sqrt{2}} \\ \frac{2\sqrt{3}\hbar}{4\sqrt{2}} + \frac{\sqrt{3}\hbar}{4\sqrt{2}} \\ \frac{3\hbar}{4\sqrt{2}} \end{bmatrix}$$

$$= \left(\frac{3}{2}\hbar\right)\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \qquad \therefore \text{ qed}$$

For the state $\left|\frac{3}{2},\frac{1}{2}\right\rangle$:

$$\hat{S}_x = \left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{2}\hbar\left|\frac{3}{2},\frac{1}{2}\right\rangle$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

$$= \left(\frac{1}{2}\hbar\right)\frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix} \qquad \therefore \text{ qed}$$

For the state $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$:

$$\hat{S}_x\left|\frac{3}{2},-\frac{1}{2}\right\rangle = -\frac{1}{2}\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix}$$

$$= \left(-\frac{1}{2}\hbar\right)\frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix} \qquad \therefore \text{ qed}$$

For the state $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$:

$$S_x\left|\frac{3}{2},-\frac{3}{2}\right\rangle = -\frac{3}{2}\hbar\left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix}$$

$$= \left(-\frac{3}{2}\hbar\right)\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix} \qquad \therefore \text{ qed}$$

i.e. the four proposed eigenvectors in the question are valid matrix representations of the eigenstates of $\hat{S}_x$, since they yield the appropriate eigenvalues for a spin $\frac{3}{2}$ particle.

To verify the states are orthonormal, simply compute their inner product(s).

For the states $\left|\frac{3}{2},\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2},\frac{1}{2}\right\rangle$, the inner product:

$$\left\langle\frac{3}{2},\frac{3}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle_x$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix} = 0$$

For the states $\left|\frac{3}{2},\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$, the inner product:

$$\left\langle\frac{3}{2},\frac{3}{2}\middle|\frac{3}{2},-\frac{1}{2}\right\rangle_x$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix} = 0$$

For the states $\left|\frac{3}{2},\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$, the inner product:

$$\left\langle\frac{3}{2},\frac{3}{2}\middle|\frac{3}{2},-\frac{3}{2}\right\rangle_x$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix} = 0$$

i.e. the states are clearly orthonormal (I didn't *explicitly* show the computations for the remaining three combinations of eigenvectors—since the LaTeX is tedious—but by a similar method their inner products are also demonstrably zero). This is the expected result, since the amplitude to find a state which has $S_x = m\hbar$ with $S_x = m'\hbar$ is zero for $m \neq m'$, i.e. demonstrating the general result:

$$\langle s, m' | s, m\rangle = \delta_{m'm}$$

III.iii – Spin $\frac{3}{2}$ (c)

Townsend 3.24
A spin-$\frac{3}{2}$ particle is in the state

$$|\psi\rangle \underset{S_z}{\longrightarrow} N\begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix}$$

(a) Determine a value for N so that $|\psi\rangle$ is appropriately normalised.

(b) What is $\langle S_x\rangle$ for this state? *Suggestion:* the matrix representation of $\hat{S}_x$ is given in example 3.4.

(c) What is the probability that a measurement of S_x will yield the value $\frac{\hbar}{2}$ for this state? *Suggestion:* See problem 3.23.

Solution

(a) The state $|\psi\rangle$ is normalised if $\langle\psi|\psi\rangle = 1$. In the matrix representation, this condition can be expressed:

$$N^*\begin{bmatrix} -i & 2 & 3 & -4i \end{bmatrix} N\begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix} = 1$$

i.e.

$$N^*N\Big(-(i^2) + 4 + 9 - 4(i^2)\Big) = 1$$

$$|N|^2 \cdot 30 = 1$$

$$\therefore\ N = \frac{1}{\sqrt{30}}$$

(b) The expectation value of S_x is given by:

$$\langle S_x\rangle = \langle\psi|\,\hat{S}_x\,|\psi\rangle$$

where $\hat{S}_x$ is the rotation generator. For a spin $\frac{3}{2}$ particle, $\hat{S}_x$ is:

$$\hat{S}_x \longrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(given in III.b and Townsend example 3.4).

