MFES/CSI — Relation Calculus Reference Sheet

Relational composition

Pointwise def.

$$\begin{tikzcd} B & A \arrow[l, "R"'] & C \arrow[l, "S"'] \arrow[ll, bend left, "R \cdot S"] \end{tikzcd} \quad b(R \cdot S)c \equiv \langle \exists\, a : b\, R\, a : a\, S\, c \rangle \tag{5.11}$$

Associativity

$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \tag{5.12}$$

Identity

$$\begin{cases} R = R \cdot id_A \\ R = id_B \cdot R \end{cases} \tag{5.13}$$

Converse

Pointwise def.

$$b\, R\, a \quad \Leftrightarrow \quad a\, R^{\circ}\, b \tag{5.14}$$

Universal-°

$$X^{\circ} \subseteq Y \quad \equiv \quad X \subseteq Y^{\circ} \tag{5.136}$$

Involution

$$(R^{\circ})^{\circ} = R \tag{5.15}$$

Contravariance

$$(R \cdot S)^{\circ} = S^{\circ} \cdot R^{\circ} \tag{5.16}$$

Isomorphism

$$R \subseteq S \equiv R^{\circ} \subseteq S^{\circ} \tag{5.137}$$

"Guardanapo"

$$b(f^{\circ} \cdot R \cdot g)a \quad \equiv \quad (f\, b)R(g\, a) \tag{5.17}$$

Relation inclusion

Pointwise

$$R \subseteq S \quad \mathit{iff} \quad \langle \forall\, a,b :: b\, R\, a \Rightarrow b\, S\, a \rangle \tag{5.19}$$

Reflexion

$$R \subseteq R \tag{5.21}$$

Transitivity

$$R \subseteq S \wedge S \subseteq T \quad \Rightarrow \quad R \subseteq T \tag{5.22}$$

Top and bottom

$$\bot \subseteq R \subseteq \top \tag{5.25}$$

Absorption

$$R \cdot \bot = \bot \cdot R = \bot \tag{5.26}$$

$\bot^{\circ} = \bot \quad \top^{\circ} = \top \quad \top \cdot \ldots \cdot \top = \top \quad \bot \cdot \ldots \cdot \bot = \bot$

Relation equality

Pointwise

$$R = S \quad \mathit{iff} \quad \langle \forall\, a,b : a \in A \wedge b \in B : b\, R\, a \Leftrightarrow b\, S\, a \rangle \tag{5.18}$$

Indirect equality

$$\begin{aligned} R = S \quad & \equiv \quad \langle \forall\, X :: (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ & \equiv \quad \langle \forall\, X :: (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{aligned} \tag{5.24}$$

"Ping-pong"

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{5.20}$$

RELATION TAXONOMY

Kernel $$\ker R = R^\circ \cdot R$$ (5.32)

Image $$\mathrm{img}\ R \stackrel{\mathrm{def}}{=} R \cdot R^\circ$$ (5.33)

Duality
$$\begin{array}{rcl} \ker (R^\circ) & = & \mathrm{img}\ R \\ \mathrm{img}\ (R^\circ) & = & \ker R \end{array}$$
(5.34, 5.35)

Criteria (5.36)

	Reflexive	*Coreflexive*
ker R	entire R	injective R
img R	surjective R	simple R

FUNCTIONS

Shunting rules
$$\begin{array}{rcl} f \cdot R \subseteq S & \equiv & R \subseteq f^\circ \cdot S \\ R \cdot f^\circ \subseteq S & \equiv & R \subseteq S \cdot f \end{array}$$
(5.46, 5.47)

Equality $$f \subseteq g \equiv f = g \equiv f \supseteq g$$ (5.48)

CONSTANT FUNCTIONS

Natural property $$\underline{k} \cdot R \subseteq \underline{k}$$ (5.39)

Corollary $$\ker\ ! = \frac{!}{!} = \top$$ (5.—)

Truth functions
$$\begin{array}{l} \mathit{true} = \underline{\textsc{True}} \\ \mathit{false} = \underline{\textsc{False}} \end{array}$$
(5.40, 5.41)

FUNCTION DIVISION

Definition $$\frac{f}{g} = g^\circ \cdot f \quad cf. \quad \begin{tikzcd} B & & A \arrow[ll, "\frac{f}{g}"'] \arrow[dl, "f"] \\ & C \arrow[from=ul, "g"'] & \end{tikzcd}$$ (5.49)

Properties
$$\begin{array}{rcl} \frac{f}{id} & = & f \\ \left(\frac{f}{g}\right)^\circ & = & \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} & = & k^\circ \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} & = & \ker f \\ a \neq b & \Leftrightarrow & \frac{\underline{a}}{\underline{b}} = \bot \end{array}$$
(5.50 → 5.54)

