# problem set

## I — Hermitian Hamiltonian

*Townsend 4.1*
Show that unitarity of the infinitesimal time-evolution operator (4.4) requires that the Hamiltonian $\hat{H}$ be Hermitian.

***Solution***
The infinitesimal time-evolution operator is:

$$\hat{U}(\mathrm{d}t) = 1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t$$

where $\hat{H}$ is a generator of time translations known as the Hamiltonian.

Note that the time evolution operator, which translates kets forward in time (i.e. $\hat{U}(t)\left|\psi(0)\right\rangle = \left|\psi(t)\right\rangle$ ), must be unitary in order that time evolution does not affect the normalisation of a state (i.e. to conserve probability):

$$\langle\psi(t)|\psi(t)\rangle = \left\langle\psi(0)\right|\hat{U}^\dagger(t)\hat{U}(t)\left|\psi(0)\right\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$$\therefore \quad \hat{U}^\dagger(t)\hat{U}(t) = 1$$

Naturally the *infinitesimal* time evolution operator must also be unitary, i.e. $\hat{U}^\dagger(\mathrm{d}t)\hat{U}(\mathrm{d}t) = 1$. Note the product $\hat{U}^\dagger(\mathrm{d}t)\hat{U}(\mathrm{d}t)$ can be expanded as follows:

$$\left(1 + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t\right)\left(1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t\right) = 1$$
$$1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t - \frac{i^2}{\hbar^2}\hat{H}^\dagger\hat{H}(\mathrm{d}t)^2 = 1$$

where $(\mathrm{d}t)^2$ term is negligible (since it's infinitesimal) and thus:

$$1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t = 1$$

Subtracting 1 from each side and rearranging gives:

$$\frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t = \frac{i}{\hbar}\hat{H}\mathrm{d}t$$

And, cancelling like terms leaves:

$$\hat{H}^\dagger = \hat{H}$$

which is consistent with the condition for a Hermitian operator (that the matrix representation of the operator is equal to its adjugate transpose).

Thus it follows that the Hamiltonian operator $\hat{H}$ is Hermitian.

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# II — Expectation Values for Stationary States

*Townsend 4.3*
Use (4.16) to verify that the expectation value of an observable $A$ does not chage with time if the system is in an energy eigenstate (a stationary state) and $\hat{A}$ does not depend explicitly on time.

***Solution***
The expectation value of an observable $A$ in terms of its corresponding operator $\hat{A}$ can be expressed:

$$\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle$$

Under time evolution, the time-dependence of the expectation value can thus be expressed:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t) | \hat{A} | \psi(t) \rangle$$

Expanding the derivative (by using the product rule) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A \rangle = \left( \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t) | \right) \hat{A} | \psi(t) \rangle + \langle \psi(t) | \hat{A} \left( \frac{\mathrm{d}}{\mathrm{d}t} | \psi(t) \rangle \right) + \langle \psi(t) | \frac{\partial \hat{A}}{\partial t} | \psi(t) \rangle$$

where the last term on the RHS is included in case the operator $\hat{A}$ explicitly depends on time. In this problem we assume it does not, and thus the last term can be neglected.

Note that from the Schrodinger equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\therefore\ \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \frac{1}{i\hbar}\hat{H}|\psi(t)\rangle$$

Using this we can simplify the derivative above to read:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle &= \left(\frac{1}{-i\hbar}\langle\psi(t)|\,\hat{H}\right)\hat{A}\,|\psi(t)\rangle + \langle\psi(t)|\,\hat{A}\left(\frac{1}{i\hbar}\hat{H}\,|\psi(t)\rangle\right)\\ &= \frac{i}{\hbar}\langle\psi(t)|\,\hat{H}\hat{A} - \hat{H}\hat{A}\,|\psi(t)\rangle\\ &= \frac{i}{\hbar}\langle\psi(t)|\,[\hat{H},\hat{A}]\,|\psi(t)\rangle\end{aligned}$$

This is valid for any observable $A$ as long as $\hat{A}$ does not depend explicitly on time.

