Modeling and Simulation — Lesson 3
The Mathematics of Epidemics: SIS, SIR, SEIR and other epidemic models

Václav B. Petrák

Faculty of Biomedical Engineering
Czech Technical University

April 17, 2025

Introduction to Compartmental Models

Compartment Models

What are Compartmental Models?

- Mathematical approach for describing and analyzing complex systems
- System divided into **discrete compartments**representing specific states or stages
- Transition rates govern the movement of individuals between compartments
- They are often applied to modelling of infectious diseases. The population is assigned to compartments with labels **S**usceptible, **I**nfectious and **R**ecovered).

What are Compartmental Models?

- Mathematical approach for describing and analyzing complex systems
- System divided into **discrete compartments**representing specific states or stages
- Transition rates govern the movement of individuals between compartments
- They are often applied to modelling of infectious diseases. The population is assigned to compartments with labels **S**usceptible, **I**nfectious and **R**ecovered).

Example of a compartmental model

$$\mathrm{S} \xrightarrow{\beta \mathrm{IS}} \mathrm{I} \xrightarrow{\gamma \mathrm{I}} \mathrm{R}$$

Design approach to Compartment Epidemiological Models

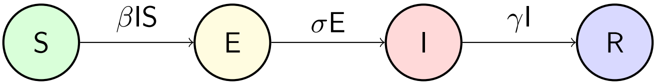
- **Abstract Models:**
  - *Key question?:* What are general patterns and relationships in epidemics?
  - *Aims:* Investigate general disease behavior. Explore various modes of transmission and countermeasures
  - *Characteristics:* High level overview. Models are simple but not trivial. Direct application to real-world situations may be limited, but it still may be a reasonable starting point.

Design approach to Compartment Epidemiological Models

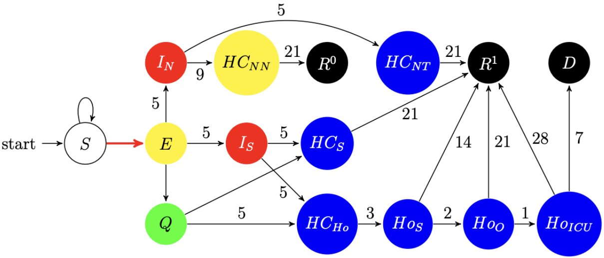
- **Abstract Models:**
  - *Key question?:* What are general patterns and relationships in epidemics?
  - *Aims:* Investigate general disease behavior. Explore various modes of transmission and countermeasures
  - *Characteristics:* High level overview. Models are simple but not trivial. Direct application to real-world situations may be limited, but it still may be a reasonable starting point.
- **Concrete Models:**
  - *Key question?:* What factors are influencing spread of disease in a specific outbreak?
  - *Aims:* Models detailed information about the situation.
  - *Characteristics:* Directly relevant to real-world applications. But lessons learned from one epidemic outbreak may not translate to new situations.

Abstract and concrete model

Basic form of SEIR model



Model used in the Czech Republic at early stages of COVID-19 pandemics



Susceptible-Infected (SI) Model: Overview

- The Susceptible-Infected (SI) model is the simplest form of all disease models. Individuals are born into the simulation with no immunity (susceptible). Once infected and with no treatment, individuals stay infected and infectious throughout their life, and remain in contact with the susceptible population.
- Two compartments: Susceptible (S) and Infected (I)
- Transition rate: infection rate (β)
- Assumes no recovery or immunity
- Model matches the behavior of diseases like cytomegalovirus

SI Model: Equations

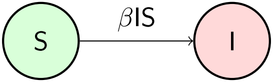
- Susceptible (S): Individuals who are not yet infected but can become infected.
- Infected (I): Individuals who are infected with the disease.

Equations:

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

Diagram:



Susceptible-Infected-Susceptible (SIS) Model

- The Susceptible-Infected-Susceptible (SIS) model extends the SI model by allowing infected individuals to recover and become susceptible again. Assumes no immunity after recovery, allowing for reinfection
- This model is appropriate for diseases that commonly have repeat infections, for example, the common cold (rhinoviruses) or sexually transmitted diseases like gonorrhea or chlamydia
- Three states:
  - Susceptible (S): Individuals who are not yet infected but can become infected or have recovered from infection and are susceptible again.
  - Infected (I): Individuals who are currently infected with the disease.
- Epidemic occurs the population reaches an equilibrium between susceptible and infective individuals.

