Synthesis of Computability and Analysis

This report is about summarizing the book [Bra18].

1 Introduction and notation

- $\mathbb{N} = \{0, 1, 2, \ldots\}$. Identification of real numbers with $0 = \varnothing$ and $n + 1 = \{0, \ldots, n\}$. $X^0 = \{\varnothing\} = 1$.
- $f :\subseteq X \rightrightarrows Y := (\Phi, X, Y)$ is a partial multi-valued function, and $\Phi \subseteq X \times Y$. The inverse is defined as $f^{-1} = (\Phi^{-1}, X, Y)$, where $\Phi^{-1} = \{(y, x) : (x, y) \in \Phi\}$.
- $f(x) = g(x)$ means that either both $f(x)$ and $g(x)$ are defined and equal, or that they are both undefined.
- A word over X is a partial function $w :\subseteq \mathbb{N} \to X$ such that $\mathrm{dom}(w) = \{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$.
- $\varepsilon = (\varnothing, \mathbb{N}, X)$ is identified with $\varnothing = 0$.
- for $u, v \in X^*$, $u \sqsubseteq v$ means that $\mathrm{graph}(u) \subseteq \mathrm{graph}(v)$
- for $p \in X^{\mathbb{N}}$, we write $p|_n := p(0) \ldots p(n) \in X^*$. For words, our terminology allows to define $p|_n$ even if n is bigger than the size of the word.
- $\hat{a} := a^{\mathbb{N}} := aaa \ldots$.
- A preorder $\leq$ is a relation such that is reflexive and transitive. A partial order is a preoder that is anti-symmetric.
- A equivalence relation $\equiv$ is a reflexive, transitive and symmetric relation. We naturally defined the equivalence classes in the usual meaning.
- From any preorder, one can define an equivalence relation by saying $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. A preorder induces a partial order on $X/\equiv$.
- In an partially ordered set $(X, \leq)$, we say that $x, y \in X$ have a supremum or a join $\sup(x, y) = x \vee y \in X$, if $x \leq \sup(x, y)$ and $y \leq \sup(x, y)$, and for all $z \in X$ satisfying $x \leq z$ and $y \leq z$, then $\sup(x, y) \leq z$. Similar definition with infinimum of meet $\inf, \wedge$. We say that $(X, \leq)$ is an upper semi-lattice if every pair of elements have a join, and a lower semi-lattice if every pair of elements has a meet. A lattice is both an upper semi-lattice and a lower semi-lattice.
- An upper semi-lattice is called distributive if for all $x, y, z \in X$,

  $$x \leq y \vee z \Rightarrow x = y' \vee z',$$

  for some $y' \leq y, z' \leq z$. If it is a lattice, then we call it distributive if

  $$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

- A map $f : X \to Y$ on preordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ is called monotone if $x \leq_X y$ implies $f(x) \leq_Y f(y)$.

- A map $C : X \to X$ on a preordered set $(X, \leq)$ is called a closure operator if it is extensive, monotone and idempotent. Interior operator satisfies contrary definitions.
- Let X, Y be preordered sets, $U : X \to Y$ and $L : Y \to X$. (L, U) is called a Galois connection if
$$L(y) \leq x \Leftrightarrow y \leq U(x).$$

2 Computability and limit computability

Turing machines are considered to be operating on $\mathbb{N}$ (in the cells).

Definition *D2.1.1 - {10}: (Discrete computable function).* A function $f :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is called computable if there is a Turing machine that halts on every $w \in \mathrm{dom}(f)$ and produces $f(w)$ on the output tape. Also, this Turing machine must not halt on $w \notin \mathrm{dom}(f)$.

Definition *D2.1.2 - {10}: (Computably enumerable and computable sets).* Let $A \subseteq \mathbb{N}^*$. A is called computably enumerable if $A = \mathrm{dom}(f)$ for some computable function. A is computable or decidable if A and $\mathbb{N}^* \backslash A$ are c.e.

Definition *D2.1.3 - {11}: (Computable function).* A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called computable if there exists a Turing machine with one-way output tape such that on input $p \in \mathrm{dom}(F)$, it produces $F(p)$ on the output tape in the long run.

Lemma *P2.1.4 - {11}: (Restriction).* $F|_A$ is computable if F is computable.

Lemma *P2.1.5 - {11}: (Composition).* $G \circ F$ is computable if F, G are computable.

Lemma *P2.1.6 - {11}: (Computable points).* $p \in \mathbb{N}^{\mathbb{N}}$ is computable if and only if the constant function $c : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with value p is computable.

Lemma *P2.1.7 - {11}: (Computable invariance).* Let F computable and $p \in \mathrm{dom}(F)$ is computable. Then, $F(p)$ is computable.

The topology we consider on $\mathbb{N}^{\mathbb{N}}$ is the product of the discrete topology on $\mathbb{N}$, i.e. generated by the cylinder sets $n_1 n_2 \ldots n_k \mathbb{N}^{\mathbb{N}}$.

Theorem *T2.1.9 - {14}: (Continuity theorem).* Any computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous.

Definition *D2.1.10 - {14}: (Tupling functions).*

(i) for $n, k \in \mathbb{N}$, $\langle n, k \rangle$ denotes the Cantor pairing function,

(ii) for $p, q \in \mathbb{N}^{\mathbb{N}}$, $\langle p, q \rangle \in \mathbb{N}^{\mathbb{N}}$ is defined as
$$\langle p, q \rangle(2n) := p(n)$$
$$\langle p, q \rangle(2n+1) := q(n)$$

(iii) for $p_0, p_1, \ldots \in \mathbb{N}^{\mathbb{N}}$, $\langle p_1, p_2, \ldots \rangle \in \mathbb{N}^{\mathbb{N}}$ is defined as
$$\langle p_1, p_2, \ldots \rangle(\langle n, k \rangle) := p_n(k)$$

(iv) for $n \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$, $\langle n, p \rangle := np$.

Example 2.1.11 in page {15} gives many examples of computable functions, based on the pairing functions.

Example 2.1.12 in page {15} gives the definition of the limit map, that is not continuous,

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \ldots \rangle \mapsto \lim_{i\to\infty} p_i.$$

Definition *D2.1.13* - {15}*: (Computable sequence).* A sequence $(p_i)_{i\in\mathbb{N}}$ of elements $p_i \in \mathbb{N}^{\mathbb{N}}$ is called computable if $\langle p_1, p_2, \ldots \rangle$ is computable.

Definition *D2.1.14* - {16}*: (Parallelization).* Let $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a function. Then, the parallelization $\langle \hat{F} \rangle :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ of F is defined by

$$\langle \hat{F} \rangle \langle p_1, p_2, \ldots \rangle := \langle F(p_1), F(p_2), \ldots \rangle$$

for every sequence p_i in $\operatorname{dom} F$.

Lemma *P2.1.15* - {16}*: (Computable parallelization).* If $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable, then its parallelization $\langle \hat{F} \rangle :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable too.

Lemma *C2.1.16* - {16}*: (Sequential invariance).* If $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable, and (p_i) be a computable sequence in $\operatorname{dom}(F)$. Then, $(F(p_i))$ is computable too.

References

[Bra18] Vasco Brattka. *Computability and Analysis.* Version 2.1 edition, 2018.