by Václav Blažej, Tomáš Valla, & Pavel Dvořák

Faculty of Information Technology, Czech Technical University in Prague, Prague, Czech Republic

Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

The Game rules

- Two players, Builder and Painter, play against each other in turns.
- They are given a graph H and a playfield of an infinite set of independent vertices.
- Each round Builder draws an edge and Painter colors it either red or blue.

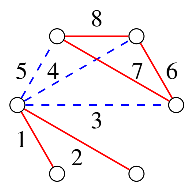
Goals of the players are:

- Builder's goal is to create the target graph H as a **monochromatic subgraph**.
- Painter's goal is to delay Builder for as many rounds as possible.

This game is closely related to **Ramsey theory**.

We want as few moves as possible

The *online Ramsey number* is the minimum number of rounds such that Builder has a winning strategy in the online Ramsey game if both players play optimally.



For example, a K_3 can be forced in 8 moves!

Known results

- Grytczuk, et al. [2008]: Created an upper bound construction for paths.
- Conlon [2009]: Shows that for cliques the game setting is easier for Builder.
- Haxell, Kohayakawa, Łuczak [1995]: non-constructive proof of a linear upper bound for cycles.
- (Many non-optimal bounds for other variants of the game.)

Our results

All of the following graphs will appear **strongly induced** as a subgraph and are asymptotically optimal (see picture ↗).

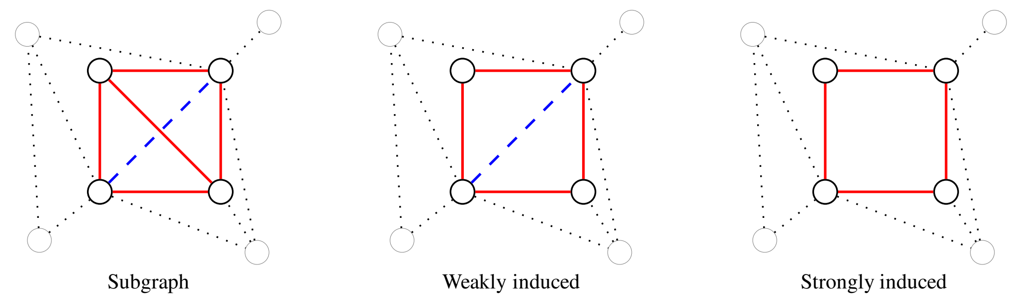
- Paths
- Cycles (constructive)
- Spiders (paths with a common endpoint)
- Centipedes (stars with centers on a path)

Open problems

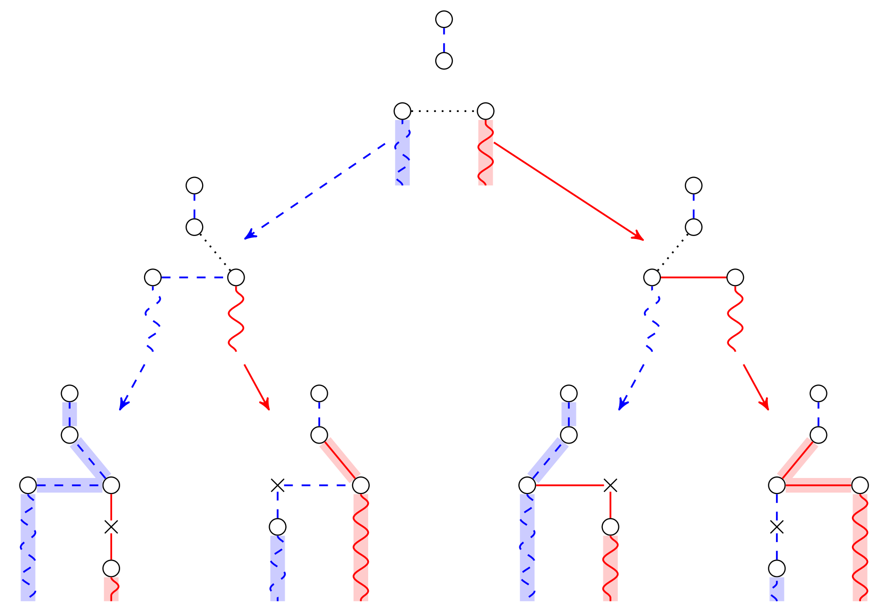
- Generalize path and cycle strategy for k colors.
- Generalize path strategy for three colors.
- Investigate more restricted variants of the online Ramsey game.

More details on the strategies are on the right side of the poster. Thank you for reading this far. If you want to learn more, it would be my pleasure to discuss the results with you in person, or you can check the paper and this poster via `QR code` (here →).

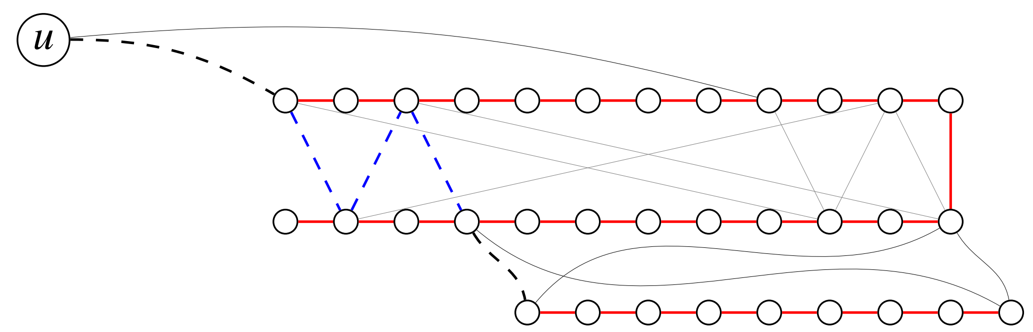
We **solved** a Builder & Painter **game** for paths, cycles, and some trees in asymptotically **optimal** number of moves.