SIS Model: Equations and diagram

Transition rates:

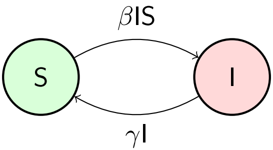
- infection rate (β)
- recovery rate (γ)

Equations:

$$\frac{dS}{dt} = \gamma I - \beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

Diagram:



SI Model with vital dynamics

- Extended Susceptible-Infected (SI) model
- Compartmental model in epidemiology considering birth and deaths
- Two compartments: Susceptible (S) and Infected (I)
- Assumes no recovery or immunity
- In addition to the simple model, it captures the impact of population dynamics on disease transmission

Equations and model diagram

Transition Rates

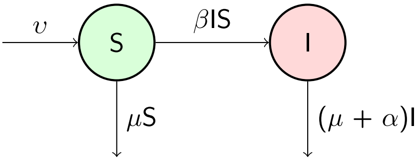
- infection rate (β)
- birth rate (υ)
- datural death rate (μ)
- disease death rate (α)

Equations

$$\frac{dS}{dt} = \upsilon - \beta SI - \mu S$$

$$\frac{dI}{dt} = \beta SI - (\mu + \alpha)I$$

Model Diagram



Susceptible-Infected-Recovered (SIR) Model: Overview

- Widely used compartmental model in epidemiology. Assumes individuals gain immunity after recovery
- Compartments: Susceptible (S), Infected (I), Recovered (R)
- Applications:
  - Studying the spread of diseases where recovery and immunity are possible
  - Evaluating the impact of intervention strategies, such as vaccination or social distancing and predicting the course of an outbreak

SIR Model: Equations and diagram

Transition rates

- Infection rate (β)
- Recovery rate (γ)

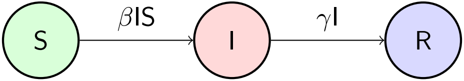
Equations

$$\frac{dS}{dt} = -\beta S I$$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Model Diagram



Reproduction Numbers in the SIR Model

Basic Reproduction Number, R_0:

- Expected number of cases generated by one case in a population **where all individuals are susceptible to infection**.
- An epidemic will only occur if $R_0 > 1$.
- R_0 **does not change over the course of an outbreak**, as it is a property of the pathogen and the interaction between the host population and the environment.
- Derived from: $R_0 = \frac{\beta}{\gamma}$ where β is the transmission rate and γ is the recovery rate.

Reproduction Numbers in the SIR Model

Basic Reproduction Number, R_0:

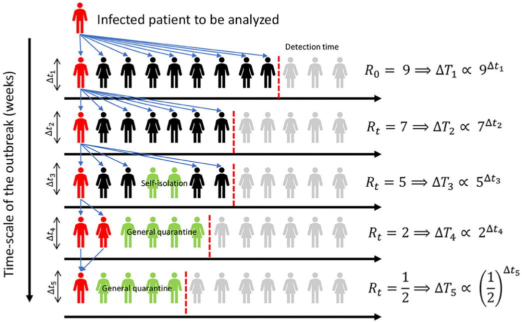
- Expected number of cases generated by one case in a population **where all individuals are susceptible to infection**.
- An epidemic will only occur if $R_0 > 1$.
- R_0 **does not change over the course of an outbreak**, as it is a property of the pathogen and the interaction between the host population and the environment.
- Derived from: $R_0 = \frac{\beta}{\gamma}$ where β is the transmission rate and γ is the recovery rate.

Effective Reproduction Number, R_t:

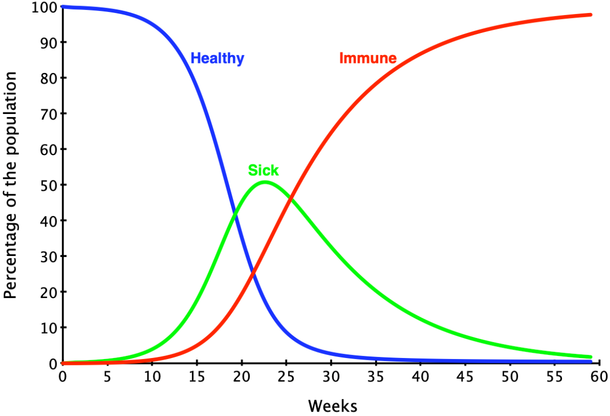
- The number of secondary infections. Not all contacts will become infected and the average number of secondary cases per infectious will be lower than R_0.
- R_t is **not fixed** and varies over the course of an epidemic due to interventions and immunizations.
- When R_t falls below 1, the outbreak is under control, leading to a decline in new cases. R_t 1 means endemic
- The calculation of R_t is complex due to changing dynamics.

