TW eq's 9.127, 9.128, 9.129

- the position-space representation of $\hat{L}$:

$$\hat{L} = \hat{\vec{r}} \times \hat{\vec{p}} \rightarrow \hat{\vec{r}} \times \frac{\hbar}{i}\vec{\nabla}$$

- expressing $\vec{\nabla}$ in spherical coordinates:

$$\hat{L} \rightarrow r\vec{u}_r \times \frac{\hbar}{i}\left(\vec{u}_r\frac{\partial}{\partial r} + \vec{u}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{\hbar}{i}\left(\vec{u}_\phi \frac{\partial}{\partial\theta} - \vec{u}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

- to find $\hat{L}_x$, take the x component of the unit vectors $\vec{u}_\phi$ and $\vec{u}_\theta$

↳ since $\underline{\vec{u}_\phi = -\vec{u}_x \sin\phi + \vec{u}_y \cos\phi}$

& $\underline{\vec{u}_\theta = \vec{u}_x \cos\theta\cos\phi + \vec{u}_y \cos\theta\sin\phi}$

→ it follows that

$$\hat{L}_x \rightarrow \frac{\hbar}{i}\left(-\sin\theta\frac{\partial}{\partial\theta} - \underbrace{\frac{\cos\theta\cos\phi}{\sin\theta}}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{\cos\theta}{\sin\theta}\cos\phi = \cot\theta\cos\phi$$

thus

$$\hat{L}_x \rightarrow \frac{\hbar}{i}\left(-\sin\theta\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

- similarly, substituting the y-component of $\vec{u}_\phi$ and $\vec{u}_\theta$:

$$\hat{L}_y \rightarrow \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

→ continued

• for 9.117: $\hat{L}_z \to \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

• combining 9.117, 9.127, and 9.128:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$= -\hbar^2\left(\sin^2\phi \frac{\partial^2}{\partial\theta^2} + \cot^2\theta\cos^2\phi\frac{\partial^2}{\partial\phi^2} + \cancel{2\sin\phi\frac{\partial}{\partial\theta}\cot\theta\cos\phi\frac{\partial}{\partial\phi}}\right)$$

$$+ -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\sin^2\phi\frac{\partial^2}{\partial\phi^2} - \cancel{2\cos\phi\frac{\partial}{\partial\theta}\cot\theta\sin\phi\frac{\partial}{\partial\phi}}\right)$$

$$+ -\hbar^2\frac{\partial^2}{\partial\phi^2}$$

$$= -\hbar^2\left[\underbrace{\sin^2\phi\frac{\partial^2}{\partial\theta^2} + \cos^2\phi\frac{\partial^2}{\partial\theta^2}}_{=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)} + \frac{\cos^2\theta}{\sin^2\theta}\cdot\cos^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\cos^2\theta}{\cancel{\sin^2\theta}}\cdot\cancel{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2}\right]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \underbrace{\left(\frac{\cos^4\theta}{\sin^2\theta} + \cos^2\theta + 1\right)}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\frac{(1-\sin^2\theta)(1-\sin^2\theta)}{\sin^2\theta} + (1-\sin^2\theta) + 1$$

$$= \frac{1}{\sin^2\theta} - \cancel{\frac{2\sin^2\theta}{\sin^2\theta}} + \frac{\sin^4\theta}{\cancel{\sin^2\theta}}\,{}^{\sin^2\theta} + 1 - \sin^2\theta + 1$$

$$= \frac{1}{\sin^2\theta} - \cancel{2} + \cancel{\sin^2\theta} + \cancel{1} - \cancel{\sin^2\theta} + \cancel{1}$$

$$= \frac{1}{\sin^2\theta}$$

$$\Rightarrow \hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Tw 9.12

• wave function $\psi(\vec{r}) = (x+y+z) f(r)$

• spherical coordinates r, θ, ϕ

$$\Rightarrow \psi(\vec{r}) = (r\sin\theta\cos\phi + r\sin\theta\sin\phi + r\cos\theta) f(r)$$

$$= r f(r)\left[\sin\theta \frac{e^{i\phi}+e^{-i\phi}}{2} + \sin\theta \frac{e^{i\phi}-e^{-i\phi}}{2i} + \cos\theta\right]$$

$$= r f(r)\left[\left(\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\right)\sin\theta + \cos\theta\right]$$

$$= r f(r)\left[\frac{1}{2}e^{i\phi}\sin\theta + \frac{1}{2}e^{-i\phi}\sin\theta + \frac{1}{2i}e^{i\phi}\sin\theta - \frac{1}{2i}e^{-i\phi}\sin\theta + \cos\theta\right]$$

$$= r f(r)\left[\left(\frac{1}{2}+\frac{1}{2i}\right)e^{i\phi}\sin\theta + \left(\frac{1}{2}-\frac{1}{2i}\right)e^{-i\phi}\sin\theta + \cos\theta\right]$$