Since the Hamiltonian is the energy operator, i.e. $\langle E\rangle = \langle\psi|\,\hat{H}\,|\psi\rangle$, it follows the following eigenvalue equation:

$$\hat{H}\,|E\rangle = E\,|E\rangle$$

where $|E\rangle$ is an energy eigenstate of $\hat{H}$ with eigenvalue $E$.

Thus if the system is in an energy eigenstate, its Hamiltonian $\hat{H}$ can be replaced by the energy eigenvalue $E$. Substituting $E$ for $\hat{H}$ in the equation for $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle$ gives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle &= \frac{i}{\hbar}\langle\psi(t)|\,[\hat{H},\hat{A}]\,|\psi(t)\rangle\\ &= \frac{i}{\hbar}\langle\psi(t)|\,[E,\hat{A}]\,|\psi(t)\rangle\\ &= \frac{i}{\hbar}\langle\psi(t)|\,E\hat{A} - \hat{A}E\,|\psi(t)\rangle\\ &= \frac{i}{E}\langle\psi(t)|\,\hat{A} - \hat{A}\,|\psi(t)\rangle = 0\end{aligned}$$

Thus if the system is in an energy eigenstate and $\hat{A}$ does not depend explicitly on time, the expectation value of the observable $A$ does not depend on time either, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = 0$.

Note that if $\hat{A}$ does depend on time, i.e. $\frac{\partial\hat{A}}{\partial t} \neq 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{\partial\hat{A}}{\partial t}$, i.e. in this case the expectation value of the observable $A$ is governed by the explicit time-dependence of $\hat{A}$ (provided, of course, that the system is in an energy eigenstate).

# III — Spin $\frac{1}{2}$ in a Magnetic Field (a)

*Townsend 4.5*
A beam of spin-$\frac{1}{2}$ particles in the $|+z\rangle$ state enters a uniform magnetic field $B_0$ in the $xz$ plane oriented at an angle $\theta$ with respect to the $z$ axis. At time $T$ later, the particles enter an SG$y$ device. What is the probability the particles will be found with $S_y = \frac{\hbar}{2}$? Check your result by evaluating the special cases $\theta = 0$ and $\theta = \frac{\pi}{2}$.

***Solution***
The magnetic field in this problem can be written in terms of its components in the $x$ and $z$ directions as:

$$\boldsymbol{B}_0 = B_0 \sin\theta \boldsymbol{i} + B_0 \cos\theta \boldsymbol{k} = B_0 \boldsymbol{n}$$

where $\boldsymbol{n}$ is a unit vector in the direction of the magnetic field.

The Hamiltonian of a spin-$\frac{1}{2}$ particle in this magnetic field is:

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}_0 = -\frac{gq}{2mc} \hat{\boldsymbol{S}} \cdot \boldsymbol{B}_0 = \frac{ge}{2mc} \hat{S}_n B_0 = \omega_0 \hat{S}_n$$

where the charge of the particle is $q = -e$, and $\omega_0 = \frac{geB_0}{2mc}$.

The time evolution operator in this case is:

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0 \hat{S}_n t/\hbar}$$

Note since $\omega_0 t = \phi$:

$$e^{-i\omega_0 \hat{S}_n t/\hbar} = e^{-i\hat{S}_n \phi/\hbar} = \hat{R}(\phi \boldsymbol{n})$$

i.e. the Hamiltonian causes the spin to precess about $\boldsymbol{n}$, the direction of the magnetic field.

