RELATING THE CUT DISTANCE AND THE WEAK* TOPOLOGY FOR GRAPHONS

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ABSTRACT. The theory of graphons is ultimately connected with the so-called cut norm. In this paper, we approach the cut norm topology via the weak* topology. We show that compactness of the cut distance follows from compactness of the Vietoris topology of the hyperspace over functions equipped with the weak* topology.

From these proofs a new order on the space of graphons naturally emerges. This order allows to compare how structured two graphons are. We establish basic properties of this «structurdness order».

Last, we introduce several new graphon parameters that can be used to identify a cut distance limit of a sequence of graphons in the space of weak* limits.

1. INTRODUCTION

Graphons emerged from work of Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztergombi [17, 2] on limits of sequences of finite graphs. We write $\mathcal{W}$ for the space of all *graphons*, i.e., all symmetric measurable functions from Ω^2 to $[0,1]$ considered, after identifying graphons that are equal almost everywhere. Here as well as in the rest of the paper, Ω is an arbitrary separable atomless probability space with probability measure ν. While it is meaningful to investigate the space $\mathcal{W}$ with respect to several metrics and topologies, the two that relate the most to graph theory are the metrics $d_\square$ and $\delta_\square$ defined below. Given $U, W \in \mathcal{W}$ we set[a]

$$d_\square(U,W) := \sup_{S,T\subset\Omega} \left| \int_{S\times T} U - \int_{S\times T} W \right| \text{ , and}$$
$$\delta_\square(U,W) := \inf_{\phi} d_\square\left(U, W^\phi\right) \text{ ,}$$

where ϕ ranges over all measure-preserving bijections of Ω and the graphon W^ϕ is defined by

$$W^\phi(x,y) = W(\phi(x),\phi(y)) \text{ .} \tag{1.1}$$

We call $d_\square$ the *cut norm distance* and $\delta_\square$ the *cut distance*. We call graphons of the form W^ϕ *versions* of W. Graphons U and W are called *weakly isomorphic* if $\delta_\square(U,W) = 0$. Note that in this case there need not exist one measure-preserving bijection ϕ for which $d_\square\left(U, W^\phi\right) = 0$; see [16, Figure 7.1].

Recall that the key property of the space $\mathcal{W}$ is its compactness with respect to the cut distance $\delta_\square$. The result was first proven by Lovász and Szegedy [17] using the regularity lemma,[b] and

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[a] All the sets and functions below are tacitly assumed to be measurable.

[b] see also [18] and [19] for variants of this approach

then by Elek and Szegedy [9] using ultrafilter techniques, by Austin [1] and Diaconis and Janson [7] using the theory of exchangeable random graphs, and finally by Doležal and Hladký [8] by optimizing a suitable parameter over the set of weak* limits.

Theorem 1.1. *Every sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ of graphons there is a subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ and a graphon Γ such that $\delta_\square(\Gamma_{n_i}, \Gamma) \to 0$.*

Recall a sequence of graphons $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ *converges weak** to a graphon W if for every $S, T \subset \Omega$ we have

$$\int_{S\times T} \Gamma_n - \int_{S\times T} W \overset{n\to\infty}{\longrightarrow} 0 \ .$$

The weak* topology is weaker than the topology generated by $d_\square$, of which the former can be viewed as a certain uniformization. Indeed, recall that a sequence of graphons $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ converges to W in the cut norm if

$$\sup_{S,T\subset\Omega} \left\{ \int_{S\times T} \Gamma_n - \int_{S\times T} W \right\} \overset{n\to\infty}{\longrightarrow} 0 \ .$$

The main message of this paper is that we can make use of the weak* convergence to prove Theorem 1.1. To this end, we look at all weak* accumulation points of sequences

$$\left\{ \Gamma'_1, \Gamma'_2, \Gamma'_3, \ldots : \Gamma'_n \text{ is a version of } \Gamma_n \right\} \ .$$

This set is non-empty by the Banach–Alaoglu Theorem.[c] Now, in this set we cleverly select one graphon Γ. The selection is done so that in addition to being a weak* accumulation point, Γ is also an cut distance accumulation point. In [8], Doležal and Hladký carried out such a program when Γ was chosen as the maximizer of an arbitrary graphon parameter of the form

$$\mathrm{INT}_f(W) := \int_x \int_y f\left(W(x,y)\right) \tag{1.2}$$

where $f : [0,1] \to \mathbb{R}$ is a fixed but arbitrary continuous strictly convex function.[d]

In this paper we take a more abstract approach and show how to select Γ without use of any graphon parameter. In doing so, in Section 3 we build a clear link between the weak* convergence and the cut distance. Also, our approach naturally gives rise to a certain «structuredness order» on graphons which we study in Section 3.4. Last, returning to the original program from [8], in Section 4 we find two further ways — a spectral one and one using subgraph densities in the spirit of Sidorenko's conjecture — to identify Γ via an optimization problem.

2. Preliminaries

2.1. **Graphon basics.** Recall that $\mathcal{W}$ is the space of all graphons, that is, meaurable functions from Ω^2 to $[0,1]$, modulo differences on null-sets. The probability measure underlying Ω is ν. We write $\nu^{\otimes k}$ for the product measure on Ω^k.

We write P_k for a path on k vertices and C_k for a cycle on k vertices.

As usual, given a finite graph H and a graphon W, we write $t(H,W)$ for the density of H in W. We call the quantity $t(P_2, W) = \int_x \int_y W(x,y)$ the *edge density of* W. Recall also that for

[c]Let us emphasize that the Banach–Alaoglu Theorem is a elementary and follows from a countable version of Tychonoff's theorem. Also, let us note in applying the Banach–Alaoglu Theorem we rely on the fact that Ω is separable.

[d]Most of [8] deals with *minimizing* $\mathrm{INT}_f(W)$ for a fixed continuous strictly *concave* function. This is obviously equivalent.

$x \in \Omega$, we have the *degree of* x in W defined as $\deg_W(x) = \int_y W(x,y)$. Recall that measurability of W gives that $\deg_W(x)$ exists for almost each $x \in \Omega$. We say that W is *p-regular* if for almost every $x \in \Omega$, $\deg_W(x) = p$.

Next, we recall basic spectral theory for graphons, details and proofs can be found in [16, §7.5]. Given a graphon $W : \Omega^2 \to [0,1]$, we can associate to it an operator $T_W : L^2(\Omega) \to L^2(\Omega)$,

$$(T_W f)(x) := \int_y W(x,y) f(y) \ .$$

T_W is a Hilbert–Schmidt operator, and hence has a discrete spectrum $\{\lambda_1(W), \lambda_2(W), \lambda_3(W), \ldots\}$ of finitely or countably many non-zero eigenvalues (with possible multiplicities). All these eigenvalues are real, bounded in modulus by 1, and their only possible accumulation point is 0. Thus, we will assume that the eigenvalues are ordered as

$$|\lambda_1(W)| \geq |\lambda_2(W)| \geq |\lambda_3(W)| \geq \ldots \ . \tag{2.1}$$

If W has only finitely many eigenvalues, say k, we shall define $\lambda_{k+1}(W) = \lambda_{k+2}(W) = \ldots := 0$.

Recall also that the eigenspaces are orthogonal.

Last, in Section 4.3 we shall use the following formula connecting eigenvalues and cycle densities. For any graphon U with spectrum $\{\lambda_i\}_{i \in \mathbb{N}}$ and for any $k \geq 3$, we have by [16, eq. (7.22), (7.23)],

$$t(C_k, U) = \sum_i \lambda_i^k \ . \tag{2.2}$$

2.2. **Topologies on $\mathcal{W}$.** There are several natural topologies on $\mathcal{W}$. The topology given by $\|\cdot\|_\infty$ norm, $\|\cdot\|_1$ norm, $\|\cdot\|_\square$ norm and the weak* topology. $\mathcal{W}$ is closed in each of these topologies. Recall that by the Banach–Alaoglu theorem, $\mathcal{W}$ is closed. We write $d_1(\cdot,\cdot)$ for the distance derived from the $\|\cdot\|_1$ norm and similarly for the other norms. Recall that the weak* topology on $\mathcal{W}$ is metrizable. We shall denote by $d_{\mathrm{w}^*}(\cdot,\cdot)$ any metric compatible with this topology.

The following fact summarizes the relation of the above topologies.

Fact 2.1. *The following identity maps are continuous* $(\mathcal{W}, d_\infty) \to (\mathcal{W}, d_1) \to (\mathcal{W}, d_\square) \to (\mathcal{W}, d_{w^*})$.

2.3. **Approximating graphons in L^1.**

Definition 2.2. Suppose that $\Gamma : \Omega^2 \to [0,1]$ is a graphon. For a finite partition $\mathcal{P}$ partition of Ω, $\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \ldots \sqcup \Omega_k$, we define a graphon $\Gamma^{\Join\mathcal{P}}$ by setting it on the rectangle $\Omega_i \times \Omega_j$ to be the constant $\frac{1}{\nu^{\otimes 2}(\Omega_i \times \Omega_j)} \int_{\Omega_i} \int_{\Omega_j} \Gamma(x,y)$. We allow graphons to have not well-defined values on null sets which handles the cases $\nu(\Omega_i) = 0$ or $\nu(\Omega_j) = 0$.

Lemma 2.3. *For every graphon* $\Gamma : \Omega^2 \to [0,1]$ *and every* $\epsilon > 0$ *there exists a finite partition* $\mathcal{P}$ *of* Ω *such that* $\left\|\Gamma - \Gamma^{\Join\mathcal{P}}\right\|_1 < \epsilon$.

For the proof of Lemma 2.3, the following use fact will be useful.

Fact 2.4. *Suppose that* $f \in L^1(\Lambda)$ *is an arbitrary function on a finite measure space* Λ *with measure* λ. *Set* $a := \frac{1}{\lambda(\Lambda)} \cdot \int_\Lambda f$. *Then for each* $b \in \mathbb{R}$ *we have that* $\|f - a\|_1 \leq 2\|f - b\|_1$.

Proof. We have

$$\begin{aligned}\|f-a\|_1 &= \int_\Lambda |f(x)-a| \leq \int_\Lambda |f(x)-b| + \int_\Lambda |a-b| = \|f-b\|_1 + \lambda(\Lambda)\cdot|a-b| \\ &= \|f-b\|_1 + \left|\int_\Lambda f(x)-b\right| \leq 2\,\|f-b\|_1 \ .\end{aligned}$$

□

Proof of Lemma 2.3. Since sets of the form $A \times B$, $A, B \subset \Omega$ generate the sigma-algebra on Ω^2, there exists a finite partition $\mathcal{P}$ of Ω and a function $S : \Omega^2 \to \mathbb{R}$ such that S is constant on each rectangle of $\mathcal{P}\times\mathcal{P}$, and such that $\|\Gamma - S\|_1 < \frac{\epsilon}{2}$. Now, for each rectangle $(A, B) \in \mathcal{P}\times\mathcal{P}$, we apply Fact 2.4 on the restricted function $\Gamma_{\restriction A\times B}$ and the constant $S_{\restriction A\times B}$. Summing up the contributions coming from these applications of Fact 2.4, we get that $\left\|\Gamma - \Gamma^{\Join\mathcal{P}}\right\|_1 \leq 2\,\|\Gamma - S\|_1 < \epsilon$. □

We call $\Gamma^{\Join\mathcal{P}}$ with properties as in Lemma 2.3 *averaged L^1-approximation of Γ by a step-graphon for precision ϵ.*

2.4. **Hyperspace $K(\mathcal{W})$.** Let X be a metrizable compact space. We denote as $K(X)$ the space of all compact subsets of X with the topology generated by sets of the forms$\{L \in K(X) : L \subset U\}$ and $\{L \in K(X) : L \cap U \neq \emptyset\}$ where $U \subseteq X$ ranges over all open sets of X. $K(X)$ is called the *hyperspace of X with the Vietoris topology.*

Fact 2.5 ((4.22) and (4.26) in [12])**.** *Let X be a metrizable compact space with compatible metric d. Then $K(X)$ is metrizable compact. Furthermore, the Hausdorff metric on $K(X)$,*

$$\rho(L, M) = \max\left\{\max_{x\in L}\{d(x, M)\}, \max_{y\in M}\{d(y, L)\}\right\}$$

is compatible with the topology on $K(X)$.

We will be interested in the situation where $X = \mathcal{W}$ is endowed with the weak* topology.

3. Weak* convergence and the cut distance

3.1. **Introducing $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$.** As outlined above, we will study the space of weak* limits or weak* accumulation points of a sequence of graphons. Let us give definitions necessary to this end. For a sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$ of graphons, we denote by $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ the set of all graphons W for which there exist versions $\Gamma_1', \Gamma_2', \Gamma_3', \ldots$ of $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ such that W is a weak* accumulation point of the sequence $\Gamma_1', \Gamma_2', \Gamma_3', \ldots$. We also denote by $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ the set of all graphons W for which there exist versions $\Gamma_1', \Gamma_2', \Gamma_3', \ldots$ of $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ such that W is a weak* limit of the sequence $\Gamma_1', \Gamma_2', \Gamma_3', \ldots$.

Let us observe some basic properties of the sets $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. We have $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) \subset \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. The set $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ can be empty (for example when $\Gamma_1 \equiv 0, \Gamma_2 \equiv 1, \Gamma_3 \equiv 0, \Gamma_4 \equiv 1, \ldots$) but $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ must be non-empty by the Banach–Alaoglu Theorem. Actually, we can describe some elements of $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ fairly easily. Let $T \subset [0, 1]$ be the set of the accumulation points of the edge densities of the graphons $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$, i.e., T is the set of the accumulation points of the sequence $\left(\int_x\int_y \Gamma_n(x, y)\right)_n$. Now, a constant c (viewed as a constant graphon) lies in $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ if and only if $c \in T$. The direction that if $c \in \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ then

$c \in T$ is obvious. Now, suppose that $c \in T$. That is, for some subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ the densities converge to c. Partition each Γ_{n_i} into i sets of measure $\frac{1}{i}$ and consider a version $\widehat{\Gamma_{n_i}}$ of Γ_{n_i} obtained by a measure-preserving bijection permuting these sets randomly. Then almost surely, $\widehat{\Gamma_{n_1}}, \widehat{\Gamma_{n_2}}, \widehat{\Gamma_{n_3}}, \ldots$ weak* converge to c. While this is included here just to get friends with $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and a proof is not needed at this point, we list it in Lemma 3.7(b) and refer to [8] for a proof.

The first non-trivial fact we will prove about the set $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ is that it is closed.

Lemma 3.1. *Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$ be a sequence of graphons. Then the following hold for the set $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$.*

(a) $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ *is weak* closed in $L^\infty\left(\Omega^2\right)$.*
(b) $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ *is weak* compact in $L^\infty\left(\Omega^2\right)$.*
(c) $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ *is closed $L^1\left(\Omega^2\right)$.*

Proof of Part (a). Suppose that $L_1, L_2, L_3, \ldots$ are elements of $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ such that $L_k \to L$ for $k \to \infty$. For every k let $\Gamma_1^k, \Gamma_2^k, \Gamma_3^k, \ldots$ be a sequence of versions of $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ converging to L_k. We find an increasing sequence $i_1, i_2, i_3, \ldots$ of integers such that for every k and for every $n \geq i_k$ we have $d_{w^*}(W_n^{k+1}, L_{k+1}) < \frac{1}{k}$. Then the following sequence of versions of $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ weak* converges to L:

$$\Gamma_1^1, \Gamma_2^1, \ldots, \Gamma_{i_1-1}^1, \Gamma_{i_1}^2, \Gamma_{i_1+1}^2, \ldots, \Gamma_{i_2-1}^2, \Gamma_{i_2}^3, \Gamma_{i_2+1}^3, \ldots, \Gamma_{i_3-1}^3, \ldots.$$

Proof of Part (b): Recall that the closed unit ball is compact in the weak* topology. Since $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ lies in this ball, it is weak* compact.

Proof of Part (c): The unit ball B of $L^\infty\left(\Omega^2\right)$ is trivially closed in $L^1\left(\Omega^2\right)$. $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ is a closed subset of B. So, $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ is closed in $L^1\left(\Omega^2\right)$. $\square$

Remark 3.2. *Note that $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ need not be closed in $L^1\left(\Omega^2\right)$ was observed by Jon Noel and his example is given in Section 7.4 of* [8]. *Actually, Noel's example shows that the set $\left\{\mathrm{INT}_f(W) : W \in \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)\right\}$ need not even achieve its supremum (here, f is a fixed continuous strictly convex function).*

3.2. **Differentiating between $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ in** [8] **and in the present paper.** In the proof of Theorem 1.1 given in [8], which is in some sense a precursor of the current work, quite some work is put into zigzagging between $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. Let us explain this in more detail. Let us fix a continuous strictly convex function f. The idea for finding the graphon Γ in Theorem 1.1 in [8] is as follows. Denoting by X either *(i)* $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ or *(ii)* $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$, we take $\Gamma \in X$ that maximizes $\mathrm{INT}_f(\Gamma)$. Using the definition of X, there exist versions $\Gamma'_{n_1}, \Gamma'_{n_2}, \Gamma'_{n_3}, \ldots$ of $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ that converge to Γ weak*.[e] The aim is to prove that $\Gamma'_{n_1}, \Gamma'_{n_2}, \Gamma'_{n_3}, \ldots$ actually converge to Γ also in the cut norm — that would obviously prove Theorem 1.1. Now, the key step in [8] is to prove that if $\Gamma'_{n_1}, \Gamma'_{n_2}, \Gamma'_{n_3}, \ldots$ do not converge to Γ in the cut norm, then there exists versions $\Gamma''_{n_{k_1}}, \Gamma''_{n_{k_2}}, \Gamma''_{n_{k_3}}, \ldots$ of a suitable subsequence of $\Gamma'_{n_1}, \Gamma'_{n_2}, \Gamma'_{n_3}, \ldots$ that weak* converge to a graphon Γ' with $\mathrm{INT}_f(\Gamma') > \mathrm{INT}_f(\Gamma)$. Since $\Gamma''_{n_{k_1}}, \Gamma''_{n_{k_2}}, \Gamma''_{n_{k_3}}, \ldots$ witnesses that $\Gamma' \in X$, this is a contradiction. Now, let us explain why we need favourable properties of both (i) and (ii) for the proof. Firstly, note that in the sentence «Since $\Gamma''_{n_{k_1}}, \Gamma''_{n_{k_2}}, \Gamma''_{n_{k_3}}, \ldots$ witnesses

[e] Note that in variant (i), we actually have $n_1 = 1, n_2 = 2, n_3 = 3, \ldots$.

that $\Gamma' \in X$» we are referring to a subsequence, so this is a correct justification only in case $X = \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. On the other hand, in the sentence «we take $\Gamma \in X$ that maximizes $\mathrm{INT}_f(\Gamma)$» we need the maximum to be achieved. Such a closeness property is enjoyed by $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ as we saw in Lemma 3.1, but not by $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ as we saw in Remark 3.2.