• using the spherical harmonics:

$$Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi}\sin\theta, \quad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

• gives

$$\Rightarrow \psi(\theta,\phi) = c_1 Y_{1,1} + c_2 Y_{1,-1} + c_3 Y_{1,0}$$

• note that

$$\langle\theta,\phi|l,m\rangle = \psi(\theta,\phi) = Y_{l,m}(\theta,\phi)$$

• recall that if a particle is in an energy eigenstate then $\hat{L}^2|\psi\rangle = l(l+1)\hbar^2|\psi\rangle$

• thus: $\langle Y_{1,1}|\hat{L}^2|Y_{1,1}\rangle = l(l+1)\hbar^2$ on $l=1$

$$= 2\hbar^2$$

• and, similarly: $\langle Y_{1,-1}|\hat{L}^2|Y_{1,-1}\rangle = 2\hbar^2$

& $\langle Y_{1,0}|\hat{L}^2|Y_{1,0}\rangle = 2\hbar^2$

$\therefore$ a measurement of $\vec{L}^2$ will yield a value of $2\hbar^2$ with probability 1.

• for L_z: $L_z|\psi\rangle = m\hbar|\psi\rangle$

POSSIBLE VALUES

$\therefore \langle Y_{1,1}|\hat{L}_z|Y_{1,1}\rangle = \hbar$ $\quad \langle Y_{1,-1}|\hat{L}_z|Y_{1,-1}\rangle = -\hbar$ $\quad \langle Y_{1,0}|\hat{L}_z|Y_{1,0}\rangle = 0$

TW 9.16

A// $Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi}\sin\theta$

• lowering operator:

$$\hat{L}_-|l,m\rangle = \sqrt{l(l+1)-m(m-1)}\,\hbar\,\underbrace{|l,m-1\rangle}_{Y_{1,0}}$$

$$\therefore\ \hat{L}_-\,Y_{1,1}(\theta,\phi) = \sqrt{2}\,\hbar\,Y_{1,0}$$

• the lowering operator in position space:

$$\hat{L}_- \rightarrow \frac{\hbar}{i}e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$$

• thus

$$\hat{L}_-Y_{1,1} = \frac{\cancel{\hbar}}{i}\cancel{e^{-i\phi}}\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)\left(-\sqrt{\frac{3}{8\pi}}\,\cancel{e^{i\phi}}\sin\theta\right) = \sqrt{2}\,\cancel{\hbar}\,Y_{1,0}$$

$$\Rightarrow\ i\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{8\pi}}\sin\theta = \sqrt{2}\,Y_{1,0}$$

$$\Rightarrow\ \sqrt{\frac{3}{8\pi}}\cos\theta = \sqrt{2}\,Y_{1,0} \Rightarrow Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

QED!

B//

• $\hat{L}^2$ in position space:

$$\hat{L}^2 \rightarrow -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

• thus

$$\hat{L}^2Y_{1,1} = \left\{\cdots\right\}\left(-\sqrt{\frac{3}{8\pi}}\,e^{i\phi}\sin\theta\right)$$

$$= \hbar^2\sqrt{\frac{3}{8\pi}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)+\frac{1}{\sin\theta}i^2\right)e^{i\phi}$$

$$= \hbar^2\sqrt{\frac{3}{8\pi}}\left(\frac{1}{\sin\theta}(\cos^2\theta-\sin^2\theta)-\frac{1}{\sin\theta}\right)e^{i\phi}$$

$$= \underbrace{\hbar^2\sqrt{\frac{3}{8\pi}}\frac{1}{\sin\theta}}\left[(\cos^2\theta-\sin^2\theta)-1\right]e^{i\phi}$$

using $\cos^2\theta = 1-\sin^2\theta$

$$= \left\{\cdots\right\}\left[-2\sin^2\theta\right]e^{i\phi}$$

$$= -2\hbar^2\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$= (2\hbar^2)\underbrace{\left[-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right]}_{Y_{1,1}}$$

$$\therefore\ \hat{L}^2Y_{1,1} = (2\hbar^2)\,Y_{1,1}$$

QED

TW 10.1

- position-space repres. of radial comp. of momentum

$$\hat{p}_r \rightarrow \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

- normalisation condition:

$$\langle\psi|\psi\rangle = \int_0^\infty r^2\, dr\, |R(r)|^2 = 1$$

- thus

$$\langle\psi|\hat{p}_r|\psi\rangle = \int_0^\infty r^2 dr\, R^*(r)\frac{\hbar}{i}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)R(r)$$