Note $\hat{S}_n = \boldsymbol{n} \cdot \hat{\boldsymbol{S}} = n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z$. In this case the magnetic field has no $y$-component, so $\hat{S}_n$ is:

$$\hat{S}_n = \frac{\hbar}{2} \begin{bmatrix} n_z & n_x \\ n_x & -n_z \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where the matrix on the RHS is the Pauli spin matrix $\sigma_n$ in the $S_z$ basis, as follows:

$$\sigma_n \xrightarrow[S_z]{} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

The time-evolution operator can thus be written in the $S_z$ basis as:

$$\hat{U}(t) = e^{-i\hat{S}_n\phi/\hbar} = \cos\frac{\phi}{2} - i\sigma_n \sin\frac{\phi}{2}$$

which gives:

$$\hat{U}(t) \xrightarrow[S_z]{} \begin{bmatrix} \cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta & -i\sin\frac{\phi}{2}\sin\theta \\ -i\sin\frac{\phi}{2}\sin\theta & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\cos\theta \end{bmatrix}$$

Thus the state of the particle at time $t$ is:

$$|\psi(t)\rangle = \hat{U}(t)\,|{+z}\rangle = \left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta\right]|{+z}\rangle - i\sin\frac{\phi}{2}\sin\theta\,|{-z}\rangle$$

The probability amplitude to find the particle in state $|{+y}\rangle$ at time $t$ can be expressed $\langle +y|\psi(t)\rangle$. Note that the state $|{+y}\rangle$ can be expressed as a superposition of the $|{\pm z}\rangle$ states:

$$|{+y}\rangle = \frac{1}{\sqrt{2}}\,|{+z}\rangle - \frac{i}{\sqrt{2}}\,|{-z}\rangle$$

Thus the probability amplitude $\langle +y|\psi(t)\rangle$ is:

$$\langle +y|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta\right] - \frac{1}{\sqrt{2}}\sin\frac{\phi}{2}\sin\theta\,|{-z}\rangle$$

And, the probability of finding the particle in $|{+y}\rangle$ reduces to:

$$\left|\,\langle +y|\psi(t)\rangle\,\right|^2 = \frac{1 - \sin(\omega t)\sin\theta}{2}$$

In the case that $\theta = 0$, i.e. the magnetic field is completely in the $z$-direction, the sine terms become zero and the probability of finding a particle in $|{+y}\rangle$ evalues to $P = \frac{1}{2}$.

In the case that $\theta = \frac{\pi}{2}$, i.e. the magnetic field is completely in the $x$-direction, and the probability of finding a particle in $|{+y}\rangle$ evalues to $P = \frac{1-\sin(\omega t)}{2}$. Note this has a time dependency. At time $T$, the specific probability is $P = \frac{1-\sin(\omega T)}{2}$.

# III — Spin $\frac{1}{2}$ in a Magnetic Field (b)

*Townsend 4.8*
A spin-$\frac{1}{2}$ particle, initially in a state with $S_n = \frac{\hbar}{2}$ with $\boldsymbol{n} = \sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{k}$, is in a constant magnetic field $B_0$ in the $z$ direction. Determine the state of the particle at time $t$ and determine how $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ vary with time. *Hint:* you can make use of the general spin states from problems 1.3 and 1.6.

***Solution***
The Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{k}$ (i.e. the $z$-direction) is given by:

$$\hat{H} = \omega_0 \hat{S}_z$$

where $\omega_0 = \frac{geB_0}{2mc}$. The time evolution operator is given by:

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0 \hat{S}_z t/\hbar} = e^{-iS_z\phi/\hbar} = \hat{R}(\phi\boldsymbol{k})$$

where $\phi = \omega_0 t$. Thus the Hamiltonian causes the spin to precess about the $z$-axis.

