problem set

I — Hermitian Hamiltonian

Townsend 4.1
Show that unitarity of the infinitesimal time-evolution operator (4.4) requires that the Hamiltonian $\hat{H}$ be Hermitian.

Solution
The infinitesimal time-evolution operator is:

$$\hat{U}(\mathrm{d}t) = 1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t$$

where $\hat{H}$ is a generator of time translations known as the Hamiltonian.

Note that the time evolution operator, which translates kets forward in time (i.e. $\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle$), must be unitary in order that time evolution does not affect the normalisation of a state (i.e. to conserve probability):

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\hat{U}^\dagger(t)\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$$\therefore\ \hat{U}^\dagger(t)\hat{U}(t) = 1$$

Naturally the *infinitesimal* time evolution operator must also be unitary, i.e. $\hat{U}^\dagger(\mathrm{d}t)\hat{U}(\mathrm{d}t) = 1$. Note the product $\hat{U}^\dagger(\mathrm{d}t)\hat{U}(\mathrm{d}t)$ can be expanded as follows:

$$\left(1 + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t\right)\left(1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t\right) = 1$$

$$1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t - \frac{i^2}{\hbar^2}\hat{H}^\dagger\hat{H}(\mathrm{d}t)^2 = 1$$

where $(\mathrm{d}t)^2$ term is negligible (since it's infinitesimal) and thus:

$$1 - \frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t = 1$$

Subtracting 1 from each side and rearranging gives:

$$\frac{i}{\hbar}\hat{H}^\dagger\mathrm{d}t = \frac{i}{\hbar}\hat{H}\mathrm{d}t$$

And, cancelling like terms leaves:

$$\hat{H}^\dagger = \hat{H}$$

which is consistent with the condition for a Hermitian operator (that the matrix representation of the operator is equal to its adjugate transpose).

Thus it follows that the Hamiltonian operator $\hat{H}$ is Hermitian.

II — Expectation Values for Stationary States

Townsend 4.3
Use (4.16) to verify that the expectation value of an observable A does not chage with time if the system is in an energy eigenstate (a stationary state) and $\hat{A}$ does not depend explicitly on time.

Solution
The expectation value of an observable A in terms of its corresponding operator $\hat{A}$ can be expressed:

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$$

Under time evolution, the time-dependence of the expectation value can thus be expressed:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\hat{A}|\psi(t)\rangle$$

Expanding the derivative (by using the product rule) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\right) + \langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle$$

where the last term on the RHS is included in case the operator $\hat{A}$ explicitly depends on time. In this problem we assume it does not, and thus the last term can be neglected.

Note that from the Schrodinger equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\therefore\ \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \frac{1}{i\hbar}\hat{H}|\psi(t)\rangle$$

Using this we can simplify the derivative above to read:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \left(\frac{1}{-i\hbar}\langle\psi(t)|\hat{H}\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left(\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle\right)$$

$$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A} - \hat{H}\hat{A}|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$$

This is valid for any observable A as long as $\hat{A}$ does not depend explicitly on time.

Since the Hamiltonian is the energy operator, i.e. $\langle E\rangle = \langle\psi|\hat{H}|\psi\rangle$, it follows the following eigenvalue equation:

$$\hat{H}|E\rangle = E|E\rangle$$

where $|E\rangle$ is an energy eigenstate of $\hat{H}$ with eigenvalue E.

Thus if the system is in an energy eigenstate, its Hamiltonian $\hat{H}$ can be replaced by the energy eigenvalue E. Substituting E for $\hat{H}$ in the equation for $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle\psi(t)|[E,\hat{A}]|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle\psi(t)|E\hat{A} - \hat{A}E|\psi(t)\rangle$$

$$= \frac{i}{E}\langle\psi(t)|\hat{A} - \hat{A}|\psi(t)\rangle = 0$$

Thus if the system is in an energy eigenstate and $\hat{A}$ does not depend explicitly on time, the expectation value of the observable A does not depend on time either, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = 0$.

Note that if $\hat{A}$ does depend on time, i.e. $\frac{\partial\hat{A}}{\partial t} \neq 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{\partial\hat{A}}{\partial t}$, i.e. in this case the expectation value of the observable A is governed by the explicit time-dependence of $\hat{A}$ (provided, of course, that the system is in an energy eigenstate).

