Distributed Algorithms
vs
Descriptive Combinatorics

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some slides (the ones with nice pictures) taken from a presentation of **Jukka Suomela**
https://jukkasuomela.fi/landscape-of-locality/

The area started by an insightful paper of **Anton Bernshteyn**

The Plan

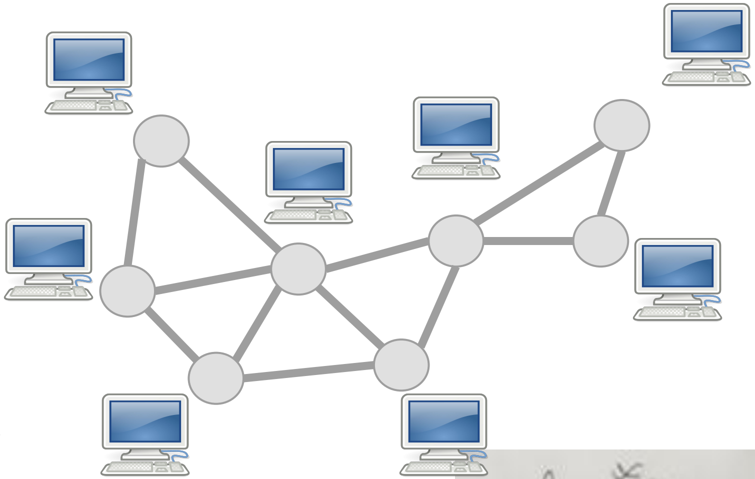
- Define the **LOCAL** model of distributed computation and give you highlights of the work done in past few years.
- Realize I don't have enough time left.
- A simple example from descriptive combinatorics, some pictures.

The **LOCAL** Model of Distributed Graph Algorithms

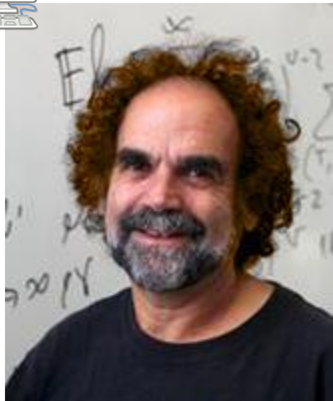
- Undirected graph on n nodes
- One computer in each node

- Synchronous message passing rounds
- Unbounded message size and computation

- Initially, nodes know only (an upper bound on) n and perhaps the max degree Δ
- Symmetry breaking:
  - either a random string (randomized algs)
  - unique identifier from $n^{O(1)}$-sized range
- In the end, each node should know its part of output
- Time complexity: number of rounds



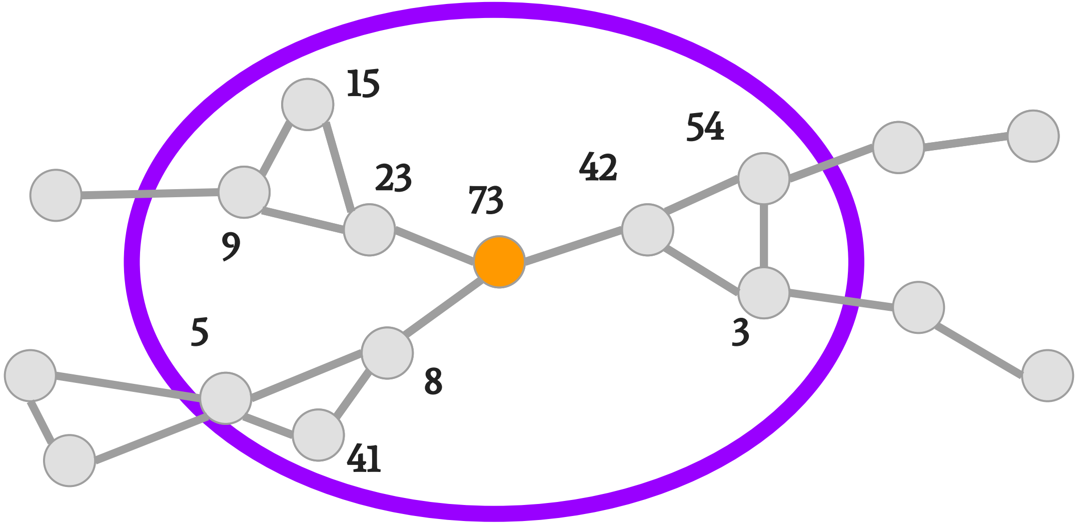
LOCAL model
[Linial FOCS'87]



The **LOCAL** Model of Distributed Graph Algorithms

"*unbounded message size and computation*":

deterministic **LOCAL** *t*-round algorithm is a function mapping *t*-hop neighbourhoods to labels.



Some Highlights

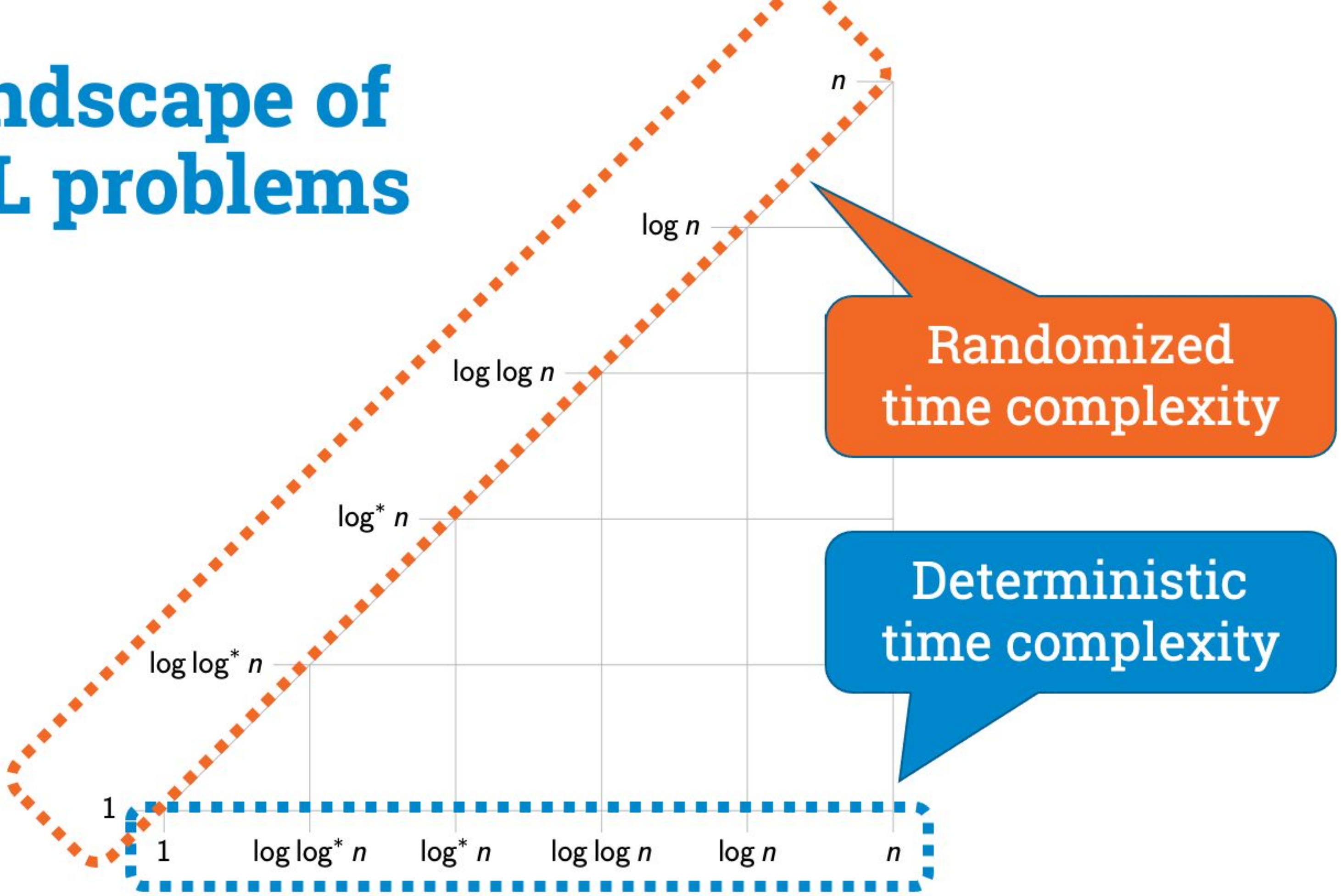
From now on:

- the maximum degree is bounded by $\Delta=O(1)$.
- we care about so-called locally checkable problems (LCLs)
  - those include vertex & edge coloring, perfect matching, maximal independent set, *list coloring*
  - in general: the solution at a node can be checked by looking at constant radius around it
- It turns out that under above conditions, we can ‘classify’ possible local complexities of LCL problems! For example: no problems of complexity $\log^{0.5} n$
- This holds especially if we further restrict ourselves to: paths, grids, trees

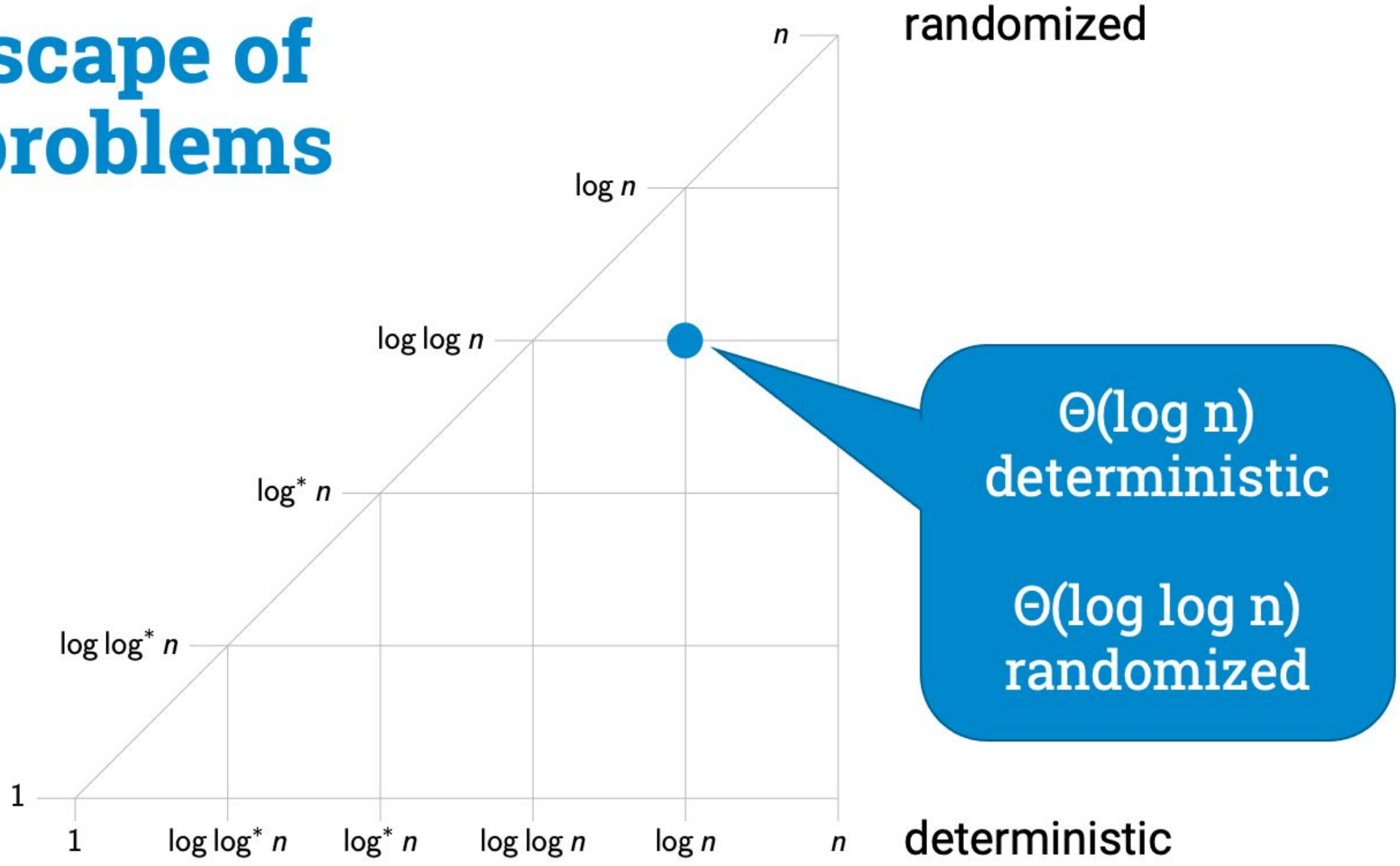
Try to guess: what are possible deterministic/randomized complexities of a local problem on an oriented path?

(2-coloring, 3-coloring, 4-coloring, maximal independent set, maximum independent set, perfect matching, list coloring, …)

