A posteriori error estimates based on multilevel decompositions with large problems on the coarsest level

Additional numerical experiments

P. Vacek, J. Papež and Z. Strakoš

This pdf present numerical results for additional test cases to supplement the experiments in the paper

P. Vacek, J. Papež and Z. Strakoš, A posteriori error estimates based on multilevel decompositions with large problems on the coarsest level,

where a detailed description and discussion is present.

2Dpeak problem

$$-\Delta u = f$$
$$u = 0 \text{ on the boundary}$$

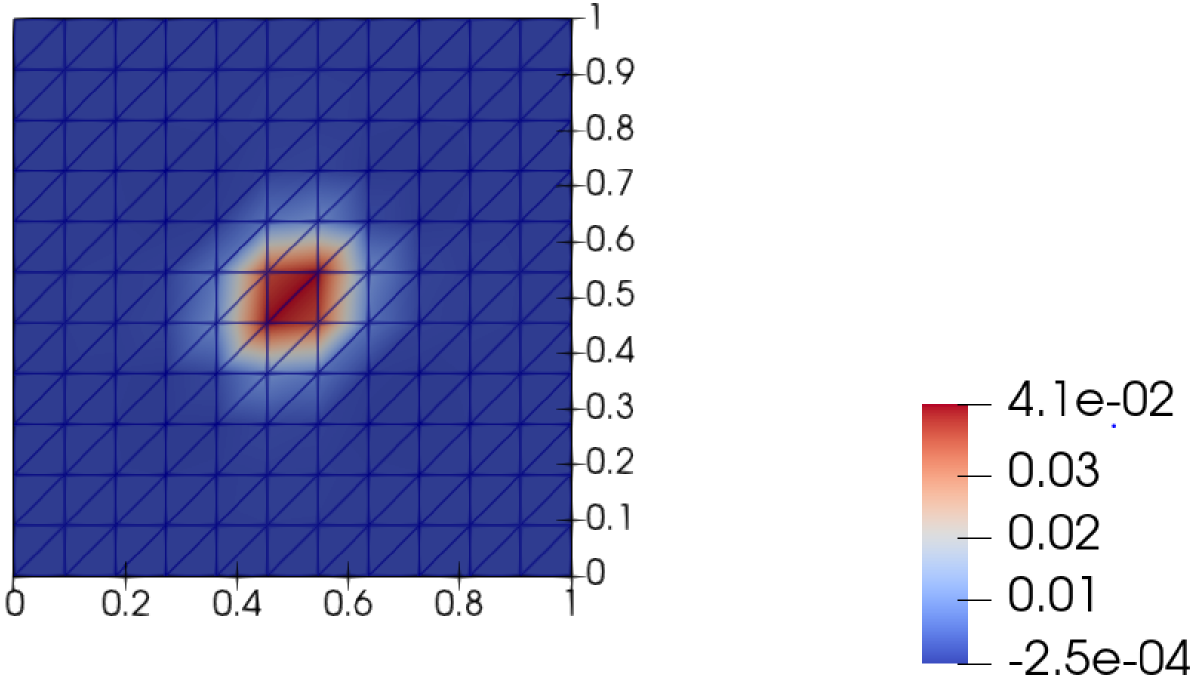
Manufactured solution:

$$u = x(x-1)y(y-1)e^{-100\left(\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\right)}$$

Discretization: continuous Galerkin Finite Elements P1

Multigrid hierarchy generated by discretizing the problem on a sequence of uniformly refined meshes

$$h_\Omega = \sqrt{2}$$
$$\min_{K\in T_0} h_K = 0.12856486930664487$$



Galerkin solution on the coarsest level

Robustness with respect to the **number of levels**

Approximations v_J generated by V-cycle iterations

```
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
stoppingCriterion.tolerance = 0.1;
coarsestLevelSolver = CGSolver(stoppingCriterion);
```

Sizes of the problem in individual settings	
coarsest-level DoFs	finest-level DoFs
100	441
100	1 849
100	7 569
100	30 625
100	123 201
100	494 209
100	1 979 649
100	7 924 225