Computing $\langle S_x \rangle$ in the matrix representation gives:

$$\langle S_x \rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} -i & 2 & 3 & -4i \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \frac{1}{\sqrt{30}} \begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix}$$

$$= \frac{\hbar}{2\sqrt{30}} \begin{bmatrix} 2\sqrt{3} & -\sqrt{3}i + 6 & 4 - 4\sqrt{3}i & 3\sqrt{3} \end{bmatrix} \frac{1}{\sqrt{30}} \begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix}$$

$$= \frac{\hbar}{60} \left(2\sqrt{3}i - 2\sqrt{3}i + 12 + 12 - 12\sqrt{3}i + 12\sqrt{3}i \right) = \frac{\hbar}{60}(24)$$

$$\therefore \ \langle S_x \rangle = \frac{2}{5}\hbar$$

(c) If a particle is in the state $|\psi\rangle$ and a measurement is carried out, the probability amplitude to find the particle in state $|\psi\rangle$ is given by $\langle\phi|\psi\rangle$. The *probability* of finding the particle in the state $|\phi\rangle$ is:

$$|\langle\phi|\psi\rangle|^2$$

From problem 3.23, the state with value $\frac{1}{2}\hbar$ has the following matrix representation in the S_x basis:

$$\frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

In this case, for the particle in state $|\psi\rangle$, if a measurement of S_x is carried out, the probability amplitude it will yield the value $\frac{1}{2}\hbar$ is given (in matrix representation) by:

$$\frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} & 1 & -1 & -\sqrt{3} \end{bmatrix} \frac{1}{\sqrt{30}} \begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}\sqrt{30}} \left(\sqrt{3}i + 2 - 3 - 4\sqrt{3}i \right)$$

And the *probability* is simply the square of the above:

$$\mathrm{P} = \left| \frac{1}{2\sqrt{2}\sqrt{30}} \left(\sqrt{3}i + 2 - 3 - 4\sqrt{3}i \right) \right|^2 = \frac{7}{60}$$

i.e. the probability a measurement of S_x will yield the value $\frac{1}{2}\hbar$ is $\frac{7}{60}$.

IV – Spin 2

Construct matrices that represent the operators $\hat{J}_x$ and $\hat{J}_y$ for particles with spin $j = 2$, in the z-basis.

Solution

For $s = 2$, there are $2s + 1 = 5$ basis states, which are $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, and $|2,-2\rangle$.

In the S_z basis, these states can be represented in matrix form as:

$$|2,2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |2,1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad ... \qquad |2,-2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Using the fact that:

$$\hat{S}_+ |s,m\rangle = \sqrt{s(s+1) - m(m+1)}\hbar \, |s, m+1\rangle$$

it follows that:

$$\hat{S}_+ |2,1\rangle = 2\hbar \, |2,2\rangle$$
$$\hat{S}_+ |2,0\rangle = \sqrt{6}\hbar \, |2,1\rangle$$
$$\hat{S}_+ |2,-1\rangle = \sqrt{6}\hbar \, |2,0\rangle$$
$$\hat{S}_+ |2,-2\rangle = 2\hbar \, |2,-1\rangle$$

Using these relations, and the matrix representations of the basis states, it is clear that the matrix elements of the raising operator must have the form:

$$\hat{S}_+ = \hbar \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(in order to satisfy the relation)

$$\hat{S}_+ |s,m\rangle = \sqrt{s(s+1) - m(m+1)}\hbar \, |s, m+1\rangle$$

The lowering operator S_- is simply the conjugate transpose of S_+, i.e.

$$\hat{S}_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Note that since:

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y$$

it follows that:

$$\hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}$$

$$\hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i}$$

Computing S_x and S_y explicitly usint the simple relations above give:

$$\hat{S}_x \rightarrow \frac{\hbar}{2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\hat{S}_y \rightarrow \frac{\hbar}{2i} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$