RELATION UNION

Pointwise definition $$b\,(R \cup S)\,a \equiv b\,R\,a \vee b\,S\,a$$ (5.57)

Universal property $$R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X$$ (5.59)

Right linearity $$R \cdot (S \cup T) = (R \cdot S) \cup (R \cdot T)$$ (5.60)

Left linearity $(S \cup T) \cdot R \;=\; (S \cdot R) \cup (T \cdot R)$ (5.61)

Converse-$\cup$ $(R \cup S)^\circ = R^\circ \cup S^\circ$ (5.65)

Top-$\cup$ $R \cup \top = \top$ (5.68)

Bottom-$\cup$ $R \cup \bot = R$ (5.69)

Union simplicity $M \cup N$ is simple $\equiv$ M, N are simple and $M \cdot N^\circ \subseteq id$ (5.70)

Relation intersection

Pointwise definition $b\,(R \cap S)\,a \;\equiv\; b\,R\,a \wedge b\,S\,a$ (5.56)

Universal property $X \subseteq R \cap S \;\equiv\; X \subseteq R \wedge X \subseteq S$ (5.58)

Distribution (1) $(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \;\Leftarrow\; \left\{ \begin{array}{c} Q \cdot \mathrm{img}\, R \subseteq Q \\ \vee \\ S \cdot \mathrm{img}\, R \subseteq S \end{array} \right.$ (5.62)

Distribution (2) $R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \;\Leftarrow\; \left\{ \begin{array}{c} (\mathrm{ker}\, R) \cdot Q \subseteq Q \\ \vee \\ (\mathrm{ker}\, R) \cdot S \subseteq S \end{array} \right.$ (5.63)

Distribution (3) $g^\circ \cdot (R \cap S) \cdot f \;=\; g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f$ (5.71)

Converse-$\cap$ $(R \cap S)^\circ = R^\circ \cap S^\circ$ (5.64)

Bottom-$\cap$ $R \cap \bot = \bot$ (5.66)

Top-$\cap$ $R \cap \top = R$ (5.67)

Misc. $k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k} \;,\; k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k}$ (5.73)

Relation division

Universal-/ $Z \cdot Y \subseteq X \;\equiv\; Z \subseteq X/Y$ (5.157)

Pointwise-/ $c\,(X/Y)\,a \equiv \langle \forall\, b \,:\, a\,Y\,b : c\,X\,b \rangle$ (5.158)

$$\begin{tikzcd} & a \arrow[dl, "X/Y"'] \\ c & b \arrow[l, "X"] \arrow[u, "Y"'] \end{tikzcd}$$

Universal- $X \cdot Z \subseteq Y \Leftrightarrow Z \subseteq X \setminus Y$ (5.159)

Pointwise- $a(X \setminus Y)c \;\equiv\; \langle \forall\, b \,:\, b\,X\,a : b\,Y\,c \rangle$ (5.160)

Misc. $X \cdot f \;=\; X/f^\circ$ (5.162)

$f \setminus X \;=\; f^\circ \cdot X$ (5.163)

$X/\bot \;=\; \top$ (5.164)

$$X / id = X \quad (5.165)$$

$$R \backslash (f^{\circ} \cdot S) = f \cdot R \backslash S \quad (5.166)$$

$$R \backslash \top \cdot S = ! \cdot R \backslash ! \cdot S \quad (5.167)$$

$$R / (S \cup P) = R / S \cap R / P \quad (5.168)$$

RELATION DIFFERENCE, IMPLICATION AND NEGATION

Universal-($-$)	$X - R \subseteq Y \equiv X \subseteq Y \cup R$	(5.138)
Pointwise-$\Rightarrow$	$b(R \Rightarrow S)a \equiv (b\,R\,a) \Rightarrow (b\,S\,a)$	(5.147)
Universal-$\Rightarrow$	$R \cap X \subseteq Y \equiv X \subseteq (R \Rightarrow Y)$	(5.148)
Distribution	$f^{\circ} \cdot (R \Rightarrow S) \cdot g = (f^{\circ} \cdot R \cdot g) \Rightarrow (f^{\circ} \cdot S \cdot g)$	(5.154)
Definition-$\neg$	$\neg R = (R \Rightarrow \bot)$	(5.—)
Pointwise-$\neg$	$b\,(\neg R)\,a \Leftrightarrow \neg\,(b\,R\,a)$	(5.—)
Complementation	$R \cup \neg R = \top$	(5.152)
Difference versus implication	$\top - R \subseteq R \Rightarrow \bot$	(5.153)
de Morgan	$\neg(R \cup S) = (\neg R) \cap (\neg S)$	(5.154)
Schröder's rule	$\neg Q \cdot S^{\circ} \subseteq \neg R \Leftrightarrow R^{\circ} \cdot \neg Q \subseteq \neg S$	(5.151)