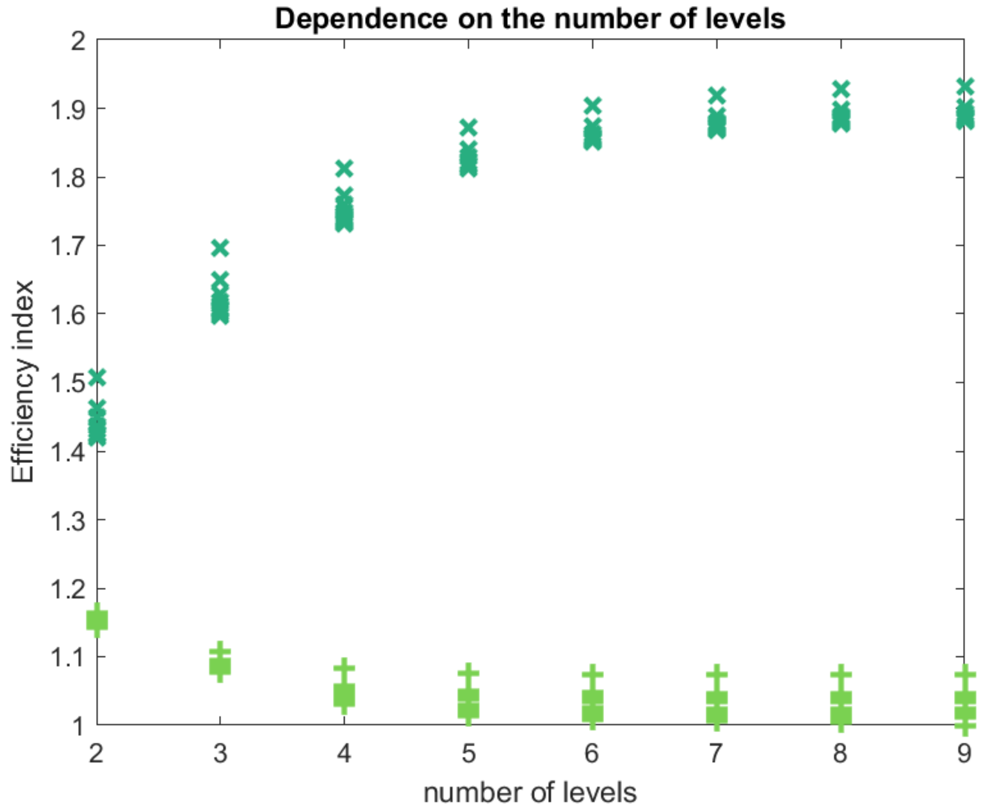
Robustness with respect to the **number of levels**

$$I_1 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0 \right)^{\frac{1}{2}}}{\|\nabla(u_J - v_J)\|}$$

$$I_2 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} (\mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j)^{\frac{1}{2}} + (\mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0)^{\frac{1}{2}} \right)}{\|\nabla(u_J - v_J)\|}$$

