Recovering Structured Signals in Noise: Comparison Lemmas and the Performance of Convex Relaxation Methods

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e. None of the above.

Outline

- **Introduction**
  - structured signal recovery
  - non-smooth convex optimization
  - LASSO and generalized LASSO
- **Comparison Lemmas**
  - Slepian, Gordon
- **Squared Error of Generalized LASSO**
  - Gaussian widths, statistical dimension
  - optimal parameter tuning
- **Generalizations**
  - other loss functions
  - other random matrix ensembles
- **Summmary and Conclusion**

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- In this setting, it is important to have signal recovery algorithms that are computationally efficient and that need not access the entire data directly (hence compressed recovery)

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- The generic problem is

$$\min_x \mathcal{L}(x,y) + \lambda f(x) \quad \text{or} \quad \min_{\mathcal{L}(x,y)\le c_1} f(X) \quad \text{or} \quad \min_{f(x)\le c_2} \mathcal{L}(x,y)$$

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  - how to choose the regularizer $\lambda \ge 0$? (or the constraint bounds c_1 and c_2?)

Example: Noisy Compressed Sensing

Consider a "desired" signal $x \in \mathcal{R}^n$, which is k-sparse, i.e., has only $k < n$ (often $k \ll n$) non-zero entries. Suppose we make m noisy measurements of x using the $m \times n$ measurement matrix A to obtain

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- Suppose each set of m columns of A are linearly independent. Then, if $m > k$, we can always find the *sparsest* solution to

$$\min_x \|y - Ax\|_2^2 ,$$

via exhaustive search of $\left(\begin{array}{c} n \\ k \end{array}\right)$ *such least-squares problems*

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- Can we do this more efficiently? And for what values of m?
- What about problems (such as low rank matrix recovery) where it is not possible to enumerate all structured signals?

LASSO

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The generalized LASSO algorithm can be used to enforce other types of structures

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- If the noise is bounded:

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Turns out *we can*. But to do so, we need to tell an earlier story....

Example

$\mathbf{X}_0 \in \mathbb{R}^{n\times n}$ is rank r. Observe, $\mathbf{y} = A \cdot \mathrm{vec}(\mathbf{X}_0) + \mathbf{z}$, solve the Matrix LASSO,

$$\min_{\mathbf{X}} \{\|\mathbf{y} - A \cdot \mathrm{vec}(\mathbf{X})\|_2 + \lambda \|\mathbf{X}\|_\star\}$$

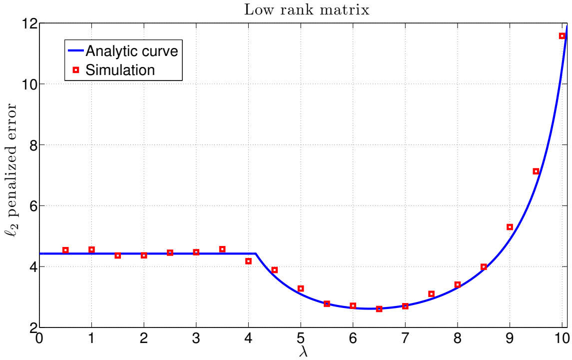

Figure: $n = 45$, $r = 6$, measurements $m = 0.6n^2$.

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- Candes and Tao showed that if A satisfies certain *restricted isometry* conditions, then ℓ_1 optimization works for small enough k
  - gives "order optimal", but **very** loose bounds

Exact Conditions for Signal Recovery

We will consider a general framework.