So, while differences between $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ were viewed in [8] as a nuisance that required a subtle and technical treatment, in this section we shall show that these differences capture the essence of cut norm convergence. Namely, we shall prove in Theorem 3.3 that each sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ of graphons contains a subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ such that

$$\mathbf{LIM}_{w*}\left(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots\right) = \mathbf{ACC}_{w*}\left(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots\right) \ , \tag{3.1}$$

and in Theorem 3.5 we shall prove that (3.1) is equivalent to cut distance convergence of $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$. Of course, a proof of Theorem 1.1 then follows immediately. We consider this the main result of the paper.

3.3. Main results: subsequences with $\mathbf{LIM}_{w*} = \mathbf{ACC}_{w*}$.

As we observed earlier we have $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) \subset \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and equality usually does not hold. The next theorem however says that we can always achieve equality after passing to a subsequence.

Theorem 3.3. *Let $\mathcal{S} = (\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ be a sequence of graphons. Then there exists a subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ such that $\mathbf{LIM}_{w*}\left(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots\right) = \mathbf{ACC}_{w*}\left(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots\right)$.*

Proof. For sequences $\mathcal{U}$ and $\mathcal{T}$ we write $\mathcal{U} \leq^\star \mathcal{T}$ if deleting finitely many terms from $\mathcal{U}$ gives us a subsequence of $\mathcal{T}$. Note that the relation $\leq^\star$ is transitive.

In the following, we construct a countable ordinal α_0 and a transfinite sequence $(\mathcal{S}_\alpha)_{\alpha \leq \alpha_0}$ of subsequences of $\mathcal{S}$ such that for every pair of ordinals $\alpha < \beta$ it holds that $S_\beta \leq^\star \mathcal{S}_\alpha$ (which trivially implies that $\mathbf{LIM}_{w*}\left(\mathcal{S}_\alpha\right) \subseteq \mathbf{LIM}_{w*}\left(\mathcal{S}_\beta\right)$) and also that $\mathbf{LIM}_{w*}\left(\mathcal{S}_\alpha\right)$ is a proper subset of $\mathbf{LIM}_{w*}\left(\mathcal{S}_\beta\right)$.

In the first step, we put $\mathcal{S}_0 = \mathcal{S}$. Now suppose that for some countable ordinal α, we have already constructed $\mathcal{S}_\beta$ for every $\beta < \alpha$. Either $\alpha = \beta + 1$ for some ordinal β or α is a limit ordinal. Suppose first that $\alpha = \beta + 1$ for some ordinal β. We distinguish two cases. If $\mathbf{LIM}_{w*}(\mathcal{S}_\beta) = \mathbf{ACC}_{w*}(\mathcal{S}_\beta)$ then we define $\alpha_0 = \beta$ and the construction is finished. Otherwise there is some graphon $W \in \mathbf{ACC}_{w*}(\mathcal{S}_\beta) \setminus \mathbf{LIM}_{w*}(\mathcal{S}_\beta)$. Then we proceed the construction by finding a subsequence $\mathcal{S}_{\beta+1}$ of $\mathcal{S}_\beta$ such that some versions of the graphons from $S_{\beta+1}$ converge to W (so that $S_{\beta+1} \leq^\star \mathcal{S}_\beta$ and $W \in \mathbf{LIM}_{w*}(\mathcal{S}_{\beta+1}) \setminus \mathbf{LIM}_{w*}(\mathcal{S}_\beta)$). Now suppose that α is a countable limit ordinal. We find an increasing sequence $\beta_1, \beta_2, \beta_3, \ldots$ of ordinals such that $\beta_i \to \alpha$ for $i \to \infty$ (this is possible as $\mathrm{cof}(\alpha) = \omega$). Now we use the diagonal method to define the sequence S_α such that $S_\alpha \leq^\star S_{\beta_i}$ for every i. Then we clearly have that $S_\alpha \leq^\star S_\beta$ for every $\beta < \alpha$, and so $\bigcup_{\beta<\alpha} \mathbf{LIM}_{w*}(\mathcal{S}_\beta) \subseteq \mathbf{LIM}_{w*}(\mathcal{S}_\alpha)$.

The obtained transfinite sequence $(\mathbf{LIM}_{w*}(\mathcal{S}_\alpha))_{\alpha \leq \alpha_0}$ is a strictly increasing sequence of closed (by Lemma 3.1) subsets of $(B_{L^\infty(\Omega^2)}, w^*)$. Therefore the sequence is at most countable, i.e. the previous construction stopped at some countable ordinal α_0, see [12, Theorem 6.9]. This means that $\mathbf{LIM}_{w*}\left(\mathcal{S}_{\alpha_0}\right) = \mathbf{ACC}_{w*}\left(\mathcal{S}_{\alpha_0}\right)$. $\square$

Remark 3.4. *Theorem 3.3 substantially extends the key Lemma 13 from* [8] *which states that any sequence of graphons* $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ *contains a subsequence* $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ *such that*

(3.2)

$$\sup\left\{\mathrm{INT}_f(\Gamma) \;:\; \Gamma \in \mathbf{LIM}_{\mathrm{w}*}(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots)\right\} = \sup\left\{\mathrm{INT}_f(\Gamma) \;:\; \Gamma \in \mathbf{ACC}_{\mathrm{w}*}(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots)\right\},$$

for a continuous strictly convex function $f : [0,1] \to \mathbb{R}$.

As promised, we shall now state that the property asserted in Theorem 3.3 is necessary and sufficient for cut distance convergence.

Theorem 3.5. *Let* $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$. *The following are equivalent:*

(a) The sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ *is Cauchy with respect to the cut distance* $\delta_\square$,

(b) $\mathbf{LIM}_{\mathrm{w}*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) = \mathbf{ACC}_{\mathrm{w}*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$.

Furthermore, in case (a) and (b) hold, we can take a maximal element W *in* $\mathbf{LIM}_{\mathrm{w}*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ *with respect to the structurdness order (defined in Section 3.4 below) and then* $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \xrightarrow{\delta_\square} W$.

We provide a proof of Theorem 3.5 in Section 3.5. In Section 3.6 we state and prove Theorem 3.24 which extends Theorem 3.5 and relates cut distance convergence to convergence in the hyperspace $K(\mathcal{W})$.

3.4. **Envelopes and the structuredness order.** Suppose that $W \in \mathcal{W}$ is a graphon. We call the set $\langle W \rangle := \mathbf{LIM}_{\mathrm{w}*}(W, W, W, \ldots)$ the *envelope* of W. Note that $\langle W \rangle \in K(\mathcal{W})$ by Lemma *3.1(b)*. We prove in this section that the map $\langle \cdot \rangle : \mathcal{W} \to K(\mathcal{W})$ is in fact a homeomorphism between $\mathcal{W}$ with the cut distance and some closed subspace of $K(\mathcal{W})$ with the Vietoris topology. Envelopes allow us to introduce structuredness order on graphons. Intuitively, less-structured graphons have smaller envelopes. Extreme examples of this are constant graphons $W \equiv c$, which are obviously the only graphons for which $\langle W \rangle = \{W\}$. This leads us to say that a graphon U is *at most as structured as a graphon* W if $\langle U \rangle \subset \langle W \rangle$. We write $U \preceq W$ in this case. Observe that $\preceq$ is a quasiorder on the space of graphons. As we shall see in Lemma 3.26, it is actually an order on the space of graphons modulo weak isomorphism. In order to prove these results we shall need several auxiliary results.

Lemma 3.6 (Lemma 7 in [8])**.** *Suppose that* $\Gamma_1, \Gamma_2, \Gamma_3, \ldots : \Omega^2 \to [0,1]$ *is a sequence of graphons. Suppose that* $W \in \mathbf{LIM}_{\mathrm{w}*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ *and that we have a partition* $\mathcal{P}$ *of* Ω *into finitely many sets. Then* $W^{\Join \mathcal{P}} \in \mathbf{LIM}_{\mathrm{w}*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$.

Lemma 3.7. *Suppose that* $W \in \mathcal{W}$. *Then*

(a) If $Q \subseteq \langle W \rangle$ then the weak* closure of Q also lies in $\langle W \rangle$,

(b) $W^{\Join \mathcal{P}} \in \langle W \rangle$ (in other words $W^{\Join \mathcal{P}} \preceq W$),

(c) $U \in \langle W \rangle$ if and only if $U \preceq W$,

(d) if $\delta_\square(W, U) = 0$ then $\langle W \rangle = \langle U \rangle$.

Proof. Item (a) follows from Lemma 3.1(a). Item (b) is a special case of Lemma 3.6.

Let us now turn to Item (c). If $U \preceq W$ then $U \in \langle W \rangle$ follows from the definition of $\preceq$ and the fact that $U \in \langle U \rangle$. To prove the opposite implication observe that if $(\varphi_n : \Omega \to \Omega)_n$ is a sequence of measure preserving bijections, $W^{\varphi_n} \xrightarrow{\mathrm{w}^*} U$ and $\psi : \Omega \to \Omega$ is a measure preserving bijection then $W^{\psi\varphi_n} \xrightarrow{\mathrm{w}^*} U^\psi$. Then we have that every version of U is in $\langle W \rangle$. Because $\langle U \rangle$ is exactly the weak* closure of the set of all versions of U we obtain that $U \preceq W$.

If $\delta_\square(W,U) = 0$ then we have a sequence $W^{\varphi_n} \xrightarrow{\|\cdot\|_\square} U$ which by Fact 2.1 implies that $W^{\varphi_n} \xrightarrow{w^*} U$. Therefore $U \preceq W$. A symmetric argument gives $W \preceq U$ and we may conclude that $\langle W \rangle = \langle U \rangle$. This gives Item (d). $\square$

Lemma 3.8. *Let $(W^{\varphi_n})_{n\in\mathbb{N}}$ be a sequence of versions of a graphon $W \in \mathcal{W}$. Then the following three properties are equivalent:*

- $W^{\varphi_n} \xrightarrow{w^*} W$,
- $W^{\varphi_n} \xrightarrow{\|\cdot\|_\square} W$,
- $W^{\varphi_n} \xrightarrow{\|\cdot\|_1} W$.

Proof. By Lemma 2.1 it is enough to show that $W^{\varphi_n} \xrightarrow{w^*} W$ implies $W^{\varphi_n} \xrightarrow{\|\cdot\|_1} W$. Let $W^{\varphi_n} \xrightarrow{w^*} W$. The following claim is crucial.

Claim. Let $B \subseteq [0,1]$ be a Borel set. Then $\nu^{\otimes 2}\left((W^{\varphi_n})^{-1}(B) \triangle (W)^{-1}(B)\right) \to 0$.

Proof of Claim. First we prove the claim for every interval of the form $[a,1]$ where $a \in [0,1]$. Let $A_a = W^{-1}([a,1])$ and $A_a^n = (W^{\varphi_n})^{-1}([a,1])$. Note that $\nu^{\otimes 2}(A_a) = \nu^{\otimes 2}(A_a^n)$ and therefore it is enough to show that $\nu^{\otimes 2}(A_a \setminus A_a^n) \to 0$. Fix some strictly increasing sequence $a_k \to a$ such that $a_0 = 0$ and define $D_{a,k}^n = (W^{\varphi_n})^{-1}([a_k, a_{k+1}))$.

Observe that φ_n naturally acts on Ω^2, so let us denote $\theta_n(x,y) = (\varphi_n(x), \varphi_n(y))$. Consider sets $C_n = \theta_n^{-1}(A_a \cap A_a^n)$ and $C_{n,k} = \theta_n^{-1}\left(A_a \cap D_{a,k}^n\right)$. Note that $C_n \subseteq A_a$ and $\nu^{\otimes 2}(A_a \setminus A_a^n) = \sum_{k\in\mathbb{N}} \nu^{\otimes 2}(C_{n,k})$. We have

$$\begin{aligned}\int_{A_a} W^{\varphi_n} &= \int_{C_n} W + \sum_{k\in\mathbb{N}} \int_{C_{n,k}} W \\ &\leq \int_{C_n} W + \sum_{k\in\mathbb{N}} a_{k+1}\nu^{\otimes 2}(C_{n,k}).\end{aligned}$$

By the weak* convergence we have

$$\begin{aligned}\left|\int_{A_a} W^{\varphi_n} - \int_{A_a} W\right| &= \left|\int_{C_n} W + \sum_{k\in\mathbb{N}} \int_{C_{n,k}} W - \int_{C_n} W - \int_{A_a\setminus C_n} W\right| \\ &= \left|\sum_{k\in\mathbb{N}} \int_{C_{n,k}} W - \int_{I_a\setminus C_n} W\right| \\ &\geq \sum_{k\in\mathbb{N}} (a - a_{k+1})\nu^{\otimes 2}(C_{n,k}) \\ &\to 0.\end{aligned}$$

Having this we may conclude the claim because for every fixed $k \in \mathbb{N}$ we must have $\nu^{\otimes 2}(C_{n,k}) \to 0$ but since $\nu^{\otimes 2}(C_{n,k}) \leq \nu^{\otimes 2}\left(D_{a,k}^n\right)$ (and this value is independent of n) and $\sum_{k\in\mathbb{N}} \nu^{\otimes 2}\left(D_{a,k}^n\right) < \infty$ we may conclude that $\lim_{n\to\infty} \sum_{k\in\mathbb{N}} \nu^{\otimes 2}(C_{n,k}) = 0$.

Denote as $\mathcal{B}$ the family of those sets Borel sets $B \subseteq [0,1]$ such that $\nu^{\otimes 2}\left((W^{\varphi_n})^{-1}(B) \triangle (W)^{-1}(B)\right) \to 0$. We show that $\mathcal{B}$ is a σ-algebra.

(a) If $B \in \mathcal{B}$ then clearly $\Omega \setminus B \in \mathcal{B}$.

(b) Let $B_0, B_1 \in \mathcal{B}$. We have

$$\nu^{\otimes 2}\left(\left(W^{\varphi_n}\right)^{-1}(B_0 \cup B_1) \triangle (W)^{-1}(B_0 \cup B_1)\right) \leq \nu^{\otimes 2}\left(\left(W^{\varphi_n}\right)^{-1}(B_0) \triangle (W)^{-1}(B_0)\right) + \nu^{\otimes 2}\left(\left(W^{\varphi_n}\right)^{-1}(B_1) \triangle (W)^{-1}(B_1)\right).$$

We conclude that $B_0 \cup B_1 \in \mathcal{B}$.

(c) Let $B_1, B_2, ...$ be sequence of disjoint sets such that $\bigcup_{i\leq m} B_i \in \mathcal{B}$ and $B = \bigcup_{i\in\mathbb{N}} B_i$. We clearly have $\sum_{i=m}^{\infty} \nu^{\otimes 2}(B_i) \to 0$ as $m \to \infty$ and therefore

$$\begin{aligned}\nu^{\otimes 2}\left(\left(W^{\varphi_n}\right)^{-1}(B) \triangle (W)^{-1}(B)\right) &\leq \nu^{\otimes 2}\left(\left(W^{\varphi_n}\right)^{-1}\Big(\bigcup_{i\leq m} B_i\Big) \triangle (W)^{-1}\Big(\bigcup_{i\leq m} B_i\Big)\right) + 2\sum_{i=m+1}^{\infty} \nu^{\otimes 2}(B_i) \\ &\to 2\sum_{i=m+1}^{\infty} \nu^{\otimes 2}(B_i)\ .\end{aligned}$$

Therefore $B \in \mathcal{B}$.

This finishes the proof because the σ-algebra generated by the intervals of the form $[a,1]$ is the σ-algebra of all Borel subsets. □

Fix $m \in \mathbb{N}$. Let us define the sequence $I_1, I_2, ..., I_{m-1}, I_m$ where $I_i = \left[\frac{i-1}{m}, \frac{i+1}{m}\right)$ for $i < m$ and $I_m = \left[\frac{m-1}{m}, 1\right]$. Define also $J_i = W^{-1}(I_i)$ and $J_i^n = \left(W^{\varphi_n}\right)^{-1}(I_i)$. Note that $\Omega^2 = J_1 \sqcup ... \sqcup J_m$. We have

$$\begin{aligned}\int_\Omega |W^{\varphi_n} - W| &= \sum_{i=1}^m \int_{J_i} |W^{\varphi_n} - W| \\ &\leq \sum_{i=1}^m \nu^{\otimes 2}(J_i \triangle J_i^n) + \sum_{i\leq m} \int_{J_i\cap J_i^n} |W^{\varphi_n} - W| \\ &\leq \sum_{i=1}^m \nu^{\otimes 2}(J_i \triangle J_i^n) + \sum_{i\leq m} \frac{1}{m}\nu^{\otimes 2}(J_i \cap J_i^n) \\ &\leq \sum_{i=1}^m \nu^{\otimes 2}(J_i \triangle J_i^n) + \frac{1}{m} \\ \boxed{\text{by Claim, } \textstyle\sum_{i=1}^m \nu^{\otimes 2}(J_i \triangle J_i^n) \to 0} \quad &\to \frac{1}{m}\ .\end{aligned}$$

We conclude that $W^{\varphi_n} \xrightarrow{\|\cdot\|_1} W$. □

Lemma 3.9. *Let $U \in \mathcal{W}$ and $\left(W^{\varphi_n}\right)_{n\in\mathbb{N}}$ be a sequence of versions of a graphon $W \in \mathcal{W}$ such that*

$$W^{\varphi_n} \xrightarrow{w^*} U \tag{3.3}$$

and $W \preceq U$. Then $U^{\varphi_n^{-1}} \xrightarrow{w^*} W$.