with

C_{numexp} =1.184153650441326



Robustness with respect to the **size of the coarsest-level problem**

Approximations v_J generated by V-cycle iterations

```
J = 3; % vcycleNumberOfLevels
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
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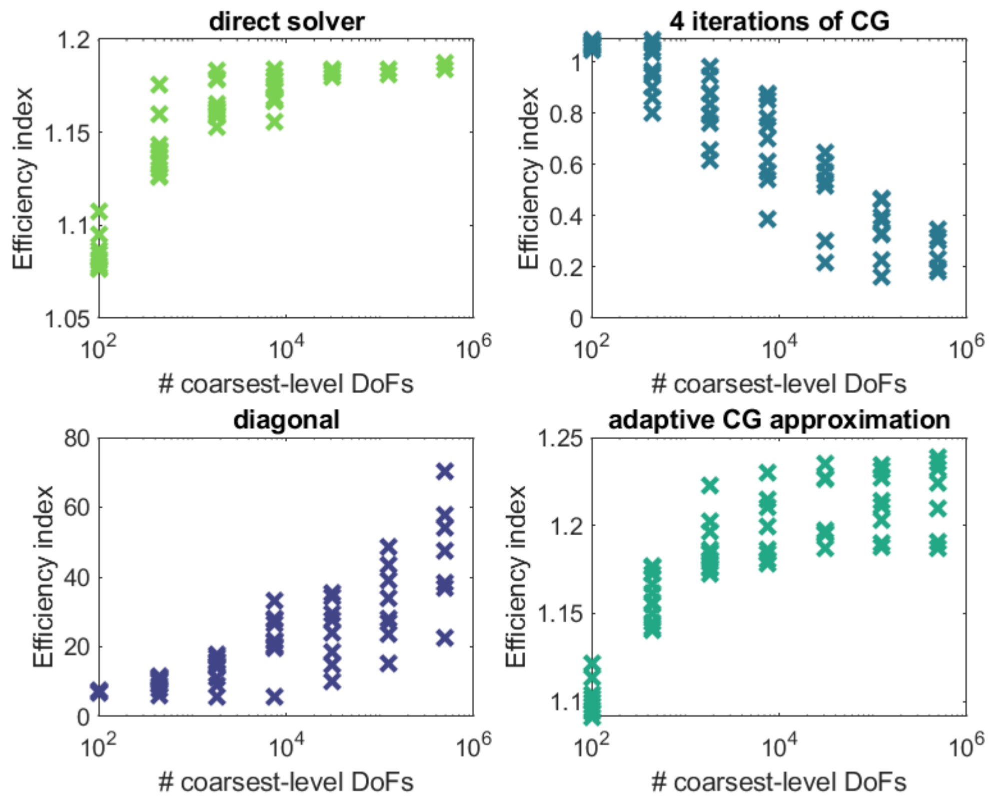
Robustness with respect to the **size of the coarsest-level problem**

$$I_3 = \frac{C_{\mathrm{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \eta \right)^{\frac{1}{2}}}{\left\| \nabla (u_J - v_J) \right\|}$$

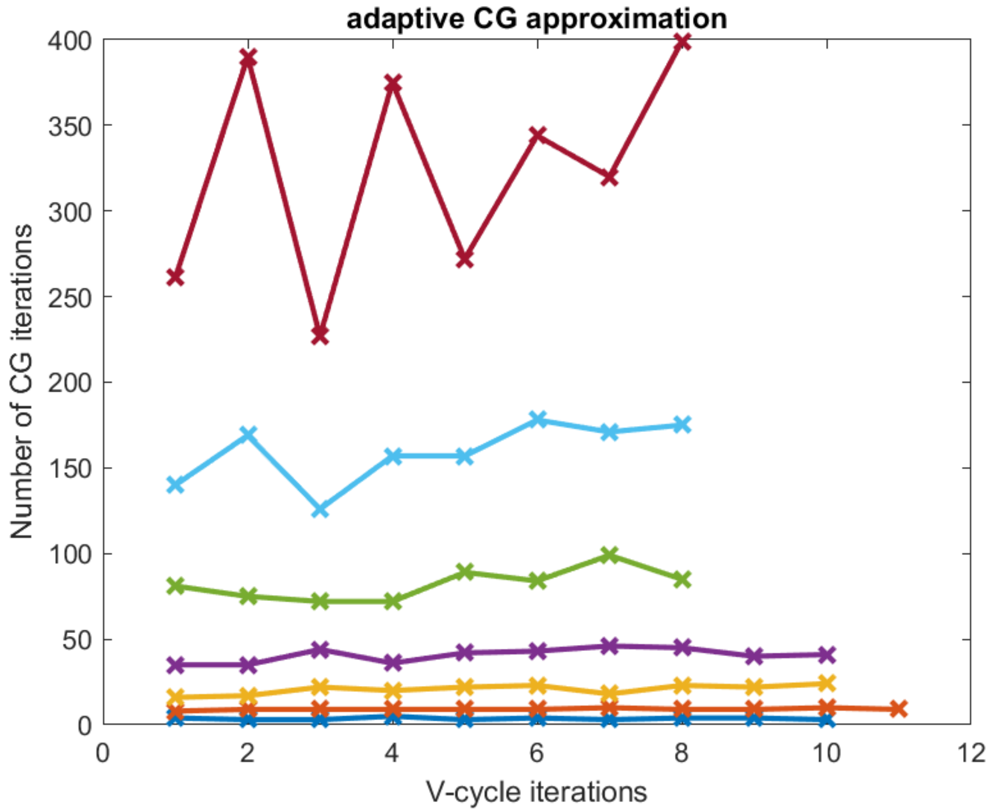
where $\eta \approx \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0$ computed using
(i) direct solver
(ii) 4 iterations of CG
(iii) by replacing $\boldsymbol{A}_0$ with its diagonal
(iv) using CG with adaptive number of iterations

C_{numexp} =1.184153650441326



Robustness with respect to the **size of the coarsest-level problem**

Number of CG iterations used when computing η



2Dshell problem

$$-\Delta u = 0$$

$u = 100$ on the inner boundary

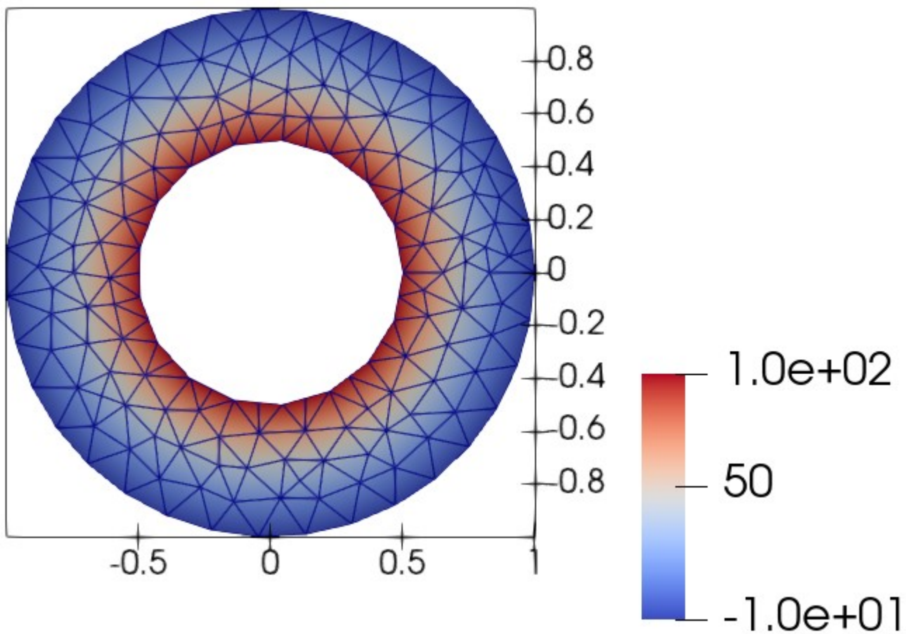
$u = -10$ on the outer boundary

Discretization: continuous Galerkin Finite Elements P1

Multigrid hierarchy generated by discretizing the problem on a sequence of uniformly refined meshes

$h_\Omega = 1$

$\min_{K \in T_0} h_K = 0.09771455584161245$



Galerkin solution on the coarsest triangulation

Robustness with respect to the **number of levels**

Approximations v_J generated by V-cycle iterations