Consider a structured signal x_0, with a structure-inducing norm $f(\cdot) = \|\cdot\|$. We have access to *linear measurements* $y = \mathcal{A}(x_0) \in R^m$, and would like to know when we can recover the signal x_0 from the convex problem

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A Bit of Geometry: Subgradients and the Polar Cone

Note that $\mathcal{N}(\mathcal{A})$ is a linear subspace and that therefore the condition can be rewritten as

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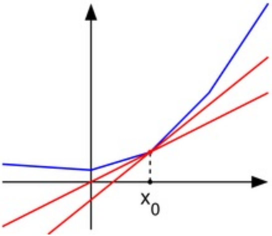
We can characterize $\text{cone}(\mathcal{U}(x_0))$ through the subgradient of the convex function $\|\cdot\|$:

$$\partial\|x_0\| = \{v | v^T(x - x_0) + \|x_0\| \leq \|x\|, \forall x\}.$$



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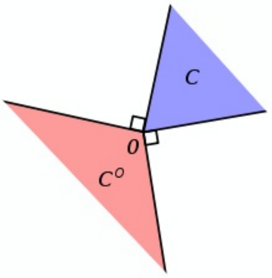
But this is simply the *polar cone* of $\partial\|x_0\|$.



Thus, we can recover x_0 from the convex problem iff:

$$\mathcal{N}(\mathcal{A}) \cap (\partial\|x_0\|)^O = \{0\}.$$

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- It is customary to assume that the measurement matrix $\mathcal{A}$ is composed of iid zero-mean unit-variance entries.

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- While computing the polar cone of the subgradient is often straightforward, checking the condition $\mathcal{N}(\mathcal{A}) \cap (\partial\|x_0\|)^O = \{0\}$ for a *specific* $\mathcal{A}$ is difficult.
- Therefore the focus has been on checking whether the condition holds for a *family* of random $\mathcal{A}$'s with high probability.
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- This makes the nullspace $\mathcal{N}(\mathcal{A})$ *rotationally-invariant*.
- The probability that a rotationally-invariant subspace intersects a cone is called the *Grassman angle* of the cone.

Phase Transitions for Convex Relaxation - Some History

- In the ℓ_1 case the subgradient cone is polyhedral and Donoho and Tanner (2005) computed the Grassman angle to obtain the minimum number of measurements required to recover a k-sparse signal
  - very cumbersome calculations, required considering exponentially many inner and outer angles, etc.

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Replica-based analysis:

- Guo, Baron and Shamai (2009), Kabashima, Wadayama, Tanaka (2009), Rangan, Fletecher, Goyal (2012), Vehkapera, Kabashima, Chatterjee (2013), Wen, Zhang, Wong, Chen (2014)

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Where does all this come from?

Tribute to Dave Slepian (1923-2007)

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d. Will you be joining the French table?
e. ✓None of the above. *Would you care to compare our beers?*

Slepian's Comparison Lemma (1962)

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Let X_i and Y_i be two Gaussian processes with the same mean μ_i and variance σ_i^2, such that $\forall\ i, i'$

- $E(X_i - \mu_i)(X_{i'} - \mu_{i'}) \geq E(Y_i - \mu_i)(Y_{i'} - \mu_{i'})$

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Slepian's Comparison Lemma (1962)

- proof not too difficult, but not trivial, either
- lemma not generally true for non-Gaussian processes

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Let $A \in \mathcal{R}^{m \times n}$ be a matrix with iid $N(0,1)$ entries and consider its maximum singular value:

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Define the two Gaussian processes

$$X_{uv} = u^T A v + \gamma \quad \text{and} \quad Y_{uv} = u^T g + v^T h,$$

where $\gamma \in \mathcal{R}$, $g \in \mathcal{R}^m$ and $h \in \mathcal{R}^n$ have iid $N(0,1)$ entries.

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$$EX_{uv}X_{u'v'} - EY_{uv}Y_{u'v'} = u^T u' v^T v' + 1 - u^T u' - v^T v' = (1 - u^T u')(1 - v^T v') \geq 0.$$

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Therefore from Slepian's lemma:

$$\underbrace{\text{Prob}\left(\max_{\|u\|=1}\max_{\|v\|=1} u^T A v + \gamma \geq c\right)}_{\geq \frac{1}{2}\text{Prob}(\|A\| \geq c)} \leq \underbrace{\text{Prob}\left(\max_{\|u\|=1}\max_{\|v\|=1} u^T g + v^T h \geq c\right)}_{\text{Prob}(\|g\|+\|h\|\geq c)}.$$

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Since $\|g\| + \|h\|$ concentrates around $\sqrt{m} + \sqrt{n}$, this implies that the probability that $\|A\|$ (significantly) exceeds $\sqrt{m} + \sqrt{n}$ is very small.

