A VERSION OF THE LOEBL-KOMLÓS-SÓS CONJECTURE FOR SKEWED TREES

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Abstract. Loebl, Komlós, and Sós conjectured that any graph with at least half of its vertices of degree at least k contains every tree with at most k edges. We propose a version of this conjecture for *skewed* trees, i.e., we consider the class of trees with at most k edges such that the sizes of the colour classes of the trees have a given ratio. We show that our conjecture is asymptotically correct for dense graphs. The proof relies on the regularity method. Our result implies bounds on Ramsey number of several trees of given skew.

1. Introduction

Many problems in extremal graph theory ask whether a certain density condition imposed on a host graph guarantees the containment of a given subgraph H. Typically, the density condition is expressed by average or minimum degree. Classical examples of results of this type are Turán's Theorem which determines the average degree that guarantees the containment of the complete graph K_r and the Erdős-Stone Theorem [ES46] which essentially determines the average degree condition guaranteeing the containment of a fixed non-bipartite graph H. On the other hand, for a general bipartite graph H the problem is wide open. For H being a tree, the long-standing conjecture of Erdős and Sós from 1962 asserts that an average degree greater than $k-1$ forces a copy of any tree of order $k+1$. (Note that a trivial bound for the average degree guaranteeing containment of such a tree is $2k$, since in such a graph we can find a subgraph of minimum degree at least k and then embed the tree greedily.) A solution of this conjecture for large k, based on an extension of the Regularity Lemma, has been announced in the early 1990's by Ajtai, Komlós, Simonovits, and Szemerédi [AKSS].

The problem of containing a tree of a given size has also been studied in settings with different density requirements. Recently, Havet, Reed, Stein and Wood conjectured that a graph of minimum degree at least $\lfloor \frac{2k}{3} \rfloor$ and maximum degree at least k contains a copy of any tree of order $k+1$ and provided some evidence for this conjecture.

Another type of density requirement, on which we focus in this paper, is considered in the Loebl-Komlós-Sós conjecture. The conjecture asserts that at least half of the vertices of degree at least k guarantees containment of a tree of order $k+1$. In other words, the requirement of average degree in Erdős-Sós conjecture is replaced by a median degree condition.

The conjecture has been solved exactly for large dense graphs [Coo09, HP16] and proved to be asymptotically true for sparse graphs [HKP+17a, HKP+17b, HKP+17c, HKP+17d] (see [HPS+15] for an overview).

All these conjectures are known to be best possible. In particular, the Loebl-Komlós-Sós conjecture is tight for paths. To observe this, consider a graph consisting of a disjoint union of copies of a graph H of order $k+1$ consisting of a clique of size $\lfloor \frac{k+1}{2} \rfloor - 1$, an independent set on the remaining vertices, and the complete bipartite graph between the two sets. Almost half of the vertices of this graph have degree k, but it does not contain a path on $k+1$ vertices as a subgraph.

A natural question is whether fewer vertices of degree k suffice when one considers only a restricted class of trees. Specifically, Simonovits asked [personal communication], whether it is the case for trees

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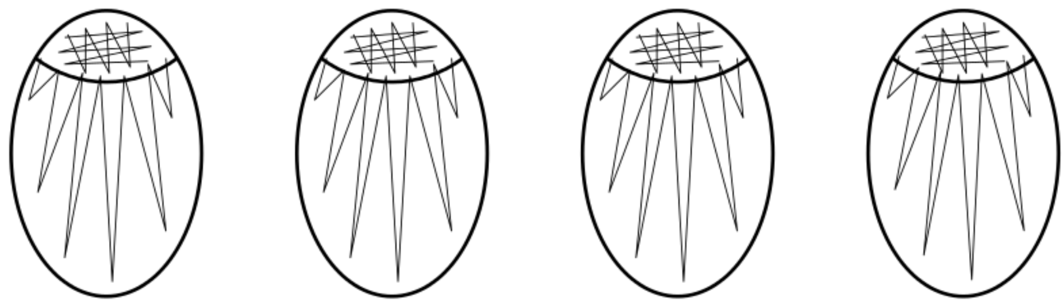

FIGURE 1.1. The graph showing the tightness of Conjecture 1 is a disjoint union of graphs of order $k+1$.

of given *skew*, that is, the ratio of sizes of the smaller and the larger colour classes is bounded by a constant smaller than 1. We propose the following conjecture.

Conjecture 1. *Any graph of order n with at least rn vertices of degree at least k contains every tree of order at most $k+1$ with colour classes V_1, V_2 such that $|V_1| \leq r \cdot (k+1)$.*

If true, our conjecture is best possible for the similar reason as the Loebl-Komlós-Sós conjecture. Indeed, given $r \in (0, 1/2]$, consider a graph consisting of a disjoint union of copies of a graph H with $k+1$ vertices consisting of a clique of size $\lfloor r(k+1) \rfloor - 1$, an independent set on the remaining vertices and the complete bipartite graph between the two sets (see Figure 1.1). Such a graph does not contain a path on $2\lfloor r(k+1) \rfloor$ vertices (or, to give an example of a tree of maximal order, a path on $2\lfloor r(k+1) \rfloor$ vertices with one end-vertex identified with the centre of a star with $k+1-2\lfloor r(k+1) \rfloor$ leaves).

We verified that the conjecture is true both for paths and for trees of diameter at most five [Roz18]. In this paper we prove that Conjecture 1 is asymptotically correct for dense graphs.

Theorem 1. *Let $0 < r \leq 1/2$ and $q > 0$. Then for any $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $k \geq qn$, any graph of order n with at least rn vertices of degree at least $(1+\eta)k$ contains every tree of order at most k with colour classes V_1, V_2 such that $|V_1| \leq rk$.*

This extends the main result of [PS12], which is a special case of Theorem 1 for $r = 1/2$. While we use and extend some of their techniques, our analysis is more complex. As in [PS12], we partition the tree into small rooted subtrees, which we then embed into regular pairs of the host graph. In order to connect those small rooted trees, we need two adjacent clusters with adquate average degree to those regular pairs, which typically will be represented by a matching in the cluster graph. Hence, we need a matching in the cluster graph that is as large as possible. For this aim, we use disbalanced regularity decomposition (see [HLT02]), placing large degree vertices into smaller clusters than the remaining vertices, hence covering as many low degree vertices as possible by this matching. We then consider several possible embedding configurations in the regularity decomposition, depending on the structure of the cluster graph, in particular depending on the properties of the adjacent clusters with suitable average degree to the optimal matching.

The structure of the rest of the paper is the following; in Section 2, we introduce notions and results related to regularity. In Section 3 we introduce tools necessary for the proof of Theorem 1 which we present in Section 4. In Sections 5 , 6 and 7 we prove our main tools from Section 3; Propositions 10 and 11. In Section 8 we discuss implications of our conjecture for Ramsey numbers of trees and further research directions.

2. REGULARITY

In this section we introduce a notion of *regular pair*, state the regularity lemma and introduce a standard method of embedding a tree into a regular pair.

Let G be a graph and let X, Y be disjoint subsets of its vertices and $\varepsilon > 0$. We define $E(X, Y)$ as the set of edges of G with one end in X and one end in Y and the *density* of the pair (X, Y) as $d(X, Y) = \frac{|E(X,Y)|}{|X||Y|}$. The *degree* $\deg(x)$ of a vertex x is the number of its neighbours. By $\deg(x, X)$ we denote the number of neighbours of x in the set X. We say that (X, Y) is an *ε-regular pair*, if for every $X' \subseteq X$ and $Y' \subseteq Y$, $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, $d(X', Y') - d(X, Y) \leq \varepsilon$.

The following lemma states a well-known fact that subsets of a regular pair to some extent 'inherit' regularity of the whole pair.

Lemma 2. *Let G be a graph and (X, Y) be an ε-regular pair of density d in G. Let $X' \subseteq X$ and $Y \subseteq Y$ such that $|X'| \geq \alpha|X|$ and $|Y'| \geq \alpha|Y|$. Then, (X', Y') is an ε'-regular pair of density at least $d - \varepsilon$, where $\varepsilon' = \max(\varepsilon/\alpha, 2\varepsilon)$.*

We say that a partition $\{V_0, V_1, \dots, V_N\}$ of $V(G)$ is an *ε-regular partition*, if $|V_0| \leq \varepsilon|V(G)|$ all but at most εN^2 pairs (V_i, V_j), $i < j$, $i, j \in [N]$, are ε-regular. We call V_0 a *garbage set.* We call a regular partition *equitable* if $|V_i| = |V_j|$ for every $i, j \in [N]$.

Lemma 3 (Szeméredi regularity lemma)**.** *For every $\varepsilon > 0$ and $N_{min} \in \mathbb{N}$ there exists $N_{max} \in \mathbb{N}$ and $n_R \in \mathbb{N}$ such that every graph G on at least n_R vertices admits an ε-regular partition $\{V_0, \dots, V_N\}$, where $N_{min} \leq N \leq N_{max}$.*

Given an ε-regular pair (X, Y), we call a vertex $x \in X$ *typical* with respect to a set $Y' \subseteq Y$ if $\deg(x, Y') \geq (d(X, Y) - \varepsilon)|Y'|$. Note that from the definition of regularity it follows that all but at most $\varepsilon|X|$ vertices of X are typical with respect to any subset of Y of size at least $\varepsilon|Y|$. This observation can be strengthened as follows.