```
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
stoppingCriterion.tolerance = 0.1;
coarsestLevelSolver = CGSolver(stoppingCriterion);
```

Sizes of the problem in individual settings	
coarsest-level DoFs	finest-level DoFs
111	515
111	2 202
111	9 092
111	36 936
111	148 880
111	597 792
111	2 395 712
111	9 591 936

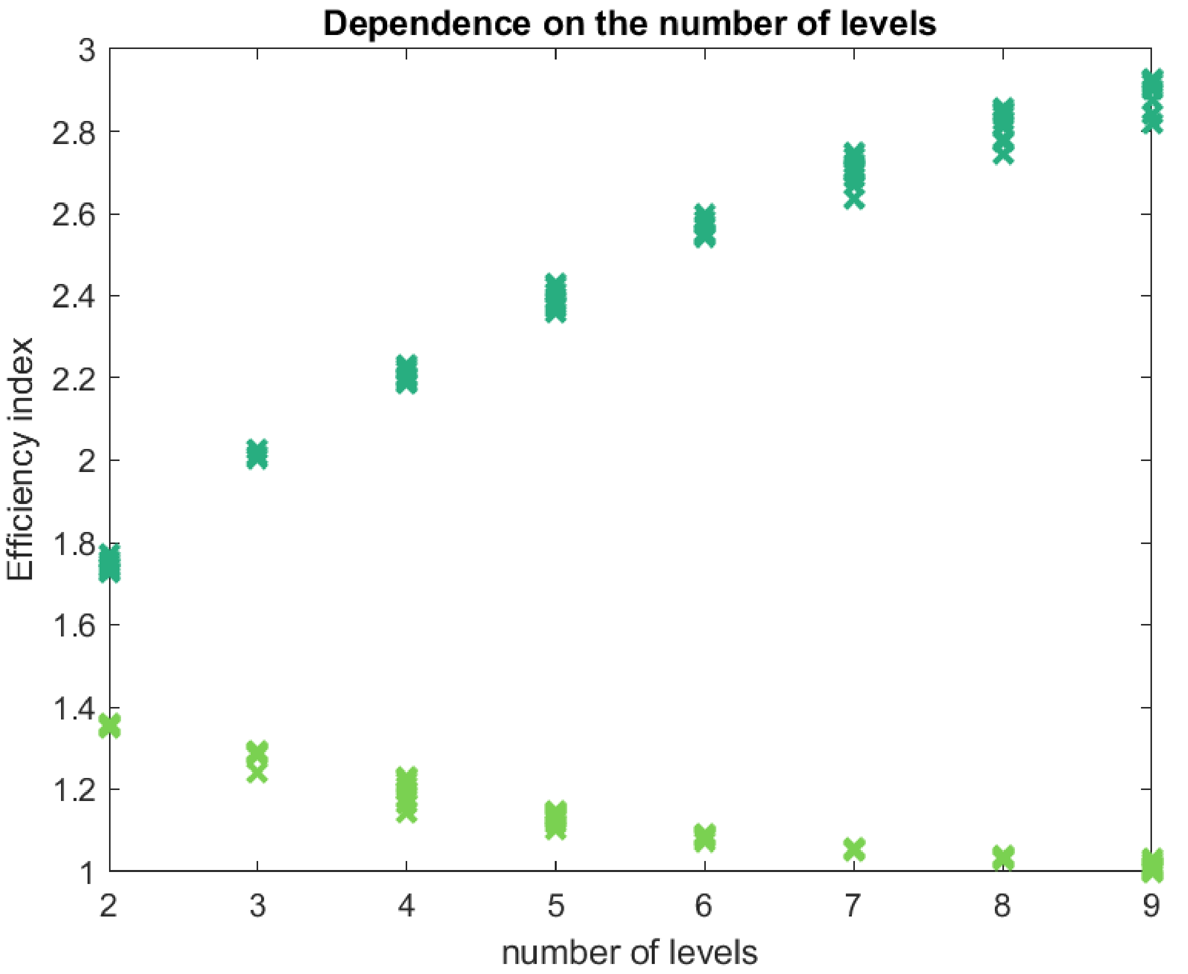
Robustness with respect to the **number of levels**

$$I_1 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0 \right)^{\frac{1}{2}}}{\| \nabla (u_J - v_J) \|}$$

$$I_2 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} (\mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j)^{\frac{1}{2}} + (\mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0)^{\frac{1}{2}} \right)}{\| \nabla (u_J - v_J) \|}$$

with

C_{numexp} = 1.386793086895968



Robustness with respect to the **size of the coarsest-level problem**

Approximations v_J generated by V-cycle iterations

