: Mathematics : 22/08/24.

Implicit form of 1^{st} order ODE: $f(x, y, y') = 0$

$x^{-3} y' - y^2 = 0$ (derivative)

Explicit form of 1^{st} order ODE: $y' = f(x)$

$y' = y^2 x^3$

$\frac{dy}{dx} = e^{-x^2}$; $\int dy = \int e^{-x^2} dx$; $x^2 = t$; $\frac{dt}{dx} = 2x$; $dt = 2x\,dx$

$y = \frac{e^{-x^2}}{-2x} + c$; $y = e^{-x^2} \cdot \int x^2 dx$ $= -\frac{e^{-x^2}}{2x\,dx}$

$0 = -\frac{3e^0}{2\times 0} + c = 0$

$y = \frac{e^{-x^2}}{-2x}$; $y(1) = \frac{e^{-1}}{-2(1)} = \frac{-1}{2e}$

$y = \frac{e^{-1}}{-2} = \frac{1}{-2e}$

$y' + \frac{y^2}{2} + c = 0$

$y + \frac{y^3}{}$

$m_1 e^{-m_1 x} + m_2 e^{-m_2 x}$

$(D^2+1)y = 0$

$(D^2+1) = 0$

$(m^2+1) = 0$

$m = -1 i$

$y = e^{-(1)x} = e^{-x}$

$\frac{1}{y} dy = k\,dx$

$\ln y = kx + \ln c$

$\ln(y/c) = kx$

$y/c = e^{kx}$

$y = c e^{kx}$.

$y_0 = c e^{kx_0}$

⇒ **Concept of solution**: (21/08/24)

A function $y = h(x)$ is called a solⁿ. of a given ODE (1) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the eqⁿ become an identity if y and y' are replaced with h and h', resp.

$\frac{dy}{dx} = y + x$, $y(0) = 0$

$y = e^x - x - 1$

$\frac{dy}{dx} = e^x - 1$

$\frac{dy}{dx} = 1 + y^2$.

⇒ $y' = f(x, y)$ ---- (1)

(1) **Variable separable form**: $\boxed{g(y)dy = f(x)\,dx}$ ---- (2)

$g(y) = \frac{1}{1+y^2}$

$f(x) = 1$.

(2) **Exact form**: $\boxed{M(x,y)dx + N(x,y)dy = 0}$ ---- (3)

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \text{ ---- (9)}$$

Let $u(x,y) = c$; $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$ ---- (4)

$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ -- (7)

compare (3) & (4) ; $\boxed{M(x,y) = \frac{\partial u}{\partial x}}$ -- (5) ; $\boxed{N(x,y) = \frac{\partial u}{\partial y}}$ -- (6)

$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ -- (8)

solution to Exact form:

$u(x,y) = \int M(x,y)dx + k_1(y)$

$u(x,y) = \int N(x,y)dy + k_2(x)$

Initial Value Problem:

$y' = f(x,y)$, $y(x_0) = y_0$

Eg: $\cos(x+y)dx + (3y^2+2y+\cos(x+y))dy = 0$

$M = \cos(x+y)$

$N = 3y^2+2y+\cos(x+y)$

$u(x,y) = \int \cos(x+y)dx + k_1(y)$

$= \sin(x+y) + k_1(y) \quad \cdots (1)$

$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk_1}{dy}$

$\therefore \frac{\partial u}{\partial y} = N$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$3y^2+2y+\cos(x+y) = \cos(x+y) + \frac{dk_1}{dy}$

$k_1 = y^3+y^2+c_1$

put $c_1 = k_2$

$u(x,y) = \sin(x+y) + y^3+y^2+k_2(x)$

Eg: $2xyy' = y^2-x^2$ $\Rightarrow$ Non seperable form & Non exact.

$(x^2-y^2)dx + 2xydy = 0$

check: Homogeneous f^n or Not. $\Rightarrow$ $\boxed{f(y/x) \quad y/x = u \quad y' \quad y = ux}$

$\frac{x^2-y^2}{2xy} = \frac{\lambda^2x^2-\lambda^2y^2}{2\lambda x\lambda y} = \frac{\lambda^2(x^2-y^2)}{\lambda^2 2xy}$

$\boxed{\frac{M(x,y)}{N(x,y)} = \frac{M(\lambda x,\lambda y)}{N(\lambda x,\lambda y)}}$ $\therefore$ Don't check for Trigonometric Exponential f^n.

$\therefore$ Homogeneous f^n. $\uparrow$ condition.

$\Rightarrow y = ux$; $\frac{dy}{dx} = \frac{du}{dx}x + u$

$y' = \frac{y^2-x^2}{2xy} \Rightarrow \left(\frac{du}{dx}x+u\right) = \frac{u^2x^2-x^2}{2xux}$

$\left(\frac{du}{dx}x+u\right) = \frac{u^2-1}{2u}$

$\frac{du}{dx}x = \frac{-1-u^2}{2u}$ $\therefore$ Reduced to VS form.

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$-ydx + xdy = 0$ $\quad M(x,y)dx + N(x,y)dy = 0$

$y' = f(x,y)$. $\quad y = cx$.

$\frac{1}{x^2}(-ydx+xdy) = 0$

$= \underbrace{-\frac{y}{x^2}}dx + \frac{1}{x}dy = d(y/x) = d(u) = 0 \Rightarrow u = c$

$M = -y/x^2$; $N = \frac{1}{x}$

$\frac{\partial M}{\partial y} = -1/x^2$; $\frac{\partial N}{\partial x} = -1/x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Exact

try with $\frac{1}{xy}$; $\frac{1}{x^2+y^2}$

$d(g/f) =$

$d(g/f) = \frac{g'f - gf'}{}$

$\Rightarrow +\frac{1}{y^2}(-ydx + xdy)$

$= -\frac{1}{y}dx + \frac{x}{y^2}dy$

$M = -\frac{1}{y}$ $\quad N = \frac{x}{y^2}$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = \frac{1}{y^2}$ $\quad \frac{\partial N}{\partial x} = \frac{1}{y^2}$; Exact.

$\frac{1}{xy}(-y\,dx + x\,dy) = 0 \quad -d\left(\ln\frac{x}{y}\right)$ $\quad \frac{1}{x^2+y^2}(-y\,dx+x\,dy)=0 \Rightarrow d\left(\tan^{-1}\frac{y}{x}\right)$

$\therefore$ exact.