Lemma 4 (Variant of Proposition 4.5 in [Zha11])**.** *Let $\{V_0, V_1, \dots, V_N\}$ be an ε-regular partition of $V(G)$ and let $X = V_j$ for some $j \in [N]$. Then all but at most $\sqrt{\varepsilon}|X|$ vertices of a cluster X are typical w. r. t. all but at most $\sqrt{\varepsilon}N$ sets V_i, $i \in [N] \setminus j$. We call such vertices of X* ultratypical.

Proof. Suppose, for a contradiction, that there are more than $\sqrt{\epsilon}|X|$ vertices of a cluster X that are not typical to more than $\sqrt{\epsilon}N$ clusters. Then we have at least $\sqrt{\epsilon}|X| \cdot \sqrt{\epsilon}N = \epsilon|X|N$ pairs formed by a cluster and a vertex from X not typical to that cluster. This in turn means that there is a cluster Y such that the number of vertices not typical to Y is at least $\epsilon|X|$. But then the set of these vertices contradicts the regularity of the pair XY. □

Next lemma states that a tree can be embedded in a sufficiently large subset of a regular pair, each of the colour classes being embedded in one 'side'. Moreover we can prescribe embedding of a few vertices.

Lemma 5. *Let T be a tree with colour classes F_1 and F_2. Let $R \subseteq F_1$, $|R| \leq 2$ such that vertices of R do not have a common neighbour in T (if $|R| = 2$).*

Let $\varepsilon > 0$ and $\alpha > 2\varepsilon$. Let (X, Y) be an ε-regular pair in a graph G with density $d(X, Y) > 3\alpha$ such that $|F_1| \leq \varepsilon|X|$ and $|F_2| \leq \varepsilon|Y|$. Let $X' \subseteq X, Y' \subseteq Y$ be sets satisfying $|X'| > 2\frac{\varepsilon}{\alpha}|X|, |Y'| > 2\frac{\varepsilon}{\alpha}|Y|$.

Let φ be any injective mapping of vertices of R to vertices of X' with degree greater than $3\varepsilon|Y|$ in Y'. Then there exists extension of φ that is an injective homomorphism from T to (X, Y) satisfying $\varphi(F_1) \subseteq X'$ and $\varphi(F_2) \subseteq Y'$.

Proof. We embed vertices of $V(T) \setminus R$ into vertices of X' and Y' which are typical to Y' and X', respectively. Assume that we have already embedded some part of the tree in this way. We claim that every vertex of this partial embedding in X is incident with more than $\varepsilon|Y|$ vertices typical with respect to X' which have not been used for the partial embedding. Similarly, every vertex of the partial embedding in Y is incident with more than $\varepsilon|X|$ vertices typical with respect to Y', which have not been used for the partial embedding.

We give arguments only for vertices embedded into X, arguments for vertices embedded into Y are symmetric. For $\varphi(r) \in X$, $r \in R$, the claim follows from the fact that $\varphi(r)$ has more than $3\varepsilon|Y|$ neighbors in Y' and out of them, at most $\varepsilon|Y|$ are not typical with respect to X' and at most $\varepsilon|Y|$

have already been used for the partial embedding. Let $\varphi(v)$, $v \in V(T) \setminus R$ be a vertex of the partially constructed embedding and without loss of generality assume $\varphi(v) \in X'$. Since $\varphi(v)$ was chosen to be typical with respect to Y', it is adjacent to at least $(d-\varepsilon)|Y'|$ vertices of Y'. Again, out of these vertices, at most $\varepsilon|Y|$ are not typical with respect to X' and at most $\varepsilon|Y|$ have already been used for the partial embedding. Thus, $\varphi(v)$ is typical to at least $(d-\varepsilon)|Y'| - 2\varepsilon|Y| > ((d-\varepsilon)2\frac{\varepsilon}{\alpha} - 2\varepsilon)|Y|$. This is strictly greater than $\varepsilon|Y|$, since $d > 3\alpha$ and $\alpha > 2\varepsilon$.

It follows that if $|R| < 2$, we can construct embedding greedily.

If $|R| = 2$, $R = \{u, v\}$, we first embed vertices of a path connecting u and v, starting from u and embedding all but the last two internal vertices of a path into typical vertices, last embedded vertex being u'. Then we find an edge between the sets the set X'' of vertices of $N(u')$ which are typical to Y' and set Y'' of vertices of $N(v)$ which are typical to X'. Since, X'' and Y'' have size greater than $\varepsilon|X|$ and $\varepsilon|Y|$, respectively by our previous argument, from ε-regularity of (X, Y), it follows that there is an edge xy between X'' and Y''. We embed the last two internal vertices to x and y.

□

3. Preliminaries

We shall switch freely between a graph H and its corresponding cluster graph $\mathbf{H}$. For example $A \subseteq V(H)$ may as well denote a cluster in an original graph, as $A \in V(\mathbf{H})$ a vertex in the corresponding cluster graph. We shall freely use the term *clusters* in a cluster graph $\mathbf{H}$ to denote vertices of $\mathbf{H}$. If $\mathcal{S} \subseteq V(\mathbf{H})$ denotes a set of clusters, then $\bigcup \mathcal{S}$ denotes the corresponding union of vertices in the original graph H. If $A \in V(\mathbf{H})$ is a cluster and $\mathcal{S} \subseteq V(\mathbf{H})$ a set of clusters, then $\overline{\deg}(A, \mathcal{S})$ denotes the average degree of vertices in A to $\bigcup \mathcal{S}$ and $\overline{\deg}(A)$ stands short for $\overline{\deg}(A, V(\mathbf{H}))$.

We shall use the following notation. The class of all trees of order k is denoted as $\mathcal{T}_k$. For a graph G and two sets $A \in V(G)$ and $B \in V(G)$ let $G[A, B]$ denote the subgraph of G induced by all edges with one endpoint in A and the other in B.

Definition 6. Let $r \leq 1/2$. We say that a graph H is an *r-skew LKS-graph with parameters* $(k, \eta, \varepsilon, d)$ if there exists a partition $\{L_1, \ldots, L_{m_L}, S_1, \ldots, S_{m_S}\}$ of $V(H)$ satisfying the following

1. $m_L \geq (1+\eta)m_S$,
2. all sets L_i have the same size and all sets S_i have the same size,
3. $r|S_j| = (1-r)|L_i|$ for all i, j,
4. each (L_i, L_j), $i, j \in [m_L]$ and each (L_i, S_j), $i \in [m_L], j \in [m_S]$ is an ε-regular pair of density either 0 or at least d,
5. there are no edges inside the sets and no edges between S_i and S_j for $i \neq j$,
6. average degree of vertices in each L_i is at least $(1+\eta)k$.

We call the sets L_i, $i \in [m_L]$, the L-clusters. Similarly, we call the sets S_i, $i \in [m_S]$, the S-clusters.

Let $\mathbf{H}$ be the graph with vertex set $\{L_1, \ldots, L_{m_L}, S_1, \ldots, S_{m_S}\}$ and with an edge (L_i, L_j), (L_i, S_j) whenever (L_i, L_j) or (L_i, S_j), respectively forms an ε-regular pair of positive density in H. Observe that for any edge (L_i, L_j) we have $\overline{\deg}(L_i, L_j) = \overline{\deg}(L_j, L_i)$, but for any edge (L_i, S_j) we have $r \cdot \overline{\deg}(L_i, S_j) = (1-r) \cdot \overline{\deg}(S_j, L_i)$. We call $\mathbf{H}$ the r-skew LKS-cluster graph. We use a dot instead of an explicit parameter when the value of the parameter is not relevant in the given context.

Proposition 7. *Let H be an r-skewed LKS graph of order n with parameters $(\cdot, \cdot, \varepsilon, \cdot)$ and let $\mathbf{H}$ be its corresponding cluster graph.*

(1) Let C and D be an L-cluster and an S-cluster of $\mathbf{H}$, respectively. Then $|C| \leq n/|V(\mathbf{H})|$ and $|D| \leq \frac{n}{r|V(\mathbf{H})|}$.

(2) If $v \in V(H)$ is an ultratypical vertex and $\mathcal{S} \subseteq V(\mathbf{H})$, then $\deg(v, \bigcup \mathcal{S}) \geq \overline{\deg}(C, \mathcal{S}) - 2\sqrt{\varepsilon}n/r$, where C is the cluster of $\mathbf{H}$ containing v.

Proof.

(1) The first inequality follows from the fact that the size of L-clusters is always at most the size of S-clusters. Then we compute $|D| = \frac{1-r}{r}|C| \leq \frac{n}{r|V(\mathbf{H})|}$.
(2) If v is ultratypical, there are at most $\sqrt{\varepsilon}|V(\mathbf{H})|$ clusters D in $\mathbf{H}$ such that v is not typical to D. Denote by $\mathcal{D}$ the set of those clusters. Then by (1) we have $|\bigcup \mathcal{D}| \leq |\mathcal{D}| \cdot n/(r|V(\mathbf{H})|) \leq \sqrt{\varepsilon}n/r$. Then

$$\begin{aligned}\deg(v, \bigcup \mathcal{S}) &\geq \deg(v, \bigcup(\mathcal{S} \setminus \mathcal{D})) \\ &\geq \overline{\deg}(C, \bigcup(\mathcal{S} \setminus \mathcal{D})) - \varepsilon n \\ &\geq \overline{\deg}(C, \bigcup \mathcal{S}) - |\bigcup \mathcal{D}| - \varepsilon n \\ &\geq \overline{\deg}(C, \bigcup \mathcal{S}) - 2\sqrt{\varepsilon}n/r \ .\end{aligned}$$

□

Definition 8. [HKP+17d, Definition 3.3] Let $T \in \mathcal{T}_{k+1}$ be a tree rooted at r. An *ℓ-fine partition of T is a quadruple $(W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$, where $W_A, W_B \subseteq V(T)$ and $\mathcal{D}_A$ and $\mathcal{D}_B$ are families of subtrees of T such that*

