Topology toolkit

All the definitions that follow are directly taken from the book **?**. We specify on which page the definition is given with brackets, for example we write {12} for page 12.

1 Open sets

Definition {18} *(Topology).* Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a topology if

(a) $\varnothing, X \in \mathcal{T}$,

(b) closure under finite intersections: $U_1, \dots, U_n \in \mathcal{T} \;\Rightarrow\; U_1 \cap \dots \cap U_n \in \mathcal{T}$,

(c) closure under arbitrary unions: $(U_\alpha) \in \mathcal{T} \;\Rightarrow\; \bigcup_\alpha U_\alpha \in \mathcal{T}$.

The elements of $\mathcal{T}$ are call open sets. $(X, \mathcal{T})$ is called a topological space.

Definition {18} *(Neighborhood).* Let $(X, \mathcal{T})$ be a topological space. For all $q \in X$, a neighborhood of q is an open set $A \in \mathcal{T}$ s.t. $q \in A$.

Definition {25} *(Interior).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. We define

$$\operatorname{Int} A := \cup \{U \subset X : U \textit{ is open}\}. \tag{1}$$

Lemma *Pb2.9* - {37} *(Disjoint union topology).* Let $\{X_\alpha\}_{\alpha \in A}$ be a sequence of disjoint topological spaces. Then, we define a topology on $\cup_{\alpha \in A} X_\alpha$ as being the set which intersection wiht each X_α is open in X_α.

2 Closed sets

Definition {24} *(Closed set).* Let $(X, \mathcal{T})$ be a topological space. We say that $A \subset X$ is closed if there exists $U \in \mathcal{T}$ such that $A = X \backslash U$.

Lemma {24} *(Topology of closed sets).* Let $(X, \mathcal{T})$ be a topological space.

1. $\varnothing, X$ are closed,
2. Finite unions of closed sets are closed,
3. Arbitrary intersections of closed sets are closed.

Definition {25} *(Closure, Exterior, Boundary).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. We define

$$\bar{A} := \cap \{C \subset X : C \textit{ is closed}\}, \tag{2}$$
$$\operatorname{Ext} A := X \backslash \bar{A} \tag{3}$$
$$\partial A := X \backslash (\operatorname{Int} A \cup \operatorname{Ext} A) \tag{4}$$

Definition {26} *(Limit point).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. We say that $q \in X$ is a limit point of A if every neighborhood of q contains a point in A that is not q.

Lemma *E2.11* - {26} *(Sequential characterization of closed sets).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. A is closed if and only if it contains all its limit points.

Definition {27} *(Dense set).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. We say that A is dense in X if $\bar{A} = X$.

3 Convergence and continuity

Definition {20} *(Convergence).* Let $(X, \mathcal{T})$ be a topological space, and (x_i) be a sequence in X. We say that (x_i) converges towards x, if for every neighborhood A of x, there exists $N > 0$ such that for all $i \geq N$, $x_i \in A$.

Definition {20} *(Continuity).* Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, and $f : X \to Y$. We say that f is continuous if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

Lemma *L2.1* - {21} *(Examples of continuous maps).* Constant map, identity map, restriction of a continuous function to an open subset, composition of continuous functions are continuous.

Definition {22} *(Homeomorphism).* Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, and $\varphi : X \to Y$. φ is said to be a homeomorphism if it is a continuous bijection with a continuous inverse. If such a map exists, then X and Y are said to be homeomorphic, and we write $X \simeq Y$

Lemma *E2.5* - {22} *(Homeomorphic sets).* $\simeq$ is an equivalence relation.

Definition {24} *(Open map).* Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, and $f : X \to Y$. f is said to be an open map if $f(U) \in \mathcal{T}$, for all $U \in \mathcal{T}$.

Definition {27} *(Closed map).* Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, and $f : X \to Y$. f is said to be a closed map if $f(C)$ is closed, for all closed C.

4 Bases

Definition {27} *(Base).* Let X be a set. A basis in X is a collection $\mathcal{B}$ of subsets of X, satisfying:

(a) $\bigcup_{B \in \mathcal{B}} B = X$,

(b) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, there exists $x \in B_3 \subseteq B_1 \cap B_2$.

Lemma *P2.9* - {27} *(Topology generated by a basis).* Let X be a set, $\mathcal{B}$ be a basis in X, and define $\mathcal{T}$ to be the collection of all unions of elements of $\mathcal{B}$. Then, $\mathcal{T}$ is a topology on X. $\mathcal{T}$ is called the topology generated by $\mathcal{B}$.