III — Spin $\frac{1}{2}$ in a Magnetic Field (a)

A beam of spin-$\frac{1}{2}$ particles in the $|+z\rangle$ state enters a uniform magnetic field B_0 in the xz plane oriented at an angle θ with respect to the z axis. At time T later, the particles enter an SGy device. What is the probability the particles will be found with $S_y = \frac{\hbar}{2}$? Check your result by evaluating the special cases $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Solution
The magnetic field in this problem can be written in terms of its components in the x and z directions as:

$$\boldsymbol{B}_0 = B_0 \sin\theta \boldsymbol{i} + B_0 \cos\theta \boldsymbol{k} = B_0 \boldsymbol{n}$$

where $\boldsymbol{n}$ is a unit vector in the direction of the magnetic field.

The Hamiltonian of a spin-$\frac{1}{2}$ particle in this magnetic field is:

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}_0 = -\frac{gq}{2mc}\hat{\boldsymbol{S}} \cdot \boldsymbol{B}_0 = \frac{ge}{2mc}\hat{S}_n B_0 = \omega_0 \hat{S}_n$$

where the charge of the particle is $q = -e$, and $\omega_0 = \frac{geB_0}{2mc}$.

The time evolution operator in this case is:

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0 \hat{S}_n t/\hbar}$$

Note since $\omega_0 t = \phi$:

$$e^{-i\omega_0 \hat{S}_n t/\hbar} = e^{-i\hat{S}_n \phi/\hbar} = \hat{R}(\phi\boldsymbol{n})$$

i.e. the Hamiltonian causes the spin to precess about $\boldsymbol{n}$, the direction of the magnetic field.

Note $\hat{S}_n = \boldsymbol{n} \cdot \hat{\boldsymbol{S}} = n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z$. In this case the magnetic field has no y-component, so $\hat{S}_n$ is:

$$\hat{S}_n = \frac{\hbar}{2}\begin{bmatrix} n_z & n_x \\ n_x & -n_z \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where the matrix on the RHS is the Pauli spin matrix σ_n in the S_z basis, as follows:

$$\sigma_n \xrightarrow[S_z]{} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

The time-evolution operator can thus be written in the S_z basis as:

$$\hat{U}(t) = e^{-i\hat{S}_n \phi/\hbar} = \cos\frac{\phi}{2} - i\sigma_n \sin\frac{\phi}{2}$$

which gives:

$$\hat{U}(t) \xrightarrow[S_z]{} \begin{bmatrix} \cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta & -i\sin\frac{\phi}{2}\sin\theta \\ -i\sin\frac{\phi}{2}\sin\theta & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\cos\theta \end{bmatrix}$$

Thus the state of the particle at time t is:

$$|\psi(t)\rangle = \hat{U}(t)|+z\rangle = \left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta\right]|+z\rangle - i\sin\frac{\phi}{2}\sin\theta|-z\rangle$$

The probability amplitude to find the particle in state $|+y\rangle$ at time t can be expressed $\langle +y|\psi(t)\rangle$. Note that the state $|+y\rangle$ can be expressed as a superposition of the $|\pm z\rangle$ states:

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$$

Thus the probability amplitude $\langle +y|\psi(t)\rangle$ is:

$$\langle +y|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\cos\theta\right] - \frac{1}{\sqrt{2}}\sin\frac{\phi}{2}\sin\theta|-z\rangle$$

And, the probability of finding the particle in $|+y\rangle$ reduces to:

$$|\langle +y|\psi(t)\rangle|^2 = \frac{1 - \sin(\omega t)\sin\theta}{2}$$

In the case that $\theta = 0$, i.e. the magnetic field is completely in the z-direction, the sine terms become zero and the probability of finding a particle in $|+y\rangle$ evalues to $P = \frac{1}{2}$.

In the case that $\theta = \frac{\pi}{2}$, i.e. the magnetic field is completely in the x-direction, and the probability of finding a particle in $|+y\rangle$ evalues to $P = \frac{1-\sin(\omega t)}{2}$. Note this has a time dependency. At time T, the specific probability is $P = \frac{1-\sin(\omega T)}{2}$.