Proof. Since $W \preceq U$, we have a sequence $\left(U^{\psi_m}\right)_{m\in\mathbb{N}}$ of versions of U such that $U^{\psi_m} \xrightarrow{w^*} W$. Therefore, for each fixed n, we have

$$U^{\varphi_n\psi_m} \xrightarrow{w^*} W^{\varphi_n} \text{ as } m \to \infty\ . \tag{3.4}$$

Combining (3.3) and (3.4), we can find a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$U^{\varphi_n\psi_{f(n)}} \xrightarrow{w^*} U \tag{3.5}$$

and such that for each $A, B \subset \Omega$,

$$\left| \int_{\varphi_n(A)\times\varphi_n(B)} U^{\varphi_n \psi_{f(n)}} - W^{\varphi_n} \right| \to 0 \,. \tag{3.6}$$

Lemma 3.8 applied to (3.5) gives that

$$U^{\varphi_n \psi_{f(n)}} \xrightarrow{\|\cdot\|_\square} U \,. \tag{3.7}$$

Now, for each $A, B \subset \Omega$, and $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\left| \int_{A\times B} U^{\varphi_n^{-1}} - W \right| &= \left| \int_{\varphi_n(A)\times\varphi_n(B)} U - W^{\varphi_n} \right| \\ &\leq \left| \int_{\varphi_n(A)\times\varphi_n(B)} U - U^{\varphi_n\psi_m} \right| + \left| \int_{\varphi_n(A)\times\varphi_n(B)} U^{\varphi_n\psi_m} - W^{\varphi_n} \right| .\end{aligned}$$

In particular,

$$\begin{aligned}\left| \int_{A\times B} U^{\varphi_n^{-1}} - W \right| &\leq \left| \int_{\varphi_n(A)\times\varphi_n(B)} U - U^{\varphi_n\psi_m} \right| + \left| \int_{\varphi_n(A)\times\varphi_n(B)} U^{\varphi_n\psi_m} - W^{\varphi_n} \right| \\ &\leq \left\| U - U^{\varphi_n\psi_{f(n)}} \right\|_\square + \left| \int_{\varphi_n(A)\times\varphi_n(B)} U^{\varphi_n\psi_{f(n)}} - W^{\varphi_n} \right| \\ \boxed{\text{by (3.7) and (3.6)}} \quad &\to 0 \,.\end{aligned}$$

We conclude that $U^{\varphi_n^{-1}} \xrightarrow{w^*} W$. □

Lemma 3.10. *Suppose that $(W_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_\square$-Cauchy sequence of graphons which weak* converges to a graphon Γ. Then $(W_n)_{n\in\mathbb{N}}$ converges to Γ also in the cut norm distance.*

Proof. Suppose for a contradiction that $W_n \overset{\|\cdot\|_\square}{\not\longrightarrow} \Gamma$. Then there exists an $\epsilon > 0$ and an infinite set $I \subset \mathbb{N}$ for each $i \in I$, $\|W_i - \Gamma\|_\square > \epsilon$. That means that for each $i \in I$ there is a set $B_i \subset \Omega$ so that $\left| \int_{B_i\times B_i} W_i - \Gamma \right| > \epsilon$.

Using that $(W_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\square$-Cauchy, we can find a number N_0 so that for each $k, \ell \geq N_0$, $\|W_k - W_\ell\|_\square \leq \frac{\epsilon}{2}$.

Let us fix an arbitrary $i \in I \cap [N_0, \infty)$. Then for an arbitrary $m \geq N_0$ we have

$$\begin{aligned}\left| \int_{B_i\times B_i} W_m - \Gamma \right| &\geq \left| \int_{B_i\times B_i} W_i - \Gamma \right| - \left| \int_{B_i\times B_i} W_m - W_i \right| \\ &\geq \epsilon - \|W_m - W_i\|_\square \geq \frac{\epsilon}{2} \,.\end{aligned}$$

In other words, the set $B_i \times B_i$ witnesses that $W_n \overset{w^*}{\not\longrightarrow} \Gamma$, a contradiction. □

Lemma 3.11. *Let $U, W \in \mathcal{W}$. Then $\langle W \rangle = \langle U \rangle$ if and only if $\delta_\square(U, W) = 0$.*

Proof. One implication is Lemma 3.7(d). For the other implication, assume that $\langle W \rangle = \langle U \rangle$ and consider a sequence $(W^{\varphi_n})_{n\in\mathbb{N}}$ such that

$$W^{\varphi_n} \xrightarrow{w^*} U \,. \tag{3.8}$$

We know from Lemma 3.9 that

$$U^{\varphi_n^{-1}} \xrightarrow{\text{w}^*} W \,. \tag{3.9}$$

Combining (3.8), (3.9) and a simple consideration about metric spaces, we get that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every function $g : \mathbb{N} \to \mathbb{N}$ with $g \geq f$ we have $U^{\varphi_n \varphi_{g(n)}^{-1}} \xrightarrow{\text{w}^*} U$. We claim that for every $\epsilon > 0$ there is $n(\epsilon) \in \mathbb{N}$ such that

$$\sup_{m \geq f(n(\epsilon))} \left\{ \left\| U - U^{\varphi_{n(\epsilon)} \varphi_m^{-1}} \right\|_\square \right\} < \epsilon.$$

Suppose that such $n \in \mathbb{N}$ does not exist. Then we can construct a function $g \geq f$ such that $U^{\varphi_n \varphi_{g(n)}^{-1}} \xrightarrow{\text{w}^*} U$ but $\left\| U - U^{\varphi_n \varphi_{g(n)}^{-1}} \right\|_\square \geq \epsilon$. This is a contradiction with Lemma 3.8.

Given now this $n(\epsilon) \in \mathbb{N}$ and the fact that every measure-preserving bijection acts by isometry on $\mathcal{W}$ with the $\|\cdot\|_\square$-norm we have

$$\sup_{m,k \geq f(n)} \left\{ \left\| U^{\varphi_m^{-1}} - U^{\varphi_k^{-1}} \right\|_\square \right\} < 2\epsilon$$

and therefore $\left\{ U^{\varphi_m^{-1}} \right\}$ is $\|\cdot\|_\square$-Cauchy. Plugging in (3.9) into Lemma 3.10, its limit must be W. $\square$

Lemma 3.12. *Suppose $U_1, U_2, U_3, \ldots$ is a sequence of graphons that converges weak* to U, and suppose that W is a graphon. Suppose that for each $n \in \mathbb{N}$ we have that $U_n \preceq W$. Then $U \preceq W$.*

Proof. We need to prove that for each $\epsilon > 0$ there exists a version W^π of U so that $d_{\text{w}*}(W^\pi, U) < \epsilon$. Let n be such that we have $d_{\text{w}*}(U_n, U) < \frac{\epsilon}{2}$. Since $U_n \preceq W$, there exists a version W^π such $d_{\text{w}*}(W^\pi, U_n) < \frac{\epsilon}{2}$. Then we have

$$d_{\text{w}*}(W^\pi, U) \leq d_{\text{w}*}(U_n, U) + d_{\text{w}*}(W^\pi, U_n) < \epsilon \,,$$

as was needed. $\square$

Definition 3.13. By an *ordered partition* $\mathcal{P}$ of a set S, we mean a finite partition of S, $S = P_1 \sqcup P_2 \sqcup P_3 \sqcup \ldots \sqcup P_k$, $\mathcal{P} = (P_1, P_2, P_3, \ldots, P_k)$ in which the sets $P_1, P_2, P_3, \ldots, P_k$ are linearly ordered (in the way they are enumerated in $\mathcal{P}$).

Definition 3.14. In this definition, we will assume that the probability space Ω is the unit interval $I = [0,1]$. For an ordered partition $\mathcal{J}$ of I into finitely many sets $C_1, C_2, \ldots, C_k$, we define mappings $\alpha_{\mathcal{J},1}, \alpha_{\mathcal{J},2}, \ldots, \alpha_{\mathcal{J},3} : I \to I$, and a mapping $\gamma_{\mathcal{J}} : I \to I$ by

$$\begin{aligned}
\alpha_{\mathcal{J},1}(x) &= \int_0^x \mathbf{1}_{C_1}(y)\,\mathrm{d}(y) \,, \\
\alpha_{\mathcal{J},2}(x) &= \alpha_{\mathcal{J},1}(1) + \int_0^x \mathbf{1}_{C_2}(y)\,\mathrm{d}(y) \,, \\
&\vdots \\
\alpha_{\mathcal{J},k}(x) &= \alpha_{\mathcal{J},1}(1) + \alpha_{\mathcal{J},2}(1) + \ldots + \alpha_{\mathcal{J},k-1}(1) + \int_0^x \mathbf{1}_{C_k}(y)\,\mathrm{d}(y) \,, \\
\gamma_{\mathcal{J}}(x) &= \alpha_{\mathcal{J},i}(x) \quad \text{if } x \in C_i, \quad i = 1, 2, \ldots, k \,.
\end{aligned} \tag{3.10}$$

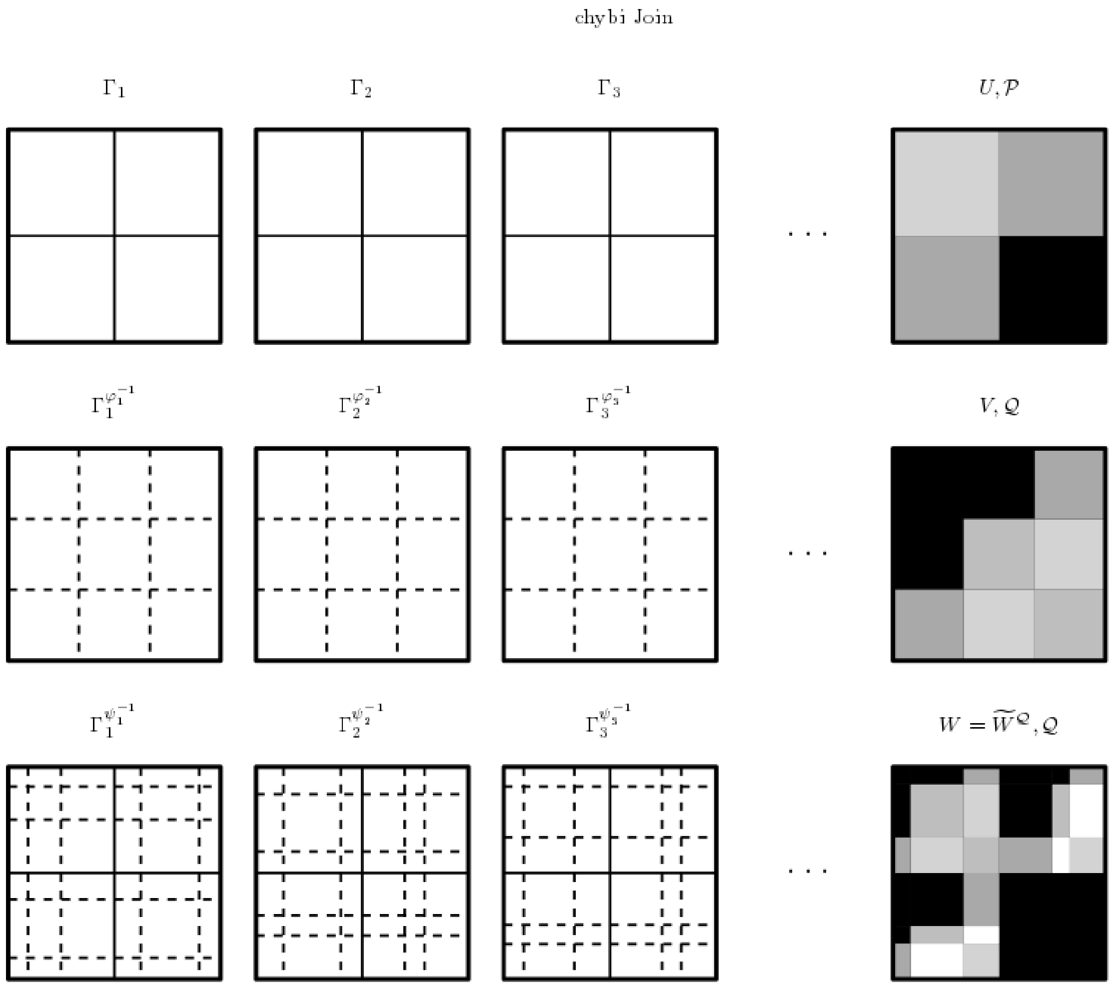

FIGURE 3.1. Two graphons U, V are stepgraphons with partitions $\mathcal{P}$ and $\mathcal{Q}$. The sequence $\Gamma_1^{\psi_1^{-1}}, \Gamma_2^{\psi_2^{-1}}, \ldots$ converges to $\widetilde{W}$ and the corresponding partitions converge to the partition $\mathcal{R}$ that refines both $\mathcal{P}$ and $\mathcal{Q}$. The graphon $W = \widetilde{W}^{\Join\mathcal{R}}$ is the desired stepgraphon that is more structured than both U and V.

Informally, $\gamma_{\mathcal{J}}$ is defined in such a way that it maps the set C_1 to the left side of the interval I, the set C_2 next to it, and so on. Finally, the set C_k is mapped to the right side of the interval I. Clearly, $\gamma_{\mathcal{J}}$ is a measure preserving almost-bijection.

Lemma 3.15. *Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$ be a sequence of graphons for which $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) \neq \emptyset$. Suppose that $U, V \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ are two step graphons. Then there exists a step graphon $W \in \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \ldots)$ that refines U and such that $U, V \preceq W$.*

Proof. We at first assume that the ordered partition $\mathcal{P} = (P_1, \ldots, P_m)$ of U and the ordered partition $\mathcal{Q} = (Q_1, \ldots, Q_n)$ of V are composed of intervals, since each partition $\mathcal{J}$ can be reordered to intervals by the measure preserving almost-bijection $\gamma_{\mathcal{J}}$. We further assume without loss of generality that the sequence of measure preserving bijections certifying that

$U \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \dots)$ contains only identities, i.e., that $\Gamma_1, \Gamma_2, \dots \xrightarrow{w^*} U$. Let $\varphi_1, \varphi_2, \dots$ be measure preserving bijections such that $\Gamma_1^{\varphi_1^{-1}}, \Gamma_2^{\varphi_2^{-1}}, \dots \xrightarrow{w^*} V$.

We now describe a sequence of measure preserving bijections $\psi_1, \psi_2, \dots$ such that the weak star limit of $\Gamma_1^{\psi_1^{-1}}, \Gamma_2^{\psi_2^{-1}} \dots$ gives the desired graphon W. For the ℓ-th graphon Γ_ℓ we define its partition $\mathcal{H}^{(\ell)} = \left(H_{1,1}^{(\ell)}, H_{1,2}^{(\ell)}, \dots, H_{1,n_1}^{(\ell)}, \dots, H_{n_1,n_2}^{(\ell)}\right)$ where $H_{i,j}^{(\ell)} = P_i \cap \varphi_\ell^{-1}(Q_j)$ and set $\psi_\ell = \gamma_{\mathcal{H}_\ell}$ using Definition 3.14. The intuition behind ψ_ℓ is that it refines each block P_i of the partition $\mathcal{P}$ with the partition $\mathcal{Q}$ – it can be, indeed, seen that for each i we have $\psi_\ell(P_i) = \psi_\ell\left(\bigcup_j H_{i,j}^{(\ell)}\right) = P_i$, where the equalities hold up to a null set.

We pass to a subsequence $n_1 \cdot n_2$ times to get that both enpoints of intervals $\psi_\ell(H_{i,j}^{(\ell)})$ converge to some fixed numbers from $[0,1]$, thus giving us a limit partition $\mathcal{R} = (R_{1,1}, \dots, R_{m,n})$ into intervals. Note that as we know that for each i we have $\psi_\ell\left(\bigcup_j H_{i,j}^{(\ell)}\right) = P_i$, it is also true that $\bigcup_j R_{i,j} = P_i$. We also have for all j that $\nu\left(\bigcup_i H_{i,j}^{(\ell)}\right) = \nu(Q_j)$, hence $\nu\left(\bigcup_i R_{i,j}\right) = \nu(Q_j)$. Now we use the fact that the set of accumulation points of our sequence is non-empty due to Banach-Alaoglu theorem, thus after passing to a subsequence yet again we get a subsequence $\Gamma_{k_1}^{\psi_{k_1}^{-1}}, \Gamma_{k_2}^{\psi_{k_2}^{-1}}, \dots \xrightarrow{w^*} \widetilde{W}$. Define W as $\widetilde{W}^{\ltimes\mathcal{R}}$. We apply 3.1 (a) to $\mathbf{LIM}_{w*}(\Gamma_{k_1}, \Gamma_{k_2}, \dots)$ and 3.7 (b) to $\widetilde{W}$ and $\widetilde{W}^{\ltimes\mathcal{R}}$ to get that that $\widetilde{W}^{\ltimes\mathcal{R}} \in \mathbf{ACC}_{w*}$.