MONOTONICITY

Composition	$R \subseteq S \wedge T \subseteq U \Rightarrow R \cdot T \subseteq S \cdot U$	(5.78)
Converse	$R \subseteq S \Rightarrow R^{\circ} \subseteq S^{\circ}$	(5.79)
Intersection	$R \subseteq S \wedge U \subseteq V \Rightarrow R \cap U \subseteq S \cap V$	(5.80)
Union	$R \subseteq S \wedge U \subseteq V \Rightarrow R \cup U \subseteq S \cup V$	(5.81)

ENDO-RELATION TAXONOMY

Reflexive	$id \subseteq R$	(5.84)
Coreflexive	$R \subseteq id$	(5.85)
Transitive	$R \cdot R \subseteq R$	(5.86)
Symmetric	$R \subseteq R^{\circ} (\equiv R = R^{\circ})$	(5.87)

Coreflexive $\Rightarrow$ Difuncional   Difuncional: $\boxed{R \cdot R^{\circ} \cdot R = R}$

(5234)   transitive $\Leftrightarrow$ difuncional $\Leftarrow \begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

Anti-symmetric	$R \cap R^\circ \subseteq id$	(5.88)
Irreflexive	$R \cap id = \bot$	(5.89)
Connected	$R \cup R^\circ = \top$	(5.90)

PAIRING ("SPLITS")

Pairing-pointwise	$(a,b)\ \langle R,S\rangle\ c \Leftrightarrow a\ R\ c \wedge b\ S\ c$	(5.101)
Pairing-def	$\langle R,S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S$	(5.102)
Universal-pairing	$X \subseteq \langle R,S\rangle \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{array} \right.$	(5.103)
Pairing-fusion	$\langle R,S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \Leftarrow R \cdot (\mathsf{img}\ T) \subseteq R \vee S \cdot (\mathsf{img}\ T) \subseteq S$	(5.104)
Pairing-fusion (functions)	$\langle R,S\rangle \cdot f = \langle R \cdot f, S \cdot f\rangle$	(5.105)
Pairing and converse	$\langle R,S\rangle^\circ \cdot \langle X,Y\rangle = (R^\circ \cdot X) \cap (S^\circ \cdot Y)$	(5.108)
Kernel of pairing	$\mathsf{ker}\ \langle R,S\rangle = \mathsf{ker}\ R \cap \mathsf{ker}\ S$	(5.111)
Functor-$\times$ def	$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle$	(5.107)
Functor-$\times$-id	$id \times id = id$	(5.112)
Functor-$\times$-composition	$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$	(5.113)
Functor-$\times$ absorption	$(R \times S) \cdot \langle P,Q\rangle = \langle R \cdot P, S \cdot Q\rangle$	(5.106)
Functor-$\times$-division	$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$	(5.127)
Pairing division	$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f,h\rangle}{\langle g,k\rangle}$	(5.109)

COPRODUCTS

Universal	$X = [R,S] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} X \cdot i_1 = R \\ X \cdot i_2 = S \end{array} \right.$	(5.114)
Definition	$[R,S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ$	(5.117)
Reflexion	$\mathsf{img}\ i_1 \cup \mathsf{img}\ i_2 = id$	(5.115)
Disjointness	$i_1^\circ \cdot i_2 = \bot$	(5.116)
Either and converse	$[R,S] \cdot [T,U]^\circ = (R \cdot T^\circ) \cup (S \cdot U^\circ)$	(5.121)
Image of either	$\mathsf{img}\ [R,S] = \mathsf{img}\ R \cup \mathsf{img}\ S$	(5.124)
Exchange law	$[\langle R,S\rangle, \langle T,V\rangle] = \langle [R,T], [S,V]\rangle$	(5.122)

Functor-+-def	$R+S=[i_1\cdot R, i_2\cdot S]$	(5.119)
Functor-+-converse	$(R+S)^\circ = R^\circ + S^\circ$	(5.123)
Functor-+-division	$\frac{f}{g}+\frac{h}{k}=\frac{f+h}{g+k}$	(5.128)

