# Embedding trees in dense graphs

V. Rozhoň

Extra seminar, 2018

# Erdős-Sós conjecture

How many edges in a graph force a certain structure there to appear?
E.g. more than $n^2/4$ edges force a graph to contain a triangle.

# Erdős-Sós conjecture

How many edges in a graph force a certain structure there to appear?
E.g. more than $n^2/4$ edges force a graph to contain a triangle.
How many edges force a graph to contain a specific tree?

# Erdős-Sós conjecture

How many edges in a graph force a certain structure there to appear?
E.g. more than $n^2/4$ edges force a graph to contain a triangle.
How many edges force a graph to contain a specific tree?

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain *any* tree on $k+1$ vertices.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Tightness: take union of $K_k$.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Tightness: take union of $K_k$.
The conjecture holds if we replace $k-1$ by $2k$.

# Erdős-Sós conjecture

## Conjecture (Erdős, Sós)

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Tightness: take union of $K_k$.
The conjecture holds if we replace $k-1$ by $2k$.

## Observation

Every graph with **minimum** degree greater than $k-1$ contain any tree on $k+1$ vertices.

# Erdős-Sós conjecture

## Conjecture (Erdős, Sós)

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Tightness: take union of $K_k$.
The conjecture holds if we replace $k-1$ by $2k$.

## Observation

Every graph with **minimum** degree greater than $k-1$ contain any tree on $k+1$ vertices.

## Observation

Every graph with average degree $l$ contain a subgraph with average degree at least $l$ and minimum degree at least $l/2$.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k - 1$ contain any tree on $k + 1$ vertices.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

**Theorem (Ajtai, Komlós, Simonovits, Szemerédi 18+)**

The Erdős-Sós conjecture holds for $k > k_0$.

The proof is very complicated.

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Can we get better?

# Erdős-Sós conjecture

**Conjecture (Erdős, Sós)**

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Can we get better?

**Theorem (Erdős, Gallai)**

Any graph with $\Delta(G) \geq k$ and $\delta(G) \geq k/2$ contains a tree $T$ on $k+1$ vertices, if $T$ is a path ...

# Erdős-Sós conjecture

## Conjecture (Erdős, Sós)

Every graph with average degree greater than $k-1$ contain any tree on $k+1$ vertices.

Can we get better?

## Theorem (Erdős, Gallai)

Any graph with $\Delta(G) \geq k$ and $\delta(G) \geq k/2$ contains a tree $T$ on $k+1$ vertices, if $T$ is a path ...

## Theorem (Wagner)

... but this is not true if $T$ is a random tree w.h.p.

With some additional assumptions, we can still get it to work:

**Theorem**

For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ such that $\Delta(T) \geq \gamma k$. If $\delta(G) \geq k/2 + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

With some additional assumptions, we can still get it to work:

**Theorem**

For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ such that $\Delta(T) \geq \gamma k$. If $\delta(G) \geq k/2 + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

**Corollary**

For any $\eta > 0$ there exists $n_0$ and $\gamma > 0$ such that for every $n > n_0$, any graph of order $n$ with average degree $\overline{\deg}(G) \geq k + \eta n$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

# Erdős-Sós conjecture

## Corollary

For any $\eta > 0$ there exists $n_0$ and $\gamma > 0$ such that for every $n > n_0$, any graph of order $n$ with average degree $\overline{\deg}(G) \geq k + \eta n$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

# Erdős-Sós conjecture

## Corollary

For any $\eta > 0$ there exists $n_0$ and $\gamma > 0$ such that for every $n > n_0$, any graph of order $n$ with average degree $\overline{\deg}(G) \geq k + \eta n$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

More honest version:

## Corollary

For any $\eta > 0$ and $q > 0$ there exists $n_0$ and $\gamma > 0$ such that for every $n > n_0$ and $k > qn$, any graph of order $n$ with average degree $\overline{\deg}(G) \geq k + \eta k$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

## Theorem

For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ such that $\Delta(T) \geq \gamma k$. If $\delta(G) \geq k/2 + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

We can actually prove a more general version:

## Theorem

For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ such that $\Delta(T) \geq \gamma k$. If $\delta(G) \geq k/2 + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

We can actually prove a more general version:

## Theorem

For any $r, \eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ with two colour classes $T_1$, $T_2$ such that $|T_1| \leq rk$ and $\Delta(T_2) \leq \gamma k$. If $\delta(G) \geq rk + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

This is actually tight in some sense.