où $R(r) = \frac{u(r)}{r}$

$$\therefore \frac{\hbar}{i}\int_0^\infty r^2 \frac{u^*(r)}{r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{u(r)}{r}\, dr$$

$$= \frac{\hbar}{i}\int_0^\infty r\, dr\, u^*(r)\left(\frac{u'}{r} - \cancel{\frac{u}{r^2}} + \cancel{\frac{u}{r^2}}\right)$$

$$= \frac{\hbar}{i}\int_0^\infty u^*(r)\, u'(r)\, dr$$

$$= \frac{\hbar}{i}\Big[\underbrace{u^* u\Big|_0^\infty}_{\text{goes to zero if } u(0)=0} - \int_0^\infty u^*(r)'\, u(r)\, dr\Big]$$

goes to zero if $u(0) = 0$, leaving:

$$= \int_0^\infty u^*(r)'\left[-\frac{\hbar}{i}\right]u(r)\, dr$$

$$= \left[\int_0^\infty r^2 \frac{u^*(r)}{r}\left[\frac{\hbar}{i}\frac{\partial}{\partial r} + \frac{1}{r}\right]\frac{u(r)}{r}\, dr\right]^*$$

$$= \langle\psi|\hat{p}_r|\psi\rangle^*$$

$\therefore$ $u(0) = 0$ for the expected value of p_r is to be real, i.e. $\langle\psi|\hat{p}_r|\psi\rangle = \langle\psi|\hat{p}_r|\psi\rangle^*$

TW 10.2

- an electron in the Coulomb field of the proton is in the state

$$|\psi\rangle = \frac{4}{5}|1,0,0\rangle + \frac{3i}{5}|2,1,1\rangle$$

A/

$$\langle E\rangle = P(E_1)\cdot E_1 + P(E_2)\cdot E_2$$

$\therefore$ $P(E_1) = |\langle 1,0,0|\psi\rangle|^2 = \frac{16}{25}$

$P(E_2) = |\langle 2,1,1|\psi\rangle|^2 = \frac{9}{25}$

- the energy levels for hydrogen are given by

$$E_n = \frac{-13.6\,eV}{n^2}$$

$\therefore$ $E_1 = -13.6$ $\quad E_2 = \frac{-13.6}{4} = -3.4$

$$\therefore \langle E\rangle = \frac{16}{25}(-13.6) + \frac{9}{25}(-3.4) = \underline{-9.928\,eV}$$

$$\langle \hat{L}^2\rangle = P(l_1)L_1^2 + P(l_2)L_2^2$$

$\therefore$ $P(l_1) = \frac{16}{25}$ $\quad P(l_2) = \frac{9}{25}$

$$L_1^2 = l(l+1)\hbar^2 \xrightarrow[l=0]{} 0$$

$$L_2^2 = l(l+1)\hbar^2 \xrightarrow[l=1]{} 2\hbar^2$$

$$\therefore \langle L^2\rangle = 2\hbar^2\cdot\frac{9}{25} = \frac{18\hbar^2}{25}$$

$$\langle L_z\rangle = P(l_1)L_{z_1} + P(l_2)L_{z_2}$$

$\therefore$ $P(l_1)$ & $P(l_2)$ are the same as above

and $L_{z_1} = m\hbar \xrightarrow[m=0]{} 0$ $\quad L_{z_2} = m\hbar \xrightarrow[m=1]{} \hbar$

$$\therefore \langle L_z\rangle = \frac{9\hbar}{25}$$

10.2 B)

$$|\psi(t)\rangle = e^{-iE_n t/\hbar}|\psi(0)\rangle$$

$$= e^{-iE_1 t/\hbar}\frac{4}{5}|1,0,0\rangle + e^{-iE_2 t/\hbar}\frac{3i}{5}|2,1,1\rangle$$

or, from part (a), $E_1 = -13.6eV$ &
$E_2 = -3.4eV$

$$\therefore |\psi(t)\rangle = e^{i\cdot 13.6 t/\hbar}\cdot\frac{4}{5}|1,0,0\rangle + e^{i\cdot 3.4 t/\hbar}\cdot\frac{3i}{5}|2,1,1\rangle$$

• none of the expected values vary with time since $\langle E(t)\rangle = \langle\psi(t)|\hat{H}|\psi(t)\rangle$ which involves multiplying $\psi^*(t)$ by $\psi(t)$, removing the exponential terms, and carrying time-independence of the expectation values.

• for the same reason, the expected values of $\vec{L}^2$ and L_z are also time-independent.