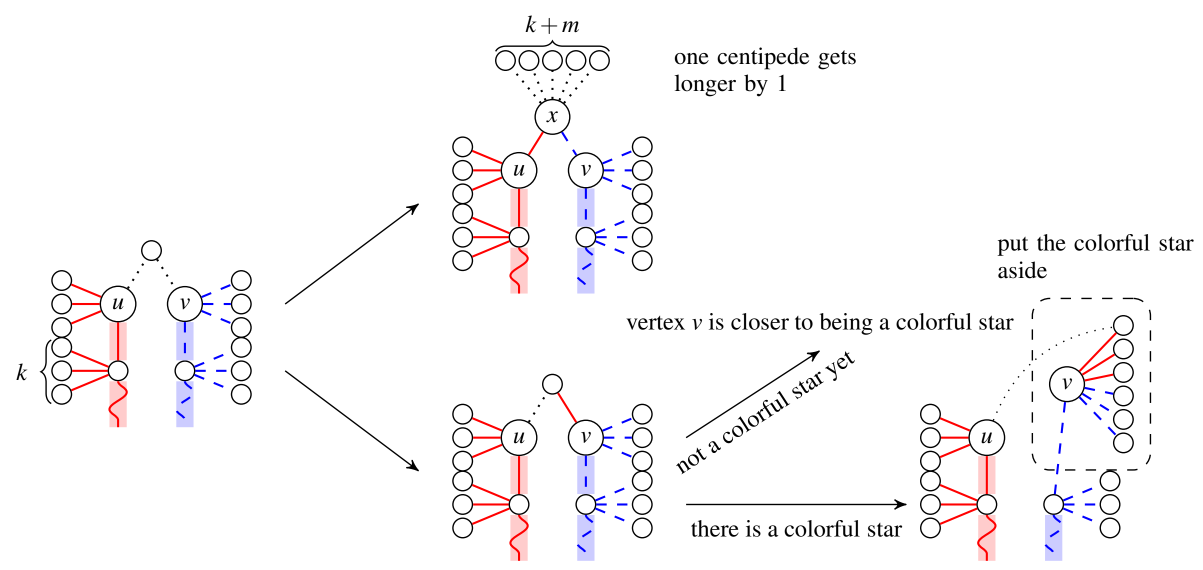
The various notions of *induced subgraph* used in the online Ramsey game.



Our strategy for induced paths creates many edges first and then uses potential over path-lengths.



This construction repeated three times forces an even cycle.



A step in creating centipedes either makes one longer or brings us closer to have many colorful stars.

Induced online Ramsey numbers

The induced results upper bound the non-induced variant as well.

$$\mathrm{ORN}(H) \le \mathrm{ORN}_{weak}(H) \le \mathrm{ORN}_{strong}(H)$$

Our strongly induced upper bounds are as follows:

- paths in $28n-27$
- even cycles in $367n-27$, odd in $735n-27$
- ℓ-subdivided stars S_k in $O(k^2\ell)$
- ℓ stars S_k on a common path in $O(k\ell)$

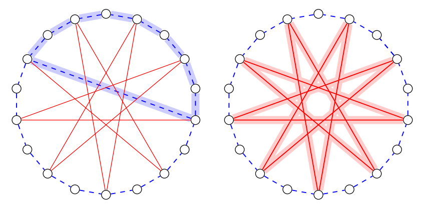
↑ has size-Ramsey lower bound $\Omega(k^2\ell)$ Beck [1990]

Paths

- Build many isolated edges, say blue is prevalent.
- Use process shown on the picture (←) to ensure that potential $3b+4r$ strictly increases each round.
- High potential implies that one path has be long.

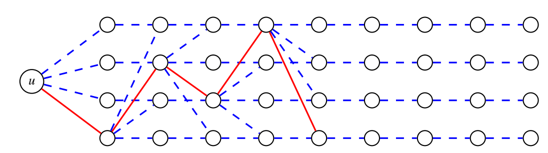
Cycles

- Construct a long induced path.
- Connecting vertices $n-2$ apart could make the desired cycle.
- Use that to force a $\frac{n}{2}$ path starting in u, ending in one of two vertices (see picture ↙).
- Repeat three times to guarantee two of those paths connect into an even cycle.
- For an odd cycle build $2n$ cycle and use connect odd vertices in the following way.



Spiders

- Build a very long induced path
- Connecting a vertex to k vertices of the path which are far apart will create the spider.
- Using that we can force the creation of a ℓ path from u.
- Repeat the process k times.



Centipedes

- We use an intricate potential to count lengths, bad edges, and colorful stars.
- We either arrive at a point where one centipede is long enough of we have many colorful stars.
- Run induced path strategy on the colorful stars.

Additionally, this result shows that the online Ramsey number for centipedes is asymptotically smaller than their size-Ramsey number. This is due to lower bound by Beck [1990] of $\Omega(k^2\ell)$. A similar result was shown by Conlon [2009].

The **size-Ramsey number** is the non-game variant of this problem. It is the smallest number of edges which have to be created in order for any coloring to contain a monochromatic subgraph. See the following example for a counterexample that a K_3 might not be found in $K_6 - e$.