# Loebl-Komlós-Sós conjecture

**Conjecture (Loebl, Komlós, Sós)**

If at least $n/2$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices.

# Loebl-Komlós-Sós conjecture

**Conjecture (Loebl, Komlós, Sós)**

If at least $n/2$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices.

**Theorem (Ajtai, Hladký, Komlós, Piguet, Stein, Szemerédi)**

For any $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, any graph of order $n$ with at least $(1+\eta)n/2$ vertices of degree at least $k + \eta k$ contains every tree of order at most $k$.

# Loebl-Komlós-Sós conjecture – simplified history

**Theorem (Piguet, Stein)**

The conjecture holds for trees with diameter at most 5.

**Theorem (Piguet, Stein)**

For any $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, any graph of order $n$ with at least $n/2$ vertices of degree at least $k + \eta n$ contains every tree of order at most $k$.

**Theorem (Ajtai, Hladký, Komlós, Piguet, Stein, Szemerédi)**

For any $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, any graph of order $n$ with at least $(1 + \eta)n/2$ vertices of degree at least $k + \eta k$ contains every tree of order at most $k$.

# Loebl-Komlós-Sós(-Simonovits) conjecture – more history

**Conjecture (Loebl)**

If at least $n/2$ vertices of a graph have degree at least $n/2$, then it contains every tree on $n/2+1$ vertices.

# Loebl-Komlós-Sós(-Simonovits) conjecture – more history

## Conjecture (Loebl)

If at least $n/2$ vertices of a graph have degree at least $n/2$, then it contains every tree on $n/2+1$ vertices.

## Conjecture (Komlós, Sós)

If at least $n/2$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices.

# Loebl-Komlós-Sós(-Simonovits) conjecture – more history

## Conjecture (Loebl)

If at least $n/2$ vertices of a graph have degree at least $n/2$, then it contains every tree on $n/2+1$ vertices.

## Conjecture (Komlós, Sós)

If at least $n/2$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices.

## Conjecture (Simonovits)

Let $0 < r < 1/2$. If at least $rn$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices with at most $r(k+1)$ vertices in one colour class.

# Loebl-Komlós-Sós-Simonovits conjecture

**Conjecture (Loebl, Komlós, Sós, Simonovits)**

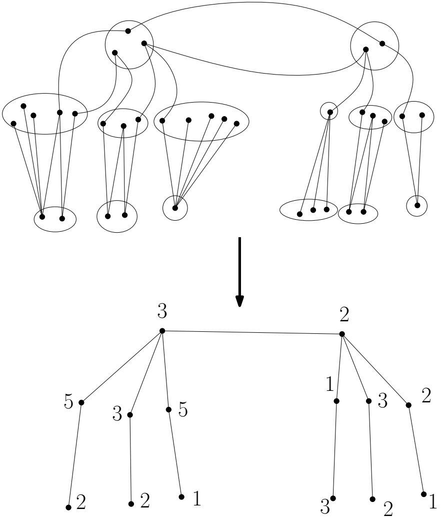
Let $0 < r < 1/2$. If at least $rn$ vertices of a graph have degree at least $k$, then it contains every tree on $k+1$ vertices with at most $r(k+1)$ vertices in one colour class.

**Theorem (Klimošová, Piguet, R.)**

Let $0 < r \leq 1/2$ and $\eta > 0$. There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, any graph of order $n$ with at least $rn$ vertices of degree at least $k + \eta n$ contains every tree of order at most $k$ such that the size of its smaller colour class is at most $rk$.

# Some proof ideas

2) Clusterization




1) Regularity lemma