III — Spin $\frac{1}{2}$ in a Magnetic Field (b)

A spin-$\frac{1}{2}$ particle, initially in a state with $S_n = \frac{\hbar}{2}$ with $\boldsymbol{n} = \sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{k}$, is in a constant magnetic field B_0 in the z direction. Determine the state of the particle at time t and determine how $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ vary with time. *Hint:* you can make use of the general spin states from problems 1.3 and 1.6.

Solution
The Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{k}$ (i.e. the z-direction) is given by:

$$\hat{H} = \omega_0 \hat{S}_z$$

where $\omega_0 = \frac{geB_0}{2mc}$. The time evolution operator is given by:

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0 \hat{S}_z t/\hbar} = e^{-iS_z \phi/\hbar} = \hat{R}(\phi\boldsymbol{k})$$

where $\phi = \omega_0 t$. Thus the Hamiltonian causes the spin to precess about the z-axis.

In this case the particle is initially in the state $|+n\rangle$, i.e. $|\psi(0)\rangle = |+n\rangle$. Since the magnetic field points in the z-direction, it is useful to express the initial state $|+n\rangle$ as a superposition of states $|\pm z\rangle$ (the eigenstates of the Hamiltonian):

$$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + \sin\frac{\theta}{2}|-z\rangle$$

Thus the initial state of the particle can be expressed:

$$|\psi(0)\rangle = |+n\rangle = \cos\frac{\theta}{2}|+z\rangle + \sin\frac{\theta}{2}|-z\rangle$$

The state of the particle at time t is:

$$\begin{aligned} |\psi(t)\rangle &= \hat{U}(t)|\psi(0)\rangle \\ &= e^{-i\hat{H}t/\hbar}\left(\cos\frac{\theta}{2}|+z\rangle + \sin\frac{\theta}{2}|-z\rangle\right) \\ &= e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|+z\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|-z\rangle \end{aligned}$$

The expected value of S_x is given by $\langle S_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle$, where $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Expanding this product:

$$\langle S_x \rangle = \left(e^{i\omega_0 t/2} \cos \frac{\theta}{2} \langle +z| + e^{-i\omega_0 t/2} \sin \frac{\theta}{2} \langle -z| \right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot$$

$$\left(e^{-i\omega_0 t/2} \cos \frac{\theta}{2} |+z\rangle + e^{i\omega_0 t/2} \sin \frac{\theta}{2} |-z\rangle \right)$$

$$= \frac{\hbar}{2} \left(e^{i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + e^{-i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)$$

$$= \frac{\hbar}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot 2 \cos(\omega_0 t) = \frac{\hbar}{2} \sin \theta \cos(\omega_0 t)$$

$$\therefore \quad \langle S_x \rangle = \frac{\hbar}{2} \sin \theta \cos(\omega_0 t)$$

The expected value of S_y is given by $\langle S_y \rangle = \langle \psi(t)| \hat{S}_y |\psi(t)\rangle$, where $\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Expanding this product, using a similar method to above:

$$\langle S_y \rangle = (...) \hat{S}_y (...)$$
$$= \frac{i\hbar}{2} \left(e^{-i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} - e^{-i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)$$
$$= \frac{i\hbar}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot (-2i \sin(\omega_0 t))$$
$$= \frac{\hbar}{2} \sin \theta \sin(\omega_0 t)$$

The expected value of S_z is given by $\langle S_z \rangle = \langle \psi(t)| \hat{S}_z |\psi(t)\rangle$, where $\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Expanding this product, using a similar method to above:

$$\langle S_z \rangle = (...) \hat{S}_z (...)$$
$$= \frac{i\hbar}{2} \left(e^{-i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} - e^{-i\omega_0 t} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)$$
$$= \frac{\hbar}{2} \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right)$$
$$= \frac{\hbar}{2} \cos \theta$$

Thus:

$$\langle S_x \rangle = \frac{\hbar}{2} \sin \theta \cos(\omega_0 t) \quad \langle S_y \rangle = \frac{\hbar}{2} \sin \theta \sin(\omega_0 t) \quad \langle S_z \rangle = \frac{\hbar}{2} \cos \theta$$

Naturally, since $\hat{S}_z$ commutes with the Hamiltonian, the expectation value of S_z is constant and does not depend on time. But the converse is true for the expectation values of S_x and S_y, whose operators do not commute with the Hamiltonian, and thus exhibit time-dependence.