(1) the three sets W_A, W_B and $\{V(T^*)\}_{T^* \in \mathcal{D}_A \cup \mathcal{D}_B}$ partition $V(T)$ (in particular, the trees in $T^* \in \mathcal{D}_A \cup \mathcal{D}_B$ are pairwise vertex disjoint),
(2) $r \in W_A \cup W_B$,
(3) $\max\{|W_A|, |W_B|\} \leq 336k/\ell$,
(4) for $w_1, w_2 \in W_A \cup W_B$ the distance $\mathrm{dist}(w_1, w_2)$ is odd if and only if one of them lies in W_A and the other one in W_B,
(5) $v(T^*) \leq \ell$ for every tree $T^* \in \mathcal{D}_A \cup \mathcal{D}_B$,
(6) $V(T^*) \cap N(W_B) = \emptyset$ for every $T^* \in \mathcal{D}_A$ and $V(T^*) \cap N(W_A) = \emptyset$ for every $T^* \in \mathcal{D}_B$,
(7) for each tree $T^* \in \mathcal{D}_A \cup \mathcal{D}_B$ $N_T(V(T^*)) \setminus V(T^*) \subseteq W_A \cup W_B$,
(8) $|N(V(T^*)) \cap (W_A \cup W_B)| \leq 2$ for each $T^* \in \mathcal{D}_A \cup \mathcal{D}_B$,
(9) if $N(V(T^*)) \cap (W_A \cup W_B)$ contains two distinct vertices z_1 and z_2 for some $T^* \in \mathcal{D}_A \cup \mathcal{D}_B$, then $\mathrm{dist}_T(z_1, z_2) \geq 6$,

Here we did not list all properties from [HKP+17d], only the ones we need.

Proposition 9. [HP16, Lemma 5.3] *Let $T \in \mathcal{T}_{k+1}$ be a tree rooted at a vertex R and let $\ell \in \mathbb{N}, \ell < k$. Then the rooted tree (T, R) has an ℓ-fine partition.*

Finally, we state two propositions that will be proved in Sections 5 and 7, respectively. The first proposition says that every LKS-graph contains one the four configurations, while the second proposition asserts that occurrence of these configurations implies containment of a given tree. Note that the first proposition is concerned only with the structure of the cluster graph, not the underlying graph, and could be stated in terms of weighted graphs instead.

Proposition 10. *Let H be a r'-skew LKS-graph $\mathbf{H}$ with parameters $(k, \eta, \cdot, \cdot)$ and let $\mathbf{H}$ be the corresponding cluster graph. We denote by $\mathcal{L}$ and $\mathcal{S}$, respectively, its set of L-clusters and S-clusters, respectively. For any numbers $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$ with $a_2 + b_1 = \tilde{r}k$, $\tilde{r} \leq r'$, there is a matching $\mathbf{M}$ in $\mathbf{H}[\mathcal{L}, \mathcal{S}]$ and two adjacent clusters $X, Y \in V(\mathbf{H})$ such that, setting $\mathcal{S}_M = \mathcal{S} \cap V(\mathbf{M})$ and $\mathcal{S}_1 = \{Z \in \mathcal{S} : \overline{\deg}(Z) \geq (\tilde{r} + r'\eta)k\} \setminus \mathcal{S}_{\mathcal{M}}$, one of the four following configurations occurs.*

A) $\overline{\deg}(X, \mathcal{S}_1 \cup \mathcal{S}_M) \geq a_2 \cdot (1 - \tilde{r})/\tilde{r} + \eta k/4$, and $\overline{\deg}(Y, \mathcal{L}) \geq \tilde{r}k + \eta k/4$,
B) $\tilde{r}a_1 > (1 - \tilde{r})a_2$, $\overline{\deg}(X, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \geq k + \eta k/4$ and $\overline{\deg}(Y, \mathcal{L}) \geq \tilde{r}k + \eta r'k/4$,
C) $\tilde{r}a_1 \leq (1 - \tilde{r})a_2$, $\overline{\deg}(X, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \geq k + \eta k/4$ and $\overline{\deg}(Y, \mathcal{L}) \geq b_1 + \eta r'k/4$,
D) $\tilde{r}a_1 \geq (1 - \tilde{r})a_2$, $b_1 \leq \tilde{r}^2k/(1 - \tilde{r})$, $\overline{\deg}(X, \mathcal{S}_M \cup \mathcal{L}) \geq k + \eta k/4$ and $\overline{\deg}(Y, \mathcal{L}) \geq b_1 + \eta k/4$, and moreover, the neighbourhood of X does not contain both endpoints of any edge from $\mathbf{M}$.

Proposition 11. *For each $\delta, q, d > 0$ and $\tilde{r}, r' \in \mathbb{Q}^+$ with $\tilde{r} \leq r' \leq 1/2$ there is $\varepsilon = \varepsilon(\delta, q, d, r') > 0$ such that for any $\tilde{N}_{max} \in \mathbb{N}$ there is a $\beta = \beta(\delta, q, r', \varepsilon, \tilde{N}_{max}) > 0$ and an $n_0 = n_0(\delta, q, \tilde{r}, \beta) \in \mathbb{N}$ such* *that for any $n \geq n_0$ and $k \geq qn$ the following holds.Let $\mathcal{D} = (W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$ be an βk-fine partition of a tree $T \in \mathcal{T}_k$ with colour classes T_1 and T_2 such that $|T_1| = \tilde{r}k$. Let H be an r'-skewed LKS-graph of order n,with parameters $(k, \delta, \varepsilon, d)$, let $\mathbf{H}$ be its corresponding cluster graph with $|V(\mathbf{H})| \leq \tilde{N}_{max}$ and $\mathcal{L}, \mathcal{S} \subseteq V(\mathbf{H})$ are sets of L-clusters and S-clusters, respectively.Let $\mathbf{M}$ be a matching in $\mathbf{H}$, let $\mathcal{S}_M = \mathcal{S} \cap V(\mathbf{M})$, $\mathcal{S}_1 := \{C \in \mathcal{S} \setminus V(\mathbf{M}) : \overline{\deg}(C) \geq (1+\delta)\tilde{r}k\}$. Let A and B be two clusters of $\mathbf{H}$ such that $AB \in E(\mathbf{H})$ and one of the following holds.*

A) $\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M) \geq a_2 \frac{1-\tilde{r}}{\tilde{r}} + \delta k$ *and* $\overline{\deg}(B, \mathcal{L}) \geq (\tilde{r} + \delta)k$

B) $\tilde{r}|V(\mathcal{D}_A) \cap V(T_2)| \geq (1-\tilde{r})|V(\mathcal{D}_A) \cap V(T_1)|$,
$\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \geq (1+\delta)k$, *and* $\overline{\deg}(B, \mathcal{L}) \geq (\tilde{r} + \delta)k$,

C) $\tilde{r}|V(\mathcal{D}_A) \cap V(T_2)| \leq (1-\tilde{r})|V(\mathcal{D}_A) \cap V(T_1)|$,
$\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \geq (1+\delta)k$, *and* $\overline{\deg}(B, \mathcal{L}) \geq |V(\mathcal{D}_B) \cap V(T_1)| + \delta k$,

D) $\tilde{r}|V(\mathcal{D}_A) \cap V(T_2)| \geq (1-\tilde{r})|V(\mathcal{D}_A) \cap V(T_1)|$, $|V(\mathcal{D}_B) \cap V(T_1)| \leq \frac{\tilde{r}^2}{(1-\tilde{r})}k$
$\overline{\deg}(A, \mathcal{S}_M \cup \mathcal{L}) \geq (1+\delta)k$, $\overline{\deg}(B, \mathcal{L}) \geq |V(\mathcal{D}_B) \cap V(T_1)| + \delta k$, *and moreover, the neighbourhood of A does not contain both endpoints of any edge from $\mathbf{M}$.*

Then $T \subseteq H$.

4. Proof of the theorem

Suppose r, q and η are fixed. If $r = 1/2$, then set $r' := r \in \mathbb{Q}$, $s := 1$, and $t := 2$. Otherwise, let $\rho := 1/2 - r > 0$ and $r' \in \mathbb{Q}$ be such that $r \leq r' \leq r(1 + \frac{\eta\rho q}{12})$ with $r' = s/t$, $s, t \in \mathbb{N}$ and $t \leq 12/(\eta\rho q r)$. Observe that $r' \leq 1/2$. Let $d := \frac{\eta^2 q^2 r'}{100}$. Let $\varepsilon = \min\{\frac{\eta d^2 q^2}{40}, \frac{1}{t}\varepsilon_{P11}(\frac{\eta r' q}{400}, q, d/2, r')\}$. Lemma 3 (Szemerédi regularity lemma) with input parameter $\varepsilon_{L3} := \varepsilon$ and $N_{min} := 1/\varepsilon$ outputs $n_R, N_{max} \in \mathbb{N}$. Set $\beta := \beta_{P11}(\frac{\eta r' q}{400}, q, r', t \cdot \varepsilon, tN_{max})$. Let $n_0 = \max\{2n_R, 2t \cdot N_{max}/\varepsilon, n_{0,P11}(\frac{\eta r' q}{400}, q, r', \beta)\}$ and let $n \geq n_0$. Suppose $k \geq qn$ is fixed. Let G be any graph on n vertices that has at least rn vertices of degree at least $(1+\eta)k$.