Tw 10.3

- at $t=0$

$$|\psi(0)\rangle = \frac{1}{2}|1,0,0\rangle + \frac{1}{\sqrt{2}}|2,1,1\rangle + \frac{1}{2}|2,1,0\rangle$$

- Hamiltonian:

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2\mu} - \frac{e^2}{|\hat{\vec{r}}|} + \omega_0 \hat{L}_z$$

- the wave fn at time t:

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}\cdot\frac{1}{2}|1,0,0\rangle + e^{-iE_{21}t/\hbar}\cdot\frac{1}{\sqrt{2}}|2,1,1\rangle + e^{-iE_{20}t/\hbar}\frac{1}{2}|2,1,0\rangle$$

as the energy eigenvalues depend on the eigenvalues of both $\hat{\vec{p}}^2$ and $\hat{L}_z$

$$\Rightarrow E_{n,m} = \frac{-13.6}{n^2}\frac{m_\pi}{m_e} + \omega_0 m\hbar$$

as m_π is the mass of the pion

$\Rightarrow$ ..

$$\langle E\rangle = \underbrace{P(E_{1,0})}_{\left(\frac{1}{2}\right)^2}E_{1,0} + \underbrace{P(E_{2,1})}_{\left(\frac{1}{\sqrt{2}}\right)^2}E_{2,1} + \underbrace{P(E_{2,0})}_{\left(\frac{1}{2}\right)^2}\underbrace{E_{2,0}}_{-E_{1,0}/4}$$

$$= \frac{1}{4}E_{1,0} + \frac{1}{2}E_{2,1} + \frac{1}{4}$$

$$\frac{-E_{1,0}}{4} + \omega\hbar$$

$$= \frac{1}{4}E_{1,0} + \frac{1}{2}\left(\frac{E_{1,0}}{4} + \omega\hbar\right) + \frac{1}{4}\left(\frac{E_{1,0}}{4}\right)$$

where $E_{1,0}$ can be evaluated as outlined here

$$\langle L_x\rangle = \langle\psi(t)|\hat{L}_x|\psi(t)\rangle = \langle\psi(t)|\frac{\hat{L}_+ + \hat{L}_-}{2}|\psi(t)\rangle$$

$$= \left[\frac{1}{2}\langle 1,0,0|e^{iE_1 t/\hbar} + \frac{1}{\sqrt{2}}\langle 2,1,1|e^{iE_{21}t/\hbar} + \frac{1}{2}\langle 2,1,0|e^{-iE_{20}t/\hbar}\right]\frac{\hat{L}_+ + \hat{L}_-}{2}\cdot$$

$$\cdot\left[\frac{1}{2}|1,0,0\rangle e^{-iE_{1,0}} + \dots\right]$$

$$= \frac{1}{4\sqrt{2}}\left[e^{i(E_{2,1}-E_{2,0})t/\hbar} + e^{-i[E_{2,1}-E_{2,0}]t/\hbar}\right]$$

$$= \frac{1}{2\sqrt{2}}\cos[(E_{2,1}-E_{2,0})t/\hbar] = \frac{1}{2\sqrt{2}}\cos\omega t$$

TW 10.3 continued

$$\langle L_z \rangle = \langle \psi | \hat{L}_z | \psi \rangle$$

$$= \left[\frac{1}{2}\langle 1,0,0| + \frac{1}{\sqrt{2}}\langle 2,1,1| + \frac{1}{2}\langle 2,1,0|\right] \hat{L}_z \left[\frac{1}{2}|1,0,0\rangle + \dots\right]$$

↳ since $\hat{L}_z |\psi\rangle = m\hbar |\psi\rangle$

it follows that

$$\langle L_z \rangle = \left[\frac{1}{2}\langle 1,0,0| + \frac{1}{\sqrt{2}}\langle 2,1,1| + \frac{1}{2}\langle 2,1,0|\right]\left[\frac{1}{\sqrt{2}}\hbar |2,1,1\rangle\right]$$

$$= \frac{\hbar}{2}$$

TW 10.5

- Bohr radius of a ground state ^1H atom:

$$r_B = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529\,\text{Å} = a_0$$

- Energy

$$E_1 = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} = -13.6\,eV$$

- For any two particle systems, the Bohr radius & ground state energy can be obtained by substituting the reduced mass, $\mu = \frac{m_1 m_2}{m_1+m_2}$, into the formulae above ↑.