Definition $\{27\}$ *(Basis criterion).* Let X be a set, and $\mathcal{B}$ be a collection of subsets of X. We say that $U \subseteq X$ satisfies the basis criterion with respect to B if for all $x \in U$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

Lemma *L2.10* - $\{27\}$ *(Identification of the topology generated by a basis through the basis criterion).* Let X be a set, $\mathcal{B}$ be a basis in X, and define $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then, $U \in \mathcal{T}$ iff U satisfies the basis criterion with respect to $\mathcal{B}$.

Lemma *L2.11* - $\{29\}$ *(Characterization of an open basis for a generating a topology).* Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B} \subseteq \mathcal{T}$. If for all $U \in \mathcal{T}$, U satisfies the basis criterion with respect to $\mathcal{B}$, then $\mathcal{B}$ generates $\mathcal{T}$.

Lemma *E2.15* - $\{29\}$ *(Examples of basis).*

1. Let (M, ρ) be a metric space. The set of open balls is a basis for the topology induced by ρ,

2. Let $(X, \mathcal{T})$ be a discrete topological space. $\{\{x\} : x \in X\}$ is a basis generating $\mathcal{T}$.

Lemma *L2.12* - $\{30\}$ *(Basis characterization of continuity).* Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, and $\mathcal{B}$ be a basis generating $\mathcal{T}_Y$. A map $f : X \to Y$ is continuous iff for all $U \in \mathcal{B} \cap \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Lemma *Pb2.8* - $\{30\}$ *(Basis generation through a homeomorphism).* Let X, Y be two topological spaces, $\mathcal{B}$ be a basis in X, and f be a surjective open map. Then, $f(\mathcal{B})$ is a basis in Y.

5 Manifolds

Definition $\{30\}$ *(Locally Euclidean space of dimension n).* A topological space $(X, \mathcal{T})$ is locally Euclidean of dimension n if every point $q \in M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Lemma *L2.13* - $\{30\}$ *(Characterization of locally Euclidean space through open balls).* A topological space $(X, \mathcal{T})$ is locally Euclidean of dimension n iff either

1. every point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$,

2. every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Definition $\{31\}$ *(Hausdorff spaces).* A topological space $(X, \mathcal{T})$ is said to be a Hausdorff space if for all $x, y \in X$, there exists respective neighborhoods U, V of x, y such that $U \cap V = \varnothing$.

Lemma *L2.14* - $\{31 - 32\}$ *(Properties of Hausdorff spaces).* Let $(X, \mathcal{T})$ be a Hausdorff space.

1. Every one-point set is closed,

2. If a sequence $\{x_i\}$ converges, the limit is unique.

Definition {32} *(Countability).* A topological space $(X, \mathcal{T})$ is said to be second countable if it admits a countable basis, and first countable if each points admits a neighborhood having a countable basis.

Definition {32} *(Cover).* A collection $\mathcal{B}$ of subsets of X is a cover if $\cup_{B\in\mathcal{B}} B = X$, and an open cover if B is open for all $B \in \mathcal{B}$ (if there is a topology on X).

Lemma *L2.15* - {32} *(Countable subcovers).* Let $(X, \mathcal{T})$ be a second countable topological space. Then, every open cover has a countable subcover.

Definition {33} *(Manifold).* An n-dimentional topological manifold (or n-manifold) is a second countable Hausdorff space that is locally Euclidean of dimension n.

Lemma *L2.16* - {34} *(Stability through open sets).* Any open subset of an n-manifold is an n-manifold.

Definition {34} *(Manifold with boundary).* An n-dimentional topological manifold (or n-manifold) is a second countable Hausdorff space that is locally homeomorphic to the half-open set $[0, \infty)^n$.

6 Combination of topological spaces

6.1 Subspace topology

Lemma {39} *(Subspace topology).* Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. Let the subspace topology on A be defined as

$$\mathcal{T}_A := \{U \subset A : U = A \cap V \text{ for some open set } V \subset X\}. \tag{5}$$

Definition {40} *(Topological embedding).* An injective continuous map that is a homeomorphism onto its image is called a topological embedding.