We first find a subgraph H of G of size $n'' \geq (1 - \eta q/2)(1 - 2\varepsilon)n$ which is an r'-skew LKS-graph with parameters $(k, \frac{\eta q}{100}, t \cdot \varepsilon, \frac{d}{2})$ and construct the corresponding LKS-cluster graph $\mathbf{H}$.

Erase $\eta \cdot qn/2$ vertices from the set of vertices that have degree smaller than $(1+\eta)k$ and let G' be the resulting graph of order $n' = n(1 - \eta q/2)$. Observe that for all $v \in V(G')$, we have $\deg_{G'}(v) \geq \deg_G(v) - \eta k/2$ and hence at least $rn \geq r'n'(1 + \eta q/4)$ vertices of G' have degree at least $(1 + \eta/2)k$.

We apply Szemerédi regularity lemma (Lemma 3) on G' and obtain an ε-regular equitable partition $V(G') = V_0 \cup V_1 \cup \cdots \cup V_N$. Erase all edges within sets V_i, between irregular pairs, and between pairs of density lower than d. Hence, we erase at most $N \cdot \binom{n'/N}{2} \leq \varepsilon (n')^2/2$ edges within the sets V_i, at most $\varepsilon N^2 \cdot \left(\frac{n'}{N}\right)^2 = \varepsilon (n')^2$ edges in irregular pairs, and at most $\binom{N}{2} \cdot d \cdot \left(\frac{n'}{N}\right)^2 \leq \frac{d}{2} \cdot (n')^2$ edges in pairs of density less than d. In total we have thus erased less than $d \cdot (n')^2 = \frac{\eta^2 q^2 r'}{100} \cdot (n')^2$ edges.

Call a set V_i an L-set if the average degree of its vertices is at least $(1 + \eta q/4)k$ and otherwise an S-set. We have at least $(1 + \frac{\eta q}{20})r'N$ L-sets. Indeed, during the erasing process, less than $\eta r' q n'/6$ vertices dropped their degree by more than $\eta k/8$. Therefore, now there are at least $(1 + \frac{\eta q}{12})r'n'$ vertices of degree at least $(1 + 3\eta/8)k$. By regularity, in each S-set V_i there are at most $\varepsilon|V_i|$ of those vertices, as otherwise they form a subset of V_i of substantial size and thus the S-set V_i would have average degree at least $(1 + 3\eta/8)k - \varepsilon n' > (1 + \eta/4)k$. So we can have at most $\varepsilon n'$ vertices of degree at least $(1 + 3\eta/8)k$ distributed among all S-sets and at most $\varepsilon n'$ of them contained in V_0. Hence, at least $(1 + \frac{\eta q}{20})r'n'$ vertices of degree at least $(1 + 3\eta/8)k$ must be contained in L-sets, producing thus at least $(1 + \frac{\eta q}{20})r'N$ L-sets.

We subdivide any L-set into $t - s$ sets of the same size, which we call *L-clusters*, adding at most $t - s - 1$ leftover vertices to the garbage set V_0. Similarly, we subdivide any S-set into s sets, which we call *S-clusters*. In this way we have $(1 - r')|C| = r'|D|$ for any L-cluster C, and any S-cluster D. By Lemma 2, if (V_i, V_j) is ε-regular and $C \subseteq V_i$ and $D \subseteq V_j$ are L or S clusters, then (C, D), is a ε'-regular pair for $\varepsilon' = t\varepsilon$ with density at least $d' := d - \varepsilon$. Observe that by the choice of n_0, we added in total less than $t \cdot N \leq \varepsilon n'$ vertices to the garbage set V_0. We delete at most $2\varepsilon n'$ vertices of the enlarged set V_0. Any L-cluster is a relatively large subset of the L-set it comes from, and thus basically inherits the

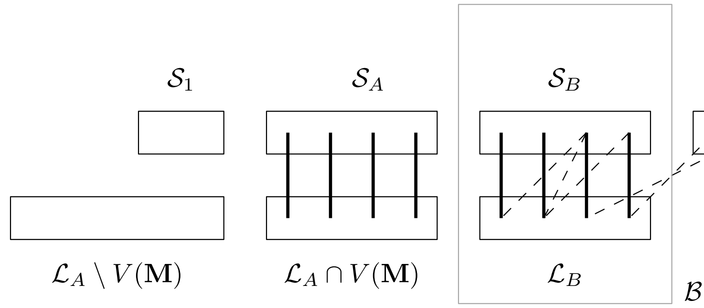

FIGURE 5.1. Various subsets of $\mathbf{H}$ used in the proof of Proposition 10.

average degree of the set it comes from. Together with the deletion of the enlarged garbage set, we obtain that each L-cluster has now average degree at least $(1 + \eta q/4)k - 3\varepsilon n' \geq (1 + \eta q/5)k$.

Denote by m_L the number of L-clusters and by m_S the number of S-clusters. We have $m_L \geq (1 + \frac{\eta q}{20})r'N \cdot (t-s)$, as each L-set divided in $t-s$ L-clusters. Similarly, we obtain $m_S < (1-r')sN$. Therefore,

$$\begin{aligned} m_L &\geq (1+\eta q/100)m_L/2 + (1-\eta q/100)(1+\eta q/20)\cdot r'N \cdot (t-s)/2 \\ &> (1+\eta q/100)m_L/2 + (1+\eta q/100)\cdot \frac{s}{t}\cdot \frac{m_S}{s(1-s/t)}\cdot (t-s)/2 \\ &= (1+\eta q/100)m_L/2 + (1+\eta q/100)\cdot \frac{m_S}{t-s}\cdot (t-s)/2 \\ &= (1+\eta q/100)(m_L+m_S)/2\ . \end{aligned}$$

Finally, we delete all edges between S-clusters. We denote by L the set of vertices contained in L-clusters and by S the set of vertices contained in S-clusters.

Let H be the resulting graph. By construction, it is an r'-skew LKS-graph of order n'', where $(1-2\varepsilon)n' \leq n'' \leq n'$, with parameters $(k, \frac{\eta q}{100}, \varepsilon', d/2)$. The vertex set of the corresponding cluster graph $\mathbf{H}$ consists of the L- and S-clusters defined above, with edges corresponding to ε'-regular pairs of density at least $d/2$ in H. Observe that $|V(\mathbf{H})| \leq t \cdot N_{max}$.

After having processed the host graph, we turn our attention to the tree. Let T be any tree of order k with colour classes T_1 and T_2 and $|T_1| \leq rk \leq r'k$. Pick any vertex $R \in V(T)$ to be the root of T. Applying Proposition 9 on T with parameter $\ell_{P9} := \beta k$, we obtain its βk-fine partition $\mathcal{D} = (W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$. Without loss of generality, assume that $W_A \subseteq V(T_2)$. Let $\tilde{r} := |V(T_1) \setminus W_B|/k$. We then apply Proposition 10 with $\eta_{P10} := \eta q/100$, $r'_{P10} := r'$, $k_{P10} := k$, $n_{P10} := n''$, $H_{P10} := \mathbf{H}$, for $vu \in \mathbf{H}$, $a_1 := |V(\mathcal{D}_A) \cap V(T_2)|$, $a_2 := |V(\mathcal{D}_A) \cap V(T_1)|$, $b_1 := |V(\mathcal{D}_B) \cap V(T_1)|$, $b_2 := |V(\mathcal{D}_B) \cap V(T_2)|$, $\tilde{r}_{P10} := \tilde{r}$. We obtain a matching $\mathbf{M} \subseteq E(\mathbf{H})$ and two adjacent clusters $A, B \in V(\mathbf{H})$ satisfying one of four configurations.

For any of these four possible configurations, Proposition 11 with input $\delta_{P11} := \frac{\eta r' q}{400}$, $q_{P11} := q$, $d_{P11} := d/2$, $\varepsilon_{P11} := \varepsilon'$, $\tilde{N}_{max,P11} := tN_{max}$, $H_{P11} := H$, $\mathbf{H}_{P11} := \mathbf{H}$, and further input as in Proposition 10, gives an embedding of T in $H \subseteq G$, proving Theorem 1.

5. Proof of Proposition 10

We will prove Proposition 10 in several steps. We start by defining the desired matching $\mathbf{M}$ as well as several other subsets of $\mathbf{H}$.

Let $\mathbf{M} \subseteq \mathbf{H}[\mathcal{L}, \mathcal{S}]$ be a matching minimising the number of vertices in the set $\mathcal{S}_0 := \{X \in \mathcal{S} : \overline{\deg}(X) < (\tilde{r} + r'\eta/2)k\}$. It follows that $\mathcal{S}_1 = \mathcal{S} \setminus (\mathcal{S}_M \cup \mathcal{S}_0)$.

We define $\mathcal{B} \subseteq V(\mathbf{M})$ as the set of those clusters X, for which there is an alternating path $P = X_1X_2 \ldots X_k$, such that $X_1 \in \mathcal{S}_0$, $X_k = X$, $X_{2i} \in \mathcal{L}$, $X_{2i+1} \in \mathcal{S}_M$, $\{X_{2i}, X_{2i+1}\} \in \mathbf{M}$. Also let $\mathcal{L}_B = \mathcal{L} \cap \mathcal{B}$ and $\mathcal{S}_B = \mathcal{S}_M \cap \mathcal{B}$. Then we define $\mathcal{A} = V(\mathbf{M}) \setminus \mathcal{B}$, $\mathcal{L}_A = \mathcal{L} \setminus \mathcal{L}_B$, $\mathcal{S}_A = \mathcal{S}_M \setminus \mathcal{S}_B$.