Integrating factor:

$M(x,y)dx + N(x,y)dy = 0 \quad \ldots (1)$

$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \ldots (2)$

$FP\,dx + FQ\,dy = 0 \quad \ldots (3)$

Theorem: If $P(x,y)dx + Q(x,y)dy = 0$ is such that the $\boxed{R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$ depends on x, $\quad F = e^{\int R(x)\,dx} \quad \therefore F = f(x)$

$$\frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x} \qquad F_y = \frac{\partial F}{\partial y}$$

Assume $\therefore F = F(x)$

$F_yP + P_yF = F_xQ + FQ_x$

$0 + FP_y = F'Q + FQ_x$

$F'Q = F\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ $\quad$ Assume $\therefore R = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$

$\frac{F'}{F} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ $\quad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$F'/F = R \qquad$ check 'R'

$\Rightarrow$ Eq. $\quad F = F(y) \qquad F_yP + FP_y = F_xQ + FQ_x$

$F'P + FP_y = FQ_x$

$\frac{F'}{F} = \boxed{\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = R^*} \qquad R^* = f(y) \qquad F(y) = e^{\int R^*(y)dy}$

$(e^{x+y} + ye^y)\,dx + (xe^y - 1)\,dy = 0, \quad y(0) = 1$

$\cancel{M = e^x . e^y + ye^y}$ $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$e^y(e^x + y)\,dx = e^y\left(\frac{1}{e^y} - x\right)dy$

$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= \frac{1}{(xe^y - 1)}\left(\frac{\partial (e^{x+y} + ye^y)}{\partial y} - \frac{\partial (xe^y - 1)}{\partial x}\right)$

$= \frac{1}{(xe^y-1)}\left[\left[\frac{\partial}{\partial y}(e^x e^y) + \frac{\partial}{\partial y}(ye^y)\right] - \left[\frac{\partial x}{\partial y} - \frac{\partial(1)}{\partial x}\right]\right]$

$= \frac{1}{xe^y - 1}\left[e^{x+y} + e^y + ye^y\right] - (e^y)$

$\frac{\partial}{\partial y}(ye^y)$

$y\frac{\partial}{\partial y}(e^y) + e^y\left(\frac{\partial y}{\partial y}\right)$

$(ye^y + e^y)$

$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \qquad R^* = f(y)$

$= \frac{1}{e^{x+y} + ye^y}\left[\frac{\partial}{\partial x}(xe^y - 1) - \frac{\partial}{\partial y}(e^{x+y} + ye^y)\right]$

$= \frac{1}{e^{x+y} + ye^y}\left[e^y - \left[e^{x+y} + ye^y + e^y\right]\right]$

$= \frac{-(e^{x+y} + ye^y)}{e^{x+y} + ye^y}$

$R^* = -1 \qquad IF = e^{\int -1\,dy}$

$IF = e^{-y}$

$e^{-y}\left[(e^{x+y} + ye^y)dx + (xe^y - 1)\,dy = 0\right]$

$(e^x + y)\,dx + (x - e^{-y})\,dy = 0.$

$\frac{\partial(e^x + y)}{\partial y} = \frac{\partial}{\partial x}(x - e^{-y})$

$1 = 1 \qquad \therefore$ True.

$u(x,y) = e^x + yx + k_1(y)$

$\frac{\partial u}{\partial y} = x + \frac{dk_1}{dy}$

$(x - e^{-y}) = x + \frac{dk_1}{dy}$

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Linear ODE : 28/08/24

$\Rightarrow y' + P(x) y = q(x)$

$f(x) y' + P(x) y = Q(x)$

$\Rightarrow y' + \tilde{P}(x) y = \tilde{q}(x)$

Operator $L : \frac{d}{dx} + p(x)$

Linearity : $L(y_1 + \alpha y_2) = L(y_1) + \alpha L(y_2)$

$L(\alpha y) = \alpha L(y)$

Case(1) : $q(x) \equiv 0$

Homogeneous Linear ODE.

$y' + p(x) y = 0$

$y(x) = e^{-\int p(x) dx} + c$.

$y' = -p(x) y$

$\frac{y'}{y} = -p(x)$

$\frac{dy}{y} = -p(x) dx$.

$\ln|y| = -\int p(x) dx + c$

$y(x) = e^{-\int p(x) dx} + c$

Case(2) : $q(x) \neq 0$

$y' + p(x) y = q(x)$

$F y' + p(x) F y = F q$ ---(3)

$(F y)' = F' y + F y'$ --- (4)

$F' y = p(x) F y$ $\qquad \frac{F'}{F} = p(x)$

$\frac{F'}{F} = p(x)$.

$\boxed{F = e^{\int p(x) dx}}$.

put $\boxed{h = \int p(x) dx}$

$h' = p(x)$

Sub. in Eq(3)

$e^h y' + h' e^h y = e^h q$

$\because x'y + yx' = (xy)'$

$(e^h y)' = e^h q$.

$e^h y = \int e^h q \, dx + c$.

$$\boxed{y = e^{-h} \left(\int e^h q(x) \, dx + c \right)}$$

Solve IVP : $y' + y \tan x = \sin 2x$; $y(0) = 1$.

$P(x) = \tan x \qquad q(x) = \sin 2x$.

$F(y') + P(x) F y = F q$.

$F y' + \tan x . F . y = F . \sin 2x$

$(F y)' = F' y + F y'$.

$F' y = \tan x . F . y$

$\frac{F'}{F} = \tan x$

$F = e^{\int \tan x \, dx}$.

$F = \sec x$.

$h = \int \tan x = \ln|\sec x|$

$e^h = \sec x \qquad e^{-h} = \cos x$.

$e^h q = 2 \sin x$ $\quad \because \sec x \times 2 \sin x \cos x$

$y(x) = \cos x (2 \int \sin x \, dx + c)$

$y(x) = c . \cos x - 2 \cos^2 x$.

$\because y(0) = 1$; $1 = c - 2$

$\boxed{c = 3}$

Reduction to linear form:

$y' + p(x)\, y = g(x)\, y^a$ "a" any real number

$a = 0$, Non-homogeneous

$a = 1$, ~~non~~- homogeneous

Let $z = [y(x)]^{1-a}$

$dz = (1-a)\, y^{1-a-1}\, dy$

$z' = (1-a)\, y^{-a} y'$

$= (1-a)\, y^{-a} [g(x)\, y^a - p(x)\, y]$

$z' = (1-a)\, [g(x) - p(x)\, z]$

$\boxed{z' + (1-a)\, p(x) z = (1-a)\, g(x)}$

↳ Reduced linear equation.

$xy' = y - 1$; $y(0) = 1$ $\quad P = (-1/x) \quad g = (-1/x)$

$y' - y/x = -1/x$

$h = -\int 1/x\, dx = -\ln(x) \Rightarrow e^h = 1/x$

$e^{-h} = x$.

$y = x \int (1/x)(-1/x)\, dx + c$.