We claim that $\widetilde{W}^{\ltimes\mathcal{R}}$ refines U. It suffices to prove that for any $\varepsilon > 0$ and any step $P_i \times P_j$ we have

$$\left| \int_{P_i \times P_j} U - \int_{P_i \times P_j} \widetilde{W}^{\ltimes\mathcal{R}} \right| < \varepsilon.$$

Take ℓ sufficiently large, so that

$$\left| \int_{P_i \times P_j} U - \int_{P_i \times P_j} \Gamma_\ell \right| < \frac{\varepsilon}{2}$$

and

$$\left| \int_{P_i \times P_j} \widetilde{W} - \int_{P_i \times P_j} \Gamma_\ell^{\psi_\ell^{-1}} \right| < \frac{\varepsilon}{2}.$$

Putting this together with the facts that $P_i = \psi(P_i)$ up to a null set and $R_{i,j} \subseteq P_i$ for all i, j, we get that

$$\int_{P_i \times P_j} U = \int_{P_i \times P_j} \Gamma_\ell \pm \frac{\varepsilon}{2} = \int_{\psi^{-1}(P_i) \times \psi^{-1}(P_j)} \Gamma_\ell \pm \frac{\varepsilon}{2} = \int_{P_i \times P_j} \Gamma_\ell^{\psi_\ell^{-1}} \pm \frac{\varepsilon}{2} = \int_{P_i \times P_j} \widetilde{W} \pm \varepsilon = \int_{P_i \times P_j} \widetilde{W}^{\ltimes\mathcal{R}} \pm \varepsilon.$$

This gives that $\widetilde{W}^{\ltimes\mathcal{R}}$ refines U.

Now we prove that $\widetilde{W}^{\ltimes\mathcal{R}} \succeq V$. It suffices to prove that for any $\varepsilon > 0$ and any step $Q_i \times Q_j$ of V we have

$$\left| \int_{Q_i \times Q_j} V - \sum_{\substack{1 \le g \le m \\ 1 \le h \le n}} \int_{R_{g,i} \times R_{h,j}} \widetilde{W}^{\ltimes\mathcal{R}} \right| < \varepsilon.$$

Take ℓ sufficiently large so that

$$\left|\int_{Q_i \times Q_j} V - \int_{Q_i \times Q_j} \Gamma_\ell^{\varphi_\ell^{-1}}\right| < \frac{\varepsilon}{3},$$
$$\sum_{g,h}\left|\int_{R_{g,i} \times R_{h,j}} \widetilde{W}^{\ltimes \mathcal{R}} - \int_{R_{g,i} \times R_{h,j}} \Gamma_\ell^{\psi_\ell^{-1}}\right| < \frac{\varepsilon}{3}$$

and, moreover, the length of each interval $\psi\left(H^{\ell}_{i,j}\right)$ differs from the length of interval $R_{i,j}$ by at most $\frac{\varepsilon}{12m^2n^2}$. Now we can bound the measure of overlap of each pair of rectangles $R_{g,i} \times R_{h,j}$ and $\psi\left(H^{(\ell)}_{g,i}\right) \times \psi\left(H^{(\ell)}_{h,j}\right)$; specifically we have

$$\nu^{\otimes 2}\left(R_{g,i} \times R_{h,j} \triangle \psi\left(H^{(\ell)}_{g,i}\right) \times \psi\left(H^{(\ell)}_{h,j}\right)\right) < 4mn \cdot \frac{\varepsilon}{12m^2n^2} = \frac{\varepsilon}{3mn},$$

where $4mn$ comes from the facts that we bound the displacement of all four sides of the rectangles and that their displacement depends on the displacement of all preceding intervals. Putting all of this together, we get that

$$\begin{aligned}
\sum_{g,h}\int_{R_{g,i} \times R_{h,j}} \widetilde{W}^{\ltimes \mathcal{R}} &= \sum_{g,h}\int_{R_{g,i} \times R_{h,j}} \Gamma_\ell^{\psi_\ell^{-1}} \pm \frac{\varepsilon}{3} \\
&= \sum_{g,h}\int_{\psi\left(H^{(\ell)}_{g,i}\right) \times \psi\left(H^{(\ell)}_{h,j}\right)} \Gamma_\ell^{\psi_\ell^{-1}} \pm \frac{2\varepsilon}{3} \\
&= \sum_{g,h}\int_{\left(P_g \cap \varphi_\ell^{-1}(Q_i)\right) \times \left(P_h \cap \varphi_\ell^{-1}(Q_j)\right)} \Gamma_\ell \pm \frac{2\varepsilon}{3} \\
&= \sum_{g,h}\int_{\left(\varphi_\ell(P_g) \cap Q_i\right) \times \left(\varphi_\ell(P_h) \cap Q_j\right)} \Gamma_\ell^{\varphi_\ell^{-1}} \pm \frac{2\varepsilon}{3} \\
&= \int_{Q_i \times Q_j} \Gamma_\ell^{\varphi_\ell^{-1}} \pm \frac{2\varepsilon}{3} \\
&= \int_{Q_i \times Q_j} V \pm \varepsilon\,.
\end{aligned}$$

□

Lemma 3.16. *Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$ be a sequence of graphons for which $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) \neq \emptyset$. Then $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ contains a maximum element with respect to the structuredness order.*

Proof. The space $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ is separable metrizable since the space $\mathcal{W}$ with the weak* topology is separable metrizable and therefore we may find a countable set $P \subseteq \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ such that its weak* closure is $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. For each $W \in P$ and $k \in \mathbb{N}$, consider a suitable graphon, denoted by $W(k)$, that is a averaged L^1-approximation of W by a step-graphon for precision $\frac{1}{k}$. Such a graphon $W(k)$ exists by Lemma 2.3. Note than $W(k) \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ by Lemma 3.6.

Let us now consider then set $Q := \{W(k) : W \in P, k \in \mathbb{N}\}$. Then the set Q is countable, contained in $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and its weak* closure is $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. Let $U_1, U_2, U_3, \ldots$ be an enumeration of the elements of Q. Let $M_1 := U_1$. Having defined a graphon M_n, let M_{n+1} be given by Lemma 3.15 for graphons M_n and U_{n+1}. Observe that the graphons $M_1, M_2, M_3, \ldots$

form a sequence for refining graphons. In particular, there is only one weak* accumulation point of this sequence (guaranteed to exist by the Banach–Alaoglu Theorem), say M, and that accumulation point is also a weak* limit. By Lemma 3.12,

$$M \succeq U_n \tag{3.11}$$

for each $n \in \mathbb{N}$.

We claim that M is a maximum element of $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. Indeed, let $\Gamma \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ be arbitrary. Since Q is weak* dense in $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$, we can find a sequence $U_{n_1}, U_{n_2}, U_{n_3}, \ldots$ weak* converging to Γ. Recall that by (3.11), $U_{n_i} \preceq M$ for each $i \in \mathbb{N}$. Lemma (3.12) now gives us that $\Gamma \preceq M$, as was needed. □

Corollary 3.17. *Let $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) = \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and $W \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ be a maximal element of $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ guaranteed to exist by Lemma 3.16. Then $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) = \langle W \rangle$. Moreover if $U \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ is another maximal element then $\delta_\square(U, W) = 0$.*

3.4.1. *Values and degrees with respect to the structurdness order.* Given a graphon $W : \Omega^2 \to [0,1]$, we can defined a pushforward probability measure on $[0,1]$ by

$$\Phi_W(A) := \nu^{\otimes 2}\left(W^{-1}(A)\right) , \tag{3.12}$$

for a set $A \subset [0,1]$. The measure Φ_W gives us the distribution of the values of W. Similarly, we can take the pushforward measure of the degrees,

$$\Upsilon_W(A) := \nu\left(\deg_W^{-1}(A)\right) , \tag{3.13}$$

for a set $A \subset [0,1]$. The measure Φ_W and Υ_W do not characterize W in general but certainly give us substantial information about W. Therefore, given two graphons $U \preceq W$ it is natural to ask how Φ_U compares to Φ_W and how Υ_U compares to Υ_W. To this end, we introduce the following concept.

Definition 3.18. Suppose that Λ_1 and Λ_2 are two probability measures on $[0,1]$. We say that Λ_1 is at *least as flat* as Λ_2 if there exists a probability measure Ψ on $[0,1]^2$ such that Λ_1 is the marginal of Ψ on the first coordinate, Λ_2 is the marginal of Ψ on the second coordinate, and for each $D \subset [0,1]$ we have

$$\int_{(x,y)\in D\times[0,1]} x \,\mathrm{d}\Psi = \int_{(x,y)\in D\times[0,1]} y \,\mathrm{d}\Psi . \tag{3.14}$$

In addition, we say that Λ_1 is *strictly flatter* than Λ_2 if $\Lambda_1 \neq \Lambda_2$.

The next proposition answers the original problem using the above concept of flatter measures.

Proposition 3.19. *Suppose that we have two graphons $U \preceq W$. Then the measure Φ_U is at least as flat as the measure Φ_W. Similarly, the measure Υ_U is at least as flat as the measure Υ_W. Lastly, if $U \prec W$ then Φ_U is strictly flatter than Φ_W.*

Let us remark that we cannot conclude that Υ_U is strictly flatter than Υ_W if $U \prec W$. To this end, it is enough to take U constant-p graphon (for some $p \in (0,1)$) and W some p-regular but non-constant graph. Then Υ_U and Υ_W are both equal to the Dirac measure on p.

In the proof of Proposition 3.19, we will need some basic facts about the weak* convergence of probability measures on $[0,1]^2$ (this convergence is also often called weak convergence or narrow convergence in the literature) which we recall here. We say that a sequence of probability measures $\Psi_1, \Psi_2, \Psi_3, \dots$ on $[0,1]^2$ converges in the weak* topology to a probability measure Ψ if for every continuous real function f defined on $[0,1]^2$ we have

$$\lim_{n\to\infty} \int_{[0,1]^2} f(x,y) d\Psi_n(x,y) = \int_{[0,1]^2} f(x,y) d\Psi(x,y).$$

This definition has many equivalent reformulations but we will need only the following one: A sequence $\Psi_1, \Psi_2, \Psi_3, \dots$ converges to Ψ if and only if $\lim_{n\to\infty} \Psi_n(A) = (A)$ for every Borel subset A of $[0,1]^2$ which satisfies that the Ψ-measure of its boundary is 0. Recall also that every sequence of probability measures has a weak* convergent subsequence.

Proof of Proposition 3.19. Let $W^{\pi_1}, W^{\pi_2}, W^{\pi_3}, \dots$ be a sequence of versions of W that converges to U. For every natural number n and every measurable subset A of $[0,1]^2$ we define

$$\Psi_n(A) = \nu^{\otimes 2}(\{(x,y) \in [0,1]^2 \colon (U(x,y), W^{\pi_n}(x,y)) \in A\}).$$

Clearly every Ψ_n is σ-additive and so it is a probability measure. Let Ψ be some weak* accumulation point of the sequence $\Psi_1, \Psi_2, \Psi_3, \dots$. Without loss of generality, we may assume that the sequence $\Psi_1, \Psi_2, \Psi_3, \dots$ converges to Ψ. Let Z be the set consisting of all $z \in [0,1]$ for which either $\Psi(\{z\} \times [0,1]) > 0$ or $\Psi([0,1] \times \{z\}) > 0$. Then Z is at most countable. So if $\mathcal{I}$ is the system of all intervals $I \subset [0,1]$ whose endpoints do not belong to Z then $\mathcal{I}$ generates the σ-algebra of all Borel subsets of $[0,1]$. Note also that whenever $I \in \mathcal{I}$ then the boundaries of both sets $I \times [0,1]$ and $[0,1] \times I$ are of Ψ-measure 0. Denote by Ψ^x, Ψ^y the marginals of Ψ on the first and on the second coordinate, respectively. Then for every $I \in \mathcal{I}$ we have

$$\begin{aligned}
&\Psi^x(I) = \Psi(I \times [0,1]) = \lim_{n\to\infty} \Psi_n(I \times [0,1]) \\
&= \lim_{n\to\infty} \nu^{\otimes 2}(\{(x,y) \in [0,1]^2 \colon (U(x,y), W^{\pi_n}(x,y)) \in I \times [0,1]\}) \\
&= \lim_{n\to\infty} \Phi_U(I) = \Phi_U(I).
\end{aligned}$$

As this is true for every $I \in \mathcal{I}$, it clearly follows that $\Psi^x = \Phi_U$. On the other hand, for every $I \in \mathcal{I}$ we have (using the obvious fact that $\Phi_{W^{\pi_n}} = \Phi_W$ for every n in the last equality) that

$$\begin{aligned}
&\Psi^y(I) = \Psi([0,1] \times I) = \lim_{n\to\infty} \Psi_n([0,1] \times I) \\
&= \lim_{n\to\infty} \nu^{\otimes 2}(\{(x,y) \in [0,1]^2 \colon (U(x,y), W^{\pi_n}(x,y)) \in [0,1] \times I\}) \\
&= \lim_{n\to\infty} \Phi_{W^{\pi_n}}(I) = \Phi_W(I),
\end{aligned}$$

so again $\Psi^y = \Phi_W$. To finish the proof of our first assertion it remains to show that

$$\int_{(x,y)\in B\times[0,1]} x d\Psi(x,y) = \int_{(x,y)\in B\times[0,1]} y d\Psi(x,y) \tag{3.15}$$

for every Borel measurable subset B of $[0,1]$. As the system $\mathcal{I}$ is closed under finite intersections and generates the Borel σ-algebra on $[0,1]$, it is enough to show (3.15) for every $B \in \mathcal{I}$. So fix $B \in \mathcal{I}$, $\varepsilon > 0$ and find some partition $\{I_1, I_2, \dots, I_n\}$ of the interval $[0,1]$ into intervals from $\mathcal{I}$ of lengths smaller than ε. By additivity of integration, we may also assume that the interval B is of length smaller than ε. Fix some points $x_0 \in B$ and $y_j \in I_j$ (for every j). Then we have (in the following, the approximation $\overset{\varepsilon}{\approx}$ is up to ε) that

$$\begin{aligned}
&\int_{(x,y)\in B\times[0,1]} yd\Psi(x,y) = \sum_{j=1}^{n}\int_{(x,y)\in B\times I_j} yd\Psi(x,y) \overset{\varepsilon}{\approx} \sum_{j=1}^{n} y_j\Psi(B\times I_j) = \sum_{j=1}^{n} y_j \lim_{n\to\infty}\Psi_n(B\times I_j)\\
&= \sum_{j=1}^{n} y_j \lim_{n\to\infty} \nu^{\otimes 2}(\{(x,y)\in[0,1]^2\colon (U(x,y),W^{\pi_n}(x,y))\in B\times I_j\})\\
&= \sum_{j=1}^{n}\lim_{n\to\infty}\int_{(x,y)\in U^{-1}(B)\cap(W^{\pi_n})^{-1}(I_j)} y_j d\nu^{\otimes 2}(x,y)\\
&\overset{\varepsilon}{\approx} \sum_{j=1}^{n}\lim_{n\to\infty}\int_{(x,y)\in U^{-1}(B)\cap(W^{\pi_n})^{-1}(I_j)} W^{\pi_n}(x,y)d\nu^{\otimes 2}(x,y)\\
&= \lim_{n\to\infty}\int_{(x,y)\in U^{-1}(B)} W^{\pi_n}(x,y)d\nu^{\otimes 2}(x,y) = \int_{(x,y)\in U^{-1}(B)} U(x,y)d\nu^{\otimes 2}(x,y)\\
&\overset{\varepsilon}{\approx} \int_{(x,y)\in U^{-1}(B)} x_0 d\nu^{\otimes 2}(x,y) = x_0\nu^{\otimes 2}(U^{-1}(B)) = x_0\Phi_U(B) = x_0\Psi^x(B)\\
&= \int_{(x,y)\in B\times[0,1]} x_0 d\Psi(x,y) \overset{\varepsilon}{\approx} \int_{(x,y)\in B\times[0,1]} xd\Psi(x,y).
\end{aligned}$$

As this is true for every $\epsilon > 0$, this is what we needed.