Claim 12. *For all $X \in \mathcal{S}_B$ we have $\overline{\deg}(X) < (\tilde{r} + r'\eta/2)k$. Also, there are no edges between clusters from $\mathcal{L}_A$ and $\mathcal{S}_0 \cup \mathcal{S}_B$.*

Proof. If the first statement was not true, the symmetric difference of $\mathbf{M}$ and an alternating path between X and a vertex in $\mathcal{S}_0$ would yield a matching contradicting the choice of $\mathbf{M}$ as a matching minimising the size of $\mathcal{S}_0$.

If the second statement was not true, we would have an alternating path ending at X which is a contradiction with the definition of $\mathcal{L}_A$. □

Now we are going to define yet another subsets of $\mathcal{L}$ based on the average degrees of the clusters.

$$\mathcal{L}^* := \{X \in \mathcal{L} : \overline{\deg}(X, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k\},$$

$$\mathcal{L}^+ := \{X \in \mathcal{L} \setminus \mathcal{L}^* : \overline{\deg}(X, \mathcal{S}_M \cup \mathcal{S}_1) \geq (1 - \tilde{r} + \eta/2)k\}.$$

Next, we define $\mathcal{L}_A^* := \mathcal{L}^* \cap \mathcal{L}_A$ and $\mathcal{L}_A^+ := \mathcal{L}^+ \cap \mathcal{L}_A$. We have $\mathcal{L}_A^* = \mathcal{L}_A \setminus \mathcal{L}_A^+$ by Claim 12.We define $\mathcal{L}_B^+$ and $\mathcal{L}_B^*$ in a similar way. Finally, let

$$\mathcal{N} = N(\mathcal{L}_A^*) \cap \mathcal{L}.$$

Now suppose that none of the four configurations from statement of the theorem occurs in the cluster graph $\mathbf{H}$. We are going to gradually constrain the structure of $\mathbf{H}$ until we find a contradiction.

Claim 13. *Let X and Y be two clusters such that $X \in \mathcal{L}$ and $\overline{\deg}(X, \mathcal{S}_0) = 0$ and $\overline{\deg}(Y, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k$. Then X and Y are not connected by an edge.*

Proof. If there is $X \in \mathcal{L}$ such that $\overline{\deg}(X, \mathcal{S}_0) = 0$, then we have $\overline{\deg}(X, \mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_M) \geq (1 + \eta)k$. Now suppose that there is an edge between such a cluster X and a cluster Y with $\overline{\deg}(Y, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k$. If $\tilde{r}a_1 > (1 - \tilde{r})a_2$, we have found Configuration B. If, on the other hand, $\tilde{r}a_1 \leq (1 - \tilde{r})a_2$, recall that $b_1 \leq a_2 + b_1 = \tilde{r}k$, meaning that we have found Configuration C. □

Corollary 14. *We have:*

(1) $e(\mathcal{L}_A, \mathcal{L}^* \cup \mathcal{S}_1) = 0$, thus $\mathcal{N}$ is a subset of $\mathcal{L}_B$,
(2) $\forall X \in \mathcal{N} : \overline{\deg}(X, \mathcal{L}) < (\tilde{r} + r'\eta/2)k$,
(3) $\forall X \in \mathcal{S}_A : \overline{\deg}(X) = \overline{\deg}(X, \mathcal{L}) < (\tilde{r} + r'\eta/2)k$.

Proof.

(1) Suppose that there is an edge between $X \in \mathcal{L}_A$ and $Y \in \mathcal{L}^* \cup \mathcal{S}_1$. From Claim 12 we get that $\overline{\deg}(X, \mathcal{S}_0) = 0$. From the definition of $\mathcal{L}^*$ and $\mathcal{S}_1$ we have $\overline{\deg}(Y, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k$. Thus we can apply Claim 13 for X and Y.
(2) Each vertex $Y \in \mathcal{N}$ has a neighbour $X \in \mathcal{L}_A^*$. If $\overline{\deg}(Y, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k$ we are in the situation of the first part of this claim.
(3) Each vertex $Y \in \mathcal{S}_A$ is matched to a vertex $X \in \mathcal{L}_A$. If $\overline{\deg}(Y, \mathcal{L}) \geq (\tilde{r} + r'\eta/2)k$, we are, yet again, in the situation of the first part of the claim.

□

Claim 15. *Every cluster in $\mathcal{N}$ has average degree at least $(\tilde{r} + \eta/2)k$ in $\mathcal{S}_0$.*

Proof. Suppose that it is not so. Then we have a cluster $Y \in \mathcal{N}$ such that

$$\begin{aligned}\overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) &\geq (1 + \eta)k - (\tilde{r} + \eta/2)k \\ &\geq (1 - \tilde{r} + \eta/2)k\ .\end{aligned}$$

Now we consider separately three cases:

(1) Suppose that $\tilde{r}a_1 \leq (1 - \tilde{r})a_2$. Then either

$$\overline{\deg}(Y, \mathcal{L}) \geq b_1 + \eta k/4,$$

which leads to the Configuration C (consider Y and its neighbour in $\mathcal{L}_A^*$), or we have

$$\begin{aligned}
\overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M) &= \overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) - \overline{\deg}(Y, \mathcal{L}) \\
&\geq (1-\tilde{r})k + \eta k/2 - (b_1 + \eta k/4) \\
&= \frac{1-\tilde{r}}{\tilde{r}}\tilde{r}k - b_1 + \eta k/4 \\
&= \frac{1-\tilde{r}}{\tilde{r}}(b_1 + a_2) - b_1 + \eta k/4 \\
&= \frac{1-2\tilde{r}}{\tilde{r}}b_1 + \frac{1-\tilde{r}}{\tilde{r}}a_2 + \eta k/4 \\
&\geq \frac{1-\tilde{r}}{\tilde{r}}a_2 + \eta k/4,
\end{aligned}$$

where we used the bound on the average degree of Y and then the facts that $b_1 + a_2 = \tilde{r}k$ and $\tilde{r} \leq r' \leq 1/2$. This, on the other hand, leads to the Configuration A (again, consider Y and its neighbour in $\mathcal{L}_A^*$).

(2) Suppose that $\tilde{r}a_1 > (1-\tilde{r})a_2$ and $b_1 \leq \frac{\tilde{r}^2}{1-\tilde{r}}k$. Following the same considerations as in the previous case we get that either $\overline{\deg}(Y, \mathcal{L}) \geq b_1 + \eta k/4$ or $\overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M) \geq \frac{1-\tilde{r}}{\tilde{r}}a_2 + \eta k/4$. The second case leads, again, to the Configuration A. We now proceed with the first case.

Let X be a neighbour of Y in $\mathcal{L}_A^*$. From Claim 12 we have $\overline{\deg}(X, \mathcal{S}_0) = 0$ and from Corollary 14.1 we have $\overline{\deg}(X, \mathcal{S}_1) = 0$, thus

$$\overline{\deg}(X, \mathcal{L} \cup \mathcal{S}_M) = \overline{\deg}(X) \geq (1+\eta)k > k + \eta k/4.$$

Moreover, all the matching edges containing clusters from $\mathcal{S} \cap N(X)$ must have both ends in the set $\mathcal{A}$ because there are no edges between vertices from $\mathcal{L} \cap \mathcal{A}$ and $\mathcal{S} \cap \mathcal{B}$ (Claim 12). On the other hand, all neighbours of X in $\mathcal{L}$ have to be in $\mathcal{B}$ (Corollary 14 (1)), so all matching edges containing vertices from $\mathcal{L} \cap N(X)$ are in $\mathcal{B}$. Thus, all of the assumptions of Configuration D for X and Y are satisfied.

(3) Finally we are left with the case $\tilde{r}a_1 > (1-\tilde{r})a_2$ and $b_1 > \frac{\tilde{r}^2}{1-\tilde{r}}k$. Note that then we have

$$a_2 = \tilde{r}k - b_1 < \tilde{r}k - \frac{\tilde{r}}{1-\tilde{r}}\tilde{r}k = (1 - \frac{\tilde{r}}{1-\tilde{r}})\tilde{r}k = (1-2\tilde{r})\,\frac{\tilde{r}}{1-\tilde{r}}k.$$

Now either

$$\overline{\deg}(Y, \mathcal{L}) \geq \tilde{r}k + r'\eta k/4,$$

or

$$\begin{aligned}
\overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M) &= \overline{\deg}(Y, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) - \overline{\deg}(Y, \mathcal{L}) \\
&\geq (1-\tilde{r})k + \eta k/2 - (\tilde{r}k + r'\eta k/4) \\
&\geq (1-2\tilde{r})k + \eta k/4 \\
&= \frac{1-\tilde{r}}{\tilde{r}}\,(1-2\tilde{r})\,\frac{\tilde{r}}{1-\tilde{r}}k + \eta k/4 \\
&\geq \frac{1-\tilde{r}}{\tilde{r}}a_2 + \eta k/4.
\end{aligned}$$

The first option leads to Configuration B while the second one leads to Configuration A.

□

After restricting the structure of $\mathbf{H}$ we are ready to derive a contradiction by combining several properties of $\mathbf{H}$ together. At first we estimate the size of the set $\mathcal{L}_A$. Recall that we have $|\mathcal{L}| \geq (1+\eta)|\mathcal{S}|$, thus $|\mathcal{L}| > |\mathcal{S}|$. We also know that $|\mathcal{L}_B| = |\mathcal{S}_B|$, because the two sets are matched in $\mathbf{M}$. This means that

$$|\mathcal{L}_A| = |\mathcal{L}| - |\mathcal{L}_B| > |\mathcal{S}| - |\mathcal{S}_B| = |\mathcal{S}_A| + |\mathcal{S}_0| + |\mathcal{S}_1|. \tag{5.1}$$

Now we proceed by bounding the size of the set $\mathcal{N}$.