$y = -x \int (1/x^2)\, dx + c$

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HW: $y' = Ay - By^2$

$u = y^{1-a} = y^{-1}$ (where $a = 2$)

$u' = -y^{-2} y' = -y^{-2}(Ay - By^2) = B - Ay^{-1} = B - Au \Rightarrow u' + Au = B$

$u = e^{-h}\left(\int e^h B + c\right) \qquad \therefore h = \int A\, dx = Ax$.

$u = e^{-Ax}\left(\int B e^{Ax} + c\right) = e^{-Ax}\left(\frac{B}{A} e^{Ax} + c\right) \qquad u = \frac{B}{A} + e^{-Ax} + c$.

(1) $|y'| + |y| = 0$, $y(0) = 1$

No sl^n available for $y(0) = 1$

only sl^n is at $y(0) = 0$

(2) $y' = 2x$, $y(0) = 1$

(3) $xy' = y - 1$, $y(0) = 1$

$\int \frac{1}{(y-1)} dy = \int \frac{1}{x} dx$.

$y = 1 + cx$.

Infinitely many solutions

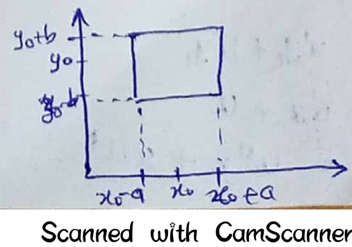
Existence & Uniqueness:

$y' = f(x, y)$, $y(x_0) = y_0$ .... (1)

Rectangle $R: \{|x - x_0| < a, |y - y_0| < b\}$

$-a + x_0 < x < a + x_0$

$-b + y_0 < y < b + y_0$



① Let the RHS of $f(x,y)$ of the ODE be continuous at all points (x,y) in some rectangle 'R'

(2) And bounded in 'R'.

(1), (2) satisfied, then f^n has atleast one sol^n.

Then the initial value problem has atleast one sol^n $y(x)$ this sol^n Exists atleast for all x in the "sub interval $|x-x_0|<\alpha$" of the interval $|x-x_0|<a$, here "α" is the smaller of the two numbers "a and b/k".

$\alpha = \min\{a, b/k\}$ $\quad$ k = upper bound $\quad |f(x,y)| \le k$

Linearly Independ. $\frac{f_1}{f_2} \neq k$.

for uniqueness:

(iii) $\left|\frac{\partial f}{\partial y}\right| \le M$

$y' = 1+y^2$, $y(0)=0$ $\quad R = \begin{Bmatrix} |x|<5, \\ |y|<3 \end{Bmatrix}$ $\quad a=5 \quad b=3 \quad x_0=0 \quad y_0=0$

$k = 1+3^2 = 10$

$|f(x,y)| < 10$ $\quad \left|\frac{\partial f}{\partial y}(x,y)\right| < M = 6$ $\quad \frac{\partial}{\partial y} y^2 = 2y = 2\times3 = 6$

$|f(x,y)|$ $\quad \alpha = \min\{a, b/k\} = \min\{5, 3/10\} = 0.3$

2nd Order Linear ODE

$\boxed{y = c_1 y_1 + c_2 y_2}$

$\boxed{y'' + p(x)y' + q(x)y = r(x)}$

$r(x) \equiv 0$ Homogeneous

$r(x) \not\equiv 0$ Non-Homogeneous

$\left.\begin{aligned} y''+y&=0 \\ y''-y&=0 \end{aligned}\right\}$ $\begin{aligned} &y_1 = \cos x,\ y_2 = \sin x \\ &y_1 = e^x,\ y_2 = e^{-x} \end{aligned}$

$y''+y = 1$ $\quad y_1 = \cos x + 1$

$y_2 = \sin x + 1$

$y = c_1 y_1 + c_2 y_2 = c_1(\cos x+1) + c_2(\sin x+1)$

$y''+y \neq 1$ $\quad \therefore$ Non-homogeneous f^n.

Wronskian:

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$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ Independent.

$y_1 \neq k y_2$ Linearly Indepen

$y_1 = k y_2$ Linearly Depende

find the basis of sol^n of the ODE

$(x^2-x)y'' - xy' + y = 0$

$x(x-2)$ Linearly Independent Except $(x=0, x=2)$

$y_1 = x$, $y_2 = ?$ $\quad y_2 = u y_1$

$y_2' = u'y_1 + u y_1'$

$= u'x + u$

$y_2'' = u''x + 2u'$

$$= (x^2-x)(u''x+2u') - x(u'x+u) + 4x = 0$$

$$= (x^2-x)(u''x+2u') - x^2u' = 0$$

$$(x^2-x)u''x + (x^2-x)2u' - x^2u' = 0$$

$$(x^2-x)u''x + 2u'x^2 - x^2u' - 2xu' = 0$$

$$(x^2-x)u''\cancel{x} + \cancel{x}(x-2)u' = 0$$

↳ 2nd order LDE.

$$(x^2-x)u'' + u'(x-2) = 0 \qquad v = u'$$

$$(x^2-x)v' + v(x-2) = 0$$

$$\frac{dv}{v} = \frac{2-x}{x(x-1)}$$

$$y'' + p(x)y' + q(x)y = 0.$$

Constant coefficients $y'' + ay' + by = 0$

$y' = ay$

$y = e^{ax}$ $\qquad y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$$

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

$\lambda^2 + a\lambda + b$ auxillary Eqn

$$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

Case I: Distinct roots

Case II: Repeated Roots $\lambda = -\frac{a}{2}, -a/2.$

$y_1 = e^{-a/2\, x}$ $\quad y_2 \neq e^{-a/2\, x}$

$y_2 = uy_1$

Case III: Complex root.

$\lambda = a \pm ib$

$$y_1 = e^{(a+ib)x} = e^{ax}.e^{ibx} = e^{ax}(\cos bx + i\sin bx)$$

$$y_2 = e^{(a-ib)x} = e^{ax}.e^{-ibx} = e^{ax}(\cos bx - i\sin bx)$$

$$\boxed{y = c_1y_1 + c_2y_2} \qquad y = e^{ax}[A\cos(bx) + B\sin(bx)]$$

Euler - Cauchy Eqn:

$$\boxed{x^2y'' + axy' + by = 0}$$

$$\boxed{y = x^m}$$

$$m(m-1)x^m + amx^m + bx^m = 0$$

$$m^2 - m + am + b = 0$$

$$m^2 + m(a-1) + b = 0$$

$$\boxed{m = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}}$$

$$x^2y'' + 3xy' + y = 0$$

$a = 3, \ b = 1.$

$$= \frac{(1-3) \pm \sqrt{(3-1)^2 - 4}}{2}$$

$$m = \frac{-2 \pm \sqrt{0}}{2} = -1.$$

$y_1 = x^{-1}$, check $y_2 = \ln(x^{-1})$

$\Rightarrow y'' + p(x)y' + q(x)y = r(x)$ 03/09/24.