Now we show that Y_U is at least as flat as Y_W, this is very similar to the previous part of the proof. For every natural number n and every measurable subset A of $[0,1]^2$ we define

$$\tilde{\Psi}_n(A) = \nu(\{x\in[0,1]\colon (\deg_U(x),\deg_{W^{\pi_n}}(x))\in A\}),$$

then every $\tilde{\Psi}_n$ is a probability measure. Let $\tilde{\Psi}$ be some weak* accumulation point of the sequence $\tilde{\Psi}_1,\tilde{\Psi}_2,\tilde{\Psi}_3,\ldots$, and assume without loss of generality that the sequence $\tilde{\Psi}_1,\tilde{\Psi}_2,\tilde{\Psi}_3,\ldots$ converges to $\tilde{\Psi}$. Let $\tilde{Z}$ be the set consisting of all $z\in[0,1]$ for which either $\tilde{\Psi}(\{z\}\times[0,1])>0$ or $\tilde{\Psi}([0,1]\times\{z\})>0$, and let $\tilde{\mathcal{I}}$ be the system of all intervals $I\subset[0,1]$ whose endpoints do not belong to $\tilde{Z}$. Denote by $\tilde{\Psi}^x,\tilde{\Psi}^y$ the marginals of $\tilde{\Psi}$ on the first and on the second coordinate, respectively. Then for every $I\in\tilde{\mathcal{I}}$ we have

$$\begin{aligned}
&\tilde{\Psi}^x(I) = \tilde{\Psi}(I\times[0,1]) = \lim_{n\to\infty}\tilde{\Psi}_n(I\times[0,1])\\
&= \lim_{n\to\infty}\nu(\{x\in[0,1]\colon (\deg_U(x),\deg_{W^{\pi_n}}(x))\in I\times[0,1]\})\\
&= \lim_{n\to\infty} Y_U(I) = Y_U(I),
\end{aligned}$$

and so $\tilde{\Psi}^x = Y_U$. On the other hand, for every $I\in\tilde{\mathcal{I}}$ we have (using the fact that $Y_{W^{\pi_n}} = Y_W$ for every n) that

$$\begin{aligned}
&\tilde{\Psi}^y(I) = \tilde{\Psi}([0,1]\times I) = \lim_{n\to\infty}\tilde{\Psi}_n([0,1]\times I)\\
&= \lim_{n\to\infty}\nu(\{x\in[0,1]\colon (\deg_U(x),\deg_{W^{\pi_n}}(x))\in[0,1]\times I\})\\
&= \lim_{n\to\infty} Y_{W^{\pi_n}}(I) = Y_W(I),
\end{aligned}$$

so again $\tilde{\Psi}^y = Y_W$. It remains to show that

$$\int_{(x,y)\in B\times[0,1]} xd\tilde{\Psi}(x,y) = \int_{(x,y)\in B\times[0,1]} yd\tilde{\Psi}(x,y)$$

for every $B \in \tilde{\mathcal{I}}$ (and so for every Borel measurable subset B of $[0,1]$). So fix $B \in \tilde{\mathcal{I}}$, $\varepsilon > 0$ and find some partition $\{I_1, I_2, \ldots, I_n\}$ of the interval $[0,1]$ into intervals from $\tilde{\mathcal{I}}$ of lengths smaller than ε. By additivity of integration, we may also assume that the interval B is of length smaller than ε. Fix some points $x_0 \in B$ and $y_j \in I_j$ (for every j). Then we have

$$\begin{aligned}
&\int_{(x,y)\in B\times[0,1]} y d\tilde{\Psi}(x,y) = \sum_{j=1}^{n} \int_{(x,y)\in B\times I_j} y d\tilde{\Psi}(x,y) \overset{\varepsilon}{\approx} \sum_{j=1}^{n} y_j \tilde{\Psi}(B\times I_j) = \sum_{j=1}^{n} y_j \lim_{n\to\infty} \tilde{\Psi}_n(B\times I_j) \\
&= \sum_{j=1}^{n} y_j \lim_{n\to\infty} \nu(\{x\in[0,1] \colon (\deg_U(x), \deg_{W^{\pi_n}}(x)) \in B\times I_j\}) \\
&= \sum_{j=1}^{n} \lim_{n\to\infty} \int_{x\in \deg_U^{-1}(B)\cap(\deg_{W^{\pi_n}})^{-1}(I_j)} y_j d\nu(x) \\
&\overset{\varepsilon}{\approx} \sum_{j=1}^{n} \lim_{n\to\infty} \int_{x\in \deg_U^{-1}(B)\cap(\deg_{W^{\pi_n}})^{-1}(I_j)} \deg_{W^{\pi_n}}(x) d\nu(x) \\
&= \lim_{n\to\infty} \int_{x\in\deg_U^{-1}(B)} \deg_{W^{\pi_n}}(x) d\nu(x) = \lim_{n\to\infty} \int_{x\in\deg_U^{-1}(B)} \int_{y\in[0,1]} W(\pi_n(x),\pi_n(y)) d\nu(y) d\nu(x) \\
&= \int_{x\in\deg_U^{-1}(B)} \int_{y\in[0,1]} U(x,y) d\nu(y) d\nu(x) = \int_{x\in\deg_U^{-1}(B)} \deg_U(x) d\nu(x) \\
&\overset{\varepsilon}{\approx} \int_{x\in\deg_U^{-1}(B)} x_0 d\nu(x) = x_0 \nu(\deg_U^{-1}(B)) = x_0 Y_U(B) = x_0 \tilde{\Psi}^x(B) \\
&= \int_{(x,y)\in B\times[0,1]} x_0 d\tilde{\Psi}(x,y) \overset{\varepsilon}{\approx} \int_{(x,y)\in B\times[0,1]} x d\tilde{\Psi}(x,y),
\end{aligned}$$

as needed.

WE STILL NEED TO PROVE THE LAST ASSERTION. $\square$

3.5. **Proof of Theorem 3.5.** Let Γ be a $\preceq$-maximal element in $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. Γ is guaranteed to exist by Lemma 3.16. Then the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ converges to Γ in the cut distance.

First we recall a several definitions and results from [8]. Assume that $W \in \mathcal{W}$ and let $B \subseteq [0,1]$ be a measurable set. Define

$$\gamma_B(x) = \int_0^x 1_B(y) dy$$

if $x \in B$ and

$$\gamma_B(x) = \alpha_B(1) + \int_0^x 1 - 1_B(y) dy$$

if $x \notin B$. Note that γ_B is a measure preserving bijection (possibly up to a null-set) which moves the mass of B accordingly to the left and therefore we may define a version of W by

$${}_B W(x,y) := W(\gamma_B^{-1}(x), \gamma_B^{-1}(y)).$$

The crucial construction from [8] is the following. Let $W, \Gamma_1, \Gamma_2, ... \in \mathcal{W}$ and assume that $\Gamma_n \xrightarrow{w^*} W$. Take some sequence of $B_1, B_2, ... \subseteq [0,1]$ of measurable sets and a subsequence such that $1_{B_{n_k}} \xrightarrow{w^*} s$ and ${}_{B_{n_k}}\Gamma_{n_k} \xrightarrow{w^*} \tilde{W}$. We define $\psi(x) = \int_0^x s(y)dy$ and $\phi(x) = \psi(1) + \int_0^x 1 - s(y)dy$.

Lemma 3.20 (Claim 1 in [8]). *For every intervals $[p_1,p_2],[q_1,q_2]\subseteq[0,1]$ we have*

$$\begin{aligned}\int_{[p_1,p_2]\times[q_1,q_2]} W(x,y) = &\int_{[p_1,p_2]\times[q_1,q_2]} \tilde{W}(\psi(x),\psi(y))s(x)s(y)\\ &+\int_{[p_1,p_2]\times[q_1,q_2]} \tilde{W}(\psi(x),\phi(y))s(x)(1-s(y))\\ &+\int_{[p_1,p_2]\times[q_1,q_2]} \tilde{W}(\phi(x),\psi(y))(1-s(x))s(y)\\ &+\int_{[p_1,p_2]\times[q_1,q_2]} \tilde{W}(\phi(x),\phi(y))(1-s(x))(1-s(y)).\end{aligned}$$

As a consequence we have that for almost every $(x,y)\in[0,1]^2$ we have

$$\begin{aligned}W(x,y) =&\tilde{W}(\psi(x),\psi(y))s(x)s(y)+\tilde{W}(\psi(x),\phi(y))s(x)(1-s(y))\\ &+\tilde{W}(\phi(x),\psi(y))(1-s(x))s(y)+\tilde{W}(\phi(x),\phi(y))(1-s(x))(1-s(y)).\end{aligned}$$

Lemma 3.21. *Let $f:[0,1]\to[0,1]$ be a continuous function. Then the map $\mathrm{INT}_f:\mathcal{W}\to[0,1]$ defined as in (1.2) is continuous for the $\|\cdot\|_1$-norm and for $U,W\in\mathcal{W}$ such that $\delta_\square(U,W)=0$ we have*

$$\mathrm{INT}_f\left(W\right)=\mathrm{INT}_f\left(U\right)\ .$$

Proof. The second part follows from the first one since it clearly holds for versions i.e. $\mathrm{INT}_f\left(W\right)=\mathrm{INT}_f\left(W^\varphi\right)$ but it is a standard fact, see for example [16], that every $U\in\mathcal{W}$ such that $\delta_\square(U,W)=0$ is $\|\cdot\|_1$-limit of versions of W.

Suppose that $W_n\to W$ in the $\|\cdot\|_1$-norm. Then there is a measure one set $A\subseteq[0,1]^2$ such that $W_{n_k}\to W$ pointwise for some subsequence. Since f is continuous we have that $f\left(W_{n_k}\right)\to f\left(W\right)$ pointwise on A. By the Lebesgue convergence theorem (the bound is constant 1) we have

$$\int_x\int_y f\left(W_{n_k}(x,y)\right)\to\int_x\int_y f\left(W(x,y)\right).$$

□

Proof of Theorem 3.5, (b)⇒(a). Let $\mathbf{LIM}_{w*}\left(\Gamma_1,\Gamma_2,\Gamma_3,\dots\right)=\mathbf{ACC}_{w*}\left(\Gamma_1,\Gamma_2,\Gamma_3,\dots\right)$ and assume that $W\in\mathbf{LIM}_{w*}\left(\Gamma_1,\Gamma_2,\Gamma_3,\dots\right)$ is a maximal element as given in Lemma 3.16. We may also assume that $\Gamma_n\xrightarrow{w^*}W$. We claim that this already implies that $\Gamma_n\xrightarrow{\|\cdot\|_\square}W$. Suppose not. Then there is an $\epsilon>0$ and a sequence $B_1,B_2,\dots$ of Borel subsets of $[0,1]$ such that $\left|\int_{B_n}\Gamma_n-W\right|>\epsilon$. We assume that $1_{B_n}\xrightarrow{w^*}s$ and ${}_{B_n}\Gamma_n\xrightarrow{w^*}\tilde{W}$ (otherwise we take a suitable subsequence).

Claim 3.22. We have $\langle W\rangle=\langle\tilde{W}\rangle$.

Proof. Let $\mathcal{I}_n=\{I_{n,1},I_{n,2},...,I_{n,n}\}$ be a partition of $[0,1]$ into n many equimeasurable intervals i.e. $I_{n,k}=[\frac{k}{n},\frac{k+1}{n})$. Define a measure preserving bijection φ_n by the following formula

$$\varphi_n(x)=\begin{cases}\frac{k}{n}+x-\psi\left(\frac{k}{n}\right) & \text{if } x\in\left[\psi(\frac{k}{n}),\psi(\frac{k+1}{n})\right) \text{ and}\\ \frac{k}{n}+\psi(\frac{k+1}{n})-\psi(\frac{k}{n})+x-\phi(\frac{k}{n}) & \text{if } x\in\left[\phi(\frac{k}{n}),\phi(\frac{k+1}{n})\right).\end{cases}$$

Define $\tilde{W}_n(x,y) = \tilde{W}(\varphi_n^{-1}(x), \varphi_n^{-1}(y))$. We claim that $\tilde{W}_n \xrightarrow{w^*} W$. To see this observe that for $m \geq n$ we have

$$\begin{aligned}\int_{I_{n,k}\times I_{n,l}} \tilde{W}_m &= \int_{\psi(\frac{k}{n})}^{\psi(\frac{k+1}{n})} \int_{\psi(\frac{l}{n})}^{\psi(\frac{l+1}{n})} \tilde{W} + \int_{\psi(\frac{k}{n})}^{\psi(\frac{k+1}{n})} \int_{\phi(\frac{l}{n})}^{\phi(\frac{l+1}{n})} \tilde{W} \\ &+ \int_{\phi(\frac{k}{n})}^{\phi(\frac{k+1}{n})} \int_{\psi(\frac{l}{n})}^{\psi(\frac{l+1}{n})} \tilde{W} + \int_{\phi(\frac{k}{n})}^{\phi(\frac{k+1}{n})} \int_{\phi(\frac{l}{n})}^{\phi(\frac{l+1}{n})} \tilde{W} \\ &= \int_{I_{n,k}\times I_{n,l}} \tilde{W}(\psi(x),\psi(y))s(x)s(y) + \int_{I_{n,k}\times I_{n,l}} \tilde{W}(\psi(x),\phi(y))s(x)(1-s(y)) \\ &+ \int_{I_{n,k}\times I_{n,l}} \tilde{W}(\phi(x),\psi(y))(1-s(x))s(y) + \int_{I_{n,k}\times I_{n,l}} \tilde{W}(\phi(x),\phi(y))(1-s(x))(1-s(y)) \\ &= \int_{I_{n,k}\times I_{n,l}} W.\end{aligned}$$

This implies that $\int_{I_{n,k}\times I_{n,l}} \tilde{W}_m \to \int_{I_{n,k}\times I_{n,l}} W$ for every $n,k,l \in \mathbb{N}$ and therefore $\tilde{W}_m \xrightarrow{w^*} W$. From the definition we have that $\tilde{W} \in \langle W \rangle$ and that finishes the proof. □

Claim 3.23. Let $f : [0,1] \to [0,1]$ be a continuous strictly concave function. Then $\mathrm{INT}_f(W) > \mathrm{INT}_f(\tilde{W})$.

Proof. The follows from the Lemma 11 and Claim 2 in [8]. □

Finally we get a contradiction since from Lemma 3.11 it follows that $\delta_\square(W,\tilde{W}) = 0$ and therefore by Lemma 3.21 we have $\mathrm{INT}_f(W) = \mathrm{INT}_f(\tilde{W})$. □

3.6. **Relating the hyperspace $K(\mathcal{W})$ and $\delta_\square$.** First, let provide a proof of an extension of Theorem 3.5, stated in Theorem 3.24 below. In addition to the original statement, we include a characterization of cut distance convergence in terms of the hyperspace $K(\mathcal{W})$, and also describe the limit graphon.

Theorem 3.24. *Let $W, \Gamma_1, \Gamma_2, \Gamma_3, \ldots \in \mathcal{W}$. The following are equivalent:*

(a) $\Gamma_n \to W$ *in the cut distance* $\delta_\square$*,*
(b) $\langle \Gamma_n \rangle \to \langle W \rangle$ *in* $K(\mathcal{W})$*,*
(c) $\mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots) = \mathbf{ACC}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots)$ *and* $W \in \mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots)$ *is the* $\preceq$*-maximal element of* $\mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots)$.

Proof of Theorem 3.24. Condition (c) implies Condition (a) by Lemma 3.5. Next, we show that Condition (c) implies Condition (b). First note that $\langle W \rangle = \mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots)$. To see this consider an open neighborhood $\mathcal{U}$ of $\langle W \rangle$ in $K(\mathcal{W})$. We may assume that $\mathcal{U}$ is given by a finite sequence $\{O_1,...,O_n\}$ of open subsets of $\mathcal{W}$ i.e. $\mathcal{U} = \left\{K \in K(\mathcal{W}) : K \subseteq \bigcup_{i\leq n} O_i \& \forall i \leq n K \cap O_i \neq \emptyset\right\}$. If there is no m such that for every $k \geq m$ we have $\langle \Gamma_k \rangle \in \mathcal{U}$ then there must be a subsequence $\langle \Gamma_{k_1} \rangle, \langle \Gamma_{k_2} \rangle, \ldots$ such that either $\langle \Gamma_{k_j} \rangle \not\subseteq \bigcup_{i\leq n} O_i$ for every $j \in \mathbb{N}$ or there is $i \leq n$ such that $\langle \Gamma_{k_j} \rangle \cap O_i = \emptyset$ for every $j < \omega$. If the first possibility happens then $\mathbf{ACC}_{w*}(\Gamma_{k_1},\Gamma_{k_2},\ldots) \cap (\mathcal{W} \setminus \bigcup_{i\leq n} O_i) \neq \emptyset$ but since $\mathbf{ACC}_{w*}(\Gamma_{k_1},\Gamma_{k_2},\ldots) = \mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots)$ and $\mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots) = \langle W \rangle \subseteq \bigcup_{i\leq n} O_i$ this is a contradiction. If the second possibility happens then $\mathbf{ACC}_{w*}(\Gamma_{k_1},\Gamma_{k_2},\ldots) \cap O_i = \emptyset$ but since $\mathbf{ACC}_{w*}(\Gamma_{k_1},\Gamma_{k_2},\ldots) = \mathbf{LIM}_{w*}(\Gamma_1,\Gamma_2,\Gamma_3,\ldots) = \langle W \rangle$ and $\langle W \rangle \cap O_i \neq \emptyset$ since $\langle W \rangle \in \mathcal{U}$ we obtain again a contradiction.

To see that Condition (b) implies Condition (c), let $U \in \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. We claim that $U \in \langle W \rangle$, if this is not the case then there is an open set $S \subseteq \mathcal{W}$ such that $\langle W \rangle \subseteq S$ and $U \notin S$ but since $\langle \Gamma_n \rangle \to \langle W \rangle$ in $K(\mathcal{W})$ there is $m \in \mathbb{N}$ such that for every $k \geq m$ we have $\langle \Gamma_k \rangle \subseteq S$ which is a contradiction with $U \in \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. Assume that $U \notin \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ then there must be an open neighborhood O of U in $\mathcal{W}$ and a subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \ldots$ such that $O \cap \langle \Gamma_{n_i} \rangle = \emptyset$ for every $i \in \mathbb{N}$. Consider now the open set $\mathcal{U} = \{K \in K(\mathcal{W}) : O \cap K \neq \emptyset\}$ and notice that $\langle W \rangle \in \mathcal{U}$ because $U \in \langle W \rangle$. Then since $\langle \Gamma_n \rangle \to \langle W \rangle$ we can find $k \in \mathbb{N}$ such that $O \cap \langle \Gamma_{n_k} \rangle \neq \emptyset$, a contradiction. Therefore we have $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) = \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. By the first part of the argument it follows that $\langle W \rangle = \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and therefore W is the $\preceq$-maximal element of $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$.

Finally assume that $\Gamma_n \to W$ in the cut distance. We may assume that in fact $\Gamma_n \xrightarrow{\|\cdot\|_\square} W$. Take any $\Gamma_{n_k}^{\varphi_k} \xrightarrow{w^*} U$. We claim that then also $W^{\varphi_k} \xrightarrow{w^*} U$. To see this fix some Borel set $A \subseteq [0,1]$ then

$$\left| \int_{A \times A} W^{\varphi_k} - U \right| \leq \left| \int_{\varphi_k(A) \times \varphi_k(A)} W - \Gamma_{n_k} \right| + \left| \int_{A \times A} \Gamma_{n_k}^{\varphi_k} - U \right| \to 0$$

where the first term tends to 0 by $\|\cdot\|_\square$-convergence and the second by the weak* convergence. By renaming we have that $W^{\varphi_n} \xrightarrow{w^*} U$ and then the same trick gives us $\Gamma_n^{\varphi_n} \xrightarrow{w^*} U$ which means that $U \in \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$. □

Let us denote the compact space of graphons after the weak equivalence factorization as $\tilde{\mathcal{W}}$ and similarly for every $W \in \mathcal{W}$ we denote its equivalence class $\tilde{W} \in \tilde{\mathcal{W}}$.