Illustration of reproduction number

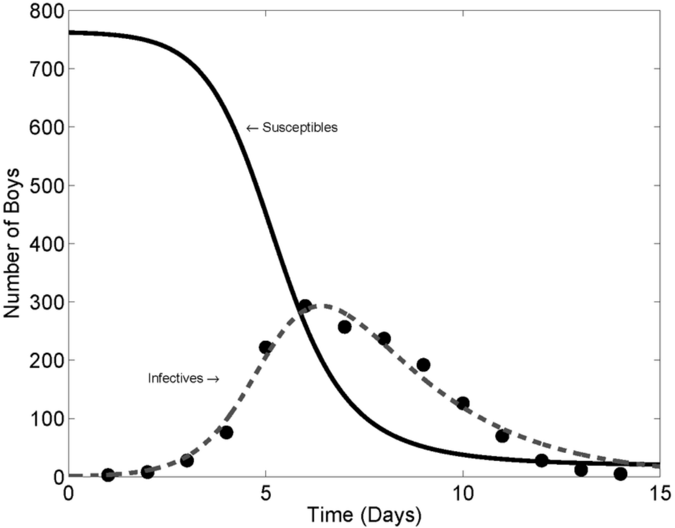
- An infected individual (red) can spread the virus among different individuals (black), not reaching part of the population (gray).
- Some individuals go into isolation (green), lowering their contagion chance.
- R_t represents the number of possible new infections caused by a single patient in each outbreak stage.
- In the first days, one individual can infect several people before isolation. As the amount of cases gets public awareness and health policies actions helps to control the outbreak, lowering R_t.



Progress of epidemic by SIR model



Epidemic in English boarding school and SIR model



Susceptible-Exposed-Infected-Recovered (SEIR) Model

- Extended compartmental model in epidemiology. Includes a separate compartment for exposed individuals who are not yet infectious. Suitable for diseases with a latent period where individuals are not infectious immediately after contracting the disease
- Compartments: Susceptible (S), Exposed (E), Infected (I), Recovered (R)
- Exposed individuals (E): have contracted the disease but are not yet infectious; differs from infected individuals (I) who are actively spreading the disease
- Applications:
  - Studying the spread of diseases with latent periods, such as COVID-19, Ebola, or measles
  - Evaluating the impact of intervention strategies, such as vaccination or social distancing and predicting the course of an outbreak

SEIR: Model Equations

Rates

- Infection rate (β): rate at which susceptible individuals become exposed
- Incubation rate (σ): rate at which exposed individuals become infected
- Recovery rate (γ): rate at which infected individuals recover and gain immunity

Equations

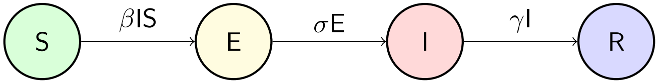
$$\frac{dS}{dt} = -\beta S I$$

$$\frac{dE}{dt} = \beta S I - \sigma E$$

$$\frac{dI}{dt} = \sigma E - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Diagram



Applications Beyond Disease Modeling

- Originally developed for epidemiology, but applicable to other fields with similar dynamics
- Key aspects: interacting populations, transitions between states, and time-dependent dynamics
- Examples of alternative applications:
  - Ecology: predator-prey interactions, invasive species spread
  - Social networks: spread of information, opinions, or behavior
  - Computer networks: malware and computer virus propagation

Agent based models

- **Definition** Agent-based modeling (ABM) is a methodology used to build formal models of real-world systems that are made up by individual units (such as e.g. people, cells, animals, or institutions) which repeatedly interact among themselves or with their environment.

Introduction to Agent-Based Models

- **Definition** Agent-based modeling (ABM) is a methodology used to build formal models of real-world systems that are made up by individual units (such as e.g. people, cells, animals, or institutions) which repeatedly interact among themselves or with their environment.
- ABM approach establishes a direct a correspondence between the individual units in the system to be modeled and the model the agents and their interactions
- In other words: Two persons meeting in real world are modeled as meeting of two agents in the model

Introduction to Agent-Based Models

- **Definition** Agent-based modeling (ABM) is a methodology used to build formal models of real-world systems that are made up by individual units (such as e.g. people, cells, animals, or institutions) which repeatedly interact among themselves or with their environment.
- ABM approach establishes a direct a correspondence between the individual units in the system to be modeled and the model the agents and their interactions
- In other words: Two persons meeting in real world are modeled as meeting of two agents in the model
- This approach contrasts with equation-based modeling, where modeled system may be represented via average properties.