In this case the particle is initially in the state $|{+n}\rangle$, i.e. $|\psi(0)\rangle = |{+n}\rangle$. Since the magnetic field points in the $z$-direction, it is useful to express the initial state $|{+n}\rangle$ as a superposition of states $|{\pm z}\rangle$ (the eigenstates of the Hamiltonian):

$$|{+n}\rangle = \cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}|{-z}\rangle$$

Thus the initial state of the particle can be expressed:

$$|\psi(0)\rangle = |{+n}\rangle = \cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}|{-z}\rangle$$

The state of the particle at time $t$ is:

$$\begin{aligned}
|\psi(t)\rangle &= \hat{U}(t)|\psi(0)\rangle \\
&= e^{-i\hat{H}t/\hbar}\left(\cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}|{-z}\rangle\right) \\
&= e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|{+z}\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|{-z}\rangle
\end{aligned}$$

The expected value of $S_x$ is given by $\langle S_x\rangle = \langle\psi(t)|\,\hat{S}_x\,|\psi(t)\rangle$, where $\hat{S}_x = \frac{\hbar}{2}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Expanding this product:

$$\begin{aligned}\langle S_x\rangle &= \left(e^{i\omega_0 t/2}\cos\frac{\theta}{2}\langle +z| + e^{-i\omega_0 t/2}\sin\frac{\theta}{2}\langle -z|\right)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\\ &\qquad\left(e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|+z\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|-z\rangle\right)\\ &= \frac{\hbar}{2}\left(e^{i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\\ &= \frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot 2\cos(\omega_0 t) = \frac{\hbar}{2}\sin\theta\cos(\omega_0 t)\end{aligned}$$

$$\therefore\quad \langle S_x\rangle = \frac{\hbar}{2}\sin\theta\cos(\omega_0 t)$$

The expected value of $S_y$ is given by $\langle S_y\rangle = \langle\psi(t)|\,\hat{S}_y\,|\psi(t)\rangle$, where $\hat{S}_y = \frac{\hbar}{2}\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$. Expanding this product, using a similar method to above:

$$\begin{aligned}\langle S_y\rangle &= (...)\hat{S}_y(...)\\ &= \frac{i\hbar}{2}\left(e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\\ &= \frac{i\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot(-2i\sin(\omega_0 t))\\ &= \frac{\hbar}{2}\sin\theta\sin(\omega_0 t)\end{aligned}$$

The expected value of $S_z$ is given by $\langle S_z\rangle = \langle\psi(t)|\,\hat{S}_z\,|\psi(t)\rangle$, where $\hat{S}_z = \frac{\hbar}{2}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Expanding this product, using a similar method to above:

$$\begin{aligned}\langle S_z\rangle &= (...)\hat{S}_z(...)\\ &= \frac{i\hbar}{2}\left(e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\\ &= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\\ &= \frac{\hbar}{2}\cos\theta\end{aligned}$$

Thus:

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\theta \cos(\omega_0 t) \quad \langle S_y \rangle = \frac{\hbar}{2} \sin\theta \sin(\omega_0 t) \quad \langle S_z \rangle = \frac{\hbar}{2} \cos\theta$$

Naturally, since $\hat{S}_z$ commutes with the Hamiltonian, the expectation value of $S_z$ is constant and does not depend on time. But the converse is true for the expectation values of $S_x$ and $S_y$, whose operators do not commute with the Hamiltonian, and thus exhibit time-dependence.

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# IV — Time Evolution of a 3-State System

*Townsend 4.13*

Let

$$\begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}$$

be the matrix representation of the Hamiltonian for a three-state system with basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

(a) if the state of the system at time $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is $|\psi(t)\rangle$? (b) if the state of the system at time $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is $|\psi(t)\rangle$?

***Solution***
The Hamiltonian operator satisfies the following eigenvalue equation:

$$\hat{H} |E\rangle = E |E\rangle$$

where $|E\rangle$ is an energy eigenstate of $\hat{H}$ with eigenvalue $E$.