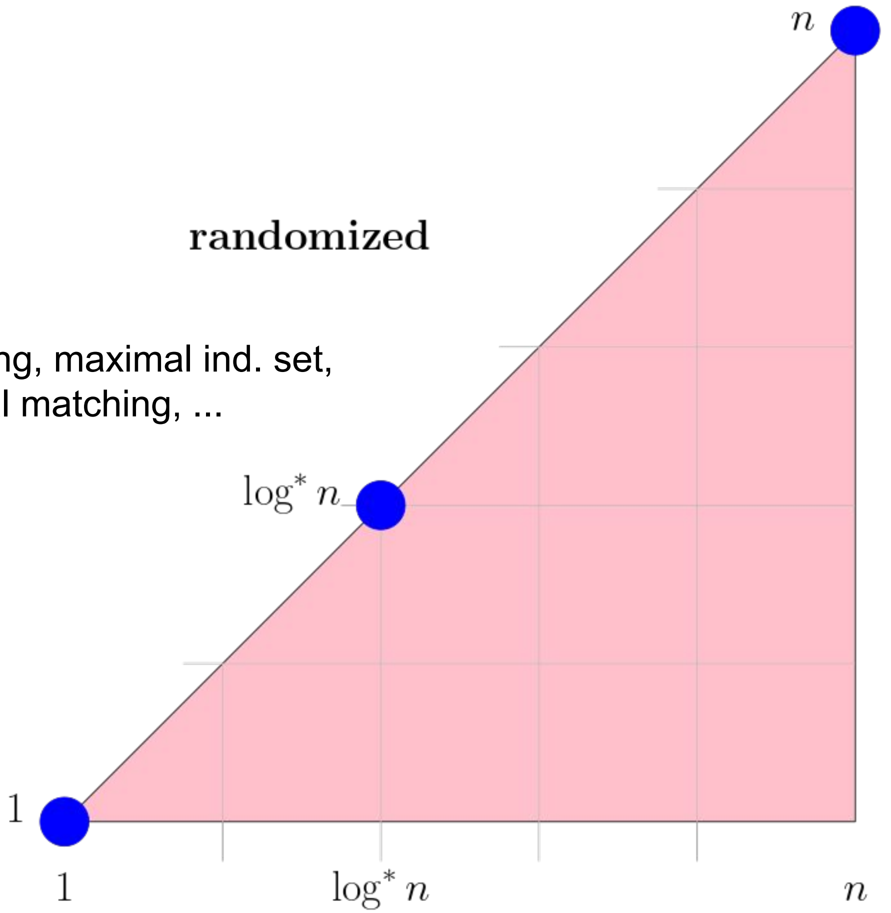
Landscape of LCL problems



Landscape of LCL problems



LOCAL, paths



randomized

deterministic

2-coloring

3 coloring, maximal ind. set,
maximal matching, ...

number of different input colors in your 2-hop neighborhood
(without inputs only trivial problems here)

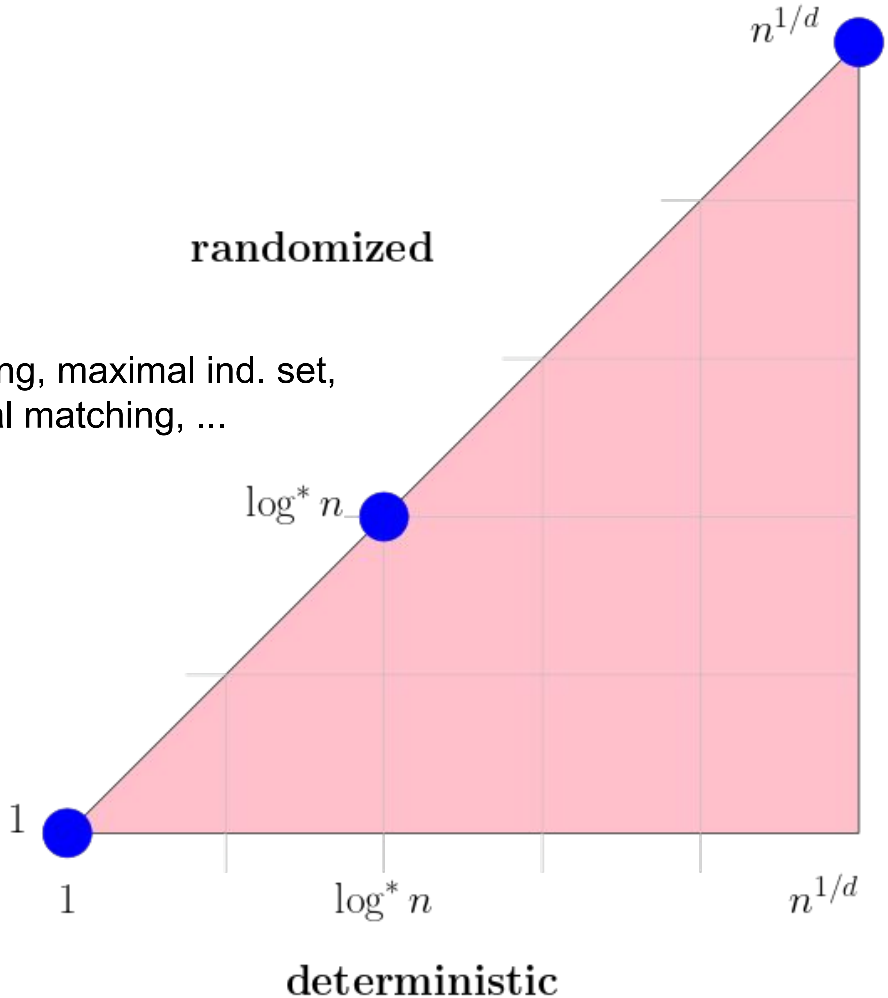
[Naor, Stockmeyer SIAM J. of Comp. 95]
[Chang, Pettie FOCS 17]
[Chang, Kopelowitz, Pettie FOCS 2016]
[Brandt et al. PODC 2017]
[Balliu et al. PODC 2019]
...

LOCAL, grids

2-coloring,
3-coloring

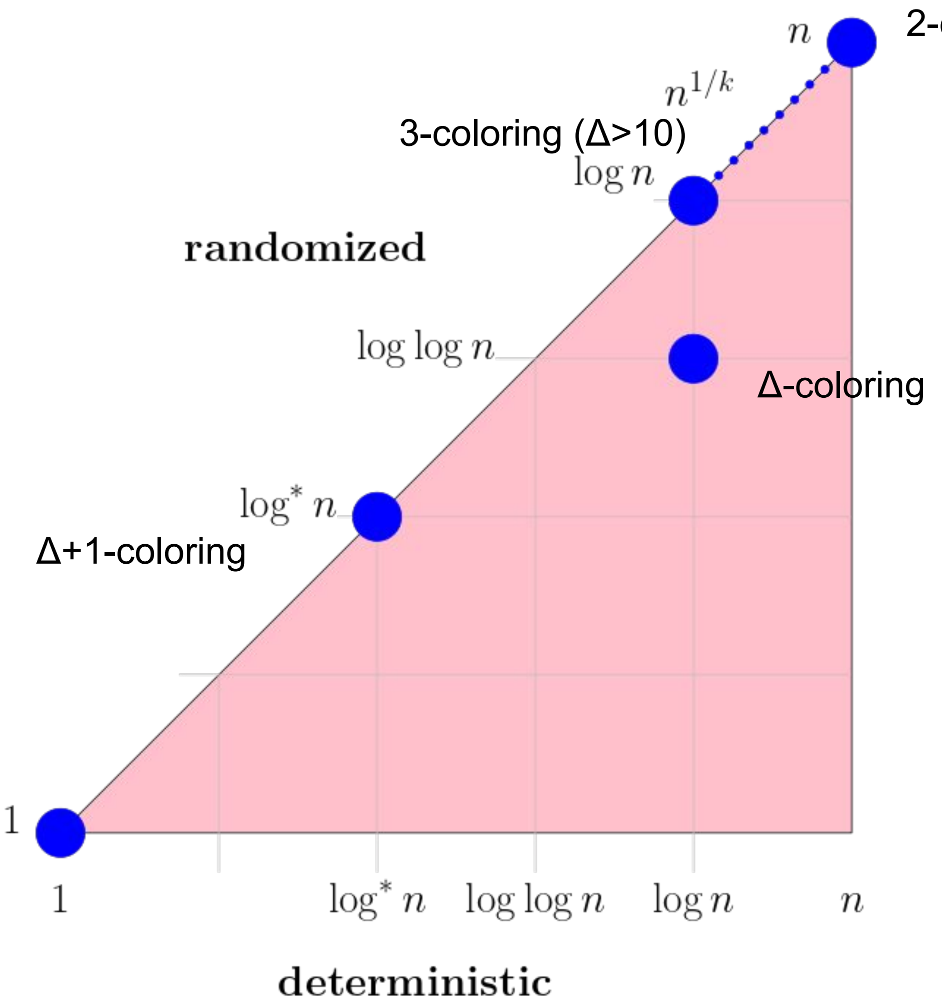
[Naor, Stockmeyer SIAM J. of Comp. 95]
[Chang, Pettie FOCS 17]
[Suomela 17]
[Chang, Kopelowitz, Pettie FOCS 2016]
[Brandt et al. PODC 2017]
[Brandt, Grunau, R 2021+]