$r(x) \equiv 0$ $y'' + ay' + by = 0$

$x^2y'' + axy' + by = 0 \rightarrow$ Euler - cauchy Eq^n.

Eg:

$x^2y'' + 3xy' + y = 0$

let $y = x^m$

$m(m-1)x^m + 3mx^m + x^m = 0$

$(m^2 - m + 3m + 1)x^m = 0$

$m^2 + 2m + 1 = 0$

$m^2 + m + m + 1 = 0$

$m(m+1) + 1(m+1) = 0$

$(m+1)^2 =$ $m = -1, -1$

$y_1 = x^{-1}$ $\because y_2 \neq x^{-1}$ $\therefore y_1 \& y_2$ linearly independent.

$y_2 = uy_1$

y_1 & y_2 are LD

$y_2 = ky_1$

$y_2 \neq k(x)y$ LI

linearly Independent . wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$\boxed{y_h = C_1y_1 + C_2y_2}$$ ---- (2)

$\Rightarrow y'' + p(x)y' + q(x)y = r(x)$ ----(1) $\because r(x) \not\equiv 0$

General sol^n. $y = y_h + y_p$

$y_h \rightarrow$ general sol^n

$y_p \rightarrow$ particular solution.

i) Method of variation parameter:

$y_1, y_2 \rightarrow$ from Homogeneous f^n

Formula

$$\boxed{y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx}$$

$u = f(x)$

solving by putting $r \equiv 0$ (HF^n)

$y_p = uy_1 + vy_2$ ---- (3)

Assume:

$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$ --- (4)

$u'y_1 + v'y_2 = 0$ --- (5)

$y_p' = uy_1' + vy_2'$ .... (6)

$y_p'' = u'y_1' + uy_1'' + v'y_2' + y_2''v$ ---- (7)

sub (3), (6), (7) in (1)

$(u'y_1' + uy_1'' + v'y_2' + vy_2'') + p(x)(uy_1' + vy_2') + q(x)(uy_1 + vy_2) = r(x)$

$(u' + p(x)u)y_1' + uy'' + vy_2'' + y_2'(v' + p(x)v) + q(x)(uy_1 + vy_2) = r(x)$

$$u(\underbrace{y_1'' + y_1 \times q + py_1}_{\substack{H.F \\ = 0}}) + v(\underbrace{y_2'' + py_2' + y_2}_{\substack{\text{Homoge } f^n \\ = 0}}) + u'y_1' + v'y_2' = r(x)$$

$u'y_1' + v'y_2' = r(x)$ ....(6)

$$\left.\begin{aligned} u'y_2 + v'y_2' &= 0 \\ u'y_1' + v'y_2' &= r \end{aligned}\right\} \Rightarrow \begin{aligned} y_2'(u'y_1 + v'y_2) &= 0 \\ y_2(u'y_1' + v'y_2') &= ry_2 \end{aligned} \Rightarrow \begin{aligned} & u'(y_1y_2') + y_2'v'\cancel{y_2} \\ & u'y_1'y_2 + v'\cancel{y_2}y_2' = ry_2 \end{aligned}$$

$$u'(y_1y_2' - y_1'y_2) = -ry_2$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$u' = \frac{-ry_2}{y_1y_2' - y_2y_1'}$$

$$u' = \frac{-ry_2}{w}$$

$$\boxed{u = -\int \frac{ry_2}{w}\,dx}$$

$$\Rightarrow \begin{aligned} y_1'(u'y_1 + v'y_2) &= 0 \\ y_1(u'y_1' + v'y_2') &= ry_1 \end{aligned}$$

$$v'(y_2y_1' - y_1y_2') = -ry_1$$

$$v' = \frac{+ry_1}{+w}$$

$$\boxed{v = \int \frac{ry_1}{w}\,dx}$$

$$\Rightarrow \boxed{y_p = uy_1 + vy_2}$$

Q: $y'' + y = \sec x$

$y_h = c_1\cos x + c_2\sin x$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$y_p = -\cos x\int \sin x \cdot \sec x\,dx + \sin x\int \cos x \sec x\,dx$

$= \cos x \ln|\sec x| + x\sin x$

$y = (c_1 + \ln|\sec x|)\cos x + (c_2 + x)\sin x$

$\Rightarrow$ solving 2nd order. 1st linearity 2nd $r = 0$ / $r \neq 0$
Homo. / Non-Homo.

Higher Order Linear ODEs

$y^n = \frac{d^ny}{dx^n}$ $\quad f(x, y, y', y'', \ldots y^n) = 0$

$$y^n + P_{n-1}(x)y^{n-1} + P_{n-2}(x)y^{n-2} + \ldots + P_1(x)y' + P_0(x)y = r(x)$$

wronskian finding is difficult for HOLODE

$$w = |\ \ |_{n \times n}$$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

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Eg#: $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$; $y(1) = 2$. $y'(1) = 1$ $y''(1) = 4$

$y = x^m$; $(m^3 - 6m^2 + 11m - 6)x^m = 0$

$m^3 - 6m^2 + 11 - 6 = 0$ $\Rightarrow (m-1)(m-2)(m-3) = 0$

$y = c_1x + c_2x^2 + c_3x^3$ by applying IC's: $c_1 = 2, c_2 = 1, c_3 = -1$

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$y^{iv} - 5y'' + 4y = 0$

$y = e^{\lambda x}$

$(\lambda^4 - 5\lambda^2 + 4)e^{\lambda x} = 0$

$\lambda^4 - 5\lambda^2 + 4 = 0$

$\lambda = -2, -1, 1, 2.$

$y = c_1e^{-2x} + c_2e^{-x} + c_3e^x + c_4e^{2x}$

$\lambda^4 - \lambda^2 - 4\lambda^2 + 4 = 0$

$\lambda^2(\lambda^2 - 1) - 4(\lambda^2 - 1) = 0$

$(\lambda^2 - 1)(\lambda^2 - 4) = 0$

$\Rightarrow e^{-2x}, e^{-x}, e^x, e^{2x}$ should satisfy above eqn indipendently when arrenged linearly.