Lemma 16. *Suppose that the set $\mathcal{L}_A^*$ (and thus also $\mathcal{N}$) is nonempty. Then the following inequality holds:*

$$|\mathcal{N}|(\tilde{r} + r'\eta/2) > |\mathcal{S}_0|(1 - \tilde{r} + \eta/2). \tag{5.2}$$

Proof. We estimate the number of edges between L_A^+ and S_A. For $\mathcal{Y}, \mathcal{Z} \subseteq V(\mathbf{H})$, we set $\vec{w}(\mathcal{Z}, \mathcal{Y}) := \sum_{Z \in \mathcal{Z}} \overline{\deg}(Z, \mathcal{Y})$. On one hand we have

$$\vec{w}(\mathcal{L}_A^+, \mathcal{S}_A) = \sum_{Z \in \mathcal{L}_A^+} \overline{\deg}(Z, \mathcal{S}) \geq |\mathcal{L}_A^+|(1 - \tilde{r} + \eta/2)k,$$

because $\vec{w}(\mathcal{L}_A^+, \mathcal{S}_B \cup \mathcal{S}_0) = 0$ (Claim 12) and $\vec{w}(\mathcal{L}_A^+, \mathcal{S}_1) = 0$ (Corollary 14). On the other hand we have

$$\begin{aligned}
\vec{w}(\mathcal{L}_A^+, \mathcal{S}_A) &= \sum_{Z \in \mathcal{L}_A^+, W \in \mathcal{S}_A} \overline{\deg}(Z, W) \\
&= \sum_{Z \in \mathcal{L}_A^+, W \in \mathcal{S}_A} \frac{1 - r'}{r'} \overline{\deg}(W, Z) \\
&= \frac{1 - r'}{r'} \vec{w}(\mathcal{S}_A, \mathcal{L}_A^+) \\
&\leq \frac{1 - r'}{r'} (|\mathcal{S}_A|(\tilde{r} + r'\eta/2)k - \vec{w}(\mathcal{S}_A, \mathcal{L}_A^*)) \\
&\leq (1 - r')|\mathcal{S}_A|(1 + \eta/2)k - \vec{w}(\mathcal{S}_A, \mathcal{L}_A^*) \\
&\leq |\mathcal{S}_A|(1 + \eta/2)(1 - \tilde{r})k - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A)\ ,
\end{aligned}$$

because all the clusters from $\mathcal{S}_A$ (if there are any) have their average degree bounded by $(\tilde{r} + r'\eta/2)k$ (Corollary 14), and $\tilde{r} \leq r' \leq 1/2$. After combining the inequalities we get

$$|\mathcal{L}_A^+|(1 - \tilde{r} + \eta/2)k \leq |\mathcal{S}_A|(1 - \tilde{r} + \eta/2)k - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A). \tag{5.3}$$

We continue by estimating the number of edges between $\mathcal{L}_A^*$ and $\mathcal{N}$. On one hand we have

$$\vec{w}(\mathcal{L}_A^*, \mathcal{N}) = \vec{w}(\mathcal{N}, \mathcal{L}_A^*) \leq |\mathcal{N}|(\tilde{r} + r'\eta/2)k$$

due to Corollary 14 (2). On the other hand we have

$$\begin{aligned}
\vec{w}(\mathcal{L}_A^*, \mathcal{N}) &= \vec{w}(\mathcal{L}_A^*, V(\mathbf{H})) - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A) - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_1 \cup \mathcal{S}_B \cup \mathcal{S}_0) \\
&= \vec{w}(\mathcal{L}_A^*, V(\mathbf{H})) - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A) \\
&\geq |\mathcal{L}_A^*|(1 + \eta)k - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A)\ ,
\end{aligned}$$

because there are neither edges between $\mathcal{L}_A^*$ and $\mathcal{S}_1$ (Corollary 14 (1)), nor edges between $\mathcal{L}_A^*$ and $\mathcal{S}_B \cup \mathcal{S}_0$ (Claim 12) and clusters in $\mathcal{L}$ have large degree.

By combining the two inequalities we get

$$|\mathcal{L}_A^*|(1 + \eta)k - \vec{w}(\mathcal{L}_A^*, \mathcal{S}_A) \leq |\mathcal{N}|(\tilde{r} + r'\eta/2)k. \tag{5.4}$$

Combining Inequalities (5.3), (5.4) and (5.1) in this order we get:

$$\begin{aligned}|\mathcal{N}|(\tilde{r}+r'\eta/2)k &\geq |\mathcal{L}_A^*|(1+\eta)k - \vec{w}(\mathcal{L}_A^*,\mathcal{S}_A)\\ &\geq |\mathcal{L}_A^*|(1+\eta/2)k + |\mathcal{L}_A^+|(1-\tilde{r}+\eta/2)k - |\mathcal{S}_A|(1-\tilde{r}+\eta/2)k\\ &= |\mathcal{L}_A|(1-\tilde{r}+\eta/2)k + |\mathcal{L}_A^*|\tilde{r}k - |\mathcal{S}_A|(1-\tilde{r}+\eta/2)k\\ &= |\mathcal{L}_A^*|\tilde{r}k + (|\mathcal{L}_A| - |\mathcal{S}_A|)(1-\tilde{r}+\eta/2)k\\ &> |\mathcal{L}_A^*|\tilde{r}k + (|\mathcal{S}_0| + |\mathcal{S}_1|)(1-\tilde{r}+\eta/2)k\\ &\geq |\mathcal{S}_0|(1-\tilde{r}+\eta/2)k\end{aligned}$$

which concludes the proof. □

Corollary 17. *The set $\mathcal{L}_A$ is empty.*

Proof. Suppose that $\mathcal{N}$ (and thus also $\mathcal{L}_A^*$) is nonempty. Then on one hand we have

$$\vec{w}(\mathcal{N},\mathcal{S}_0) = \frac{1-r'}{r'}\vec{w}(\mathcal{S}_0,\mathcal{N}) \leq \frac{1-r'}{r'}|\mathcal{S}_0|(\tilde{r}+r'\eta/2)k \leq |\mathcal{S}_0|(1+\eta/2)(1-r')k\ , \tag{5.5}$$

due to the definition of $\mathcal{S}_0$ and the fact $\tilde{r} \leq r'$. On the other hand we have

$$\vec{w}(\mathcal{N},\mathcal{S}_0) \geq |\mathcal{N}|(\tilde{r}+\eta/2)k \tag{5.6}$$

due to Claim 15. After combining the inequalities we get that

$$|\mathcal{N}|(\tilde{r}+\eta/2) \leq |\mathcal{S}_0|(1-r')(1+\eta/2) \leq |\mathcal{S}_0|(1-\tilde{r})(1+\eta/2). \tag{5.7}$$

Combining with Lemma 16 we get

$$|\mathcal{S}_0|(1-\tilde{r}+\eta/2) < |\mathcal{N}|(\tilde{r}+r'\eta/2) < |\mathcal{N}|(\tilde{r}+\eta/2) \leq |\mathcal{S}_0|(1-\tilde{r})(1+\eta/2).$$

which gives a contradiction, because

$$1-\tilde{r}+\eta/2 > 1-\tilde{r}+\eta/2-\tilde{r}\eta/2 = (1-\tilde{r})(1+\eta/2).$$

Thus $\mathcal{L}_A^*$ and $\mathcal{N}$ are empty.

Now suppose that $\mathcal{L}_A^+ = \mathcal{L}_A$ is nonempty. Then on one hand we have

$$\begin{aligned}\vec{w}(\mathcal{L}_A^+,\mathcal{S}_A) &= \frac{1-r'}{r'}\vec{w}(\mathcal{S}_A,\mathcal{L}_A^+)\\ &< \frac{1-r'}{r'}|\mathcal{S}_A|(\tilde{r}+r'\eta/2)k\\ &\leq |\mathcal{S}_A|(1+\eta/2)(1-r')k\ ,\end{aligned}$$

because of Corollary 14 (3) and on the other hand we have

$$\begin{aligned}\vec{w}(\mathcal{L}_A^+,\mathcal{S}_A) &= \vec{w}(\mathcal{L}_A^+,\mathcal{S}_M\cup\mathcal{S}_1)\\ &\geq |\mathcal{L}_A^+|(1-\tilde{r}+\eta/2)k\\ &= |\mathcal{L}_A|(1-\tilde{r}+\eta/2)k\\ &\geq |\mathcal{S}_A|(1-r'+\eta/2)k\\ &> |\mathcal{S}_A|(1+\eta/2)(1-r')k\ ,\end{aligned}$$

where we used the definition of $\mathcal{L}_A^+$, Corollary 14 (1), Claim 12, Inequality (5.1), and the fact that $\tilde{r} \leq r'$.

Combining the inequalities gives a contradiction. Thus, the set $\mathcal{L}_A$ has to be empty. □

From Corollary 17 it follows that all L-clusters are in $\mathcal{L}_B$ and thus are matched to $\mathcal{S}_M$, i.e., $|\mathcal{L}| = |\mathcal{S}_M| \leq |\mathcal{S}|$, which contradicts our assumption that $|\mathcal{L}| > |\mathcal{S}|$.

6. Embedding

We call a pair (F, R) an *anchored τ forest* if F is a forest (possibly consisting of a single tree), $R \subseteq V(F_1)$, where F_1 is one of the colour classes of F, $F - R$ decomposes into components of size at least two and at most τ, each component K in $F - R$ is adjacent in F to at least one and at most two vertices from R and each two vertices in R are of distance at least 4. We shall use the notation $K \in F - R$ to denote that the tree K is one of the components of $F - R$.