Corollary 3.25. *The map $\langle \cdot \rangle : (\tilde{\mathcal{W}}, \delta_\square) \to K(\mathcal{W})$ is a continuous injection i.e. $(\tilde{\mathcal{W}}, \delta_\square)$ is homeomorphic to some closed subspace X of $K(\mathcal{W})$. Moreover the metric $\delta_\square$ is equivalent to the hyperspace metric i.e. for $U, W \in \mathcal{W}$ defined as*

$$\rho(U, W) = \max \left\{ \sup_\varphi \left\{ \inf_\psi \{ d_{w^*}(U^\varphi, W^\psi) \} \right\}, \sup_\varphi \left\{ \inf_\psi \{ d_{w^*}(U^\psi, W^\varphi) \} \right\} \right\}.$$

As $K(\mathcal{W})$ is compact by Fact 2.5, so is X. Hence $(\tilde{\mathcal{W}}, \delta_\square)$ is compact.

Proof. The map is well-defined and injective by Lemma 3.11 and it is continuous by Theorem 3.5. Since by Theorem 3.5 and Theorem 3.3 the space $(\tilde{\mathcal{W}}, \delta_\square)$ is compact we have that the image of $(\tilde{\mathcal{W}}, \delta_\square)$ under $\langle \cdot \rangle$ is compact and the map is in fact a homeomorphism. It is a standard fact about compact spaces that if two metrics give the compact topology then they are equivalent. □

3.7. **Basic properties of the structurdness (quasi)order.** Above, we proved properties of the structurdness (quasi)order $\preceq$ that were needed for our abstract proof of Theorem 3.24. First, let us recall that $\preceq$ is actually an order modulo weak isomorphism.

Lemma 3.26. *The relation $\preceq$ on the space $\tilde{\mathcal{W}}$ is an order and as a relation, i.e. $\preceq \subseteq \tilde{\mathcal{W}} \times \tilde{\mathcal{W}}$, it is closed.*

Proof. Since by Corollary 3.25 is the space $(\tilde{\mathcal{W}}, \delta_\square)$ homeomorphic to some closed subspace of $K(\mathcal{W})$ and the relation $\preceq$ is interpreted as $\subseteq$ on $K(\mathcal{W})$ it is enough to verify the properties for the relation $\subseteq$ on $K(\mathcal{W})$. But both properties are trivially satisfied for the relation $\subseteq$. □

As it is noted on the proof of Lemma 3.26 the space $(\tilde{\mathcal{W}}, \delta_\square)$ homeomorphic to some closed subspace of $K(\mathcal{W})$ and the relation $\preceq$ is interpreted as $\subseteq$ on $K(\mathcal{W})$. This observation allows us to immediately obtain the following results.

Proposition 3.27. *(1) Suppose that $P \subseteq \mathcal{W}$ is upper-directed in the structurdness order, i.e. for every $U_0, U_1 \in P$ there is $V \in P$ such that $U_0, U_1 \preceq V$. Then there is a graphon $W \in \mathcal{W}$ such that W is the supremum of P in $\preceq$.*
(2) Suppose that $P \subseteq \mathcal{W}$ is down-directed in the structurdness order, i.e. for every $U_0, U_1 \in P$ there is $V \in P$ such that $V \preceq U_0, U_1$. Then there is a graphon $W \in \mathcal{W}$ such that W is the infimum of P in $\preceq$.

Proof. First of all consider the set $\langle P \rangle = \{\langle U \rangle : U \in P\}$. This set is upper-directed in $\subseteq$ in $K(\mathcal{W})$. It is not hard to realize that K that is defined as the weak* closure of $\bigcup_{U \in P} \langle U \rangle$ is the supremum of $\langle P \rangle$ in $\subseteq$ on $K(\mathcal{W})$. We only need to show that there is $W \in \mathcal{W}$ such that $K = \langle W \rangle$. Consider a countable set $P_0 \subseteq P$ such that $\langle P_0 \rangle$ is dense in $\langle P \rangle$, this can be done since $K(\mathcal{W})$ is separable metrizable, and take some enumeration of $U_1, U_2, ...$ of P_0. Define inductively an increasing chain $\Gamma_1, \Gamma_2, ... \subseteq P$ such that for every $n \in \mathbb{N}$ we have that $U_1, ..., U_n, \Gamma_{n-1} \preceq \Gamma_n$, note that this can be done since P is upper-directed. It is straightforward to check that $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots) = \mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and moreover $K = \mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ because the $\langle P_0 \rangle$ is dense in $\langle P \rangle$. By Lemma **??** we can pick a maximal element W of $\mathbf{LIM}_{w*}(\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ and then it must hold that $K = \langle W \rangle$.

The proof of the second part is the same. ☐

Corollary 3.28. *(1) Suppose that $W_1 \preceq W_2 \preceq W_3 \preceq \ldots$ is a sequence of graphons. Then this sequence is cut distance convergent.*
(2) Suppose that $W_1 \succeq W_2 \succeq W_3 \succeq \ldots$ is a sequence of graphons. Then this sequence is cut distance convergent.

Proof. This is a special case of Proposition 3.27 and one can easily check that $W_n \to W$ in $\delta_\square$ where W is the supremum of $W_1 \preceq W_2 \preceq W_3 \preceq \ldots$ (or the infimum in the $W_1 \succeq W_2 \succeq W_3 \succeq \ldots$ case). ☐

Corollary 3.29. *Let $K \in K(\mathcal{W})$ then there exists $W \in \mathcal{W}$ such that $K = \langle W \rangle$ if and only if K is upper-directed (for every $U_0, U_1 \in K$ there is $V \in K$ such that $U_0, U_1 \preceq V$).*

Proof. If K is upper-directed then W is the supremum of K that exists thanks to Proposition 3.27. ☐

Now we state a proposition characterising the minimal and maximal elements of the structuredness order. Its proof was suggested to us by L. M. Lovász.

Proposition 3.30. *The minimal elements of the structuredness order are exactly constant graphons. The maximal elements of the structuredness order are exactly graphons with values zero or one almost everywhere.*

Proof. The first part follows directly from the fact that an envelope of any graphon contains a constant graphon (see Lemma 3.7(b)). For the second part suppose that W is a graphon such that its value is neither 0 nor 1 on a set of positive measure. This implies that there is an ε such that W has values between ε and $1 - \varepsilon$ on a set of positive measure. Now consider a measure preserving map φ such that $\varphi(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $\varphi(x) = 2x - 1$ for $\frac{1}{2} < x \leq 1$.

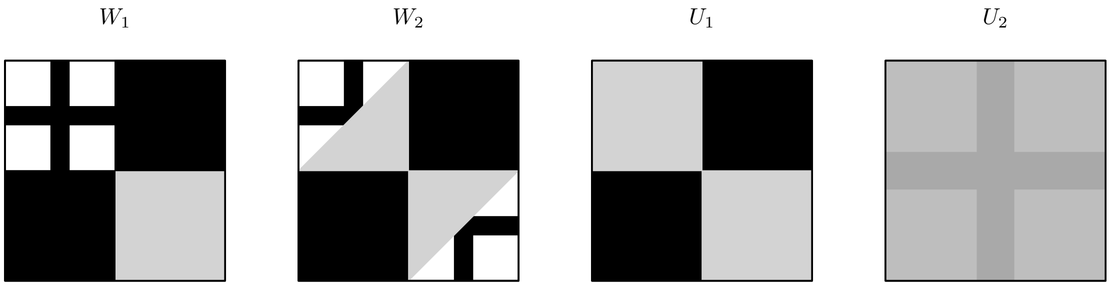

FIGURE 3.2. The four graphons W_1, W_2, U_1, U_2 witnessing that an intersection of two envelopes is not necessarily an envelope.

The graphon W^φcontains four copies of W scaled by a factor of one half. Let W^+ be a graphon such that $W^+ = W^\varphi + \varepsilon$ for $x, y \in [0, \frac{1}{2}]^2 \cup [\frac{1}{2}, 1]^2$ and $W^+ = W^\varphi - \varepsilon$ otherwise. The values of W^+ are bounded by 0 and 1 and W and $W^+$are not weakly isomorphic (compare $\mathrm{INT}_f(W)$ and $\mathrm{INT}_f(W^+)$ for any strictly convex function f). Moreover, we claim that $W \in \langle W^+ \rangle$. To see this, one can construct a sequence of measure preserving almost bijections $\psi_1, \psi_2, \ldots$, defined as $\psi_n(x) = \frac{\lfloor 2nx \rfloor}{2n} + x$ for $0 \le x \le \frac{1}{2}$ and $\psi_n(x) = \frac{\lfloor 2nx \rfloor - 2n + 1}{2n} + x$ for $\frac{1}{2} \le x \le 1$, that interlace the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and thus serve as an approximation of φ. The fact that $W_1^{+\psi_1}, W_2^{+\psi_2}, \ldots \xrightarrow{\mathrm{w}^*} W$ can be seen directly from the definition of weak* convergence. □

Finally note that intersection of two envelopes of two graphons with the same density is not necessarily an envelope of another graphon. To show this we provide a simple example of graphons W_1, W_2, U_1, U_2 such that we have $U_1, U_2 \preceq W_1, W_2$, but there is not a graphon V such that $U_1, U_2 \preceq V \preceq W_1, W_2$. For fixed $\varepsilon > 0$ we define the four graphons as follows (the definitions should be clear from Figure 3.2).

(1) Define $W_1(x, y) = 1$ if and only if $(x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon] \times [0, 1] \cup [0, 1] \times [\frac{1}{4} - \frac{\varepsilon}{2}, \frac{1}{4} + \frac{\varepsilon}{2}]$. Moreover, $W_1 = \varepsilon - \varepsilon^2$ for $(x, y) \in [\frac{1}{2}, 1]^2$ and is zero otherwise.
(2) Define W_2 as W_1 but switch its values on the triangle with vertices $[0, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, 0]$ with the triangle with vertices $[\frac{1}{2}, 1], [1, 1], [1, \frac{1}{2}]$. Note that $\int_{[0,\frac{1}{2}]^2} W_1 = \int_{[\frac{1}{2},1]^2} W_1 = \int_{[0,\frac{1}{2}]^2} W_2 = \int_{[\frac{1}{2},1]^2} W_2$.
(3) Define $U_1 = \varepsilon - \varepsilon^2$ for $(x, y) \in [0, \frac{1}{2}]^2$ and 1 otherwise. Note that $U_1 \preceq W_1, W_2$ (this follows by applying Lemma 3.7(b) twice just for the top-left and bottom-right subgraphons of the graphons).
(4) Finally, define U_2 to be such that $U_2(x, y) = \frac{1}{4} \left(W_1 \left(\frac{x}{2}, \frac{y}{2} \right) + W_1 \left(\frac{x+1}{2}, \frac{y}{2} \right) + W_1 \left(\frac{x}{2}, \frac{y+1}{2} \right) + W_1 \left(\frac{x+1}{2}, \frac{y+1}{2} \right) \right)$. We have again $U_2 \preceq W_1, W_2$ (consider a sequence of bijections interlacing the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, as in the proof of Proposition 3.30).

Claim 3.31. There does not exist a graphon V such that $U_1, U_2 \preceq V \preceq W_1, W_2$.

Proof. We will use the following observation: if for a graphon Γ we have $\sup_{|A|=a,|B|=b} \int_{A \times B} \Gamma = t$ and $\Gamma \succeq \Gamma'$, then $\sup_{|A|=a,|B|=b} \int_{A \times B} \Gamma' \le t$. This follows directly from the definition of weak* convergence.

Now assume that there is a graphon V such that $U_1, U_2 \preceq V \preceq W_1, W_2$. We apply the mentioned observation in a slightly different form; observe that

$$\sup_{|J|=\frac{1}{2}} \int_{J\times([0,1]\setminus J)\,\cup\,([0,1]\setminus J)\times J} V \geq \sup_{|J|=\frac{1}{2}} \int_{J\times([0,1]\setminus J)\,\cup\,([0,1]\setminus J)\times J} U_1 = \frac{1}{2}\,.$$

This implies that there is a set $J \subseteq [0,1]$, $|J| = \frac{1}{2}$ such that the value of V is 1 almost everywhere on the set $J \times ([0,1] \setminus J) \,\cup\, ([0,1] \setminus J) \times J$. Without loss of generality assume that $J = [0, \frac{1}{2}]$. Now from the fact that there is a sequence $\varphi_1, \varphi_2, \ldots$ such that $W_1^{\varphi_1}, W_2^{\varphi_2}, \ldots \xrightarrow{\text{w}^*} V$ and, thus, $\lim_{n\to\infty} \int_{[0,\frac{1}{2}]\times[\frac{1}{2},1]} W_n^{\varphi_n} = \int_{[0,\frac{1}{2}]\times[\frac{1}{2},1]} V$ we infer that for any ε there is n sufficiently large such that either $\nu\left(\varphi_n([0,\frac{1}{2}]) \cap [0,\frac{1}{2}]\right) \leq \varepsilon$ or $\nu\left(\varphi_n([0,\frac{1}{2}]) \cap [0,\frac{1}{2}]\right) \geq 1-\varepsilon$. From this we conclude that actually $W_1^{\varphi_1} \cap [0,\frac{1}{2}]^2, W_1^{\varphi_2} \cap [0,\frac{1}{2}]^2 \ldots$ converge weak* to either $V \cap [0,\frac{1}{2}]^2$ or $V \cap [\frac{1}{2},1]^2$. Without loss of generality assume that the sequence $W_1^{\varphi_1} \cap [\frac{1}{2},1]^2, W_1^{\varphi_2} \cap [\frac{1}{2},1]^2, \ldots$ converges weak* to $V \cap [\frac{1}{2},1]^2$. We also assume that there is a sequence $W_1^{\psi_1} \cap [0,\frac{1}{2}]^2, W_1^{\psi_2} \cap [0,\frac{1}{2}]^2, \ldots$ converging weak* to $V \cap [0,\frac{1}{2}]^2$ (the other case where the sequence $W_1^{\psi_1} \cap [\frac{1}{2},1]^2, W_1^{\psi_2} \cap [\frac{1}{2},1]^2, \ldots$ converges weak* to $V \cap [0,\frac{1}{2}]^2$ can be treated in the same way).

Note that $\int_{[0,1]\times[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]} U_2 = (2\varepsilon)\cdot 1 \cdot \frac{3}{4} + o(\varepsilon) = \frac{3}{2}\varepsilon + o(\varepsilon)$. On the other hand,

$$\begin{aligned}
\sup_{|C|=2\varepsilon} \int_{[0,1]\times C} V &= \sup_{A\subseteq[0,\frac{1}{2}],B\subseteq[\frac{1}{2},1],|A\cup B|=2\varepsilon} \int_{[0,1]\times A} V + \int_{[0,1]\times B} V \\
&= (2\varepsilon)\cdot\frac{1}{2} + \sup_{A\subseteq[0,\frac{1}{2}],B\subseteq[\frac{1}{2},1],|A\cup B|=2\varepsilon} \int_{[0,\frac{1}{2}]\times A} V + \int_{[\frac{1}{2},1]\times B} V \\
&\leq (2\varepsilon)\cdot\frac{1}{2} + \sup_{A\subseteq[0,\frac{1}{2}],B\subseteq[\frac{1}{2},1],|A\cup B|=2\varepsilon} \int_{[0,\frac{1}{2}]\times A} W_2 + \int_{[\frac{1}{2},1]\times B} W_1 \\
&= \varepsilon + \left(\frac{1}{4}\varepsilon + o(\varepsilon)\right) + o(\varepsilon) \\
&= \frac{5}{4}\varepsilon + o(\varepsilon),
\end{aligned}$$

which, for ε small enough, is smaller than the appropriate value for U_2. But this is a contradiction with the fact that $V \succeq U_2$. $\square$

4. CUT DISTANCE IDENTIFYING GRAPHON PARAMETERS

4.1. **Basics.** In the previous sections, we based our treatment of the cut distance on $\mathbf{ACC}_{\mathrm{w}*}(W_1, W_2, W_3, \ldots)$ and $\mathbf{LIM}_{\mathrm{w}*}(W_1, W_2, W_3, \ldots)$, which are sets of functions. In contrast, the key objects in [8] are sets of numerical values, $\left\{\mathrm{INT}_f(W) : W \in \mathbf{ACC}_{\mathrm{w}*}(W_1, W_2, W_3, \ldots)\right\}$ and $\left\{\mathrm{INT}_f(W) : W \in \mathbf{LIM}_{\mathrm{w}*}(W_1, W_2, W_3, \ldots)\right\}$ with notation taken from (1.2). In this section, we introduce an abstract framework to approaching the cut distance via similar optimization problems. To this end, we give the following key definitions. By a *graphon parameter* we mean any function $\theta : \mathcal{W} \to \mathbb{R}$, $\theta : \mathcal{W} \to \mathbb{R}^n$ (for some $n \in \mathbb{N}$) or $\theta : \mathcal{W} \to \mathbb{R}^{\mathbb{N}}$ such that $\theta(W_1) = \theta(W_2)$ for any two graphons W_1 and W_2 with $\delta_\square(W_1, W_2) = 0$. We say that a graphon parameter θ is a *cut distance identifying graphon parameter* if we have that $W_1 \prec W_2$ implies $\theta(W_1) < \theta(W_2)$ (here, by $<$ we understand the usual Euclidean order on $\mathbb{R}$ in case $\theta : \mathcal{W} \to \mathbb{R}$ and the lexicographic order in case $\theta : \mathcal{W} \to \mathbb{R}^n$ or

$\theta : \mathcal{W} \to \mathbb{R}^{\mathbb{N}}$). We say that a graphon parameter θ is a *cut distance compatible graphon parameter* if we have that $W_1 \preceq W_2$ implies $\theta(W_1) \leq \theta(W_2)$.

So, the main result of [8] is that for a strictly convex continuous function $f : [0,1] \to \mathbb{R}$, the parameter $\mathrm{INT}_f : \mathcal{W} \to \mathbb{R}$ is cut distance identifying (the assumption of continuity is probably only an artefact of their proof and not needed, see Remark 3.4).