Introduction to Agent-Based Models

- **Definition** Agent-based modeling (ABM) is a methodology used to build formal models of real-world systems that are made up by individual units (such as e.g. people, cells, animals, or institutions) which repeatedly interact among themselves or with their environment.
- ABM approach establishes a direct a correspondence between the individual units in the system to be modeled and the model the agents and their interactions
- In other words: Two persons meeting in real world are modeled as meeting of two agents in the model
- This approach contrasts with equation-based modeling, where modeled system may be represented via average properties.
- Applications: Epidemiology, social sciences, economics, and more.

Differences between ABMs and Differential Equation Models

- **Granularity**: ABMs simulate individual agents, differential equations models represent populations as continuous variables.

Differences between ABMs and Differential Equation Models

- **Granularity**: ABMs simulate individual agents, differential equations models represent populations as continuous variables.
- **Stochasticity**: ABMs usually include randomness in agents' actions, differential equations models are deterministic (but randomness can be also added).

Differences between ABMs and Differential Equation Models

- **Granularity**: ABMs simulate individual agents, differential equations models represent populations as continuous variables.
- **Stochasticity**: ABMs usually include randomness in agents' actions, differential equations models are deterministic (but randomness can be also added).
- **Spatial Structure**: ABMs typycally incorporate spatial structures and movement, differential equations models assume well-mixed populations.

Differences between ABMs and Differential Equation Models

- **Granularity**: ABMs simulate individual agents, differential equations models represent populations as continuous variables.
- **Stochasticity**: ABMs usually include randomness in agents' actions, differential equations models are deterministic (but randomness can be also added).
- **Spatial Structure**: ABMs typycally incorporate spatial structures and movement, differential equations models assume well-mixed populations.
- **Complexity**: ABMs handle complex interactions and adaptations, differential equations models focus on average rates of change.

Advantages of Agent-Based Models

- **Emergence**: Emergence occurs when a complex entity has properties or behaviors that its parts do not have on their own, and emerge only when they interact in a wider whole. Think of snow flakes or ant colonies. In ABMs, higher-level system properties may emerge from the interactions of lower-level agents.

Advantages of Agent-Based Models

- **Emergence**: Emergence occurs when a complex entity has properties or behaviors that its parts do not have on their own, and emerge only when they interact in a wider whole. Think of snow flakes or ant colonies. In ABMs, higher-level system properties may emerge from the interactions of lower-level agents.
- **Detail**: Ability to model systems at the level of individual agents allows for detailed exploration of wide range of behavior.

Advantages of Agent-Based Models

- **Emergence**: Emergence occurs when a complex entity has properties or behaviors that its parts do not have on their own, and emerge only when they interact in a wider whole. Think of snow flakes or ant colonies. In ABMs, higher-level system properties may emerge from the interactions of lower-level agents.
- **Detail**: Ability to model systems at the level of individual agents allows for detailed exploration of wide range of behavior.
- **Adaptability**: Agents can be programmed to adapt their behaviors in response to changing conditions or rules.

Advantages of Agent-Based Models

- **Emergence**: Emergence occurs when a complex entity has properties or behaviors that its parts do not have on their own, and emerge only when they interact in a wider whole. Think of snow flakes or ant colonies. In ABMs, higher-level system properties may emerge from the interactions of lower-level agents.
- **Detail**: Ability to model systems at the level of individual agents allows for detailed exploration of wide range of behavior.
- **Adaptability**: Agents can be programmed to adapt their behaviors in response to changing conditions or rules.
- **Customization**: Models can be easily tailored to specific situations by altering rules, behaviors, and interactions of agents.

Disadvantages of Agent-Based Models

- **Computational Demands**: High complexity and detail level can lead to significant computational costs.

Disadvantages of Agent-Based Models

- **Computational Demands**: High complexity and detail level can lead to significant computational costs.
- **Data Requirements**: Developing realistic models often requires detailed empirical data, which may not always be available.

Disadvantages of Agent-Based Models

- **Computational Demands**: High complexity and detail level can lead to significant computational costs.
- **Data Requirements**: Developing realistic models often requires detailed empirical data, which may not always be available.
- **Analysis Complexity**: The richness of the output can make analysis and interpretation challenging.

Disadvantages of Agent-Based Models

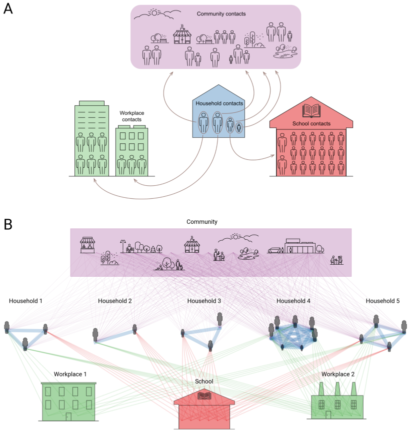
- **Computational Demands**: High complexity and detail level can lead to significant computational costs.
- **Data Requirements**: Developing realistic models often requires detailed empirical data, which may not always be available.
- **Analysis Complexity**: The richness of the output can make analysis and interpretation challenging.
- **Reproducibility Issues**: Due to stochastic elements, reproducing results can be difficult.