```
J = 3; % vcycleNumberOfLevels
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
stoppingCriterion.tolerance = 0.1;
coarsestLevelSolver = CGSolver(stoppingCriterion);
```

Sizes of the problem in individual settings	
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111	2 202
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36 936	597 792
148 880	2 395 712
597 792	9 591 936

Robustness with respect to the **size of the coarsest-level problem**

$$I_3 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \eta \right)^{\frac{1}{2}}}{\left\| \nabla (u_J - v_J) \right\|}$$

where $\eta \approx \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0$ computed using

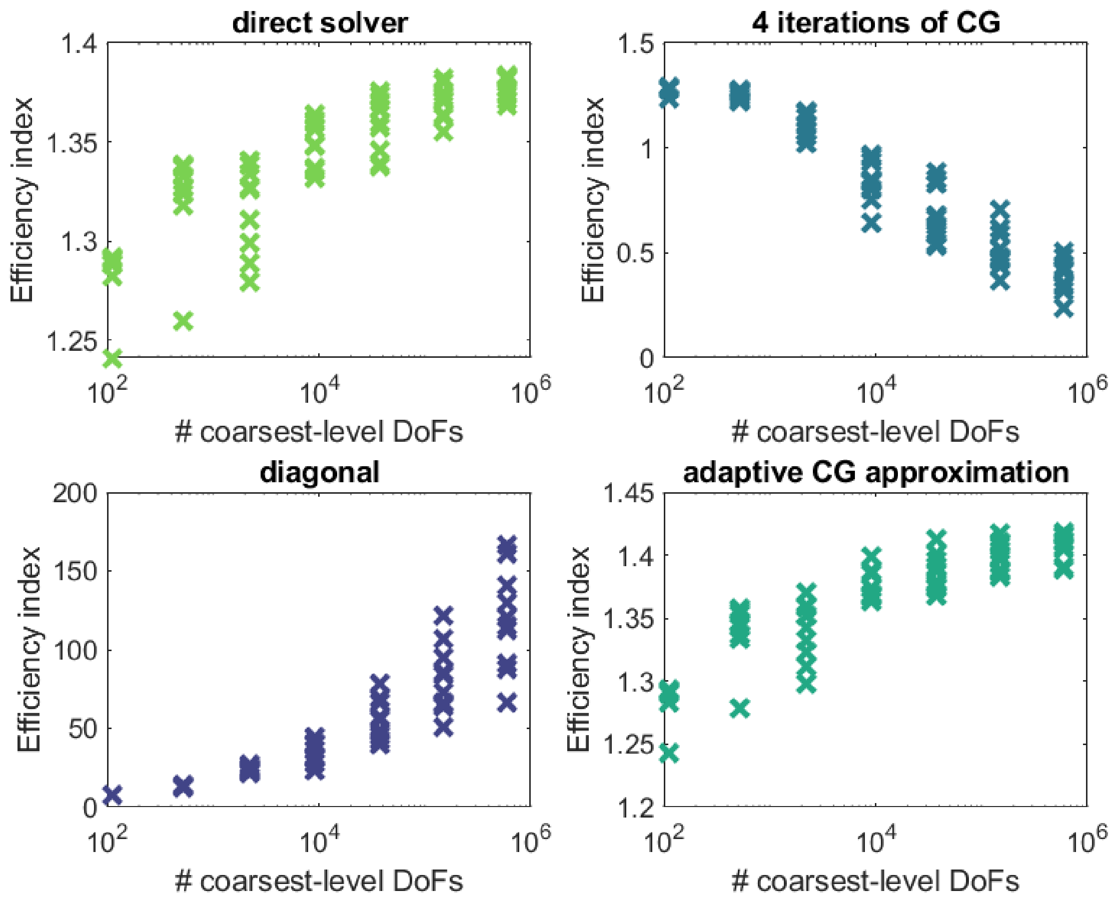
(i) direct solver

(ii) 4 iterations of CG

(iii) by replacing $\boldsymbol{A}_0$ with its diagonal