Minimum Singular Value of a Gaussian Matrix

Let $A \in \mathcal{R}^{m \times n}$ $(m \leq n)$ be a matrix with iid $N(0,1)$ entries and consider its minimum singular value:

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It took 24 years for there to be progress...

Gordon's Comparison Lemma (1988)

Let X_{ij} and Y_{ij} be two Gaussian processes with the same mean μ_{ij} and variance σ_{ij}^2, such that $\forall\ i, j, i', j'$

1. $E(X_{ij} - \mu_{ij})(X_{ij'} - \mu_{ij'}) \le E(Y_{ij} - \mu_{ij})(Y_{ij'} - \mu_{ij'})$
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- If c is a high probability lower bound on $\phi(\cdot,\cdot)$, same is true of $\Phi(\cdot,\cdot)$
- Basis for "escape through mesh" and "Gaussian width"
- Can be used to show that $\sigma_{\min}(A)$ behaves as $\sqrt{n} - \sqrt{m}$

A Stronger Version of Gordon's Lemma (TOH 2014)

$$\begin{cases} \Phi(G) & = & \min_{x\in S_x}\max_{y\in S_y} \; y^T Gx + \psi(x,y) \\ \phi(g,h) & = & \min_{x\in S_x}\max_{y\in S_y} \; \|x\| g^T y + \|y\| h^T x + \psi(x,y) \end{cases}$$

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$$Prob\left(|\Phi(G) - c| \ge \epsilon\right) \le 2Prob\left(|\phi(g,h) - c| \ge \epsilon\right).$$

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3. *If, in addition, the optimizations over x are* strongly *convex, and $\phi(g,h)$ concentrates, then for* any *norm $\|\cdot\|$, for which $\|\hat{x}_\phi\|$ concentrates, with high probability we have*

$$\|\hat{x}_\Phi\| = \|\hat{x}_\phi\|\left(1 + o(1)\right)$$

Least-Squares

Suppose we are confronted with the *noisy* measurements:

$$y = Ax + z,$$

where $A \in \mathcal{R}^{m\times n}$ is the measurement matrix with iid $N(0,1)$ entries, $y \in \mathcal{R}^m$ is the measurement vector, $x_0 \in \mathcal{R}^n$ is the unknown desired signal, and $z \in \mathcal{R}^n$ is the unknown noise vector with iid $N(0,\sigma^2)$ entries.

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where $A \in \mathcal{R}^{m \times n}$ is the measurement matrix with iid $N(0,1)$ entries, $y \in \mathcal{R}^m$ is the measurement vector, $x_0 \in \mathcal{R}^n$ is the unknown desired signal, and $z \in \mathcal{R}^n$ is the unknown noise vector with iid $N(0,\sigma^2)$ entries. In the general case, to be meaningful, we require that

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Let us analyze this using the stronger version of Gordon's lemma.

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To this end, define the estimation error $w = x_0 - x$, so that
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This satisfies all the conditions of the lemma. The simpler optimization is therefore:

$$\min_w \max_{\|u\|\leq 1} \sqrt{\|w\|^2 + \sigma^2} g^T u + \|u\| \begin{bmatrix} h_w^T & h_\sigma \end{bmatrix} \begin{bmatrix} w \\ \sigma \end{bmatrix},$$

where $g = R^m$, $h_w = R^n$ and $h_\sigma \in R$ have iid $N(0,1)$ entries.