4 coloring, maximal ind. set,
maximal matching, ...



number of different
colors in your 2-hop
neighborhood

LOCAL, trees



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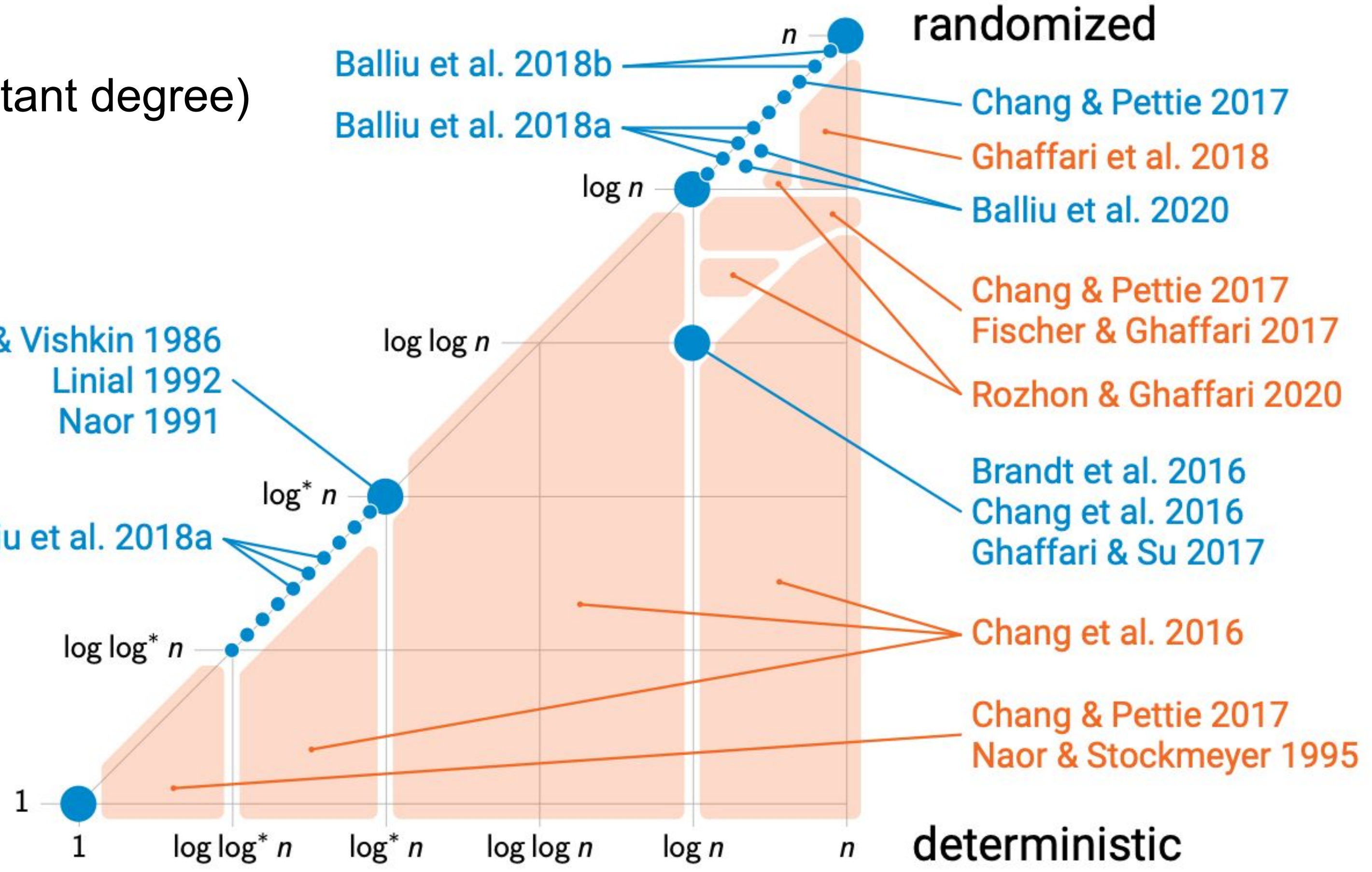
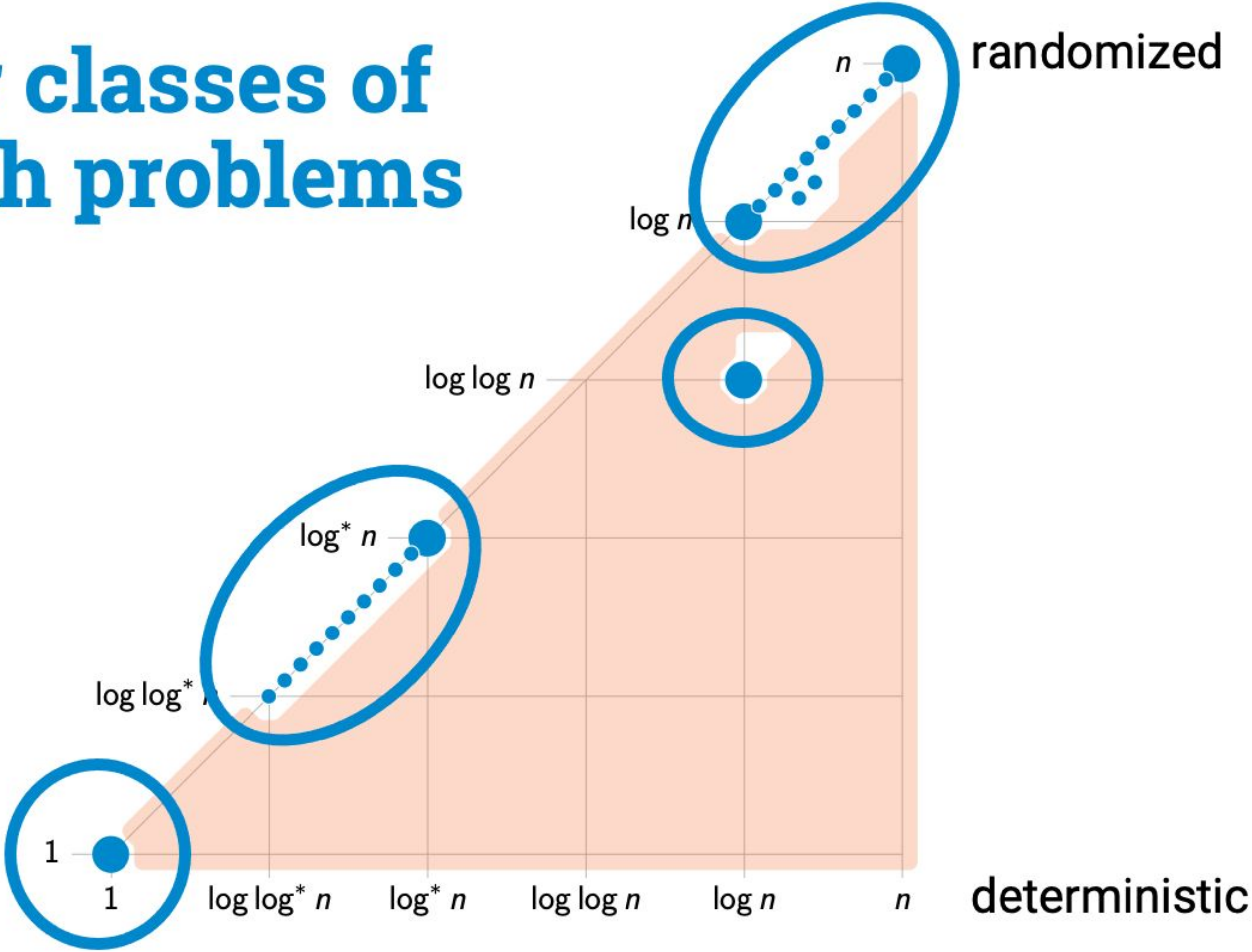
[Brandt et al. PODC 2017]
[Chang et al. SODA 2018]
[Balliu et al. Dist. Comp 2020]
[Chang DISC 2020]
[Brandt, Grunau, R 2021+]