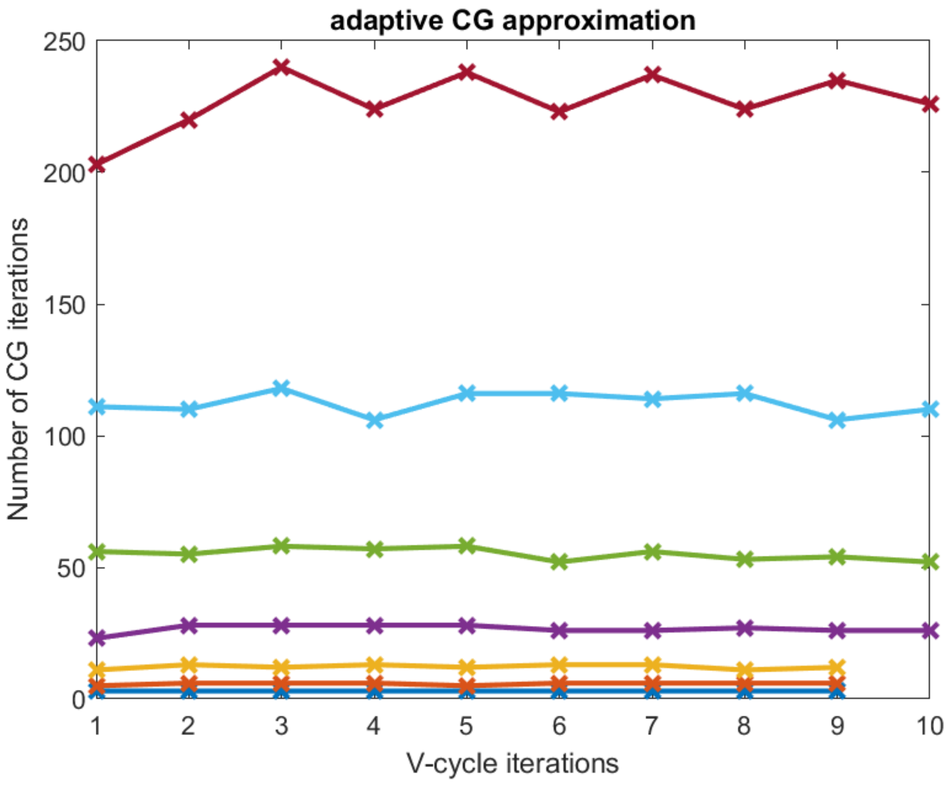
(iv) using CG with adaptive number of iterations

C_{numexp} = 1.386793086895968



Robustness with respect to the **size of the coarsest-level problem**

Number of CG iterations used when computing η



3Dshell problem

$$-\Delta u = 0$$

$u = 100$ on the inner boundary

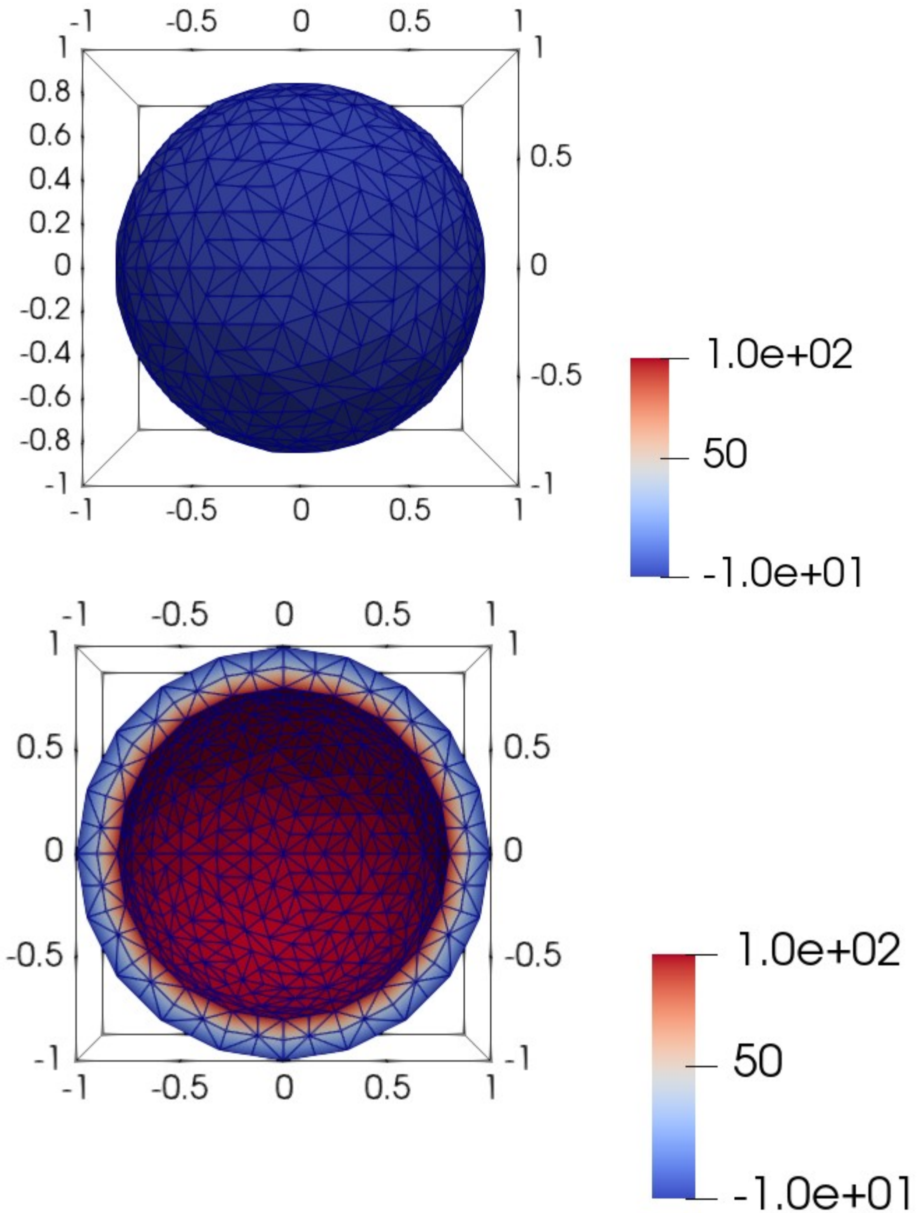
$u = -10$ on the outer boundary

Discretization: continuous Galerkin Finite Elements P1

Multigrid hierarchy generated by discretizing the problem on a sequence of uniformly refined meshes
The initial mesh was generated in HyTeG*

$$h_\Omega = 1$$

$$\min_{K \in T_0} h_K = 0.13832808829250579$$



clip of Galerkin solution on the coarsest triangulation

Robustness with respect to the **number of levels**

Approximations v_J generated by V-cycle iterations