Using the matrix representation of $\hat{H}$ given above, the eigenstates and eigenvalues can be computed by solving the eigenvalue equation. First, rearranging the equation:

$$(\hat{H} - \boldsymbol{I}E) |E\rangle = 0$$

where $\boldsymbol{I}$ is the identity matrix. In matrix form this equation is:

$$\begin{bmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Taking the determinant of the matrix $(\hat{H} - \boldsymbol{I}E)$:

$$\begin{bmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{bmatrix}_{\text{det.}}$$

$$= (E_0 - E)(E_1 - E)(E_0 - E) - A^2(E_1 - E) = 0$$

Which gives three the eigenvalues $E = \{E_1, E_0 + 1, E_0 - A\}$.

Setting $E = E_1$ and substituting:

$$\begin{bmatrix} E_0 - E_1 & 0 & A \\ 0 & E_1 - E_1 & 0 \\ A & 0 & E_0 - E_1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$(E_0 - E_1)a + Ac = 0$$
$$(E_1 - E_1)b = 0$$
$$Aa + (E_0 - E_1)c = 0$$

This is only possible if $b = 1$ and $a = c = 0$. Thus the eigenstate corresponding to the eigenvalue $E = E_1$ is:

$$|E_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note this corresponds to the basis state $|2\rangle$, thus:

$$|E_1\rangle = |2\rangle$$

Next, setting $E = E_0 + A$ and substituting:

$$\begin{bmatrix} -A & 0 & A \\ 0 & E_1 - E_0 - A & 0 \\ A & 0 & -A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$-Aa + Ac = 0$$
$$(E_1 - E_0 - A)b = 0$$
$$Aa - Ac = 0$$

Thus is only possible if $a = c$ and $b = 0$. Requiring that the vector be normalized also pulls out a factor of $\frac{1}{\sqrt{2}}$. Thus the eigenstate corresponding to the eigenvalue $E = E_0 + A$ is:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Noting that this can also be written:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

means this eigenstate is a superposition of the basis states $|1\rangle$ and $|3\rangle$:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{2}} |3\rangle$$

Next, setting $E = E_0 - A$ and substituting:

$$\begin{bmatrix} A & 0 & A \\ 0 & E_1 - E_0 + A & 0 \\ A & 0 & A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$Aa + Ac = 0$$
$$(E_1 - E_0 + A)b = 0$$
$$Aa + Ac = 0$$

This is only possible if $a = -c$ and $b = 0$. Again, requiring that the vector be normalized pulls out a factor of $\frac{1}{\sqrt{2}}$. Thus the eigenstate corresponding to the eigenvalue $E = E_0 - A$ is:

$$|E_0 - A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

i.e. this eigenstate is also a superposition of the basis states $|1\rangle$ and $|3\rangle$:

$$|E_0 - A\rangle = \frac{1}{\sqrt{2}} |1\rangle - \frac{1}{\sqrt{2}} |3\rangle$$

If the initial state is $|2\rangle$:

$$|\psi(0)\rangle = |2\rangle$$

the eigenvalue is simply $E_1$, and it follows that:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |E_1\rangle$$

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If the initial state is $|3\rangle$:

$$|\psi(0)\rangle = |3\rangle$$

This state can be written as the following superposition:

$$|3\rangle = \frac{1}{\sqrt{2}} |E_0 + A\rangle + \frac{1}{\sqrt{2}} |E_0 - A\rangle$$

Thus at time $t$ the state of the system is:

$$|\psi(t)\rangle = \frac{e^{-i(E_0+A)t/\hbar}}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{e^{-i(E_0-A)t/\hbar}}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

# V — Spin $\frac{3}{2}$ in a Magnetic Field

*Townsend 4.15*
If the Hamiltonian for a spin-$\frac{3}{2}$ particle is given by

$$\hat{H} = \omega_0 \hat{S}_x$$

and at time $t = 0$ $|\psi(0)\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$, determine the probability that the particle is in the state $\left|\frac{3}{2}, -\frac{3}{2}\right\rangle$ at time $t$. Evaluate this probability when $t = \frac{\pi}{\omega_0}$ and explain your result. *Suggestion:* See problem 3.23 for the eigenstates of $\hat{S}_x$.