Theorem *T3.3* - {41} *(Characteristic property of Subspace Topologies).* Suppose $A \subset X$ is a subspace. For any topological space Y, a map $f : Y \to A$ is continuous iff the following composite map from Y to X is continuous

$$Y \xrightarrow{f} A \xrightarrow{\iota_A} X. \tag{6}$$

Theorem *T3.9* - {47} *(Uniqueness of Subspace Topologies).* Suppose $A \subset X$ is a subset of X. Then, $\mathcal{T}_A$ is the unique topology on A satisfying the characteristic property.

Lemma *P3.4* - {41} *(Properties of Subspace topology).* Let A be a subspace of some topological space X.

(a) The inclusion map in continuous, and more precisely is a topological embedding.

(b) If $f : X \to Y$ is continuous, then so is $f_{|A}$.

(c) If $f : X \to Y$ is continuous, then so is $f : X \to f(X)$.

(d) Closed subsets of A are intersections of A with closed subsets of X.

(e) If $B \subset A$ is a subspace of A, then B is a subspace of X.

(f) If $B \subset A \subset X$ is open in A, and A is open in X, then B is open in X.

(g) $\mathcal{B}$ is a basis then $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$ is a basis in X.

(h) Any subspace of a Hausdorff space is Hausdorff.

(i) Any subspace of a scond countable space is second countable.

Lemma *L3.8* - {46} *(Gluing lemma).* Let X be a topological space, and suppose that $X = A_1 \cup \ldots A_k$, where each A_i is closed in X. For each i, let $f_i : A_i \to Y$ be a continuous map such that $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. There exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i$, for all i.

6.2 Product spaces

Definition {48} *(Basis of Cartesian product).* Let $X_1, \ldots, X_n$ be topological spaces. We let

$$\mathcal{B} = \{U_1 \times \ldots \times U_n : U_i \in \mathcal{T}_{X_i}\}. \tag{7}$$

$\mathcal{B}$ is a basis in $X_1 \times \ldots \times X_n$, and the topology it generates is called the product topology $\mathcal{T}$. $(X_1 \times \ldots \times X_n, \mathcal{T})$ is called the product space.

Theorem *T3.10/11* - {49} *(Characteristic property of Product topologies).* Let $X_1 \times \ldots \times X_n$ be a product space. A map $f : B \to X_1 \times \ldots \times X_n$ is comtinuous iff each component $f_i := \pi_i \circ f$ is continuous. The product topology is the only to satisfy it.

Definition *Munkers (Infinite product topology (cylinder set topology)).* Let $X_1, \ldots, X_n, \ldots$ be topological spaces, and let $X := \prod_{i=1}^{\infty} X_i$. We let

$$\mathcal{B} := \{U \subset X : \exists n \in \mathcal{N}, U_n \in \mathcal{T}_{X_n}, \pi_n^{-1}(U_n) = U\}. \tag{8}$$

Then, $\mathcal{B}$ generates a topology on X, this topology is the only one that makes the projection maps continuous.

For many properties of product topologies, see page {50}.

6.3 Quotient spaces

Definition {52} *(Quotient space topology).* Let X be a topological space, Y a set, and $\pi : X \to Y$ be a surjective map. We define a topology on Y by declaring $U \subset Y$ to be open iff $\pi^{-1}(U)$ is open in X. This is called the quotient topology on Y. Conversely, we say that $\pi : X \to Y$ is a quotient map if it is surjective, continuous, and Y has the quotient topology induced by π.

{52} We say that $U \subset X$ is saturated if there exist $V \subset Y$ such that $U = \pi^{-1}(V)$ (i.e. U is a union of equivalence classes). $\pi^{-1}(\{y\})$ is called a fiber. A saturated set is a union of fibers.

Lemma *L3.16* - {53} *(Characterization quotient maps).* A continuous surjective map $\pi : X \to Y$ is a quotient map iff it takes saturated open sets to open sets, and same with saturated closed sets.

Lemma *L3.17* - {53} *(Restriction of quotient maps).* The restriction of a quotient map to a saturated open or closed set is a quotient map.

{53} A surjective continuous open or closed map is a quotient map. Composition of quotient maps are quotient maps.

Theorem *T3.29/31* - {57} *(Characteristic property of Quotient topologies).* Let $\pi : X \to Y$ be a quotient map. For any space B, $f : Y \to B$ is continuous iff $f \circ \pi$ is continuous. π is a quotient map iff the characteristic property holds.

By Corollary 3.32, quotient spaces are homeomorphic to each other.