```
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
stoppingCriterion.tolerance = 0.5;
coarsestLevelSolver = CGSolver(stoppingCriterion);
```

Sizes of the problem in individual settings	
coarsest-level DoFs	finest-level DoFs
642	7 686
642	71 694
642	614 430
642	5 079 102

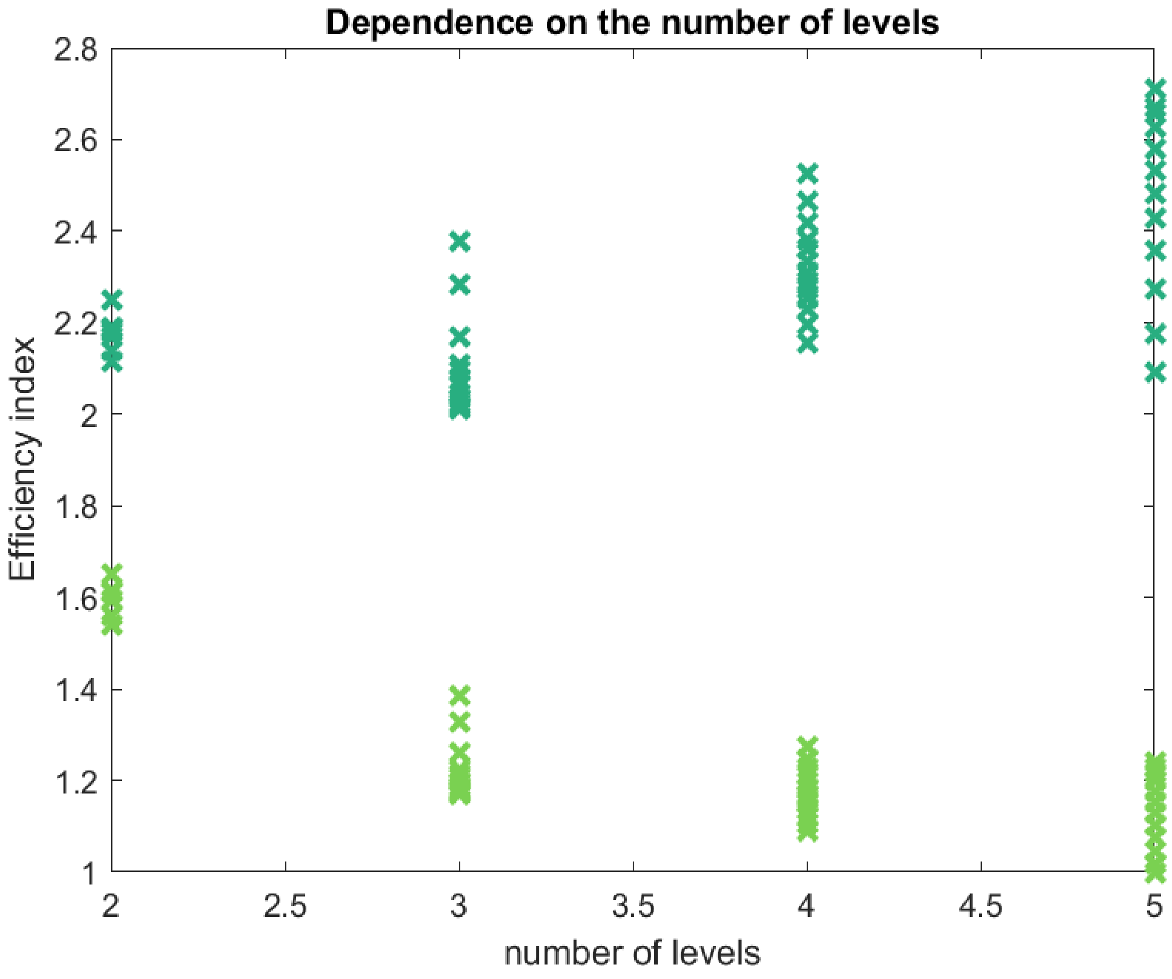
Robustness with respect to the **number of levels**

$$I_1 = \frac{C_{\mathrm{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0 \right)^{\frac{1}{2}}}{\| \nabla (u_J - v_J) \|}$$

$$I_2 = \frac{C_{\mathrm{numexp}} \left(\sum_{j=1}^{J} \left(\mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j \right)^{\frac{1}{2}} + \left(\mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0 \right)^{\frac{1}{2}} \right)}{\| \nabla (u_J - v_J) \|}$$

with

C_{numexp} = 1.842461635385764



Robustness with respect to the **size of the coarsest-level problem**

Approximations v_J generated by V-cycle iterations