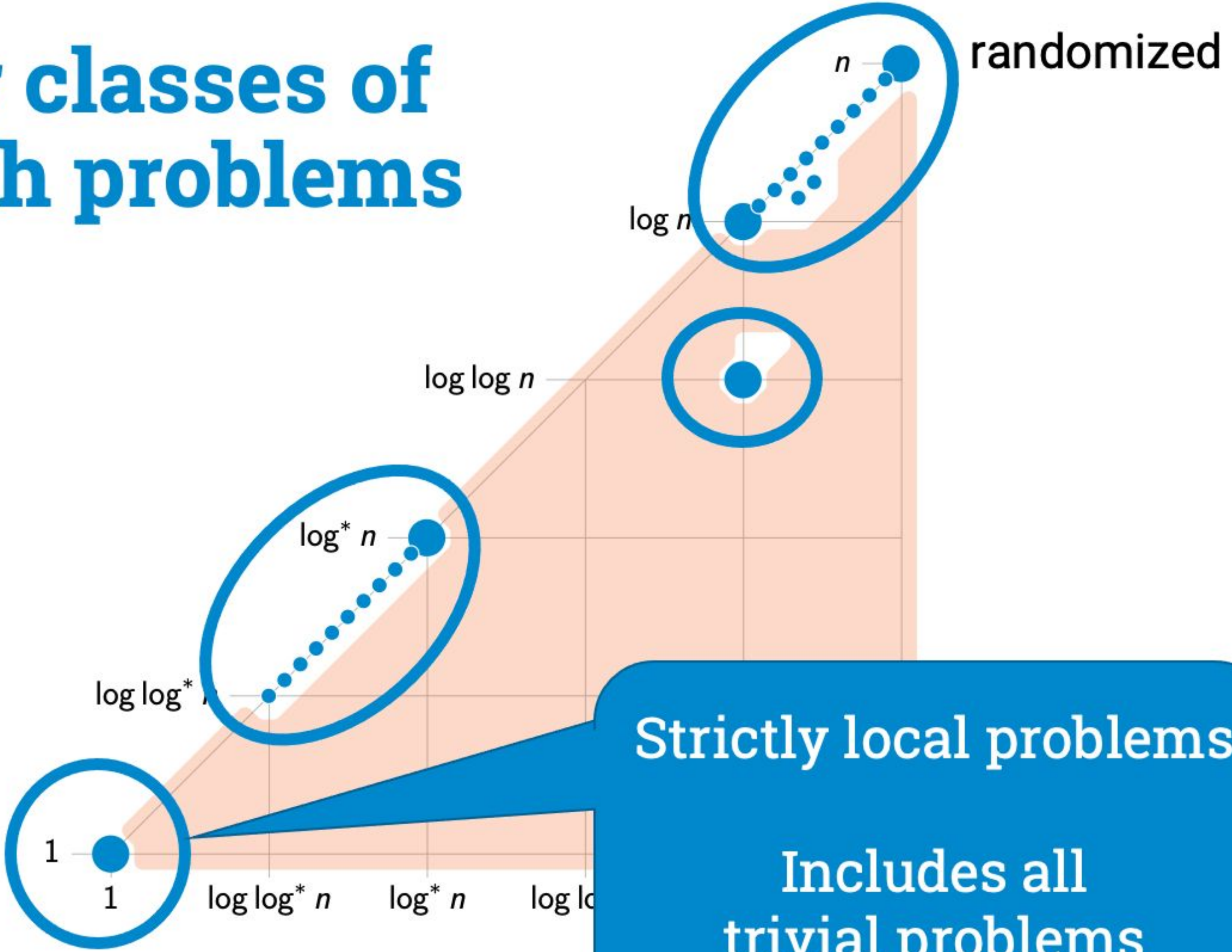
General (constant degree) graphs



Four classes of graph problems



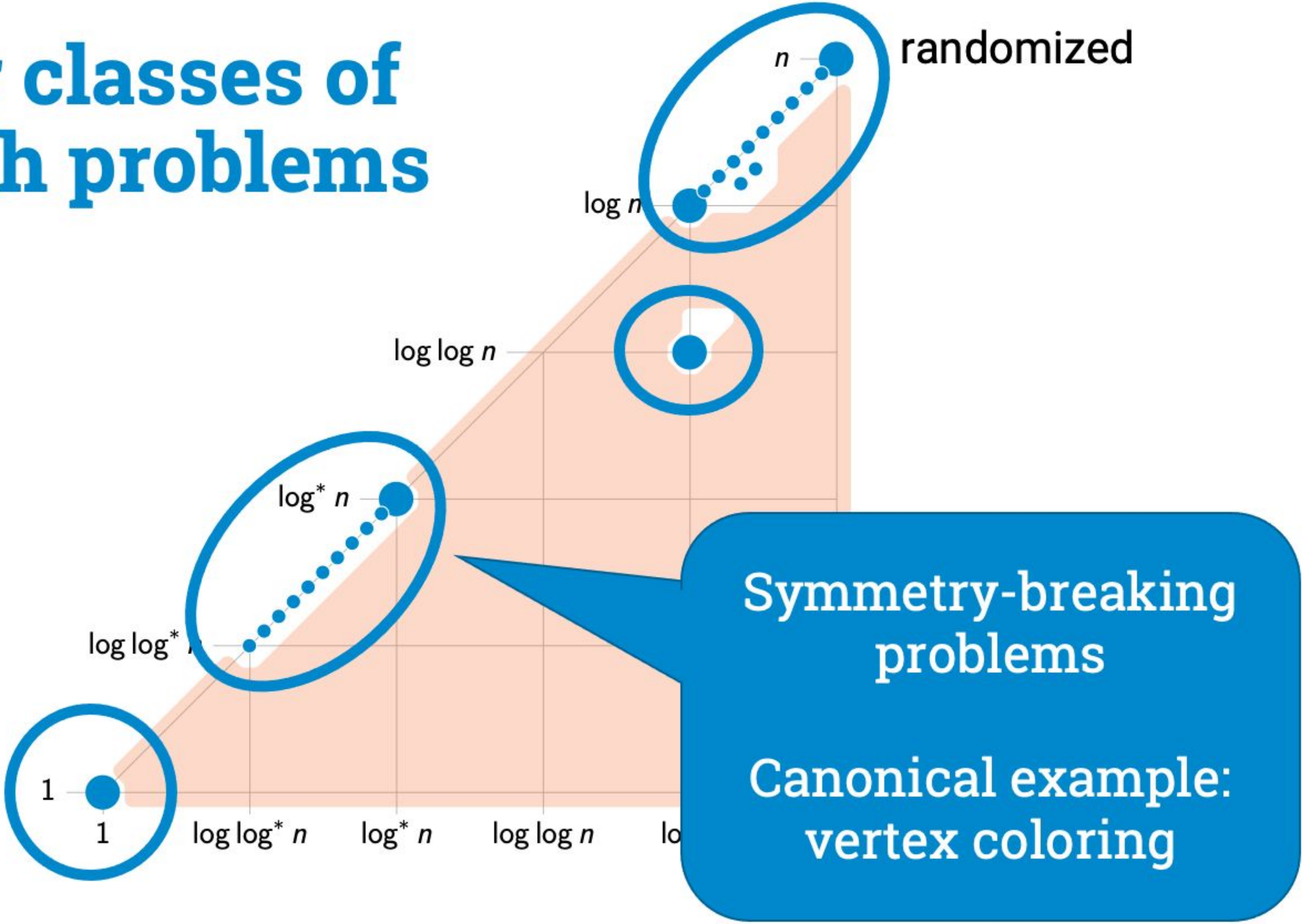
Four classes of graph problems



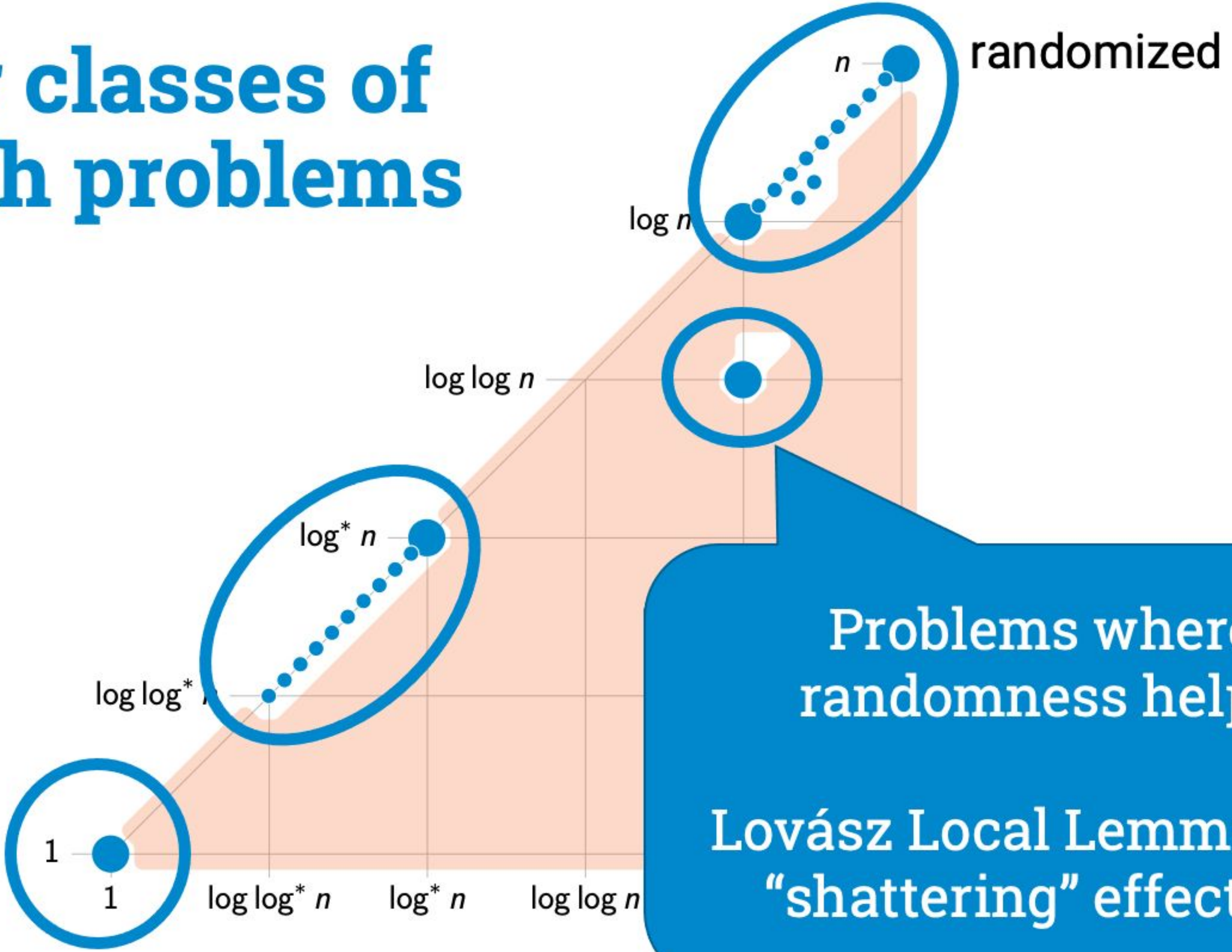
Strictly local problems

Includes all trivial problems

Four classes of graph problems



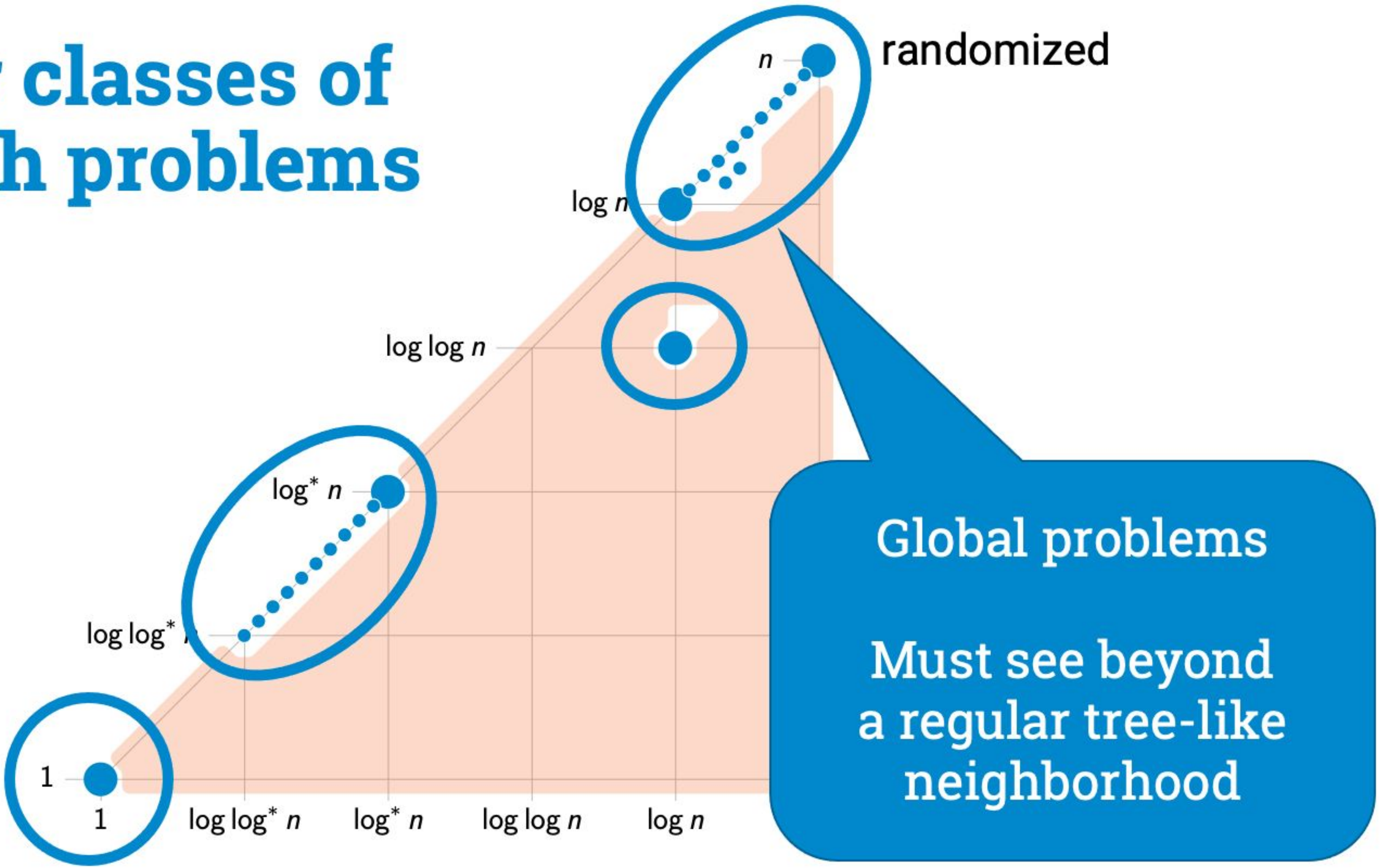
Four classes of graph problems



Problems where randomness helps

Lovász Local Lemma and "shattering" effective

Four classes of graph problems



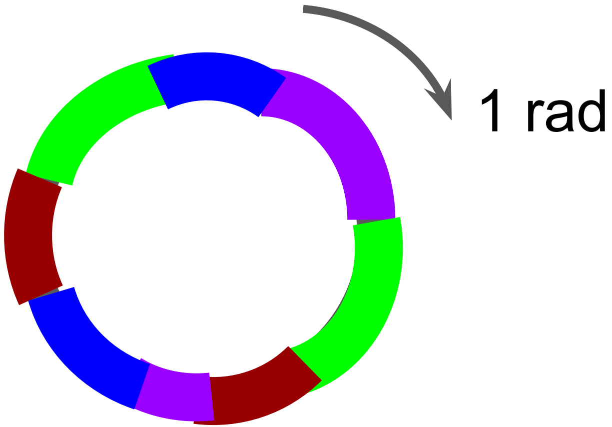
If these pictures are the only thing you remember from this talk, I will still be very happy. :)

Descriptive Combinatorics

A riddle: Color every point of a unit cycle such that

1) no two vertices 1 radian apart have the same color
2) vertices of the same color are finite union of intervals

How many colors do you need?



