Measure theory

All the definitions that follow are directly taken from the book **?**. We specify on which page the definition is given with brackets, for example we write {12} for page 12.

1 Borel measures

{81} Let $(X,\mathcal{U})$ be a topological space. It is called locally compact if each point has a compact neighborhood. It is called σ-compact if there exists an increasing sequence of compact sets (K_i) such that

$$\bigcup_{i\in\mathcal{N}} K_i = X. \tag{1}$$

We assume in this section that $(X,\mathcal{U})$ is a locally compact Hausdorff space and denote by $\mathcal{B}$ the Borel σ-algebra generated by $\mathcal{U}$.

1.1 Regular Borel Measures

Definition *D3.1* - {82} *(Borel Measure, regular measure).* A measure $\mu : \mathcal{B} \to [0,\infty]$ is called

- a Borel measure if $\mu(K) < \infty$ for every compact set $K \subset X$
- outer regular if for every Borel set,

$$\mu(B) = \inf\{\mu(U) : B \subseteq U, U \text{ is open}\} \tag{2}$$

- inner regular if for every Borel set,

$$\mu(B) = \sup\{\mu(K) : B \subseteq K, K \text{ is compact}\}. \tag{3}$$

A Radon measure is an inner regular Borel measure.

Definition {86} *(Local regularity).* Let $\mathcal{A}$ be a subclass of the Borel sets $\mathcal{B}$. We say that a measure μ is outer (inner) regular for the class $\mathcal{A}$ if (**??**) (or (**??**)) is satisfied, with the inf (sup) taken on $\mathcal{A}$.

Lemma *L3.7* - {86} *().* Let μ be an outer regular Borel measure that is inner regular for $\mathcal{U}$. Then,

- μ is inner regular for $\{B \in \mathcal{B} : \mu(B) < \infty\}$
- If X is σ-compact then μ is regular.

Theorem *T3.8* - {87} *(Riez).* Let $\mu_1 : \mathcal{B} \to [0,\infty]$ be an outer regular Borel measure that is inner regular on open sets. Let

$$\mu_0(B) := \sup\{\mu_1(K) : K \subset B \textit{ and } K \textit{ is compact}\}. \tag{4}$$

The following holds:

(i) μ_0 is a Radon measure, it agrees with μ_1 on all compact sets and all open sets, and $\mu_0(B) \le \mu_1(B)$ for all $B \in \mathcal{B}$

(ii) If X is σ-compact then $\mu_0 = \mu_1$,

(iii) If $f : X \to \mathbb{R}$ is a compactly supported continuous function then

$$\int_X f d\mu_0 = \int_X f d\mu_1. \tag{5}$$

(iv) Let $\mu : \mathcal{B} \to [0, \infty]$ be a Borel measure that is inner regular on open sets. Then, $\int_X f d\mu = \int_X f d\mu_1$ for every compactly supported continuous function iff $\mu_0 \le \mu \le \mu_1$.

1.2 Outer Borel Measures

Definition *D1.54* - {39} *(Complete measure space).* A measure space $(X, \mathcal{A}, \mu)$ is complete if

$$N \in \mathcal{A}, \mu(N) = 0, E \subset N \Rightarrow E \in \mathcal{A}. \tag{6}$$

Definition *D2.3* - {50} *(Outer measure).* Let X be a set. A function $\nu : 2^X \to [0, \infty]$ is called an outer measure if it satisfies the following axioms:

(a) $\nu(\varnothing) = 0$,

(b) If $A \subset B \subset X$ then $\nu(A) \le \nu(B)$,

(c) If $A_i \subset X$ for $i \in \mathbb{N}$, then $\nu\left(\cup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \nu(A_i)$.