```
J = 2; % vcycleNumberOfLevels
vcycleRelativeTolerance = 1e-11;
vcycleMaximumNumberOfIterations = 1000;
smoother = GSSmoother(3,3);
% define coarsest level solver
stoppingCriterion.name = 'res2norm';
stoppingCriterion.relative = true;
stoppingCriterion.tolerance = 0.5;
coarsestLevelSolver = CGSolver(stoppingCriterion);
```

Sizes of the problem in individual settings	
coarsest-level DoFs	finest-level DoFs
642	7 686
7 686	71 694
71 694	614 430
614 430	5 079 102

Robustness with respect to the **size of the coarsest-level problem**

$$I_3 = \frac{C_{\text{numexp}} \left(\sum_{j=1}^{J} \mathbf{r}_j^* \mathbf{D}_j^{-1} \mathbf{r}_j + \eta \right)^{\frac{1}{2}}}{\left\| \nabla (u_J - v_J) \right\|}$$

where $\eta \approx \mathbf{r}_0^* \boldsymbol{A}_0^{-1} \mathbf{r}_0$ computed using

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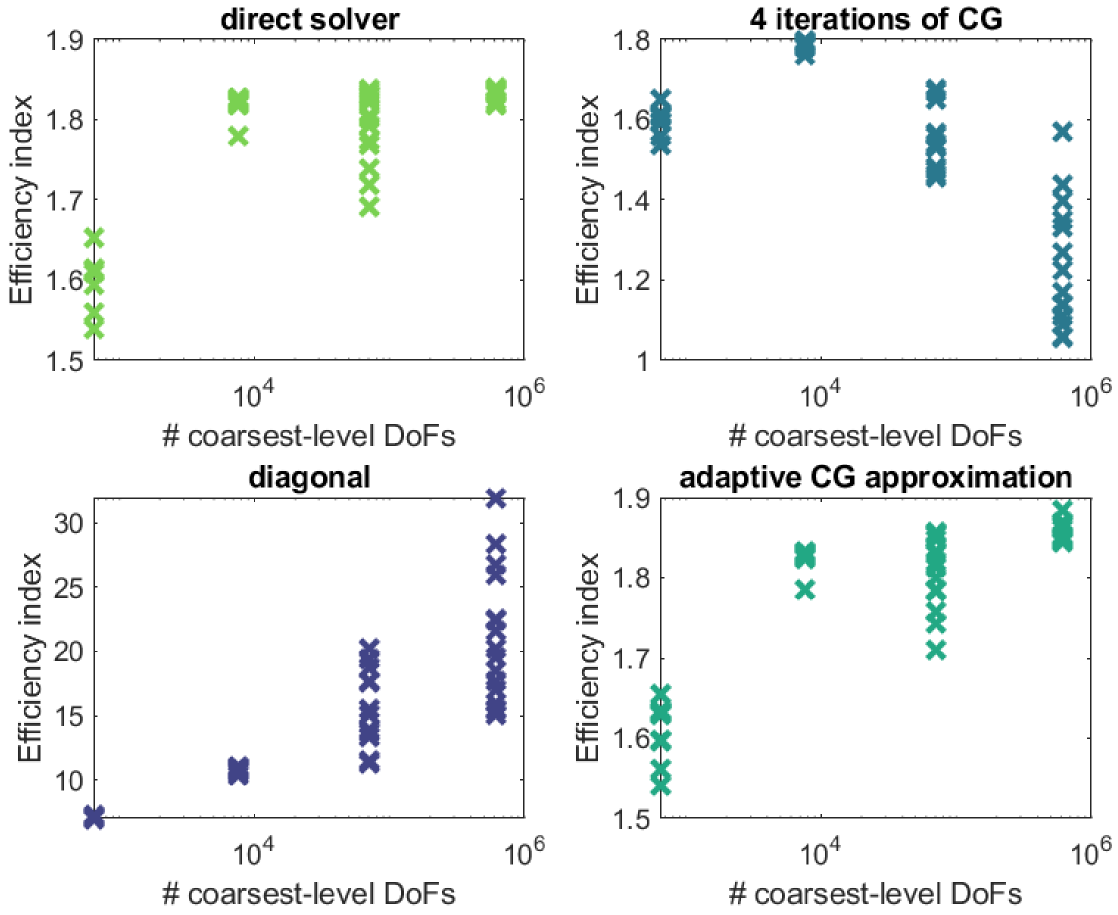

Robustness with respect to the **size of the coarsest-level problem**

Number of CG iterations used when computing η