Cut distance identifying graphon parameters can be used to prove compactness of the graphon space. This is stated in the next two theorems.

Theorem 4.1. *Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ be a sequence of graphons. Suppose that θ is a cut distance identifying graphon parameter. Then there exists a subsequence $\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots$ such that $\mathbf{ACC}_{w*}(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots)$ contains an element Γ with $\theta(\Gamma) = \inf\{\theta(W) : W \in \mathbf{ACC}_{w*}(\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \ldots)\}$.*

Theorem 4.2. *Let $W_1, W_2, W_3, \ldots$ be a sequence of graphons.Suppose that θ is a cut distance identifying graphon parameter. Suppose that $\Gamma \in \mathbf{ACC}_{w*}(W_1, W_2, W_3, \ldots)$ is such that $\theta(\Gamma) = \inf\{\theta(W) : W \in \mathbf{ACC}_{w*}(W_1, W_2, W_3, .$ Then $W_1, W_2, W_3, \ldots$ converges to Γ in the cut distance.*

Theorems 4.1 and 4.2 follow from Theorem 3.3[f] and Lemma 3.16. So, while the concepts of cut distance identifying graphon parameters do not bring any new tools compared to the structuredness order, knowing that a particular parameter is cut distance identifying allows calculations that are often more direct than working with the structurdness order.

4.1.1. *Relation to quasirandomness.* Recall that dense quasirandom finite graphs correspond to constant graphons. Thus, the key question in the area of quasirandomness is which parameters can be used to characterize constant graphons.

For $p \in [0,1]$, we write $\mathcal{W}_p = \left\{W :\in \mathcal{W} : \int_x \int_y W(x,y) = p\right\}$ for all graphons with edge density p. Let us summarize the most common properties used to characterize constant graphons.

Theorem 4.3. *Let $p \in [0,1]$. Then the constant-p graphon is the only graphon U in the family $\mathcal{W}_p$ satisfying any of the following conditions.*

(a) We have $t(C_{2\ell}, U) \leq p^{2\ell}$ for a fixed $\ell \in \{2,3,4,\ldots\}$.
(b) Order the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ of U as in (2.1). Then $|\lambda_1| \leq p$ and $|\lambda_2| = 0$.
(c) We have $\mathrm{INT}_f(U) \leq f(p)$ for a fixed strictly convex function f.

Parts (a) (for $\ell = 2$) and (b) appear in the Chung–Graham–Wilson Theorem [3]. The other values of ℓ in Part (a) relate to the Sidorenko conjecture and the Forcing conjecture, and are treated in detail in Section 4.4. Part (c) follows from the main result of [8] (for discontinuous functions, we need an extension given in Remark 3.4).

4.2. **Convex functions are cut distance compatible.** In this section we prove Theorem 4.4 and then discuss its consequences.

In the theorem we use $\mathbb{R}^n$ together with lexicographical ordering and Euclidean metric and $\mathbb{R}^{\mathbb{N}}$ together with lexicographical ordering and the following metric: for $u, v \in \mathbb{R}^{\mathbb{N}}$ we define $d(u,v)$ to be smallest number ε such that $|u_i - v_i| \leq \varepsilon$ for every $1 \leq i \leq \frac{1}{\varepsilon}$.

Theorem 4.4. *Let $f : \mathcal{W} \to \mathbb{R}$, $f : \mathcal{W} \to \mathbb{R}^n$, or $f : \mathcal{W} \to \mathbb{R}^{\mathbb{N}}$ be a graphon parameter that is convex and continuous in L_1. Then f is cut distance compatible.*

[f]Let us note that an alternative direct proof of Theorem 4.1 can be repeated mutatis mutandis from Lemma 13 in [8]. This latter proof is more elementary and does not need transfinite induction or any appeal to the Vietoris topology.

Before proving Theorem 4.4 we prove the following auxiliary Lemma:

Lemma 4.5. *Let U be a graphon and $\mathcal{P} = (P_1, P_2, \ldots, P_{|\mathcal{P}|})$ a partition of the interval $[0,1]$. Then for any ε there exists n and a sequence of measure preserving bijections $\pi_1, \pi_2, \ldots, \pi_n$ such that*

$$\left\| \frac{1}{n} \left(W^{\pi_1} + \cdots + W^{\pi_n} \right) - W^{\bowtie \mathcal{P}} \right\|_1 < \varepsilon.$$

Proof. Without loss of generality assume that the partition $\mathcal{P}$ consists of intervals; otherwise we can work with the graphon $W^{\gamma(\mathcal{P})}$, with γ defined in Definition 3.14, instead of W. We apply Lemma 2.3 to get that for any $\varepsilon > 0$ there is a partition $\mathcal{Q}$ such that $\left\| W - W^{\bowtie \mathcal{Q}} \right\|_1 < \frac{\varepsilon}{2}$. This implies that for any sequence of measure preserving transformations $\pi_1, \pi_2, \ldots, \pi_n$ we have

$$\left\| \frac{1}{n} \sum_{1 \leq \ell \leq n} \left(W^{\pi_\ell} - \left(W^{\bowtie \mathcal{Q}} \right)^{\pi_\ell} \right) \right\|_1 \leq \frac{1}{n} \sum_{1 \leq \ell \leq n} \left\| \left(W^{\pi_\ell} - \left(W^{\bowtie \mathcal{Q}} \right)^{\pi_\ell} \right) \right\|_1 < \frac{1}{n} \cdot n \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \tag{4.1}$$

Let $\mathcal{R} = \mathcal{P} \cap \mathcal{Q}$. Split each subset of $\mathcal{R}$ arbitrarily into sets of measure

$$s = \frac{\varepsilon}{10 \left| \mathcal{Q} \right| \left(\min_{1 \leq i \leq |\mathcal{P}|} \left(|P_i| \right) \right)^4}$$

such that we get at most one left-over set of smaller size for each subset of $\mathcal{R}$. This gives us a partition $\mathcal{R}'$ such that all of its subsets are of the same size except of several exceptional subsets of overall measure at most $s \cdot |\mathcal{Q}|$ that we call R_0. Now for each ℓ define π_ℓ to be a map that for each i randomly permutes all the subsets of $\mathcal{Q}$ that are contained in P_i except for the small exceptional sets. Such a map can be composed as $\gamma\left(\mathcal{Q}\right)^{-1} \pi'_\ell \gamma\left(\mathcal{Q}\right)$, where γ is a function from Definition 3.14 that maps each subset of $\mathcal{Q}$ to an interval, while π'_i permutes the intervals. For each pair of two non-exceptional sets $Q_i \subseteq P_a$, $Q_j \subseteq P_b$ consider the density of the step graphon $\left(W^{\bowtie \mathcal{Q}} \right)^{\pi_1}$ in step $Q_i \times Q_j$ that we denote by $\left(W^{\bowtie \mathcal{Q}} \right)^{\pi_1} \left(Q_i, Q_j \right)$. This density is a random variable with values between 0 and 1 and its expected value is very close to the density of the super-step $P_a \times P_b$. Indeed, the only difference between these two quantities comes from the exceptional sets:

$$\begin{aligned}
\left| \mathbf{E}\left[\left(W^{\bowtie \mathcal{Q}} \right)^{\pi_1} \left(Q_i, Q_j \right) \right] - W^{\bowtie \mathcal{P}} \left(P_a, P_b \right) \right| &= \left| \frac{\int_{(P_a \setminus R_0) \times (P_b \setminus R_0)} W^{\bowtie \mathcal{P}}}{\left| (P_a \setminus R_0) \times (P_b \setminus R_0) \right|} - \frac{\int_{P_a \times P_b} W^{\bowtie \mathcal{P}}}{\left| P_a \times P_b \right|} \right| \\
&\leq \max \left(\frac{\int_{P_a \times P_b} W^{\bowtie \mathcal{P}}}{\left| (P_a \setminus R_0) \times (P_b \setminus R_0) \right|} - \frac{\int_{P_a \times P_b} W^{\bowtie \mathcal{P}}}{\left| P_a \times P_b \right|}, \frac{\int_{P_a \times P_b} W^{\bowtie \mathcal{P}}}{\left| P_a \times P_b \right|} - \frac{\int_{(P_a \setminus R_0) \times (P_b \setminus R_0)} W^{\bowtie \mathcal{P}}}{\left| P_a \times P_b \right|} \right) \\
&\leq \max \left(\frac{1}{\left| P_a \times P_b \right|} \left(\frac{1}{1 - \frac{2|R_0|}{|P_a \times P_b|}} - 1 \right), \frac{2 \left| R_0 \right|}{\left| P_a \times P_b \right|} \right) \\
&\leq \max \left(\frac{1}{\left| P_a \times P_b \right|} \cdot 2 \frac{2 \left| R_0 \right|}{\left| P_a \times P_b \right|}, \frac{2 \left| R_0 \right|}{\left| P_a \times P_b \right|} \right) \\
&\leq \frac{4s \left| \mathcal{Q} \right|}{\left| P_a \times P_b \right|^2} \\
&\leq \frac{4\varepsilon}{10}. \qquad (4.2)
\end{aligned}$$

We now apply Hoeffding's inequality to get that for any n we have

$$P\left[\left|\mathbf{E}\left[\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_1}(Q_i,Q_j)\right]-\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}(Q_i,Q_j)\right|\ge\frac{\varepsilon}{3}\right]\le 2\mathrm{e}^{-2n\left(\frac{\varepsilon}{3}\right)^2}.$$

Thus, for n sufficiently large we can bound the expression by a number smaller that $\frac{1}{|\mathcal{Q}|^2}$, use union bound over all non-exceptional sets $Q_i\times Q_j$, and argue that there is a fixed sequence $\pi_1,\pi_2,\dots,\pi_n$ such that

$$\left|\mathbf{E}\left[\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_1}(Q_i,Q_j)\right]-\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}(Q_i,Q_j)\right|<\frac{\varepsilon}{10}\tag{4.3}$$

for all non-exceptional sets $Q_i\times Q_j$. Putting this together with the inequality 4.2 we get that

$$\left|W^{\ltimes\mathcal{P}}(P_a,P_b)-\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}(Q_i,Q_j)\right|<\frac{\varepsilon}{10}+\frac{4\varepsilon}{10}=\frac{\varepsilon}{2},\tag{4.4}$$

i.e., the difference between densities of non-exceptional steps of $\mathcal{Q}$ are similar to the appropriate supersets of $\mathcal{P}$. Now we can bound the size of the expression $\left\|\frac{1}{n}\left(\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_1}+\cdots+\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_n}\right)-W^{\ltimes\mathcal{P}}\right\|_1$ as the sum of differences over all steps of $\mathcal{Q}$. We have

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}-W^{\ltimes\mathcal{P}}\right\|_1 &=\int\left|\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}-W^{\ltimes\mathcal{P}}\right|\\
&=\sum_{Q_i,Q_j}\nu(Q_i)\,\nu(Q_j)\left|\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}(Q_i,Q_j)-W^{\ltimes\mathcal{P}}(Q_i,Q_j)\right|\\
\boxed{\text{Inequality (4.4)}}\quad &<\frac{\varepsilon}{2}. \qquad (4.5)
\end{aligned}$$

Putting this together with (4.1), we have

$$\begin{aligned}
&(4.6)\\
&\left\|\frac{1}{n}\sum_{1\le\ell\le n}W^{\pi_\ell}-W^{\ltimes\mathcal{P}}\right\|_1\le\left\|\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\pi_\ell}-\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}\right)\right\|_1+\left\|\frac{1}{n}\sum_{1\le\ell\le n}\left(W^{\ltimes\mathcal{Q}}\right)^{\pi_\ell}-W^{\ltimes\mathcal{P}}\right\|_1\\
&(4.7)\qquad\qquad <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

□

Now we are ready to prove Theorem 4.4.

Proof. We will prove the theorem just for the hardest case when $f:\mathcal{W}\to\mathbb{R}^{\mathbb{N}}$. Let f_i denote the i-th value of f. We prove that for every ε it holds that $f_i(U)\ge f_i(V)-\varepsilon$ for all $i\le\frac{1}{\varepsilon}$.

Applying Lemma 2.3 for any $\delta>0$ gives us a partition $\mathcal{Q}$ of V such that $\left\|V^{\ltimes\mathcal{Q}}-V\right\|_1<\delta$. Note that $V^{\ltimes\mathcal{Q}}\preceq V$ by Lemma 3.7(b) . By the continuity of f in L_1 we get that for any $\varepsilon>0$ we can find a partition $\mathcal{Q}$ such that

$$f_i\left(V^{\ltimes\mathcal{Q}}\right)>f_i(V)-\frac{\varepsilon}{2}\tag{4.8}$$

for all $i \leq \frac{1}{\varepsilon}$. There is a sequence of versions $U^{\varphi_1}, U^{\varphi_2}, \ldots$ converging weak* to $V^{\ltimes\mathcal{Q}}$. For any $\delta > 0$ we can take ℓ sufficiently large so that $\left|\int_{Q_i\times Q_j} U^{\varphi_\ell} - \int_{Q_i\times Q_j} V^{\ltimes\mathcal{Q}}\right| < \frac{\delta}{2|\mathcal{Q}|^2}$ for all $1 \leq i,j \leq |\mathcal{Q}|$, which implies that

$$\left\|(U^{\varphi_\ell})^{\ltimes\mathcal{Q}} - V^{\ltimes\mathcal{Q}}\right\|_1 < \frac{\delta}{2}. \tag{4.9}$$

We apply the lemma 4.5 to grahon U^{φ_ℓ}, the partition $\mathcal{Q}$ and the desired error term being equal to $\frac{\delta}{2}$, and get that there is n and a sequence of measure preserving bijections $\pi_1, \pi_2, \ldots, \pi_n$ such that

$$\left\|\frac{1}{n}\left(U^{\pi_1\varphi_\ell} + \cdots + U^{\pi_n\varphi_\ell}\right) - (U^{\varphi_\ell})^{\ltimes\mathcal{Q}}\right\|_1 < \frac{\delta}{2}. \tag{4.10}$$

We combine the two inequalities (4.9) and (4.10) to get that

(4.11)
$$\left\|\frac{1}{n}\left(U^{\pi_1\varphi_\ell} + \cdots + U^{\pi_n\varphi_\ell}\right) - V^{\ltimes\mathcal{Q}}\right\|_1 \leq \left\|\frac{1}{n}\left(U^{\pi_1\varphi_\ell} + \cdots + U^{\pi_n\varphi_\ell}\right) - (U^{\varphi_\ell})^{\ltimes\mathcal{Q}}\right\|_1 + \left\|(U^{\varphi_\ell})^{\ltimes\mathcal{Q}} - V^{\ltimes\mathcal{Q}}\right\|_1$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{4.12}$$

As this holds for any $\delta > 0$, for given $\varepsilon > 0$ we can use the continuity of f and get that

$$f_i\left(\frac{1}{n}\left(U^{\pi_1\varphi_\ell} + \cdots + U^{\pi_n\varphi_\ell}\right)\right) > f_i\left(V^{\ltimes\mathcal{Q}}\right) - \frac{\varepsilon}{2} \tag{4.13}$$

for every $i \leq \frac{1}{\varepsilon}$.

Finally we put the inequalites 4.8 and 4.13 together and use Jensen's inequality to conclude that

$$\begin{aligned}
f_i(U) &= \frac{1}{n}\left(f_i\left(U^{\pi_1\varphi_\ell}\right) + \cdots + f\left(U^{\pi_n\varphi_\ell}\right)\right)\\
\boxed{\text{convexity}} \quad &\geq f_i\left(\frac{1}{n}\left(U^{\pi_1\varphi_\ell} + \cdots + U^{\pi_n\varphi_\ell}\right)\right)\\
\boxed{\text{Equation (4.13)}} \quad &> f_i\left(V^{\ltimes\mathcal{Q}}\right) - \frac{\varepsilon}{2}\\
\boxed{\text{Equation (4.8)}} \quad &> f_i(V) - \varepsilon
\end{aligned}$$

for every $i \leq \frac{1}{\varepsilon}$. □

Theorem 4.4 has several corollaries:

(1) The function $\mathrm{INT}_f = \int f(W(x,y))\,\mathrm{dxdy}$ is cut distance compatible for any continuous and convex function f. This is Lemma [reference]. It is not hard to verify that the condition on the continuity is in this case not needed, as all convex functions defined on the interval $[0,1]$ are continuous anywhere inside the interval.

(2) The fact that the spectrum of graphons is a cut distance identifying parameter is discussed in Section 4.3 but it also follows directly from Theorem 4.4. This holds because one can easily verify that the spectrum is convex [reference?] and continuous in L_1 [reference?] and, thus, cut distance compatible. It is cut distance identifying, since the L_2 norm, that can be also written as the sum of squares of eigenvalues, is cut distance identifying by Theorem 4.14.

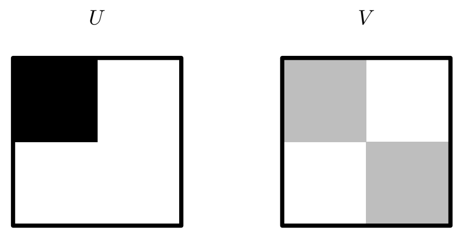

FIGURE 4.1. Two graphons U, V such that $V = \frac{U+U^{\varphi}}{2}$, but $V \not\preceq U$.

(3) The connection with Sidorenko's conjecture is further discussed in Section 4.4.