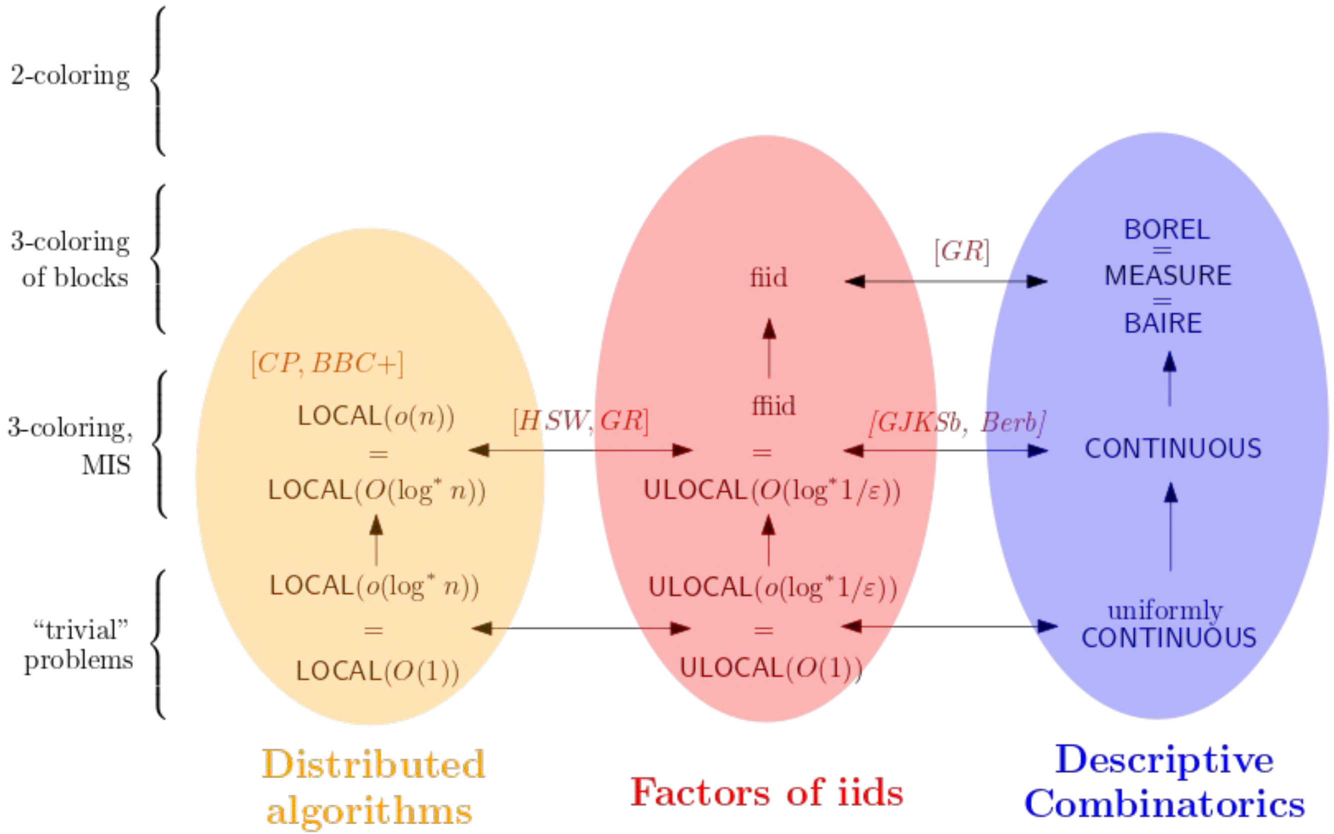
Bernshteyn's Insightful Paper

In general,

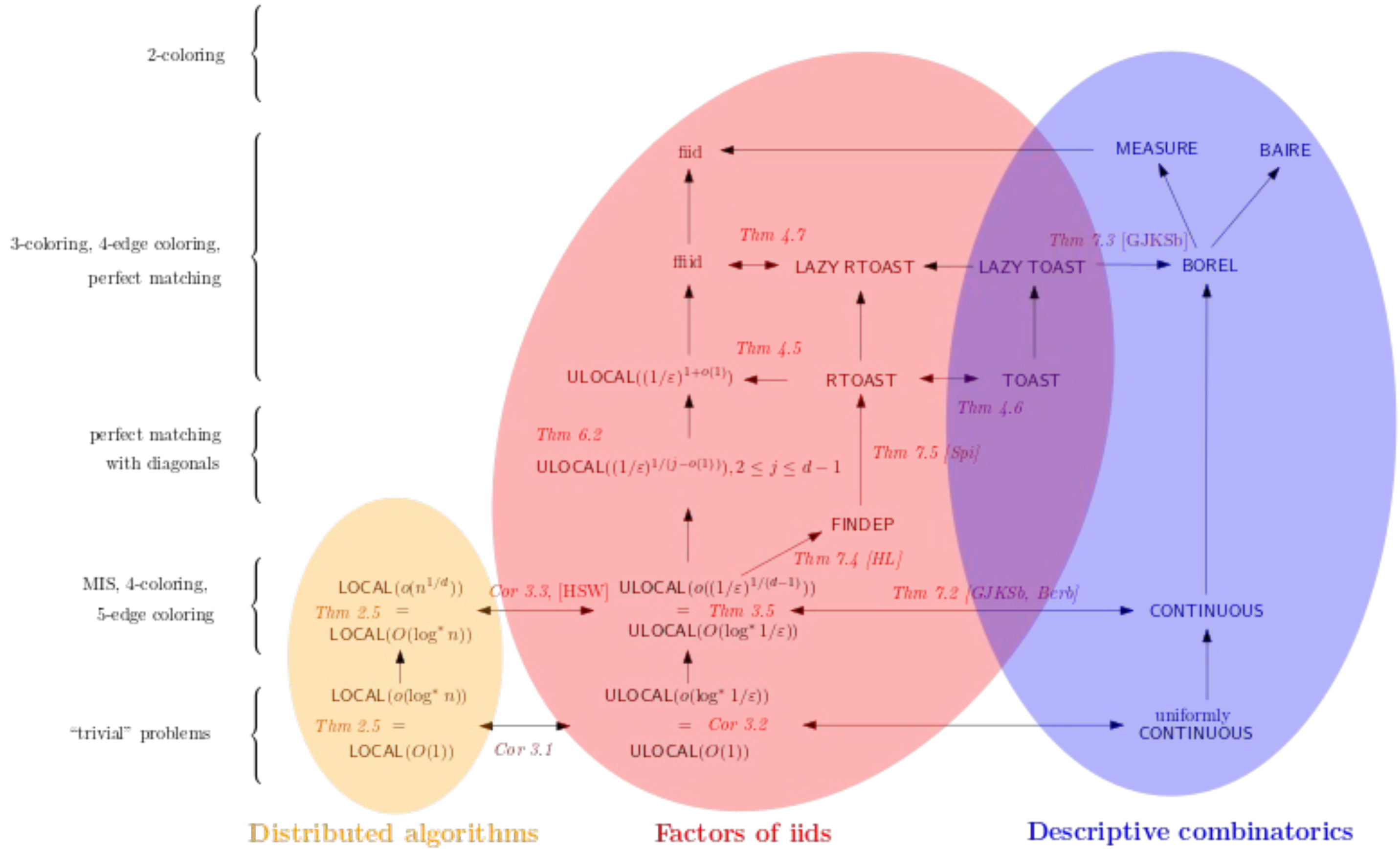
1. [Chang, Pettie] Any local problem solvable in $O(\log^* n)$ rounds can be solved by first finding a maximal independent set in $G^{O(1)}$ and then doing additional $O(1)$ distributed steps.
2. [Kechris, Solecki, Todorcevic] One can always find a maximal independent set in any Borel graph. By definition, you can do local constructions requiring $O(1)$ distributed steps.