A set A is called ν-measurable if is satisfies

$$\nu(D) = \nu(D \cap A) + \nu(D \backslash A). \tag{7}$$

Theorem *T2.4* - {50} *(Caratheodory).* Let $\mathcal{A}$ be the set of ν-measurable sets. Then, $\mathring{A}$ is a σ-algebra, $\mu := \nu_{|\mathcal{A}}$ is a measure and the measure space $(X, \mathring{A}, \mu)$ is complete.

Definition *D3.11* - {92} *(Borel outer measure).* A Borel outer measure on X is an outer measure $\nu : 2^X \to [0, \infty]$ that satisfies the following axioms:

(a) If $K \subset X$ is compact then $\nu(K) < \infty$.

(b) If $K_0, K_1 \subset X$ are disjoint compact sets then $\nu(K_0 \cup K_1) = \nu(K_0) + \nu(K_1)$.

(c) ν is outer regular on every set,

(d) ν is inner regular on every open set.

Theorem *T3.12* - {92} *(Caratheodory for Borel outer measures).* Let ν be a Borel outer measure. Then $\nu_{\mathcal{B}}$ is an outer regular Borel measure and is inner regular on open sets.

1.3 Riez representation theorem

Definition {97} *(Compactly supported function).* A function $f : X \to \mathbb{R}$ is said to be compactly supported if its support

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}} \tag{8}$$

is compact. We define:

$$C_c(X) = \{f : X \to \mathbb{R} | f \text{ is continuous and compactly supported}\}. \tag{9}$$

Definition *D3.13* - {97} *(Positive linear functional).* A linear functional $\Lambda : C_c(X) \to \mathbb{R}$ is said to be positive if

$$f \geq 0 \Rightarrow \Lambda(f) \geq 0. \tag{10}$$

Lemma {98} *(Continuous functions with bounded support are integrable).* Let $\mu : \mathcal{B} \to [0, \infty]$ be a Borel measure. For all $f \in C_c(X)$, f is μ-integrable.

We define:

$$\Lambda_\mu(f) := \int_X f d\mu. \tag{11}$$

Theorem *T3.15* - {98} *(Riez representation theorem).* Let $\Lambda : C_c(X) \to \mathbb{R}$ be a positive linear functional. Then,

(i) There exists a unique Radon measure μ_0 such that $\Lambda_{\mu_0} = \Lambda$

(ii) There exists a unique outer regular Borel measure μ_1 that is inner regular on open sets and $\Lambda_{\mu_1} = \Lambda$

(iii) The Boral measures μ_0 and μ_1 agree on compact and open sets, and $\mu_0 \leq \mu_1$

(iv) Let μ be a Boral measure that is inner regular on open sets. Then, $\Lambda_\mu = \Lambda$ implies that $\mu_0 \leq \mu \leq \mu_1$.

Lemma *C3.17* - {105} *(Outer Radon).* Radon measures are outer regular on compact sets.

Theorem *T3.18* - {105} *(Borel measures are regular).* Let X be a locally compact Hausdorff space. Then,

(a) Assume X is σ-compact. Then, every Borel measure that is inner regular on open sets is regular.

(b) Assume that every open set of X is σ-compact. Then, every Borel measure is regular.

Definition {106} *(Bases, countability).* A basis of a topological space $(X,\mathcal{U})$ is a collection $\mathcal{V}$ such that for every $U \in \mathcal{U}$, there is a subset $\mathcal{S}$ of $\mathcal{V}$ such that $U = \bigcup_{V \in \mathcal{S}} V$. A second countable topological space has a countable basis. For a first countable topological space, every point has a local countable basis, i.e. there is a countable sequence of sets W_i, that all contain x, such that every open set that contains x contains one of the W_i.

Lemma *L3.21* - $\{106\}$ *(σ-compactness and countability).* Let X be a locally compact Hausdorff space.

(i) If X is a second countable space, then every open subset is σ-compact.

(ii) If every open set of X is σ-compact, then X is first countable.

The Alexandrov double arrow space is an example of a space that is σ-compact but not second countable.

References

Dietmar A. Salamon. *Measure and Integration*. 2020.