IV — Time Evolution of a 3-State System

Townsend 4.13

Let

$$\begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}$$

be the matrix representation of the Hamiltonian for a three-state system with basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

(a) if the state of the system at time $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is $|\psi(t)\rangle$? (b) if the state of the system at time $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is $|\psi(t)\rangle$?

Solution

The Hamiltonian operator satisfies the following eigenvalue equation:

$$\hat{H} |E\rangle = E |E\rangle$$

where $|E\rangle$ is an energy eigenstate of $\hat{H}$ with eigenvalue E.

Using the matrix representation of $\hat{H}$ given above, the eigenstates and eigenvalues can be computed by solving the eigenvalue equation. First, rearranging the equation:

$$(\hat{H} - \boldsymbol{I} E) |E\rangle = 0$$

where $\boldsymbol{I}$ is the identity matrix. In matrix form this equation is:

$$\begin{bmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Taking the determinant of the matrix $(\hat{H} - \boldsymbol{I} E)$:

$$\begin{bmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{bmatrix}_{\text{det.}}$$

$$= (E_0 - E)(E_1 - E)(E_0 - E) - A^2(E_1 - E) = 0$$

Which gives three the eigenvalues $E = \{E_1, E_0 + 1, E_0 - A\}$.

Setting $E = E_1$ and substituting:

$$\begin{bmatrix} E_0 - E_1 & 0 & A \\ 0 & E_1 - E_1 & 0 \\ A & 0 & E_0 - E_1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$(E_0 - E_1)a + Ac = 0$$
$$(E_1 - E_1)b = 0$$
$$Aa + (E_0 - E_1)c = 0$$

This is only possible if $b = 1$ and $a = c = 0$. Thus the eigenstate corresponding to the eigenvalue $E = E_1$ is:

$$|E_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note this corresponds to the basis state $|2\rangle$, thus:

$$|E_1\rangle = |2\rangle$$

Next, setting $E = E_0 + A$ and substituting:

$$\begin{bmatrix} -A & 0 & A \\ 0 & E_1 - E_0 - A & 0 \\ A & 0 & -A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$-Aa + Ac = 0$$
$$(E_1 - E_0 - A)b = 0$$
$$Aa - Ac = 0$$

Thus is only possible if $a = c$ and $b = 0$. Requiring that the vector be normalized also pulls out a factor of $\frac{1}{\sqrt{2}}$. Thus the eigenstate corresponding to the eigenvalue $E = E_0 + A$ is:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Noting that this can also be written:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

means this eigenstate is a superposition of the basis states $|1\rangle$ and $|3\rangle$:

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{2}} |3\rangle$$

Next, setting $E = E_0 - A$ and substituting:

$$\begin{bmatrix} A & 0 & A \\ 0 & E_1 - E_0 + A & 0 \\ A & 0 & A \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

gives the following three equations:

$$Aa + Ac = 0$$
$$(E_1 - E_0 + A)b = 0$$
$$Aa + Ac = 0$$

This is only possible if $a = -c$ and $b = 0$. Again, requiring that the vector be normalized pulls out a factor of $\frac{1}{\sqrt{2}}$. Thus the eigenstate corresponding to the eigenvalue $E = E_0 - A$ is:

$$|E_0 - A\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

i.e. this eigenstate is also a superposition of the basis states $|1\rangle$ and $|3\rangle$:

$$|E_0 - A\rangle = \frac{1}{\sqrt{2}} |1\rangle - \frac{1}{\sqrt{2}} |3\rangle$$

If the initial state is $|2\rangle$:

$$|\psi(0)\rangle = |2\rangle$$

the eigenvalue is simply E_1, and it follows that:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |E_1\rangle$$

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If the initial state is $|3\rangle$:

$$|\psi(0)\rangle = |3\rangle$$

This state can be written as the following superposition:

$$|3\rangle = \frac{1}{\sqrt{2}} |E_0 + A\rangle + \frac{1}{\sqrt{2}} |E_0 - A\rangle$$

Thus at time t the state of the system is:

$$|\psi(t)\rangle = \frac{e^{-i(E_0+A)t/\hbar}}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{e^{-i(E_0-A)t/\hbar}}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

V — Spin $\frac{3}{2}$ in a Magnetic Field

Townsend 4.15
If the Hamiltonian for a spin-$\frac{3}{2}$ particle is given by

$$\hat{H} = \omega_0 \hat{S}_x$$

and at time $t = 0$ $|\psi(0)\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle$, determine the probability that the particle is in the state $\left| \frac{3}{2}, -\frac{3}{2} \right\rangle$ at time t. Evaluate this probability when $t = \frac{\pi}{\omega_0}$ and explain your result. *Suggestion:* See problem 3.23 for the eigenstates of $\hat{S}_x$.