3. [Bernshteyn] Hence, $\mathrm{LOCAL}(O(\log^* n)) \subseteq \mathrm{BOREL}$!

[Bernshteyn] For similar reasons, $\mathrm{RLOCAL}(\mathrm{poly}\ \mathrm{loglog}(n)) \subseteq \mathrm{MEASURE}$.

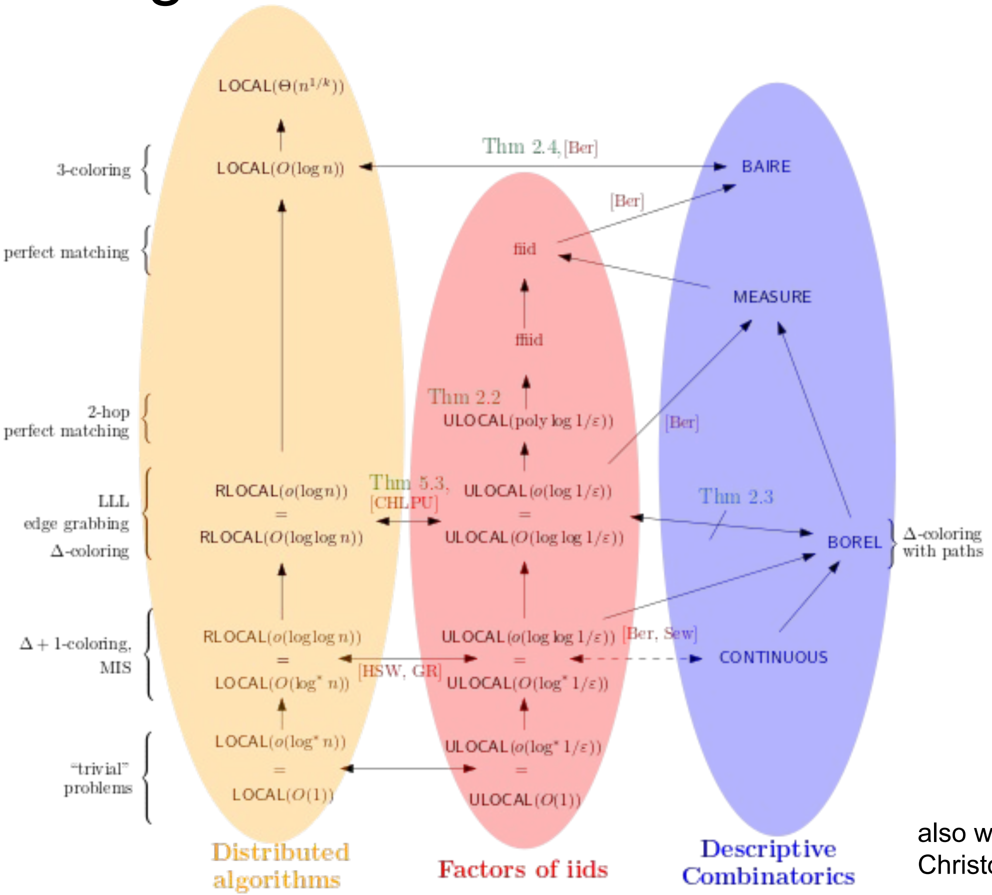
Big Picture for Oriented Paths with Inputs



Big Picture for Oriented Grids



Big Picture for Regular Trees



also with Sebastian Brandt, Yi-jun Chang, Christoph Grunau, and Zoltan Vidnyanszky

“Locality is a field. “

(paraphrasing Ronitt Rubinfeld)