$y''' - y'' + 100y' - 100y = 0$; $y(0) = 4$, $y'(0) = 11$, $y''(0) = -299$

$y = c_1e^x + A\cos(10x) + B\sin(10x)$

$y = e^{\lambda x}$

$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$

$(\lambda - 1)(\lambda^2 + 100) = 0$

$\lambda = 1, \pm 10i$

$A = c_2 + c_3$

$B = i(c_2 - c_3)$

$y = c_1y_1 + c_2y_2 + c_3y_3$

$y_1 = e^x$

$y_2 = e^{10ix}$

$y_3 = e^{-10ix}$

$= c_1e^x + c_2e^{10ix} + c_3e^{-10ix}$

$= c_1 \cdot e^x + c_2[\cos(10x) + i\sin(10x)] + c_3[\cos(10x) - i\sin(10x)]$

$= c_1e^x + A\cos(10x) + B\sin(10x).$

$y^v - 3y^{iv} + 3y''' - y'' = 0$

$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$

$\lambda^2(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0$

$\lambda = 0, 0, 1, 1, 1$

$y = (c_1 + c_2x) + (c_3 + c_4x + c_5x^2)e^x$

$y = (c_1e^{0x} + c_2xe^{0x}) + (c_3 + c_4x + c_5x^2)e^x$

$y = (c_1 + c_2x) + (c_3 + c_4x + c_5x^2)e^x$

$$y^n + P_{n-1}(x)\, y^{n-1} + P_{n-2}(x)\, y^{n-2} + \dots + P(x)\, y' + P(x)\, y = r(x) \qquad r(x) \neq 0,$$

$$\boxed{y = y_h + y_p} \qquad \boxed{y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)}\, r(x)\, dx}$$

$$y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

$$y_p = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)\, dx + \dots + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)\, dx$$

↳ Method of variation of parameters

" $W_j(x)$ " $[j = 1 \dots n]$ is obtained from " $W(x)$ " by replacing the j^{th} column of W by the column " $[0\ 0\ 0 \dots 0\ 1]^T$ "

eg:

$n=2$. $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\qquad W_1(y_1,y_2) = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$

$W_2(y_1,y_2) = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$

$n=2$

$$y_p = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)\, dx$$

$$y_p = y_1(x) \int \frac{-y_2\, r(x)}{W} dx + y_2(x) \int \frac{y_1\, r(x)}{W} dx$$

eg: $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln(x) \quad (x>0) \qquad r(x) = \frac{x^4 \ln(x)}{x^3}$

$y = x^m \qquad y' = mx^m \qquad y'' = m^2 x^m \qquad y''' = m^3 x^m$

$(m^3 - 6m + 11m - 6) x^m = 0$

$(m^3 - 6m + 11m - 6) = 0$

$m = 1, 2, 3.$

$y_h = c_1 x + c_2 x^2 + c_3 x^3$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$= x(12x^2 - 6x^2) - x^2(6x - 0) + x^3(2-0)$

$= 6x^3 - 6x^3 + 2x^3$

$= 2x^3$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = -x^2(1\cancel{2x^2} - 3x^2) + x^3(\cancel{2x})$$

$= +3x^4 + 2x^4$

$= +6x^4$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = x(-3x^2) + x^3(1)$$

$= -2x^3$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$= x(2x) - x^2(1)$

$= 2x^2 - x^2 = x^2$

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$$y_p = x\int \frac{x^{2}(x^{2}\ln x)}{2x^3}dx + x^2\int \frac{-2x^3\, x\ln x}{2x^3}dx + x^3\int \frac{x^2\cdot x\ln x}{2x^3}dx.$$

$\int x^{0}\ln x\,dx$

$u\int v\,du - \int(u'\int v\,du)\,dx$

$$= x\int \frac{x^2}{2}\ln x\,dx + x^2\int x\ln x\,dx + x^3\underline{\int \ln x\,dx}.$$

$$= x\left[\ln x\cdot\frac{x^3}{6} - \int\frac{1}{x}\cdot\frac{x^3}{6}dx\right] + x^2\left[\ln x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot x\,dx\right] + \frac{x^3}{2}(x\ln x - x) + C.$$

$$y_p = \frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right)$$

$\Rightarrow$ $y'' + e^x y' + \sin(x)\,y = 0$
$y'' + e^x y' + \ln(x)\,y = 0$ } linear problems.

$y'' + y' + y = 0$

$(x^2-4)\,y'' + (x-3)\,y' + x^2 y = 0$

singular point $\Rightarrow$ not ordinary point

{05/09/2024}

Power series Solutions: Jop

Every Polynomial = function
Every function $\neq$ Polynomial

$y'' + p(x)y' + q(x)y = 0 \;\ldots\ldots\; (1)$

DEF: A point x_0 is said to be an ordinary point of (1). if $p(x)$ & $q(x)$ are analytic at x_0
is (infinitely differentiable)

Existence of Power series soln:

If $x = x_0$ is an ordinary point of the differential eqn Eq (1) then we can always find two linearly independent power series solns centered at x_0: $y = \sum_{n=0}^{\infty} C_n (x-x_0)^n$

(method of convergent)

eg: $y' - y = 0$ $\quad x \neq 0$, To find Non-Trivial soln!
Assumed at $x_0 = 0$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$= C_0 + C_1x + C_2x^2 + \ldots$$

$$y' = C_1 + 2C_2x + \ldots = \sum_{n=1}^{\infty} nC_n x^{n-1}$$

$$y' - y = (C_1 + 2C_2x + 3C_3x^2 + \ldots) - (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots) = 0$$

$\Rightarrow$ $(C_1 - C_0) + (2C_2 - C_1)x + (3C_3 - C_2)x^2 + (4C_4 - C_3)x^3 + \ldots = 0$

$C_1 = C_0$

$C_2 = \frac{C_1}{2} = \frac{C_0}{2!}$

$C_3 = \frac{C_2}{3} = \frac{C_0}{3!}$

$C_4 = \frac{C_0}{4!}$

$y = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots$

$y = c_0(1 + x$ $\quad y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$

$$y = c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$y = c_0 e^x$$

Eg: $y'' + y = 0$ $\quad q(x) = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$y''+y = (2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots)$

$(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots) = 0$

x^0 $(2c_2 + c_0) = 0$ $\quad c_2 = -\frac{c_0}{2}$

x^1 $(6c_3 + c_1) = 0$ $\quad c_3 = -c_1/6 = \frac{-c_1}{3!}$

x^2 $(12c_4 + c_2) = 0$ $\quad c_4 = -c_2/12 = \frac{c_0}{4!}$

x^3 $(20c_5 + c_3) = 0$ $\quad c_5 = -\frac{c_3}{20} = \frac{c_1}{5!}$

x^4 $(30c_6 + c_4) = 0$ $\quad c_6 = \frac{-c_4}{30} = -\frac{c_0}{6!}$

$\Rightarrow$ $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$

$$y = c_0 + c_1x - \frac{c_0}{2!}x^2 - \frac{c_1}{3!}x^3 + \frac{c_0}{4!}x^4 + \frac{c_1}{5!}x^5 - \frac{c_0}{6!}x^6 + \cdots$$

$$y = c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$y = c_0 \cos x + c_1 \sin x$$