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When A has iid $N(0,1)$ entries, $A^T A$ is a *Wishart matrix* whose asymptotic eigendistribution is well known, from which we obtain

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Back to the Squared Error of Generalized LASSO

However, for generalized LASSO, we do not have closed form solutions and the machinery becomes very useful:

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Or:

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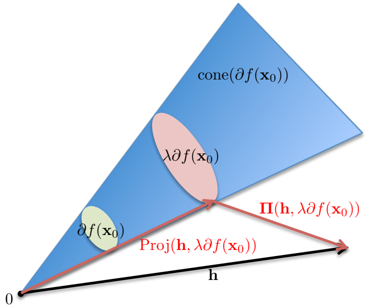
Main Result: The Squared Error of Generalized LASSO

Generate an n-dimensional vector h with iid $N(0,1)$ entries and define:

$$D_f(x_0, \lambda) = E\text{dist}^2\left(h, \lambda\partial f(x_0)\right).$$



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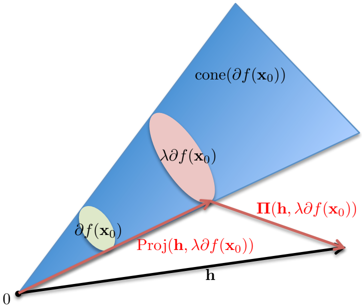

It turns out that $\text{dist}^2(h_w, \lambda\partial f(\mathbf{x}_0))$ concentrates to $D_f(x_0,\lambda)$, so that:

$$\lim_{\sigma\to 0} \frac{\|x_0-\hat{x}\|^2}{\sigma^2} \to \frac{D_f(x_0,\lambda)}{m-D_f(x_0,\lambda)}.$$

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It is easy to see that

$$D_f(x_0,\lambda^*)=E\text{dist}^2\left(h,\text{cone}(\partial f(x_0))\right)\stackrel{\Delta}{=}\omega^2.$$

Main Result

- $$\omega^2 = E\text{dist}^2\left(h, \text{cone}(\partial f(x_0))\right)$$

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- The quantity ω^2 determines the minimum number of measurements required to recover a k-sparse signal using (appropriate) convex optimization. (The so-called *recovery thresholds*.)

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- for qb-dimensional k block-sparse signals and $f(x) = \|x\|_{1,2}$:

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Example

$\mathbf{X}_0 \in \mathbb{R}^{n \times n}$ is rank r. Observe, $\mathbf{y} = A \cdot \mathrm{vec}(X_0) + \mathbf{z}$, solve the Matrix LASSO,

$$\min_{\mathbf{X}} \{ \|\mathbf{y} - A \cdot \mathrm{vec}(X)\|_2 + \lambda \|\mathbf{X}\|_\star \}$$

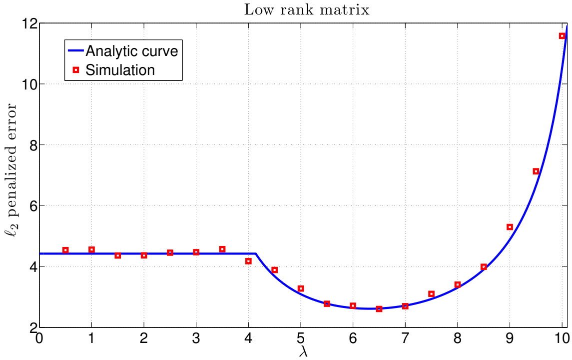

Figure: $n = 45$, $r = 6$, measurements $m = 0.6n^2$.