***Solution***
For a spin-$\frac{3}{2}$ particle there are four spin states, $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$, $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$, $\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$, and $\left|\frac{3}{2}, -\frac{3}{2}\right\rangle$. For this particle, whose Hamiltonian has eigenstates in the $S_x$ basis, the representation of these four states in the $S_z$ basis is given by:

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \qquad \left|\frac{3}{2}, \frac{1}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix} \qquad \left|\frac{3}{2}, -\frac{3}{2}\right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix}$$

For this particle, at time $t$ the state is given by:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

where in this case $|\psi(0)\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$. It is convenient to represent this state in the $S_z$ basis as:

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle_x = \frac{1}{2\sqrt{2}} \left|\frac{3}{2}, \frac{3}{2}\right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}} \left|\frac{3}{2}, \frac{1}{2}\right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}} \left|\frac{3}{2}, -\frac{1}{2}\right\rangle_z + \frac{1}{2\sqrt{2}} \left|\frac{3}{2}, -\frac{3}{2}\right\rangle_z$$

Thus at time $t$, applying the time evolution operator, the state of the particle is:

$$
\begin{aligned}
|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\Bigg( & \frac{1}{2\sqrt{2}}\left|\frac{3}{2},\frac{3}{2}\right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle_z \\
& + \frac{\sqrt{3}}{2\sqrt{2}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_z + \frac{1}{2\sqrt{2}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle_z \Bigg) \\
= & \frac{e^{-3i\omega_0 t/2}}{2\sqrt{2}}\left|\frac{3}{2},\frac{3}{2}\right\rangle_z + \frac{\sqrt{3}e^{-i\omega_0 t/2}}{2\sqrt{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle_z \\
& + \frac{\sqrt{3}e^{i\omega_0 t/2}}{2\sqrt{2}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_z + \frac{e^{3i\omega_0 t/2}}{2\sqrt{2}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle_z
\end{aligned}
$$

The probability amplitude to find the particle in state $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ is:

$$\left\langle\frac{3}{2},-\frac{3}{2}\middle|\psi(t)\right\rangle$$

where the bra vector $\left\langle\frac{3}{2},-\frac{3}{2}\right|$ in the $S_z$ basis is:

$$
\begin{aligned}
\left\langle\frac{3}{2},-\frac{3}{2}\right|_x = \Bigg( & \frac{1}{2\sqrt{2}}\left\langle\frac{3}{2},\frac{3}{2}\right|_z - \frac{\sqrt{3}}{2\sqrt{2}}\left\langle\frac{3}{2},\frac{1}{2}\right|_z \\
& + \frac{\sqrt{3}}{2\sqrt{2}}\left\langle\frac{3}{2},-\frac{1}{2}\right|_z - \frac{1}{2\sqrt{2}}\left\langle\frac{3}{2},-\frac{3}{2}\right|_z \Bigg)
\end{aligned}
$$

Taking the product $\left\langle\frac{3}{2},-\frac{3}{2}\middle|\psi(t)\right\rangle$ gives the probability amplitude as:

$$
\begin{aligned}
& \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{e^{-3i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{\sqrt{3}e^{-i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{\sqrt{3}e^{i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{e^{3i\omega_0 t/2}}{2\sqrt{2}} \end{bmatrix} \\
& = \frac{e^{-3i\omega_0 t/2}}{8} - \frac{3e^{-i\omega_0 t/2}}{8} + \frac{3e^{i\omega_0 t/2}}{8} - \frac{e^{3i\omega_0 t/2}}{8}
\end{aligned}
$$

Setting $t = \frac{\pi}{\omega_0}$, this amplitude becomes zero.

Thus the probability that the particle is in the state $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ at time $t$ is zero. This makes sense as the states $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2},\frac{3}{2}\right\rangle$ are orthogonal.