Q: $y' = 2yx$

$y' - 2yx = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1x^0 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$$

$= (1c_1x^0 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots) - 2(x)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots) = 0$

$= \cancel{-2c_0x} + (\cancel{c_2} - 2c_0)x$

$= c_1 + (2c_2 - 2c_0)x + (3c_3 - 2c_1)x^2 + (4c_4 - 2c_2)x^3 + (5c_5 - 2c_3)x^4 + \cdots$

$C_1 = 0$ $\qquad$ $C_1 = 0$ $\qquad$ $C_3 = 0$

$(1+2+3)$

$2C_2 - 2C_0 = 0$ $\qquad$ $C_2 = C_0$

$3C_3 - 2C_1 = 0$ $\qquad$ $C_4 = C_0/2$

$4C_4 - 2C_2 = 0$ $\qquad$ $C_5 = 0$

$5C_5 - 2C_3 = 0$ $\qquad$ $C_6 = \frac{2C_2}{6} = \frac{C_0}{6}$

$6C_6 - 2C_4 = 0$

$7C_7 - 2C_5 = 0$ $\qquad$ $C_7 = 0.$

$8C_8 - 2C_6 = 0$ $\qquad$ $C_8 = \frac{C_0}{24}.$

$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \cdots$

$y = C_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right)$

$y = C_0 e^{x^2}$

$y'' + xy' + y = 0$

$y = \sum_{n=0}^{\infty} C_n x^n$

$y' = \sum_{n=1}^{\infty} nC_n x^{n-1}.$

$y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}.$

$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=1}^{\infty} nC_n x^n + \sum_{n=0}^{\infty} C_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} nC_n x^n + \sum_{n=0}^{\infty} C_n x^n = 0.$

$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} = -\sum_{n=0}^{\infty}(n+1)C_n x^n$

$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n = -\sum_{n=0}^{\infty}(n+1)C_n x^n.$

$n-2 = m$

$n = m+2$

$\therefore n = 2$

$2-2 = m.$

when $n=2$ $\sum_{m=0}^{\infty}(m+2)(m+1)C_{m+2}x^n$

replace 'm' with 'n'

$\sum_{m=0}^{\infty}(n+2)(n+1)C_{n+2}x^n.$

$x^0: (2)(1)\ C_2(1) = -(1)C_0x^0$ $\qquad$ $C_2 = \frac{-C_0}{2}.$

$x^1: (3)(2)\ C_3\cancel{x} = -(2)C_1\cancel{x}.$ $\qquad$ $C_3 = -\frac{C_1}{3}$

$x^2: (4)(3)\ C_4\cancel{x^2} = -(3)C_2\cancel{x^2}$ $\qquad$ $C_4 = -\frac{C_2}{4} = \frac{C_0}{2\times4} = \frac{C_0}{2^2(1.2)}$

$x^3: (5)(4)\ C_5\cancel{x^3} = -(4)C_3\cancel{x^3}$ $\qquad$ $C_5 = -\frac{C_3}{5} = \frac{C_1}{3\times5}.$

$$C_{2k} = \frac{(-1)^k C_0}{2.4.6 \ldots (2k)} = \frac{(-1)^k C_0}{2^k (k!)}$$

$$C_{2k+1} = \frac{(-1)^k C_1}{3.5.7 \ldots (2k+1)} =$$

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

$$y = C_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} + \ldots\right) + C_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \ldots\right)$$

$$= C_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k (k!)} + C_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{3.5.7 \ldots (2k+1)}$$

Ex: find at least the "first four" non zero term in a power series expansion $t = x - \pi$ of the given IVProblem.

$t = ?$ $x = \pi$ $t = 0$

$y'' - (\sin x) y = 0 \quad y(\pi) = 1 \quad y'(\pi) = 0$ $\quad x_0 = \pi$

$y(t) = 1$ at $x = \pi$

$y(x - \pi) = 1$

$y(0) = 1.$

$t = x - \pi$

$x = t + \pi$

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n.$$

$\sin(\pi + t) =$

$- \sin(t)$

$y'' - \sin(\pi + t) y = 0.$

$y'' + \sin(t) y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$\sum c_n t^n$

$$= \sum_{n=2}^{\infty} n(n-1) C_n t^{n-2} + \left(t - \frac{t^3}{3!} + \frac{t^5}{120} - \ldots\right)(C_0 + C_1 t + C_2 t^2 + C_3 t^3 + \ldots)$$

$$= (2C_2 t^0 + 6C_3 t + 12C_4 t^2 + 20C_5 t^3 + \ldots) + \left(t - \frac{t^3}{6} + \frac{t^5}{120} + \ldots\right)(C_0 + C_1 t + C_2 t^2 + C_3 t^3 + \ldots) = 0$$

t^0: $C_2 = 0$

t: $6C_3 + C_0 = 0$ $\quad C_3 = -C_0/6$

t^2: $12C_4 + C_1 = 0$ $\quad C_4 = -C_1/12$

t^3: $20C_5 + \left(C_2 - \frac{C_0}{6}\right) = 0$ $\quad C_5 = \frac{C_0}{120}$

$$y = C_0 + C_1 t + (0 \times t^2) - \frac{C_0}{6} t^3 - \frac{C_1}{12} t^4 + \frac{C_0}{120} t^5 + \ldots$$

$$y = C_0\left(1 - \frac{t^3}{6} + \frac{t^5}{120} - \ldots\right) + C_1\left(t - \frac{t^4}{12} + \ldots\right)$$

$$1 = C_0\left(1 - \frac{0}{6} + \frac{0}{120}\right) + C_1(0 + 0 + .) \qquad \boxed{C_0 = 1}$$

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$y' = c_1 + c_2 t + c_3 t^2 + \cdots$ $\qquad$ $y(0) = 0$

$a_6 = \frac{1}{180}$

$y'(0) = 0$ $\qquad$ $\boxed{c_1 = 0.}$

$$y = \left(1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \right)$$

$$\boxed{y = \left(1 - \frac{(x-\pi)^3}{3!} + \frac{(x-\pi)^5}{5!} + \cdots\right)}$$

$y'' - xy' + 2y = \cos x$. $\qquad$ (Non Homogeneous Problem)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n.$$

$$= \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1) x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = (2c_2 + 2c_0) + \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) - n c_n + 2c_n\right] x^n.$$

$x^0 : 1 = 2c_2 + 2c_0 \Rightarrow$ $\qquad$ $c_2 = \frac{1}{2} - c_0.$

$x^1 : 0 = 6c_3 + c_1 \Rightarrow$ $\qquad$ $c_3 = -c_1/6$

$x^2 : -\frac{1}{2} = 12c_4 + \cancel{c_2}. \Rightarrow c_4 = \frac{-1}{24}$

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

$$= c_0 + c_1 x + \left(\frac{1}{2} - c_0\right) x^2 - \frac{c_1}{6} x^3 - \frac{1}{24} x^4 + \cdots$$

$$y = \underbrace{c_0(1 - x^2 + \cdots) + c_1\left(x - \frac{x^3}{3!} + \cdots\right)}_{\text{Homogeneous}} + \underbrace{\left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)}_{\text{Particular}}$$

{10/09/24

$y' = y^2 - x$, $y(0) = 1$

$\frac{dy}{dx} = y^2 - x$. $\qquad$ $dy = (y^2 - x)\,dx$.