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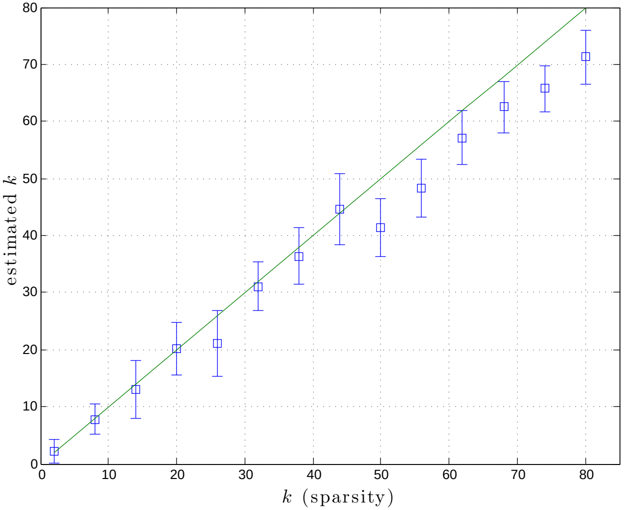
4. For this value of k find the optimal λ^*.

Estimating the Sparsity: $n = 520$, $m = 280$



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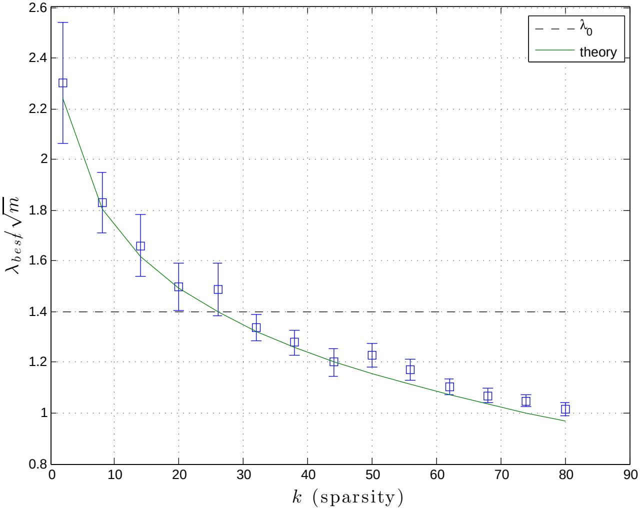
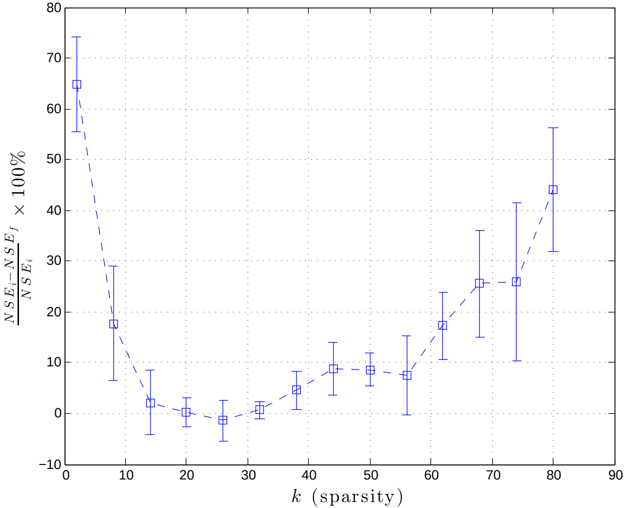


Improvement in NSE: $n = 520$, $m = 280$



Generalizations

Finite σ

When σ is not very small, we must study:

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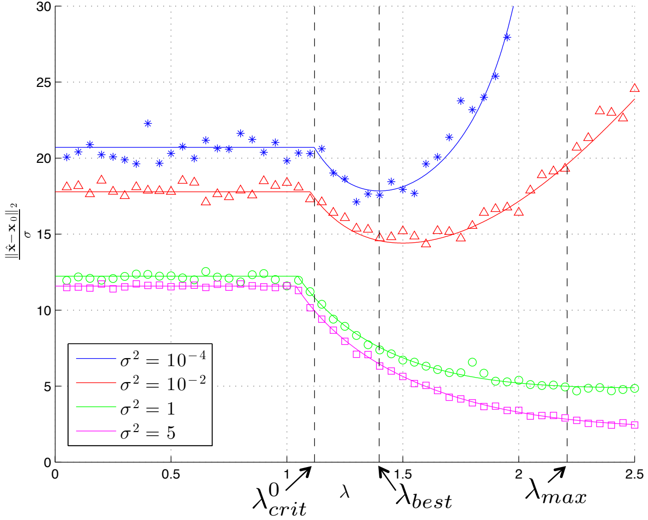
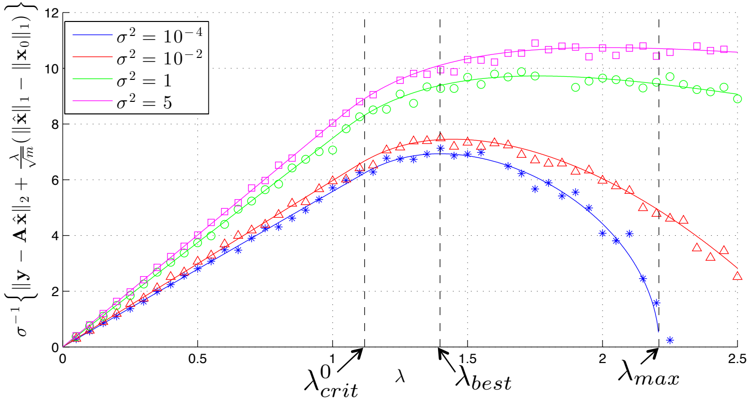
The analysis is a bit more complicated, but absolutely do-able.

NSE for Finite σ: $n = 500$, $m = 150$, $k = 20$



Cost for Finite σ: $n = 500$, $m = 150$, $k = 20$



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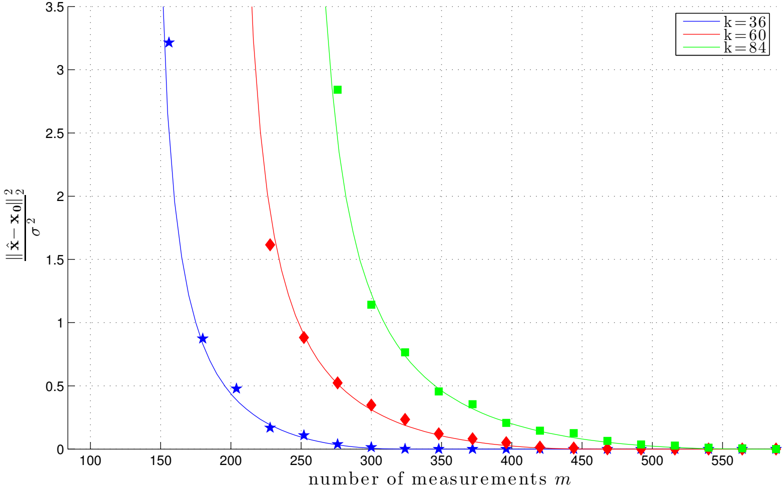
- In turns out that we now must analyze

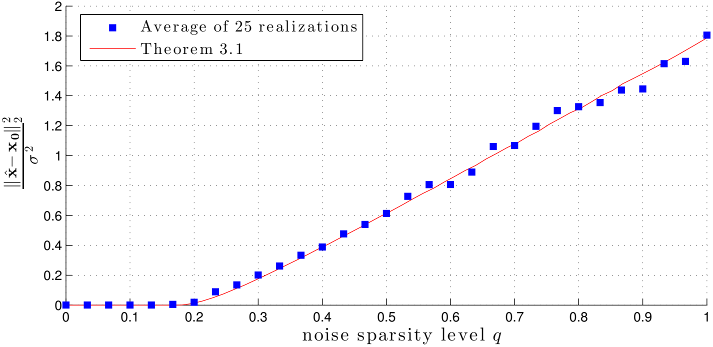
$$\phi(g,h) = \min_{\mathbf{w}} \max_{\|\mathbf{v}\|_\infty \le 1} \sqrt{\|\mathbf{w}\|^2 + \sigma^2} g^T \mathbf{v} - \|\mathbf{v}\| h^T \mathbf{w} + \sup_{\mathbf{s} \in \lambda \partial f(\mathbf{x}_0)} \mathbf{s}^T \mathbf{w}$$

  This is a bit more complicated, but still completely doable.