INJECTIVITY PREORDER

Definition	$R\leqslant S \;\equiv\; \mathsf{ker}\, S \subseteq \mathsf{ker}\, R$	(5.234)
Join	$\langle R,S\rangle \leqslant X \;\equiv\; R\leqslant X \wedge S\leqslant X$	(5.235)
Cancellation	$R\leqslant\langle R,S\rangle$ and $S\leqslant\langle R,S\rangle$.	(5.236)
Shunting	$R\cdot g\leqslant S \;\equiv\; R\leqslant S\cdot g^\circ$	(5.237)
Meet	$X\leqslant [R^\circ,S^\circ]^\circ \;\Leftrightarrow\; X\leqslant R\wedge X\leqslant S$	(5.238)
Either-injective	$id\leqslant[R,S] \Leftrightarrow id\leqslant R\wedge id\leqslant S\wedge R^\circ\cdot S=\bot$	(5.126)

THUMB RULES

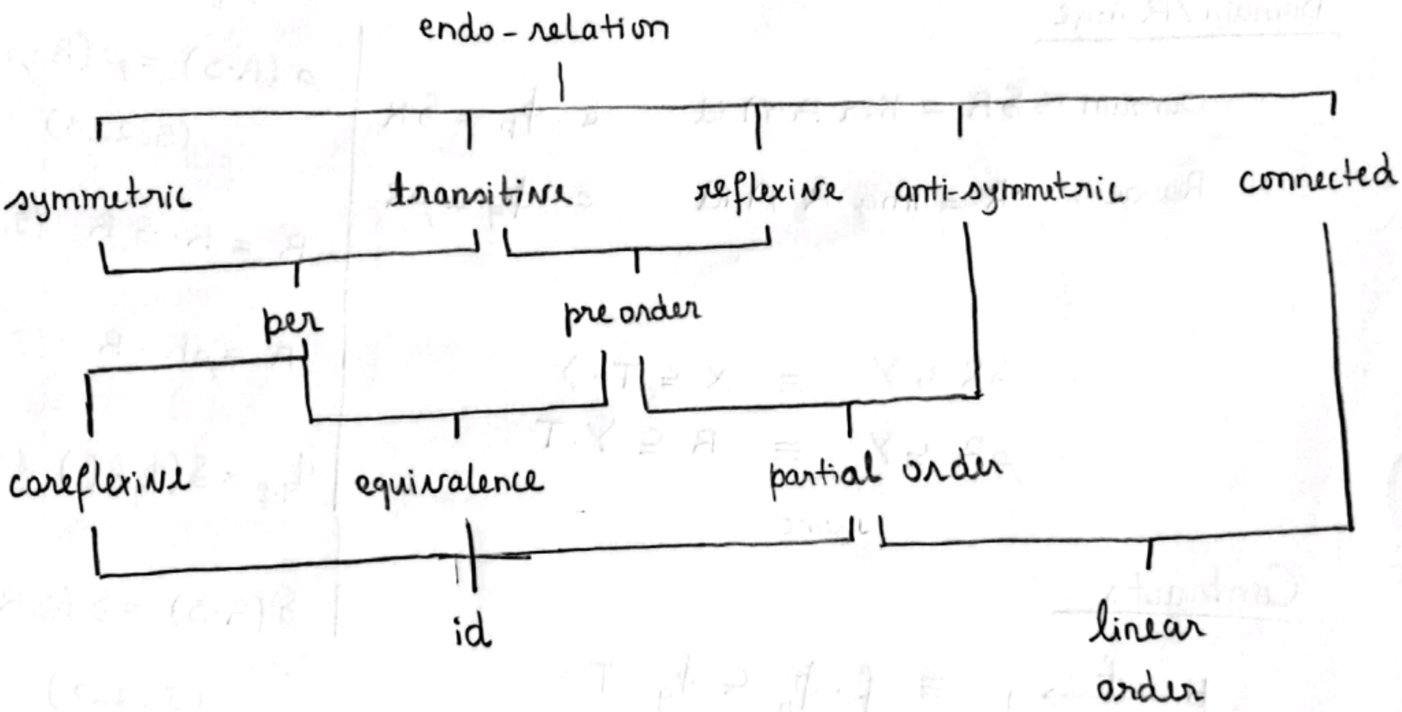
A function f is a bijection iff *its converse f° is a function g*	(5.38)
- converse of *injective* is *simple* (and vice-versa)	(5.43)
- converse of *entire* is *surjective* (and vice-versa)	(5.44)
- *smaller* than injective (simple) is injective (simple)	(5.82)
- *larger* than entire (surjective) is entire (surjective)	(5.83)
$\langle R, id\rangle$ is always *injective*, for whatever R	(5.111a)