Not solvable by Power serie as "y^2" is there.

$\frac{dy}{dx} - x = y^2$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y^2} = 1$

$y' = y^2 - x \quad , y(0) = 1$ $\qquad x_0 < \xi < x_0+h$

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots + \frac{h^n}{n!} y^n(x_0) + \frac{h^{n+1}}{(n+1)!} y^{n+1}(\xi)$$

↑ taylor series expansion. consider $h = x$

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \times 4 + \cdots$$

$\Rightarrow y'(x) = y^2 - x$ $\quad y(0)=1$

$y'(0) = 1 - 0 = 1$

$\Rightarrow y'' = 2yy' - 1$

$y''(0) = 2(1) - 1 = 1$

$\Rightarrow y''' = 2yy'' + y' \times 2 \times y'$

$y'''(0) = 2 + 2 = 4$

Numerics for ODE & PDE :

Only for 1st order.

Implicit $\quad F(x, y, y') = 0$ $\qquad$ (or) "$\underline{y' = f(x,y)}$, $\underline{y(x_0) = y_0}$" Explicit.

$h = \frac{b-a}{N}$ $\qquad h \to$ step size

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} \cdot y''(x_0) + \cdots$$

a, $x_0, x_1, x_2, x_3, x_4, x_5, \ldots$ b

$$y(x_1) \approx y(x_0) + h y'(x_0) + \underbrace{\frac{h^2}{2!} y''(\xi)}_{\text{result in local error.}}$$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_0 + 3h$

$$y_1 = y_0 + h f(x_0, y_0) + e_1$$

$$y_2 = y_1 + h f(x_1, y_1) + e_2$$

$\vdots$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Error:

$O(h^4) < O(h^3) < O(h^2)$

$\Rightarrow h < 1$

↳ Euler 1st order method.

$N \to$ No. of steps $\qquad N = \frac{b-a}{h}$

$O(\,) \Rightarrow$ order of.

Local Error: $O(h^2)$

Global Error: $N\, O(h^2) \Rightarrow \frac{1}{h} O(h^2)$

↓ accumulated local error $\qquad = O(h)$

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Eg: $y' = y + x, \quad y(0) = 0$, find $y(1)$? $\quad y = e^x - x - 1$

$h = 0.2$

n	x_n	y_n	$y(x_n)$	Error $y(x_n) - y_n$
0	0	0	0	0
1	0.2	0	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

$y_1 = y_0 + h f(x_0, y_0)$

$= y_0 + 0.2 (y_0 + x_0)$

$= 0 + 0.2 (0 + 0)$

$y_1 = 0$

0 (x_0), x_1, x_2, x_3, x_4, x_5 (1)

$x_0 = 0$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$x_3 = x_2 + h = 0.6$

$x_4 = x_3 + h = 0.8$

$x_5 = x_4 + h = 0.8$

$y_2 = y_1 + 0.2(y_1 + x_1)$
$= 0 + 0.2(0 + 0.2)$
$= 0.04$

$y_3 = y_2 + 0.2(y_2 + x_2)$
$= 0.04 + 0.2(0.04 + 0.4)$
$= 0.04 + 0.088 = \cancel{0.092} = 0.128$

$\Rightarrow$

$y(x_n) = e^x - x - 1$

$x_1 = 0.2 \quad = e^{0.2} - 0.2 - 1$

$y(x_1) = 0.021$.

Two step method

Modified Euler Method: (Predictor – Corrector method):

Global error is reduced to $O(h^3)$

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$

FD $\quad \frac{\partial u}{\partial t} \approx \frac{u_{i+1} - u_i}{\Delta t}$

BD $\quad \frac{\partial u}{\partial t} \approx \frac{u_i - u_{i-1}}{\Delta t}$

Central diffc $\quad \approx \frac{u_{i+1} - u_{i-1}}{2\Delta t}$

Taylor series exp:

$$y(x_n+h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \cdots + \frac{h^{n+1}}{(n+1)!}y^{n+1}(\xi)$$

$x_n < \xi(x_n+h)$

$$= y(x_n) + h\tilde{f}_n + \frac{h^2}{2!}\tilde{f}_n' + \ldots \quad [\tilde{f}_n = f(x_n, y_n)]$$

$$= y(x_n) + h\tilde{f}_n + \frac{h^2}{2!}\tilde{f}_n' + \frac{h^3}{3!}f_n'' + \cdots \quad (*)$$

$$y(x_n+h) - y(x_n) = h\tilde{f}_n + \frac{h^2}{2!}\tilde{f}_n' + \frac{h^3}{3!}f_n''$$

$$\cancel{y(x_n+h) = y(x_n) + \frac{h}{2}[f_n + f_{n+1}]} \quad (*)$$

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y' = f(x,y) \\ y'(x_n) = \underbrace{f(x_n, y_n)}_{\tilde{f}_n} \end{cases}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[\tilde{f}_n + \tilde{f}_{n+1}\right]$$

$$= y_n + \frac{h}{2}\left[\tilde{f}_n + \underbrace{\tilde{f}_n + h\tilde{f}_n' + \frac{h^2}{2!}\tilde{f}_n'' + \frac{h^3}{3!}\tilde{f}_n''' + \cdots}_{\text{Taylor series Expan.}}\right] \quad (**)$$

Subtract $(**) - (*) \Rightarrow \frac{h}{2}(\tilde{f}_n + \tilde{f}_n) - h\tilde{f}_n$

$\Rightarrow$ h & h^2 terms cancel.

$$= \left(\frac{h^3}{4}\tilde{f}_n'' - \frac{h^3}{6}\tilde{f}_n''\right) + \ldots$$

Error is order of $O(h^3)$

$$= \frac{1}{12}h^3\tilde{f}_n''$$

Classical Runge-Kutta Method of fourth order:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

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Euler Method:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$y' = f(x, y)$

$y(x_0) = y_0$

small segment Error = local Error

full segment Error = Global Error.