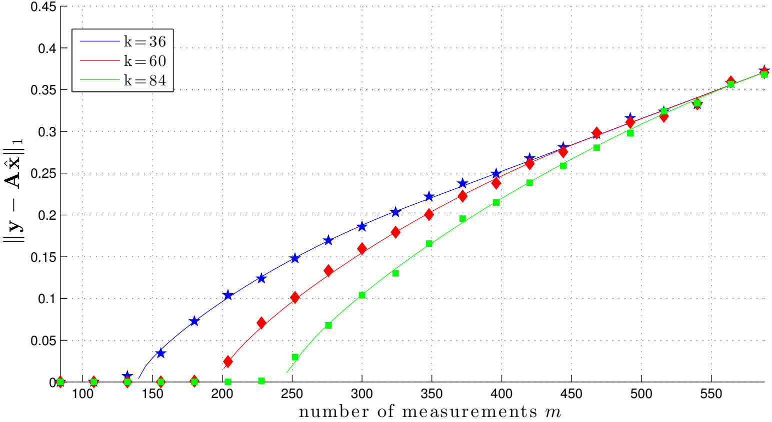
Squared Error vs Number of Measurements



Squared Error vs Sparsity of Noise



Cost vs Number of Measurements



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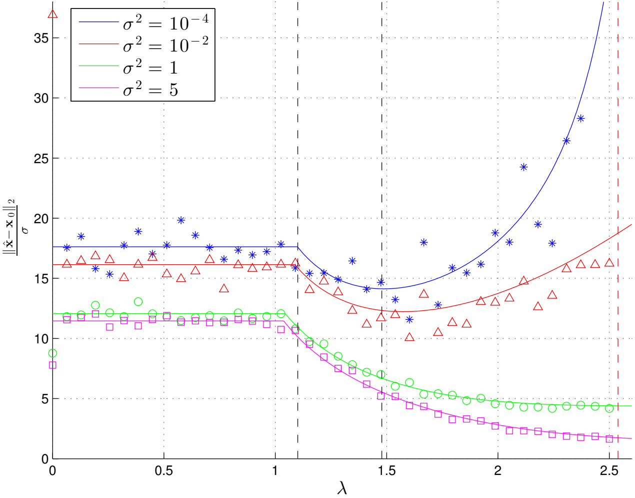
- "Close" to proving this?

NSE for iid Bernouli($\frac{1}{2}$): $n = 500$, $m = 150$, $k = 20$



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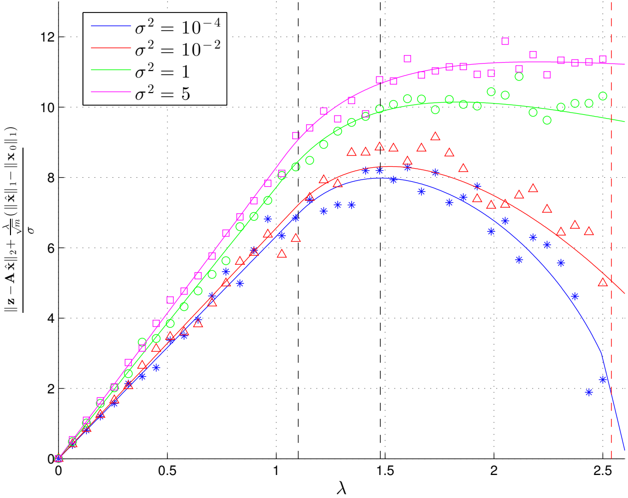

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$$QQ^T = I_m, \qquad P(\Theta Q \Omega) = P(Q),$$

  for all orthogonall Θ and Ω.
- For such random matrices, we have shown that the two optimization problems:

$$\begin{cases} \Phi(Q,z) &= \min_w \quad \|\sigma z - Qw\| + \lambda f(w) \\ \phi(g,h) &= \min_{w,l} \max_{\beta\geq 0} \quad \|\sigma v - w - l\| + \beta(\|l\|\cdot\|g\| - h^T l) + \lambda f(w) \end{cases}$$

  where z, v, h and g have iid $N(0,1)$ entries, have the same optimal costs and statistically the same optimal minimizer.