QUANTIFIER CALCULUS

Trading-∀	$\langle\forall k : R\wedge S : T\rangle = \langle\forall k : R : S\Rightarrow T\rangle$	(A.1)
Trading-∃	$\langle\exists k : R\wedge S : T\rangle = \langle\exists k : R : S\wedge T\rangle$	(A.2)
de Morgan	$\neg\langle\forall k : R : T\rangle = \langle\exists k : R : \neg T\rangle$	(A.3)
de Morgan	$\neg\langle\exists k : R : T\rangle = \langle\forall k : R : \neg T\rangle$	(A.4)
One-point-∀	$\langle\forall k : k=e : T\rangle = T[k:=e]$	(A.5)
One-point-∃	$\langle\exists k : k=e : T\rangle = T[k:=e]$	(A.6)
Nesting-∀	$\langle\forall a,b : R\wedge S : T\rangle = \langle\forall a : R : \langle\forall b : S : T\rangle\rangle$	(A.7)

Nesting-∃	$\langle \exists a,b : R \wedge S : T\rangle = \langle \exists a : R : \langle \exists b : S : T\rangle\rangle$	(A.8)
Rearranging-∀	$\langle \forall k : R \vee S : T\rangle = \langle \forall k : R : T\rangle \wedge \langle \forall k : S : T\rangle$	(A.9)
Rearranging-∀	$\langle \forall k : R : T \wedge S\rangle = \langle \forall k : R : T\rangle \wedge \langle \forall k : R : S\rangle$	(A.10)
Rearranging-∃	$\langle \exists k : R : T \vee S\rangle = \langle \exists k : R : T\rangle \vee \langle \exists k : R : S\rangle$	(A.11)
Rearranging-∃	$\langle \exists k : R \vee S : T\rangle = \langle \exists k : R : T\rangle \vee \langle \exists k : S : T\rangle$	(A.12)
Splitting-∀	$\langle \forall j : R : \langle \forall k : S : T\rangle\rangle = \langle \forall k : \langle \exists j : R : S\rangle : T\rangle$	(A.13)
Splitting-∃	$\langle \exists j : R : \langle \exists k : S : T\rangle\rangle = \langle \exists k : \langle \exists j : R : S\rangle : T\rangle$	(A.14)

Endo-relations +



Shrinking

Universal: $X \subseteq R \upharpoonright S \equiv X \subseteq R \wedge X \cdot R^\circ \subseteq S$

Definition: $R \upharpoonright S = R \cap S / R^\circ$

Fusion rules: $(S \cdot f) \upharpoonright R = (S \upharpoonright R) \cdot f$

$(f \cdot S) \upharpoonright R = f \cdot (S \upharpoonright (f^\circ \cdot R \cdot f))$

Chaotic: $R \upharpoonright \top = R$

Impossible: $R \upharpoonright \bot = \bot$

Brute force: $R \upharpoonright id$ = largest simple fragment of R

Overriding

Definition: $R \dagger S = S \cup R \cap \perp / S^{\circ}$

Universal: $X \subseteq R \dagger S \equiv X - S \subseteq R \wedge (X - S) \cdot S^{\circ} = \perp$

Predicates

$\Phi_p = id \cap \frac{true}{p}$

Pre restriction of R: $R \cdot \Phi_p = R \cap T \cdot \Phi_p$

Pos restriction of R: $\Phi_q \cdot R = R \cap \Phi_q \cdot T$

Domain / Range

Domain: $\delta R = Ker\ R \cap id$ e $\Phi_p = \delta R$

Range: $\rho R = img\ R \cap id$ e $\Phi_q = \rho R$

$\delta X \subseteq Y \equiv X \subseteq T \cdot Y$

$\rho R \subseteq Y \equiv R \subseteq Y \cdot T$

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$\rho(R \cdot S) = \rho(R \cdot \rho S)$ (5.228)

$R = R \cdot \delta R$ (5.229)

$R = \rho R \cdot R$ (5.230)

$\Phi_{q \cdot f} = \delta(\Phi_q \cdot f)$ (5.231)

$\delta(R \cdot S) = \delta(\delta R \cdot S)$ (5.227)

Contracts

$p \xrightarrow{f} q \equiv f \cdot \Phi_p \subseteq \Phi_q \cdot T$

FT

$R_{t:=N} = R_N$ (6.1)

$R_{t:=F(t_1,\dots,t_n)} = F(R_{t_1},\dots,R_{t_n})$ (6.2)

$R_{t:=t' \to t''} = R_{t'} \to R_{t''}$ (6.3)

Reynolds arrow

$f(R \leftarrow S)g = f \cdot R \subseteq S \cdot g$