6.4 Group actions

Definition {58} *(Topological group).* A topological group is a group G endowed with a topology such that the product and inverse maps are continuous. A discrete group is a topological group with the discrete topology.

Note that any group with the discrete topology is a topological group.

Lemma *L3.34* - {59} *(Topological subgroup).* Any subgroup are product of topological groups is a topological group.

Definition {59} *(Translation).* For $g \in G$, the left translation map $L_g : G \to G$ defined as $L_g(g') = gg'$ is a homeomorphism. For $g \in G$, the right translation map $R_g : G \to G$ defined as $R_g(g') = g'g$ is a homeomorphism.

Definition {59} *(Group actions).* Let G be a group and X a topological space. A left action of G on X is a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, with the following properties

(i) For any $x \in X$, and any $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$,

(ii) For all $x \in X$, $1 \cdot x = x$.

We say that the action of G on X is continuous if $G \times X \to X$ is continuous. For $x \in X$, we say that $G \cdot x := \{g \cdot x : g \in G\}$ is the orbit of x. We say that an action is transitive is the orbit is the entire space. It is said to be free if the only element satifying $g \cdot x = x$ is $g = 1$. We define as an equivalence relation all the points that are on a same orbit. We denote the quotient sapce by X/G, also called the orbit space of the action.

7 Connectedness

7.1 Generalities on connectedness

Definition {65} *(Separation and connectedness).* Let $(X, \mathcal{T})$ be a topological space. A separation of X is a pair of disjoint open sets $U, V \in \mathcal{T}$, such that $U \cup V = X$. If a separation exists, we say that X is disconnected, and connected otherwise.

Lemma *P4.2* - {66} *(Characterization of connectedness).* Let $(X, \mathcal{T})$ be a topological space. X is connected if and only if the sets that are both open and closed are X and $\varnothing$.

Theorem *T4.3* - {67} *(Connectedness theorem).* Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected as well.

Lemma *P4.4* - {67} *(Properties on connected sets).*

(a) If A is a connected subsut of $U \cup V$, then $A \subset U$ or $A \subset V$.

(b) A is connected $\Rightarrow \bar{A}$ is connected.

(c) Let A_α be a collection of connected set with one common point. Then, $\cup_\alpha A_\alpha$ is connected.

(d) Any finite product of connected spaces is connected.

(e) Any quotient space of a connected set is connected.

Theorem *P4.5* - {68} *(Connected sets are intervals).* A nonempty subset of $\mathbb{R}$ is connected iff it is an interval.

Theorem *T4.6* - {68} *(Intermediate value theorem).* Let X be a connected topological space and f a continued real-valued function. For $p, q \in X$, f takes all values between $f(p)$ and $f(q)$.

7.2 Path-connectedness

Definition {69} *(Path connectedness).* A path in a topological space $(X, \mathcal{T})$ from p to q is a continuous function $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$. We say that $(X, \mathcal{T})$ is path connected if for each $p, q \in X$, there exists a path in $(X, \mathcal{T})$ from p to q.

Theorem *T4.7* - {69} *(Path connectedness implies connectedness).* Path connectedness implies connectedness.

7.3 Components, path components

Definition {70} *(Connectivity relation).* Let $(X, \mathcal{T})$ be a topological space. We define the connectivity relation $p \sim q$ as there exists a connected subset of X containing both p and q.

Lemma *P4.11* - {70} *(Connectivity relation is equivalent).* The connectivity relation is an equivalence relation.

Definition {70} *(Components).* The elements of $X/\sim$ are called the components of X.

Lemma *L4.12* - {71} *(Maximal connected sets are components).* The components of X are exactly the maximal connected subsets of X, that is, connected sets that are not contained in any larger connected set.

Lemma *P4.14* - {71} *(Properties of components).* Let X be a topological space.

(a) the components of X are closed in X,

(b) every connected subset of X is contained in a single component.

Definition {71/72} *(Path components).* Let $(X, \mathcal{T})$ be a topological space. We define the path connectivity relation $p \underset{p}{\sim} q$ as there exists a path from p to q. The elements of $X/\underset{p}{\sim}$ are called the path components of X.

Lemma *P4.15* - {72} *(Properties of path components).* Let X be a topological space.

(a) Each path component is contained in a single component, and each component is a disjoint union of path components,

(b) If $A \subseteq X$ is path connected, then A is contained in a single path component.

Definition {72} *(Local connectedness).* A topological space X is locally connected if it admits a basis of connected open sets, and locally path connected if it admits a basis of path connected open sets.