First we state a proposition that will allow us to use matching edges in our r-skewed LKS-cluster graph to embed part of our tree T.

Specifically, in Proposition 18 we are given an anchored forest (F, R), an r-skewed LKS-graph which contains a cluster A with some nice average degree to some $L - S$-matching, and an injective mapping of R on ultratypical vertices of A. We want to extend it to an embedding of F.

Proposition 18. *For all $\eta, d > 0$ and $r \in \mathbb{Q}^+$, $0 < r \leq 1/2$, there is an $\varepsilon = \varepsilon(\eta, d, r) > 0$ such that for any $\tilde{N}_{max} \in \mathbb{N}$ there is a $\beta = \beta(\eta, r, \varepsilon, \tilde{N}_{max}) > 0$ such that for all $n \in \mathbb{N}$ the following holds.Let (F, R) be an anchored βn-forestwith colour classes F_1 and F_2 such that $R \subseteq F_2$ and for each component $K \in F - R$, we have $|F_1 \cap K| \leq |F_2 \cap K|$. Let H be an r-skewed LKS-graph of order n with parameters $(\cdot, \cdot, \varepsilon, d)$ with a corresponding cluster graph $\mathbf{H}$ of order at most $\tilde{N}_{max}$. Let $U \subseteq V(H)$ and let $\mathbf{M} \subseteq E(\mathbf{H})$ be a matching in $\mathbf{H}$ between L-clusters and S-clusters.*

If for $A \in V(\mathbf{H})$ we have

$$\overline{\deg}(A, \mathcal{S} \cap V(\mathbf{M})) \geq \frac{1-r}{r}|F_2| + \sum_{C \subseteq S\,:\,CD \in \mathbf{M}} \max\{|U \cap C|, \frac{1-r}{r}|U \cap D|\} + \eta n \ ,$$

then for any injective mapping of R on ultratypical vertices of A, there is an embedding φ of F avoiding U and extending this mapping such that $\varphi(V(F_1)) \subseteq S \cap V(\mathbf{M})$, $\varphi(V(F_2) \setminus R) \subseteq L \cap \bigcup V(\mathbf{M})$, and $V(F_2)$ are mapped on ultratypical vertices. Moreover, for any cluster $C \in V(\mathbf{H})$ where we embedded vertices from $F - R$ it holds that $|C \setminus (U \cup \varphi(F))| \geq r\eta/8|C|$.

Next, we state a proposition allowing us to use high average degree of some clusters to embed further part of our tree T.

Specifically, in Proposition 19 we are given an anchored forest (F, R), an r-skewed LKS-graph which contains a cluster A with big enough average degree to a set of clusters with high average degree, and an injective mapping of R on ultratypical vertices of A. We want to extend it to an embedding of F.

When using the proposition, we always set $\mathcal{B}$ to be the set of L-clusters in (1) and the set of S_1-clusters in (2).

Proposition 19. *For all $\eta, d > 0$ and $0 < r \leq 1/2$, there is an $\varepsilon = \varepsilon(\eta, d, r) > 0$ such that for any $\tilde{N}_{max} \in \mathbb{N}$ there is a $\beta = \beta(\eta, r, \varepsilon, \tilde{N}_{max}) > 0$ such that for all $n \in \mathbb{N}$ the following holds. Let (F, R) be an anchored βn-forest with colour classes F_1 and F_2 such that $R \subseteq F_2$. Let H be an r-skewed LKS-cluster graph with parameters $(\cdot, \cdot, \varepsilon, d)$ of order n with an associated cluster graph $\mathbf{H}$ of order at most $\tilde{N}_{max}$. Let $U \subseteq V(H)$ and let $\mathcal{B} \subseteq V(\mathbf{H})$ be a set of clusters. Let $\varphi : R \to A$ with $A \in V(\mathbf{H})$ be an injective mapping on ultratypical vertices.*

(1) If $\overline{\deg}(A, \mathcal{B}) \geq |F_1| + |\bigcup \mathcal{B} \cap U| + \eta n$, then we can extend φ to $N(R)$ so that $\varphi(N(R))$ are ultratypical vertices in $\bigcup \mathcal{B} \setminus U$ and find a set $W = W_1 \dot\cup W_2 \dot\cup \ldots \subseteq \bigcup \mathcal{B} \setminus (U \cup \varphi(R \cup N(R)))$ of reserved vertices such that $|W_i| = |(F_1 \cap K_i) \setminus N(R)|$, with $K_i \in F - R$ and such that W_i lies in the same cluster as $\varphi(K_i \cap N(R))$ and for each cluster $C \in \mathcal{B}$ with $C \cap \varphi(N(R)) \neq \emptyset$ we have $|C \setminus (U \cup W \cup \varphi(N(R)))| \geq r\eta/8 \cdot |C|$.

Moreover, for any set $\tilde{U} \subseteq V(G) \setminus (U \cup W \cup \varphi(R \cup N(R)))$, for which $\overline{\deg}(B) \geq |F_1| + |F_2| + |U \cup \tilde{U}| + \eta n$ for each $B \in \mathcal{B}$ and such that for any $C \in V(\mathbf{H})$ with $C \cap \tilde{U} \neq \emptyset$ we have $|C \setminus (U \cup W \cup \tilde{U} \cup \varphi(N(R)))| \geq r\eta/8 \cdot |C|$, we can further extend φ to the whole F avoiding $U \cup \tilde{U}$ such that $\varphi(F_1) \subseteq \bigcup \mathcal{B}$. Moreover, the extension φ is such that for any cluster $C \in V(\mathbf{H})$ with $C \cap \varphi(F - (R \cup N(R)) \neq \emptyset$, we have $|C \setminus (\tilde{U} \cup U)| \geq r\eta/8 \cdot |C|$.

(2) If $\overline{\deg}(A, \mathcal{B}) \geq |F_1| + |\bigcup \mathcal{B} \cap U| + \eta n$ *and* $\overline{\deg}(B, V(\mathbf{H}) \setminus \mathcal{B}) \geq |F_2| + |U| + \eta n$ *for each* $B \in \mathcal{B}$*, then we can extend* φ *to* F *in* $V(G)$ *avoiding* U *and such that* $\varphi(V(F_1)) \subseteq \bigcup \mathcal{B}$*,* $\varphi(V(F_2)) \subseteq \bigcup N_{\mathbf{H}}(\mathcal{B}) \setminus \mathcal{B}$*, and* $V(F_2)$ *are mapped on ultratypical vertices. Moreover, the embedding* φ *is such that for any cluster* $C \in V(\mathbf{H})$ *with* $C \cap \varphi(F - R) \neq \emptyset$*, we have* $|C \setminus (\varphi(F) \cup U)| \geq r\eta/8 \cdot |C|$*.*

We at first prove Proposition 18.

Proof of Proposition 18. Given $\eta, d > 0$ and $r \in \mathbb{Q}$ set $\varepsilon = \min\{(\frac{\eta r}{12})^2, \frac{dr\eta}{100}\}$.For any $\tilde{N}_{max} \in \mathbb{N}$ set $\beta = \frac{\varepsilon r \eta}{4\tilde{N}_{max}}$.

We shall define a set $\tilde{U}$ of vertices used for the embedding process. At the beginning $\tilde{U} = \varphi(R)$. At any time of the embedding process, let φ be the partial embedding of F. We shall embed one by one each component $K \in F - R$. The embedding φ will be defined in such a way that $\varphi(K \cap F_1) \subseteq S$ and $\varphi(K \cap F_2 \setminus R) \subseteq L$. During the whole embedding process, we shall ensure that the following holds

$$\overline{\deg}(A, \mathcal{S} \cap V(\mathbf{M})) \geq \frac{1-r}{r}(|F_2| - |\varphi(F_2)|) + \sum_{C \subseteq S\,:\,CD \in \mathbf{M}} \max\{|(U \cup \tilde{U}) \cap C|, \frac{1-r}{r}|(U \cup \tilde{U}) \cap D|\} + \eta n \ . \tag{6.1}$$

This holds at the beginning when $\tilde{U} = R$.

For each next $K \in F - R$ to be embedded, let R_K be the vertices in R adjacent to K (at least one, at most two). Let $\mathcal{S}' \subseteq \mathcal{S} \cap V(\mathbf{M})$ be such that both $\varphi(R_K)$ are typical to each cluster $C \in \mathcal{S}'$. By Lemma 4 we have that $|\mathcal{S} \cap \mathbf{M}| - |\mathcal{S}'| \leq 2\sqrt{\varepsilon}|V(\mathbf{H})|$ and thus similarly as in the proof of Proposition 7 we can calculate for $x_i \in R_K$, $i = 1, 2$ that $\deg(\varphi(x_i), \bigcup \mathcal{S}') \geq \overline{\deg}(A, \mathcal{S} \cap V(\mathbf{M})) - 3\sqrt{\varepsilon}n/r$ and thus

$$\begin{aligned}
\deg(\varphi(x_i), \bigcup \mathcal{S}') \geq{}& \frac{1-r}{r}(|F_2| - |\varphi(F_2)|) + \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \frac{1-r}{r}|\tilde{U} \cap D| \\
&+ \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \max\{|U \cap C|, \frac{1-r}{r}|U \cap D|\} + \eta n - 3\sqrt{\varepsilon}n/r \\
\geq{}& \frac{1-r}{r}(|F_2| - |\varphi(F_2)|) + \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \max\{|(U \cup \tilde{U}) \cap C|, \frac{1-r}{r}|(U \cup \tilde{U}) \cap D|\} + 3\eta n/4 \\
\geq{}& \sum_{C \subseteq \mathcal{S}'\,:\,CD \in \mathbf{M}} \left(\max\{|(U \cup \tilde{U}) \cap C|, \frac{1-r}{r}|(U \cup \tilde{U}) \cap D|\} + 3\eta n/(4|\mathcal{S}'|) \right) .
\end{aligned}$$