Note that it seems that it seems natural to believe that all cut distance compatible functions are convex. This impression is, however, false, since the two notions of linear combinations and structuredness are not the same. The example to this is shown in Figure 4.1. The graphon U is defined as $U(x,y) = 1$ if and only if $(x,y) \in [0, \frac{1}{2}]^2$ and zero otherwise, while $V(x,y) = \frac{1}{2}$ if and only if $(x,y) \in [0, \frac{1}{2}]^2 \cup [\frac{1}{2}, 1]^2$ and $V(x,y) = 0$ otherwise. If we set $\varphi(x) = 1 - x$, then clearly $V = \frac{U+U^{\varphi}}{2}$, but $V \not\preceq U$. This comes from the fact that for any measure preserving bijections π we have $\int_{[0,\frac{1}{2}]\times[\frac{1}{2},1]} U^{\pi} = \nu\left(\pi\left([0,\frac{1}{2}]\right) \cap [0,\frac{1}{2}]\right) \cdot \nu\left(\pi\left([0,\frac{1}{2}]\right) \cap [\frac{1}{2},1]\right)$. Thus, for any sequence of measure preserving bijections $\pi_1, \pi_2, \ldots$ such that $U^{\pi_1}, U^{\pi_2}, \ldots \xrightarrow{w^*} V$ it needs to hold that either $\nu\left(\pi\left([0,\frac{1}{2}]\right) \cap [0,\frac{1}{2}]\right) \to 0$ or $\nu\left(\pi\left([0,\frac{1}{2}]\right) \cap [0,\frac{1}{2}]\right) \to \frac{1}{2}$, a contradiction.

This example in turn shows that a cut distance compatible function f needs not be convex. Take any compatible f and define $f'(W) = f(W) + \left(\frac{1}{2}f(U) + \frac{1}{2}f(U^{\varphi}) - f(V) + 1\right)$ for all all W such that $W \succeq V$. Such function is clearly still cut distance compatible, but no longer convex. This example works even if we restrict ourselves to graphons lying in an envelope of a fixed graphon W, since if we set $W(x,y) = 1$ if and only if $(x,y) \in [0, \frac{1}{4}]^2 \cup [\frac{1}{4}, \frac{1}{2}]^2$ and set $U' = \frac{U}{2}, V' = \frac{V}{2}$, then we have three graphons U', V', W such that $U', V' \preceq W$, $V' = \frac{U'+U'^{\varphi}}{2}$, but $V' \not\preceq U'$.

4.3. **Spectrum as a cut distance identifying parameter.** From the spectrum of graphons, we can define a partial order $\overset{S}{\preceq}$ over the set of graphons which we call the spectral quasiorder.

Definition 4.6. For a given graphon W denote its by

$$\lambda_1(W) \geq \lambda_2(W) \geq \lambda_3(W) \geq \ldots \geq 0 \geq \ldots \geq \widetilde{\lambda_3}(W) \geq \widetilde{\lambda_2}(W) \geq \widetilde{\lambda_1}(W) .$$

We write $W \overset{S}{\preceq} U$ if $\lambda_i(W) \leq \lambda_i(U)$ and $\widetilde{\lambda}_i(W) \geq \widetilde{\lambda}_i(U)$ for all $i = 1, 2, 3, \ldots$. Further we write $W \overset{S}{\prec} U$ if $W \overset{S}{\preceq} U$ and at least one of the above inequalities is strict for at least one eigenvalue. Then $\overset{S}{\preceq}$is a quasiorder on $\mathcal{W}$, which we call the *spectral quasiorder*.

We will prove that structurdness order implies spectral quasiorder. But first we need some lemma.

Lemma 4.7. *Let W_n and U be graphons on Ω^2 such that $W_n \xrightarrow{w^*} U$. Let $u, v \in L^2(\Omega)$. Then we have $\langle W_n u, v\rangle \to \langle U u, v\rangle$.*

Proof. Since stepfunctions are dense in $L^2(\Omega)$, and since the forms $\langle W_n\cdot,\cdot\rangle$ and $\langle U\cdot,\cdot\rangle$ are obviously bilinear, it suffices to prove this for indicator functions of sets, $u = \mathbf{1}_A$, $v = \mathbf{1}_B$ (where $A, B \subset \Omega$). But in that case $\langle W_n u, v\rangle = \int_{A\times B} W_n$ and $\langle Uu, v\rangle = \int_{A\times B} U$. The statement follows since $W_n \xrightarrow{w^*} U$. □

Theorem 4.8. *If $U \prec W$, then $U \overset{S}{\prec} W$.*

Proof. Consider the sequence of versions $W^{\pi_n} \xrightarrow{w^*} U$. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0$ be the positive eigenvalues of U with associated eigenvectors $u_1, u_2, u_3, \ldots$, and let $\beta_1 \geq \beta_2 \geq \beta_3 \geq \ldots \geq 0$ be the positive eigenvalues of W. First, we will prove that for any given $\epsilon > 0$, it holds $\beta_i \geq \lambda_i - \epsilon$. By the Maxmin characterization of eigenvalues, we have

$$\beta_k = \max_{H \text{ s.t. } \dim(H)=k} \min_{g\in H, \|g\|=1} \langle Wg, g\rangle .$$

Fix the space $H = \operatorname{span}\left\{u_1^{\pi_n^{-1}}, u_2^{\pi_n^{-1}}, \ldots, u_k^{\pi_n^{-1}}\right\}$, where $u_1^{\pi_n^{-1}}(x) = u_1(\pi_n^{-1}(x))$. Then, we have

$$\beta_k \geq \min_{g\in H, \|g\|=1} \langle Wg, g\rangle . \tag{4.14}$$

Furthermore, by Lemma 4.7 we can find n large enough so that for all i and j, we have

$$\left|\langle W^{\pi_n} u_i, u_j\rangle - \langle Uu_i, u_j\rangle\right| < \frac{\epsilon}{k^2 - k}.$$

Now, for $g \in H$ that realizes the minimum in (4.14), we can write its orthogonal decomposition as $g = \sum_{i=1}^k c_i u_i^{\pi_n^{-1}}$, where $\sum_{i=1}^k c_i^2 = 1$. Thus, we obtain

$$\begin{aligned}
\langle Wg, g\rangle &= \langle W^{\pi_n} g^{\pi_n}, g^{\pi_n}\rangle \\
&= \left\langle W^{\pi_n} \sum_{i=1}^k c_i u_i, \sum_{i=1}^k c_i u_i \right\rangle \\
&= \sum_{i,j=1}^k c_i c_j \langle W^{\pi_n} u_i, u_j\rangle \\
&> \sum_{i=1}^k c_i^2 \left(\langle Uu_i, u_i\rangle - \frac{\epsilon}{k^2-k}\right) + \sum_{i\neq j}^k c_i c_j \left(\langle Uu_i, u_j\rangle - \frac{\epsilon}{k^2-k}\right) \\
&\geq \sum_{i=1}^k c_i^2 \left(\lambda_i - \frac{\epsilon}{k^2-k}\right) - \sum_{i\neq j}^k c_i c_j \left(\frac{\epsilon}{k^2-k}\right) \\
&\geq \lambda_k - \epsilon.
\end{aligned}$$

Thus, by equation (4.14) we have $\beta_k \geq \lambda_k - \epsilon$.

A similar argument can be used for the negative eigenvalues $\widetilde{\lambda_1} \leq \widetilde{\lambda_2} \leq \widetilde{\lambda_3} \leq \ldots \leq 0$ of U and $\widetilde{\beta_1} \leq \widetilde{\beta_2} \leq \widetilde{\beta_3} \leq \ldots \leq 0$ of W to show that $\widetilde{\beta_i} \leq \widetilde{\lambda_i} + \epsilon$. That implies $U \overset{S}{\preceq} W$. To show that the inequality is strict for at least one eigenvalue, assume by contradiction that

$$\|W\|_2^2 = \sum \lambda_i(W)^2 = \sum \lambda_i(U)^2 = \|U\|_2^2 .$$

Notice that $\|\cdot\|_2^2$ is a strict convex function. Thus, by Theorem (4.14) we have that $\|\cdot\|_2^2$ is cut-distance identifying. Thus, that is a contradiction which finishes the proof. □

4.4. **Subgraph densities as cut distance identifying parameters.** In this section, we investigate for which finite graphs H, the graphon parameter $t(H,\cdot):\mathcal{W}\to\mathbb{R}$ has the cut distance identifying property. As we shall see, this problem is tightly linked to the Siderenko conjecture and to the Forcing conjecture, which we shall recall now. Let us say that a graph H is *Sidorenko* if for every graphon we have $t(H,W)\geq\left(\int\int W(x,y)\right)^{e(H)}$. In other words, H is Sidorenko if and only if $t(H,W)\geq t\left(H,W^{\Join\mathcal{J}}\right)$ for the trivial partition $\mathcal{J}=[0,1]$. Further, we say that a Sidorenko graph is *forcing* if an equality in $t(H,W)\geq t\left(H,W^{\Join\mathcal{J}}\right)$ implies that W is a constant function. In this language, Sidorenko's conjecture asserts that H is Sidorenko if and only if H is bipartite. Similarly, the Forcing conjecture asserts H is forcing if an only if H is bipartite and contains a cycle. Sidorenko's conjecture was asked independently by Erdős and Simonovits and by Sidorenko, [20, 21]. The Forcing conjecture was first hinted in [22, Section 5]. The direction that if H is Sidorenko (or forcing) then H is bipartite is trivial. The other direction is open, despite being known in many special cases, see [5, 15, 13, 4, 10, 14, 23].

If H is such that $t(H,\cdot):\mathcal{W}\to\mathbb{R}$ has the cut distance identifying property, then H is Sidorenko by Lemma 3.30. As we shall see in Example 4.11, the path P_3 shows that the converse direction does not hold. We however do not know of a forcing graph H for which $t(H,\cdot):\mathcal{W}\to\mathbb{R}$ does not have the cut distance identifying property.

Problem 4.9. Characterize graphs H for which $t(H,\cdot):\mathcal{W}\to\mathbb{R}$ is a cut distance identifying graphon parameter.

We can give a partial characterization of such graphs by using the notion of so-called norm graphs from Hatami [11]. We say that a graph H is a *norm graph* if the function $t(H,\cdot):\mathcal{W}\to\mathbb{R}$ is a norm, which is basically equivalent to condition $t(H,U)+t(H,V)\geq t(H,U+V)$ for any two graphons U,V. Note that Sidorenko's conjecture is clearly true for any such H. This follows easily e.g. from Lemma 4.5. Norm graphs correspond to a natural attempt for proving Sidorenko's conjecture. Theorem 4.4 basically verifies that this intuition is correct - every norm induced by a norm graph H is continuous in L_1 and is, thus, cut distance compatible. We have, however, observed in Section 4.2 that the notion of linear combinations of graphons is not the same as the notion of structurdness. This leads to the following question.

Conjecture 4.10. *Is it true that a graph H is a norm graph if and only if it is cut distance compatible?*

The norm graphs described by Hatami in [11] include e.g. the complete graphs $K_{m,n}$. Moreover, Theorem 2.10 from the same article implies that among trees only stars $K_{1,n}$ are norm graphs. To support the possibility that the answer to our question is positive, we show that the path on three edges, the smallest bipartite graph that is not a norm graph, is not cut distance compatible too.

Example 4.11. There are two stepgraphons $U_1\succeq U_2$ such that $t(P_3,U_1)\leq t(P_3,U_2)$.

Proof. The two stepgraphons U_1,U_2 with five steps of equal sizes are shown in Figure 4.2 in the form of the corresponding graphs. The steps with density zero that include all diagonal steps correspond to non-edges and values $1,a,b$ correspond to appropriate densities. Note that $U_2\preceq U_1$ is certified by a sequence of bijections interlacing the two steps of U_1 that correspond to the top and the bottom vertex in Figure 4.2. Then we compute the weighted number of homomorphisms in U_1:

$$w(P_3,U_1)=2\cdot\left(1^3+a^3+b^3+b^2\cdot1+b\cdot1^2\right).$$

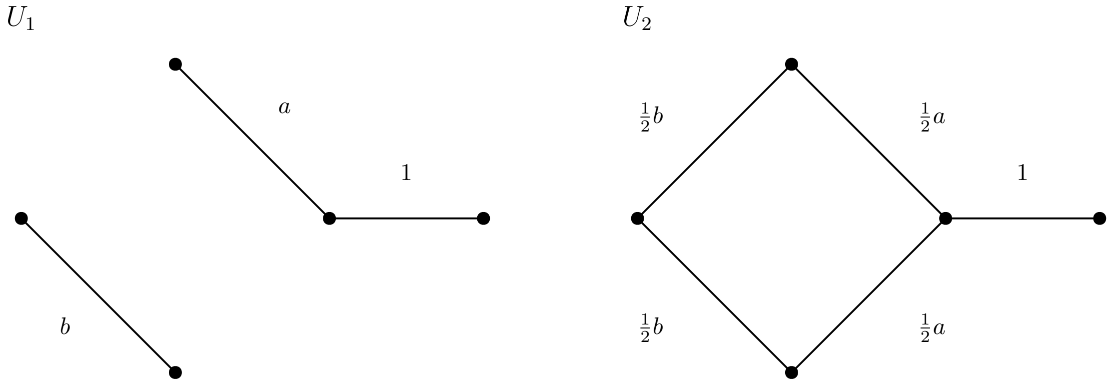

FIGURE 4.2. Two stepgraphons with five steps of equal sizes from Example 4.11.

On the other hand we compute that

$$w\left(P_3, U_2\right) \geq 2 \cdot \left(1^3 + 2 \cdot 1 \cdot \frac{b}{2} \cdot \frac{a}{2} + 2 \cdot \frac{b}{2} \cdot 1^2\right).$$

Now it suffices to set $a = 10^{-2}$ and $b = 10^{-3}$. □

We also show another simple proof of the compatibility of stars from the notion of flat measures.

Proposition 4.12. *For each $\ell \in \mathbb{N}$, the graphon parameter $t\left(K_{1,\ell}, \cdot\right) : \mathcal{W} \to \mathbb{R}$ is cut distance compatible.*

Proof. The key is to observe note that for a graphon Γ, we have $t\left(K_{1,\ell}, \Gamma\right) = \int_{x \in [0,1]} x^{\ell} \mathrm{d}\Upsilon_{\Gamma}$, where Υ_{Γ} is defined by (3.13). So, suppose that $U \preceq W$. By Proposition 3.19, we have that Υ_U is at least as flat as Υ_W. Let Λ be a measure on $[0,1]^2$ as in Definition 3.18 that witnesses this. We have

$$\begin{aligned}
t\left(K_{1,\ell}, U\right) &= \int_{x \in [0,1]} x^{\ell} \mathrm{d}\Upsilon_U \\
\boxed{\text{by (3.14)}} \quad &= \int_{x \in [0,1]} \left(\int_{y \in [0,1]} y \mathrm{d}\Phi_W\right)^{\ell} \mathrm{d}\Phi_U \\
\boxed{\text{Jensen's inequality}} \quad &\leq \int_{x \in [0,1]} \int_{y \in [0,1]} y^{\ell} \, \mathrm{d}\Phi_W \mathrm{d}\Phi_U \\
&= \int_{(x,y) \in [0,1]^2} y^{\ell} \mathrm{d}\Lambda \\
&= \int_{y \in [0,1]} y^{\ell} \mathrm{d}\Phi_W \\
&= t\left(K_{1,\ell}, W\right) \; .
\end{aligned}$$

□

Cut-distance-identifying parameters compare to cut-distance-compatible parameters as forcing graphs to Sidorenko graphs. Recall that for a forest F, the parameter $t(F, \cdot)$ is not cut-distance identifying. This is because all p-regular graphons satisfy $t\left(F, \cdot\right) = p^{e(F)}$. On the other hand, a simple corollary of the fact that graphon spectrum is cut distance identifying is that the density of all even cycles is a cut distance identifying parameter too.

Proposition 4.13. *For each $\ell \in \{2, 3, 4, \ldots\}$, the graphon parameter $t(C_{2\ell}, \cdot) : \mathcal{W} \to \mathbb{R}$ is cut distance identifying.*

Before giving a proof, let us note Lemma 11 in [6] is equivalent to the case $\ell = 2$ of the proposition. However, the proof in [6] does not seem to generalize to higher ℓ, in which cases Proposition 4.13 seems to be new.

Proof of Proposition 4.13. To prove the proposition, suppose that ℓ is fixed and $W_1 \prec W_2$ are two graphons. Theorem 4.8 tells us that $W_1 \overset{S}{\prec} W_2$. That is, the sum of the (2ℓ)-th powers of eigenvalues of W_1 is strictly smaller than that of W_2. The statement now follows from (2.2). ☐

4.5. **Revising the parameter** $\mathrm{INT}_f(\cdot)$**.** Recall that the main result of [8] implies that $\mathrm{INT}_f(\cdot)$ is cut-distance-identifying if $f : [0, 1] \to \mathbb{R}$ to is a strictly convex continuous function. As we show here, the assumption of continuity is just an artifact of the proof in [8].

Theorem 4.14. *Suppose that $f : [0, 1] \to \mathbb{R}$ is a strictly convex function. Then $\mathrm{INT}_f(\cdot)$ is cut-distance-identifying.*

Proof. The key is to observe note that for a graphon Γ, we have $\mathrm{INT}_f(\Gamma) = \int_{x\in[0,1]} f(x)\mathrm{d}\Phi_\Gamma$, where Φ_Γ is defined by (3.12). So, suppose that $U \prec W$. By Proposition 3.12, we have that Φ_U is strictly flatter than Φ_W. Let Λ be a measure on $[0, 1]^2$ as in Definition 3.18 that witnesses this. We have

$$\begin{aligned}
\mathrm{INT}_f(U) &= \int_{x\in[0,1]} f(x)\mathrm{d}\Phi_U \\
\boxed{\text{by (3.14)}} \quad &= \int_{x\in[0,1]} f\left(\int_{y\in[0,1]} y\mathrm{d}\Phi_W\right)\mathrm{d}\Phi_U \\
\boxed{\text{Jensen's inequality}} \quad &< \int_{x\in[0,1]}\int_{y\in[0,1]} f(y)\,\mathrm{d}\Phi_W\mathrm{d}\Phi_U \\
&= \int_{(x,y)\in[0,1]^2} f(y)\mathrm{d}\Lambda \\
&= \int_{y\in[0,1]} f(y)\mathrm{d}\Phi_W \\
&= \mathrm{INT}_f(W)\ .
\end{aligned}$$

☐

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