A/ DEUTERON & AN ELECTRON

$\mu = 0.9997m \simeq m$

$\therefore\ r_B \simeq 0.529\,\text{Å} = a_0$

& $E_1 \simeq -13.6\,eV$

computations not shown because they are not that interesting; done on computer

B/ POSITRONIUM

$\mu = 0.5m$

$\therefore\ r_B = 1.06\,\text{Å} = 2a_0$

$E_1 = -6.8\,eV$

ditto

C/ BOUND STATE OF PROTON & -ve MUON

$\mu = 183.6\,m$

$\therefore\ r_B = 0.0288\,\text{Å}$

$E_1 \simeq -2500\,eV$

ditto

D/ GRAVITATIONAL BOUND STATE OF 2 NEUTRONS

- The gravitational force btw. the two neutrons is equal to the centripetal force

$$\frac{Gm^2}{r^2} = \frac{mv^2}{r} \Rightarrow r = \frac{Gm}{v^2}$$

as the velocity $v = \frac{n\hbar}{mr}$, giving

$$\Rightarrow r = \frac{Gm^3 r^2}{n^2\hbar^2} \simeq 3.64\times10^{24}\,m$$

(computationally)

→ continued

- the ground state energy

$$E = T + V = \frac{1}{2}mv^2 + \frac{-Gm^2}{r}$$

$$= \frac{1}{2}\frac{Gm^2}{r} - \frac{Gm^2}{r} = -\frac{1}{2}\frac{Gm^2}{r}$$

(computationally) $\simeq 1.6 \times 10^{-68}$ eV

- wavelength of radiation

↳ the energy of the emitted photon is $E_{21} = E_1\left(\frac{1}{4} - 1\right) = -\frac{3}{4}E_1$

& the wavelength of radiation is given by

$$\frac{hc}{\lambda} = -\frac{3}{4}E_1 \Rightarrow \lambda = \frac{-4hc}{3E_1}$$

- deuterium-electron

$$\lambda = \frac{-4hc}{3 \cdot 13.6} \simeq 122\,\text{nm}$$ ultraviolet

- positronium

$$\lambda = \frac{-4hc}{3 \cdot 6.8} \simeq 243\,\text{nm}$$ ultraviolet

- proton & -ve muon

$$\lambda = \frac{-4hc}{3 \cdot 2500} \simeq 0.66\,\text{nm}$$ x ray

- gravitational bound state of neutron pair

$$\lambda = \frac{-4hc}{3 \cdot (1.6 \times 10^{-68})} \simeq 1 \times 10^{65}\,\text{nm}$$

radiowaves

TW 10.6

TW 10.43(a):

$$R_{3,0} = 2\left(\frac{Z}{3a_0}\right)^{3/2}\left[1 - \frac{2Zr}{3a_0} + \frac{2(Zr)^2}{27a_0^2}\right]e^{-Zr/3a_0}$$

• in general:

$$R_{nl}(\rho) = \rho^l \left(\sum_{k=0}^{\infty} a_k \rho^k\right) e^{-\frac{\rho}{2}}$$

• power series

$$\Rightarrow R_{3,0}(\rho) = \left(\frac{2}{3}\frac{Zr}{a_0}\right)^0 \left[\sum_{k=0}^{\infty} a_k \left(\frac{2}{3}\frac{Zr}{a_0}\right)^k\right] e^{-Zr/3a_0}$$

$$= e^{-Zr/3a_0}\left[a_0 + a_1\left(\frac{2}{3}\frac{Zr}{a_0}\right) + a_2\left(\frac{2}{3}\frac{Zr}{a_0}\right)^2 + a_3\left(\frac{2}{3}\frac{Zr}{3a_0}\right)^3 + \dots\right]$$

$$= e^{-Zr/3a_0}\, a_0 \underbrace{\left[1 - \frac{2}{3}\frac{Zr}{3a_0} + \frac{2Z^2r^2}{27a_0}\right]}_{\text{similar in form to } 10.45a}$$

the factor in front is different

TW 10.7

- the Bohr radius for ^3He is 2x smaller than that for tritium and hydrogen

$$r_B(^3He) = \frac{1}{2} a_0$$

- the system is initially in the ground state of tritium, given by:

$$\psi(^3H) : \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$. note also that $\psi(^3He) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{a_0}\right)^{3/2} e^{\frac{-2r}{a_0}}$

- the probability that the electron is found in the ground state of the new atom, ^3He can be expressed:

$$|\langle \psi(^3He) | \psi(^3H) \rangle|^2$$

$$= \left| \frac{1}{\pi}\left(\frac{\sqrt{2}}{a_0}\right)^3 \int_0^\infty e^{-3r/a_0} \, d^3r \right|^2 = \frac{2^3 \cdot 16}{a_0^6}\left(\frac{a_0}{3}\right)^6 \simeq \underline{0.7}$$