Solution
For a spin-$\frac{3}{2}$ particle there are four spin states, $\left| \frac{3}{2}, \frac{3}{2} \right\rangle, \left| \frac{3}{2}, \frac{1}{2} \right\rangle, \left| \frac{3}{2}, -\frac{1}{2} \right\rangle$, and $\left| \frac{3}{2}, -\frac{3}{2} \right\rangle$. For this particle, whose Hamiltonian has eigenstates in the S_x basis, the representation of these four states in the S_z basis is given by:

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \qquad \left| \frac{3}{2}, \frac{1}{2} \right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{bmatrix}$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{bmatrix} \qquad \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_x \longrightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{bmatrix}$$

For this particle, at time t the state is given by:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

where in this case $|\psi(0)\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle$. It is convenient to represent this state in the S_z basis as:

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle_x = \frac{1}{2\sqrt{2}} \left| \frac{3}{2}, \frac{3}{2} \right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}} \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z + \frac{1}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z$$

Thus at time t, applying the time evolution operator, the state of the particle is:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} \left(\frac{1}{2\sqrt{2}} \left| \frac{3}{2}, \frac{3}{2} \right\rangle_z + \frac{\sqrt{3}}{2\sqrt{2}} \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z \right.$$
$$\left. + \frac{\sqrt{3}}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z + \frac{1}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z \right)$$

$$= \frac{e^{-3i\omega_0 t/2}}{2\sqrt{2}} \left| \frac{3}{2}, \frac{3}{2} \right\rangle_z + \frac{\sqrt{3}e^{-i\omega_0 t/2}}{2\sqrt{2}} \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z$$
$$+ \frac{\sqrt{3}e^{i\omega_0 t/2}}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z + \frac{e^{3i\omega_0 t/2}}{2\sqrt{2}} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z$$

The probability amplitude to find the particle in state $\left| \frac{3}{2}, -\frac{3}{2} \right\rangle$ is:

$$\left\langle \frac{3}{2}, -\frac{3}{2} \middle| \psi(t) \right\rangle$$

where the bra vector $\left\langle \frac{3}{2}, -\frac{3}{2} \right|$ in the S_z basis is:

$$\left\langle \frac{3}{2}, -\frac{3}{2} \right|_x = \left(\frac{1}{2\sqrt{2}} \left\langle \frac{3}{2}, \frac{3}{2} \right|_z - \frac{\sqrt{3}}{2\sqrt{2}} \left\langle \frac{3}{2}, \frac{1}{2} \right|_z \right.$$

$$\left. + \frac{\sqrt{3}}{2\sqrt{2}} \left\langle \frac{3}{2}, -\frac{1}{2} \right|_z - \frac{1}{2\sqrt{2}} \left\langle \frac{3}{2}, -\frac{3}{2} \right|_z \right)$$

Taking the product $\left\langle \frac{3}{2}, -\frac{3}{2} \middle| \psi(t) \right\rangle$ gives the probability amplitude as:

$$\begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{e^{-3i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{\sqrt{3}e^{-i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{\sqrt{3}e^{i\omega_0 t/2}}{2\sqrt{2}} \\ \frac{e^{3i\omega_0 t/2}}{2\sqrt{2}} \end{bmatrix}$$

$$= \frac{e^{-3i\omega_0 t/2}}{8} - \frac{3e^{-i\omega_0 t/2}}{8} + \frac{3e^{i\omega_0 t/2}}{8} - \frac{e^{3i\omega_0 t/2}}{8}$$

Setting $t = \frac{\pi}{\omega_0}$, this amplitude becomes zero.

Thus the probability that the particle is in the state $\left| \frac{3}{2}, -\frac{3}{2} \right\rangle$ at time t is zero. This makes sense as the states $\left| \frac{3}{2}, -\frac{3}{2} \right\rangle$ and $\left| \frac{3}{2}, \frac{3}{2} \right\rangle$ are orthogonal.