Disadvantages of Agent-Based Models

- **Computational Demands**: High complexity and detail level can lead to significant computational costs.
- **Data Requirements**: Developing realistic models often requires detailed empirical data, which may not always be available.
- **Analysis Complexity**: The richness of the output can make analysis and interpretation challenging.
- **Reproducibility Issues**: Due to stochastic elements, reproducing results can be difficult.
- **Validation Challenges**: Validating the model against real-world outcomes can be complicated due to the complexity of the models.

Applications of Agent-Based Models - Specific Examples

Epidemiology: Modeling the spread of COVID-19 to evaluate the effectiveness of social distancing measures in various communities. Agent-based model *Coves* was developed to help policymakers make decisions based on the best available data, while taking into account the large uncertainties in terms of COVID-19 transmission dynamics, disease progression, and other aspects.



Applications of Agent-Based Models - Specific Examples

Ecology: Simulation of agent-based model of collective nest choice by the ant. Helps in testing the effects and importance of alternative parameter values and decision rules on colonial decision making.

Applications of Agent-Based Models - Specific Examples

Economics: Analyzing consumer behavior in online stores to predict reactions to changes in price or product availability. May be used for optimizing pricing strategies.

Applications of Agent-Based Models - Specific Examples

Urban Planning: Planning public transportation systems by simulating the movement patterns of individuals within a city. Helps in making public transport more efficient and accessible.

Applications of Agent-Based Models - Specific Examples

Social Sciences: Studying the spread of information and misinformation through social networks to understand the dynamics of public opinion formation.

Applications of Agent-Based Models - Specific Examples

Military: An agent-based computational model for the Battle of Trafalgar. Help to decide what outcomes were possible or what environmental factors has affected the results.

Basics of an Agent-Based SIR Model

- **Agents**: Represent individuals in a population.
- **States**: Susceptible, Infected, Recovered.

Basics of an Agent-Based SIR Model

- **Agents**: Represent individuals in a population.
- **States**: Susceptible, Infected, Recovered.
- **Rules of Transition**:
  - Susceptible to Infected: Based on proximity to infected agents and a probability of transmission.
  - Infected to Recovered: After a certain period or based on individual health dynamics.

Basics of an Agent-Based SIR Model

- **Agents**: Represent individuals in a population.
- **States**: Susceptible, Infected, Recovered.
- **Rules of Transition**:
  - Susceptible to Infected: Based on proximity to infected agents and a probability of transmission.
  - Infected to Recovered: After a certain period or based on individual health dynamics.
- **Simulation**: Runs iterative, updating the state of each agent based on rules, until a steady state is reached or for a set number of steps.

Advantages of ABMs in Epidemic Modeling

- **Flexibility**: Can model complex behaviors and interventions (e.g., social distancing, quarantine).

Advantages of ABMs in Epidemic Modeling

- **Flexibility**: Can model complex behaviors and interventions (e.g., social distancing, quarantine).
- **Heterogeneity**: May account for individual differences in susceptibility, immunity and mobility.

Advantages of ABMs in Epidemic Modeling

- **Flexibility**: Can model complex behaviors and interventions (e.g., social distancing, quarantine).
- **Heterogeneity**: May account for individual differences in susceptibility, immunity and mobility.
- **Spatial Dynamics**: Models how the disease spreads in physical space, including community structures.

Advantages of ABMs in Epidemic Modeling

- **Flexibility**: Can model complex behaviors and interventions (e.g., social distancing, quarantine).
- **Heterogeneity**: May account for individual differences in susceptibility, immunity and mobility.
- **Spatial Dynamics**: Models how the disease spreads in physical space, including community structures.
- **Data Integration**: Can incorporate empirical data at the individual level to calibrate or validate the model.

Simulation Loop in an Agentic SIR Model

- **Update States:**
  - For each susceptible agent, check proximity to infected agents.
  - Use a random draw with the infection probability to transition from susceptible to infected.
  - For each infected agent, use a random draw with the recovery probability to transition to recovered.
- **Record Results:**
  - Count the number of susceptible, infected, and recovered individuals and store these counts for visualization.
- **Visualization:**
  - Update plots showing current positions and a time-series plot tracking S, I, and R dynamics.
- **Update Positions:**
  - Move each agent using a predefined random movement rule.
  - Ensure that new positions remain within the simulation boundaries.