Isotropically Random Unitary Matrices

- Using the above result, we have been able to show that

$$\lim_{\sigma\to 0}\frac{\|x_0-\hat{x}\|^2}{\|z\|^2}\to\frac{D_f(x_0,\lambda)}{m-D_f(x_0,\lambda)}\cdot\frac{n-D_f(x_0,\lambda)}{n}.$$

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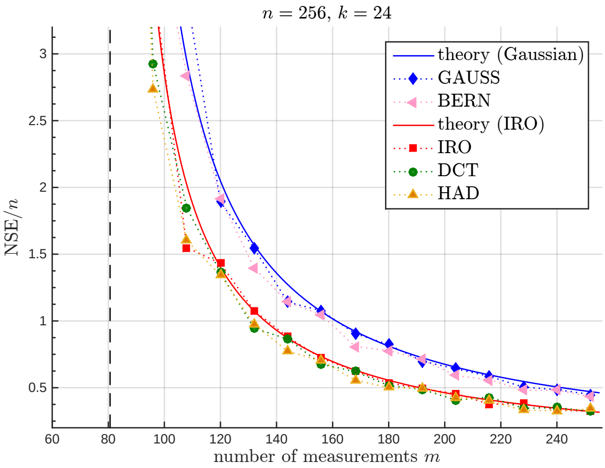
- Since $\frac{n - D_f(x_0,\lambda)}{n} < 1$, this is strictly better than the Gaussian case.

NSE for Isotropically Unitary Matrix: $n = 520$, $k = 20$



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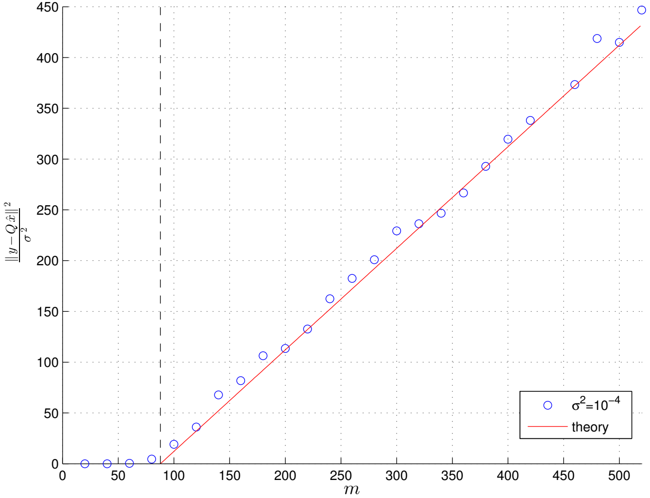

Other Measurements: Quadratic Gaussians

In certain applications, such as *graphical LASSO* and *phase retrieval*, we encounter problems of the following form:

$$\min_{S \geq 0} \quad \mathrm{trace} G^T S G + \Psi(S), \tag{0.1}$$

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Can we come up with comparison lemmas for the Gaussian process $\text{trace} U^T S G$?

Summary and Conclusion

- Developed a general theory for the analysis of a wide range of structured signal recovery problems for iid Gaussian measurement matrices
- Theory builds on a strengthening of a lemma of Gordon (whose origin is one of Slepian)
- Allows for optimal tuning of regularizer parameter
- Various loss functions and regularizers can be considered
- Results appear to be universal ("close" to a proof)
- Theory generalized to isotropically random unitary matrices
- Generalization to quadratic Gaussian measurements would be very useful (for phase retrieval, graphical LASSO, etc.)