Lemma *L4.16* - {72} *(Properties of locally conected sets).* (a) If X is locally connected, then each component of X is open,

(b) If X is locally path connected , then each component is open, the path components and components are the same, and X is connected iff it is path connected.

Theorem *P4.17* - {73} *(Path connectedness of manifolds).* Every manifold is locally path connected.

8 Compactness

Definition {73} *(Subcover).* Let $\mathcal{U}$ be a cover of X. Then, a subcover is a subset of $\mathcal{U}$ that still covers X.

Definition {73} *(Compactness).* Let X be a topological space. X is said to be compact if every open cover of X admits a finite subcover. A subset $A \subset X$ is said to be compact if it is compact with respect to the subset topology.

Theorem *T4.18* - {73} *(Compactness theorem).* Let X, Y be two topological spaces, and suppose that X is compact. Let $f : X \to Y$ be a continuous function. Then, $f(X)$ is compact.

Lemma *P4.19* - {74} *(Properties of compactness).*

(a) Every closed subset of a compact space is compact.

(b) In a Hausdorff space X, compact sets can be separatd by open sets.

(c) Every compact set of a Hausdorff space is closed.

(d) Every product of compact spaces is compact.

(e) Every quotient of a compact space is compact.

Theorem *T4.20* - {76} *(Extreme value theorem).* If X is a compact space and $f : X \to \mathbb{R}$ is continuous, then f attains its minimal and maximal values.

8.1 Limit point and sequential compactness

Definition {76} *(Limit point compactness).* A space X is said to be limit point compact if for every infinite subset $A \subseteq X$, A has a limit point in X.

Definition {77} *(Sequential compactness).* A space X is said to be sequntially compact if for every sequence in X has a subsequence converging in X.

Lemma *P4.22* - {77} *(Compact $\subset$ Limit point compact).* Compactness implies limit point compactness.

Lemma *L4.23* - {77} *(Limit point + 1st count + Hausdorff $\Rightarrow$ Sequential).* For first countable Hausdorff spaces, limit point compactness implies sequential compactness.

Lemma *P4.25* - {79} *(Closed map lemma).* Let F ba a continuous map from a compact space to a Hausdorff space.

(a) F is a closed

(b) If F is surjective, it is a quotient map.

(c) If F is injective, it is a topological embedding.

(d) If F is bijective, it is a homemorphism.

8.2 Closed map lemma

Lemma *L4.25* - {78} *(2nd count + Hausdorff ⇒ compactnesses are eq).* For metric spaces and second countable Hausdorff spaces, compactness, limit point compactness, and sequential compactness are all equivalent.

8.3 Locally compact spaces

Definition {81} *(Locally compact space).* X is locally compact if there every $q \in X$ has a compact set containing one of its neighborhoods.

Definition {82} *(Relatively compact space).* A is relatively compact in X if $\bar{A}$ is compact.

Lemma {82} *(Locally compact Hausdorff spaces).* Let X be a Hausdorff space. The followiong are iff:

(a) X is locally compact.

(b) each point of X has a relatively compact neighborhood.

(c) X has a basis of relatively compact open sets.

Lemma {82} *(Shrinking lemma).* Let X be a locally compact Hausdorff space. If $x \in X$ and U is neighborhood of x, there is a relatively compact neighborhood of c such that $\bar{V} \subseteq U$.

Definition {84} *(Proper map).* $f : X \to Y$ is a proper map if the inverse image of compact subsets are also compact subsets.

Lemma {84} *(Proper ⇒ Closed).* Let X, Y be a locally compact Hausdorff spaces and $f : X \to Y$ be continuous and proper. Then, f is closed.

Theorem {85} *(Baire category theorem).* Let X be a locally compact Hausdorff space or a complete metric space. Every countable collection of dense open subsets has a dense intersection.

Definition {85} *(Nowhere dense set).* A set $A \subset X$ is said to be nowhere dense if its closure contains no nonempty open set.

Lemma {85} *(Corollary of Baire category theorem).* Let X be a locally compact Hausdorff space or a complete metric space. Any countable union of nowhere dense set has empty interior.

Definition {86} *(Baire categories).* A first Baire category set (or meager set) is a countable union of nowhere dense sets, and a second Baire categroy set is a set that is not of first Baire category.

References

John M. Lee. *Introduction to Topological Manifolds.* Springer, 2000.