Eg: $y' = y + x$, $y(0) = 0$. Given, Exact solution $y = e^x - x - 1$, $h = 0.2$

find: $y(1)$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

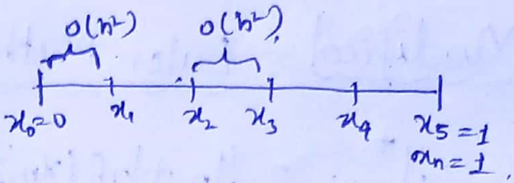
$x_2 = x_1 + h = x_0 + 2h = 0.4$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

$x_5 = x_4 + h = 0.8 + 0.2 = 1$

Error $= |y_n - y(x_n)|$



n	x_n	y_n	$y(x_n)$	Error.
0	0.0	0.0	0.0	0.0
1	0.2	0.0	0.021	
2				
3				
4				
5				

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n	x_n	y_n (Range Kutta)	Modified Euler	$y(x_n)$	Error RK	Error modified Euler	Error Euler
0	0.0	0.00000	0.00000	0.0	0.0	0.0	0.0
1	0.2	0.02140	0.02	0.021403	0.00003	0.0014	0.021
2	0.4	0.091818	0.0884	0.091825	0.000007	0.0034	0.052
3	0.6	0.222107	0.2158	0.22219	0.000011	0.0063	0.094
4	0.8	0.425521	0.4153	0.425541	0.000021	0.0102	0.152
5	1.0	0.718281	0.7027	0.718282	0.000031	0.0156	0.230

Q: $y'' + y = 0$, $y(0) = y_0$, $y'(0) = y_1$

$y = u_1$

$y' = u_2$

$y'' = u_2'$

$\frac{du_1}{dx} = u_2$

$\frac{du_2}{dx} = u_2' = y''$

$y'' = -y$

$\frac{du_2}{dx} = -y$

$\frac{du_2}{dx} = -u_1$

$u_1(0) = y(0) = y_0$

$u_2(0) = y'(0) = y_1$

$u_{1,n+1} = u_{1,n} + h f_1(x_n, u_{1,n}, u_{2,n})$

$u_{2,(n+1)} = u_{2,n} + h f_2(x_n, u_{1,n}, u_{2,n})$

$\frac{du_1}{dx} = f_1(x, u_1, u_2)$

$\frac{du_2}{dx} = f_2(x, u_1, u_2)$

$$\boxed{\frac{d\vec{u}}{dx} = \vec{f}(x, \vec{u})}$$

Q : $y'' + 10y' + 7y = 10$

$y = u_1$

$y' = u_2$

$y'' = u_2'$

$$\frac{du_1}{dx} = u_2$$

$$\frac{du_2}{dx} = 10 - 7u_1 - 10u_2$$

$\Rightarrow$

Numerical PDE's :

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$\Delta u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

↳ laplace

⇒ Any function is differentiable at (a,b) not in $[a,b]$

↳ Elliptic
↳ Parabolic
↳ Hyperbolic.
↳

$A u_{xx} + B u_{xy} + C u_{yy} + f(x, y, u, u_x, u_y) = 0.$

for laplace
$B = 0$,
$f(\,) = 0$

$B^2 - AC < 0$ Elliptical $\Rightarrow$ $AC - B^2 > 0$

$B^2 - AC = 0$ Parabolic $AC - B^2 = 0$

$B^2 - AC \geq 0$ Hyperbolic $AC - B^2 < 0$

$\Rightarrow -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u_x + 7u_y + 10u = 10xy$

$A = -1$
$C = -1$
$f(\,) = u_x + 7u_y + 10u - 10xy$

⇒ Boundary conditions : Dirrichlet, Neumann, Mixed

⇒ finding normal

Problem: Differentiable

$f'(x) = \lim_{x \to 0^+} \frac{f(x+h) - f(x)}{h}$

$= \lim_{x \to 0^-} \frac{f(x-h) - f(x)}{h}$

$[a, b]$
$(a-h]$ is not present so it can't differentiable.

Dirichlet :

$-\Delta u = f$ in Ω

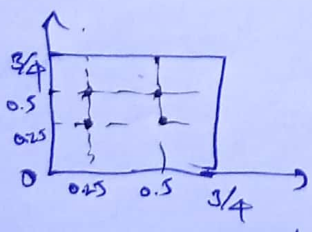
$u = 0$ on $\partial\Omega$

$\Omega = (0, 3/4) \times (0, 3/4)$

Neumann :

$-\Delta u = f$ in Ω

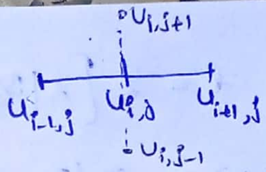
$\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$



$f(x, y) = x^2 - y^2 + 7x$

$f(x_1, y_1) = f_{11} = (0.25^2) - (0.25^2) + 7(0.25)$

value is zero on boundary
have to find inner points



$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad u(x_i, y_j) = u_{ij}$$

$$\frac{\partial u(x_i, y_j)}{\partial x} \approx \frac{u_{i+1,j} - u_{i,j}}{h} \qquad \frac{\partial u(x_i, y_j)}{\partial y} \approx \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - u_{i,j} + u_{i-1,j} - u_{i,j}}{h^2} \qquad \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right) = \frac{1}{h}\left[\frac{u_{i+1,j} - u_{i,j}}{h} - \frac{(u_{i,j} - u_{i-1,j})}{h}\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}}$$

$$-u_{i+1,j} + 4u_{ij} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = h^2 \cancel{f(x,y)} = h^2 f_{ij}$$

i.e. $h = k$ in this eq.

$i = 1,\ j = 1$

$$-u_{21} + 4u_{11} - \cancel{u_{01}}^{0} - u_{12} - \cancel{u_{10}}^{0} = (0.25)^2 f_{11}$$

$\Rightarrow$ unknown: u_{11} u_{21} u_{22} u_{12} innerpoints.

$i = 2,\ j = 1$

$$-\cancel{u_{31}}^{0} + 4u_{21} - u_{11} - u_{22} - \cancel{u_{20}}^{0} = (0.25)^2 f_{21}.$$

$i = 1,\ j = 2$

$$-u_{22} + 4u_{12} - \cancel{u_{0,2}}^{0} - \cancel{u_{1,3}}^{0} - u_{1,1} = (0.25)^2 f_{12}.$$

$i = 2,\ j = 2$

$$-\cancel{u_{3,2}}^{0} + 4u_{22} - u_{12} - \cancel{u_{2,3}}^{0} - u_{21} = (0.25^2) f_{22}$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 0 & 4 & -1 \\ -1 & 4 & 0 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} =$$

$R_2 \Leftrightarrow R_3$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$