Then there is a $C \in \mathcal{S}'$ with $CD \in \mathbf{M}$ such that

$$\deg(\varphi(x_i), C) \geq \max\{|(U \cup \tilde{U}) \cap C|, \frac{1-r}{r}|(U \cup \tilde{U}) \cap D|\} + 3\eta n/(4|\mathcal{S}'|) \ .$$

Thus,

$$\begin{aligned}
|C| - \max\{|(\tilde{U} \cup U) \cap C|, \frac{1-r}{r}|(\tilde{U} \cup U) \cap D|\} &\geq \deg(\varphi(x_i, C)) - \max\{|(\tilde{U} \cup U) \cap C|, \frac{1-r}{r}|(\tilde{U} \cup U) \cap D|\} \\
&\geq 3\eta n/(4|\mathcal{S}'|) \\
&\geq \frac{1-r}{r}\beta n + \eta n/(2|V(\mathbf{H})|) \\
&\geq |F_1 \cap K| + \eta r|C|/2 \ , \qquad (6.2)
\end{aligned}$$

where the third inequality follows from the definition of β and the last inequality follows from Proposition 7 (1). Similarly we have

$$\begin{aligned}
|D \setminus (\tilde{U} \cup U)| &\geq \frac{r}{1-r}(|C| - \max\{|(\tilde{U} \cup U) \cap C|, \frac{1-r}{r}|(\tilde{U} \cup U) \cap D|\}) \\
&\geq \beta n + \frac{r}{1-r}\eta n/(2|V(\mathbf{H})|) \\
&\geq |F_2 \cap K| + \eta r|D|/2 \ , \qquad (6.3)
\end{aligned}$$

where we again use the definition of β and Proposition 7 (1).

In particular, in the neighbourhood of each vertex $u_i \in \varphi(R_K)$, $i = 1, 2$, there are at least $|F_1 \cap K|$ unused vertices of $C \setminus U$ that are typical w.r.t. $D \setminus (\tilde{U} \cup U)$. Let $\varphi(N(x_i) \cap K) = v_i$, $i = 1, 2$, be such vertices. Hence,

$$\deg(v_i, D \setminus (\tilde{U} \cup U)) \geq (d - \varepsilon)|D \setminus (\tilde{U} \cup U)| \geq (d - \epsilon)r\eta|D|/8 > 3\varepsilon|D|$$

for $i = 1, 2$. Observe that $|K| \leq \beta n < \frac{\varepsilon rn}{|V(\mathbf{H})|} \leq \varepsilon \min\{|C|, |D|\}$. We can thus use Lemma 5 with $T_{L5} := K$, $X'_{L5} := C \setminus (\tilde{U} \cup U)$, $Y'_{L5} := D \setminus (\tilde{U} \cup U)$, $R_{L5} := \{N(x_i),\ i = 1, 2\}$ $\varepsilon_{L5} := \varepsilon$, $\alpha_{L5} := \frac{16\varepsilon}{\eta r}$, and $d_{L5} := d$ to embed K in $C \cup D$ with $\varphi(F_1 \cap K) \subseteq C \setminus (\tilde{U} \cup U) \subseteq \mathcal{S}$ and $\varphi(F_2 \cap K \setminus R) \subseteq D \setminus (\tilde{U} \cup U) \subseteq \mathcal{L}$. Add $\varphi(K)$ to $\tilde{U}$. From (6.2) and (6.3), we now have that $|C \setminus (\tilde{U} \cup U)| \geq r\eta/8|C|$, and $|D \setminus (\tilde{U} \cup U)| \geq r\eta/8|D|$. Observe also that for the partial embedding φ we have

$$\begin{aligned}
\overline{\deg}(A, \mathcal{S} \cap V(\mathbf{M})) &\geq \frac{1-r}{r}|F_2| + \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \max\{|U \cap C|, \frac{1-r}{r}|U \cap D|\} + \eta n \\
&\geq \frac{1-r}{r}((|F_2| - |\varphi(F_2)|) + |\tilde{U} \cap L|) + \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \max\{|U \cap C|, \frac{1-r}{r}|U \cap D|\} + \eta n \\
&\geq \frac{1-r}{r}(|F_2| - |\varphi(F_2)|) + \sum_{C \subseteq \mathcal{S}\,:\,CD \in \mathbf{M}} \max\{|(U \cup \tilde{U}) \cap C|, \frac{1-r}{r}|(U \cup \tilde{U}) \cap D|\} + \eta n\ ,
\end{aligned}$$

where the last inequality comes from the fact that $|F_1 \cap K| \leq |F_2 \cap K|$ for all $K \in F - R$, and that the embedding φ was defined in such a way that $\varphi(F_1) \subseteq \mathcal{S}$ and $\varphi(F_2 \setminus R) \subseteq \mathcal{L}$.

Proceeding in the same way for every $K \in F - R$, we extend $\varphi(R)$ to the whole anchored forest F in such a way that $\varphi(F_1) \subseteq \mathcal{S} \cap V(\mathbf{M})$, $\varphi(F_2 \setminus R) \subseteq \mathcal{L} \cap V(\mathbf{M})$, and for each cluster $C \in V(\mathbf{H})$ with $C \cap \varphi(F - R) \neq \emptyset$ we have $|C \setminus (\tilde{U} \cup U)| \geq r\eta/8|C|$.

□

We conclude this section by proving Proposition 19.

Proof of Proposition 19. Given $\eta, d > 0$ and $r \in \mathbb{Q}^+$, let $\varepsilon := \min\{\left(\frac{\eta r}{12}\right)^2, \frac{dr\eta}{100}\}$. Then for any $\tilde{N}_{max} \in \mathbb{N}$, set $\beta = \frac{r\eta\varepsilon}{4\tilde{N}_{max}}$.

We shall prove only the more difficult Case (1). Case (2) can be proven either analogously, or can be much simplified as F_2 will be mapped outside of $\mathcal{B}$ and thus does not need any reservation or cause any difficulties in embedding F_1.

We define a set $W = W_1 \cup W_2 \cup \ldots$ of reserved vertices by setting $W = \emptyset$ at the beginning and progressively adding vertices to it. Also we shall define the set $\tilde{W}$ as the set of vertices used by the partial embedding of $F - R$. Hence at the beginning we have $\tilde{W} = \emptyset$. Suppose that for some s, we have already embedded $K_j \in F - R$, for $j \leq s$. Suppose that $W = W_1 \cup \cdots \cup W_s$ is the corresponding set of reserved vertices, i.e., $|W \cup \tilde{W}| = \sum_{j=1}^{s} |K_j|$. For the next component $K_{s+1} \in F - R$ to be embedded, let R_{s+1} be the set of vertices in R adjacent to K_{s+1} (at least one, at most two). Let $\mathcal{B}' \subseteq \mathcal{B}$ be such that $\varphi(R_{s+1})$ are typical to each cluster $C \in \mathcal{B}'$. By Lemma 4 we have that $|\mathcal{B} \setminus \mathcal{B}'| \leq 2\sqrt{\varepsilon}|V(\mathbf{H})|$ and thus similarly as in Proposition 7 we get for $x \in R_{s+1}$ that

$$\begin{aligned}
\deg(\varphi(x), \bigcup \mathcal{B}') &\geq \overline{\deg}(A, \mathcal{B}) - 3\sqrt{\varepsilon}n/r \\
&\geq \sum_{j=1}^{s+1} |K_j \cap F_1| + |\bigcup \mathcal{B} \cap U| + \eta n - 3\sqrt{\varepsilon}n/r \\
&\geq |\bigcup_{j=1}^{s} W_j| + |\tilde{W}| + |K_{s+1} \cap F_1| + |\bigcup \mathcal{B} \cap U| + 3\eta n/4 \\
&\geq |K_{s+1} \cap F_1| + |W| + |\tilde{W}| + |\bigcup \mathcal{B} \cap U| + 3\eta n/4,
\end{aligned}$$

Hence there is a cluster $B \in \mathcal{B}'$ (not depending on the choice of vertex x in R_{s+1}) such that

$$\deg(\varphi(x), B \setminus (U \cup W \cup \tilde{W})) \geq 3\eta n/(4|\mathcal{B}'|) \geq \frac{\eta n}{4\tilde{N}_{max}} + \frac{\eta n}{2|V(\mathbf{H})|} > \beta n + \eta r/2 \cdot |B|.$$

In particular, in the neighbourhood of each vertex of R_{s+1} there are at least $|K_{s+1}|$ unused and unreserved ultratypical vertices in $B \setminus U$. For each $x \in R_{s+1}$, map its neighbor in K_{s+1} to one of these vertices and add the image to $\tilde{W}$. Choose a set of vertices of size $|K_{s+1} \cap F_1 \setminus N(R)|$ in $B \setminus (U \cup W \cup \tilde{W})$ and add it to W_{s+1} (i.e., also to W). Observe that $|B \setminus (U \cup W \cup \tilde{W})| \geq \eta r/8 \cdot |B|$. We proceed in the same way for every $K \in F - R$.

When we have embedded $N(R) \cap K$ of the last component $K \in F - R$, we have obtained an embedding of $N(R)$ and a reservation set $W = W_1 \cup W_2 \cup \dots$ for $F_1 \setminus N(R)$ such that W_j lies in the same cluster as $\varphi(N(R) \cap K_j)$ does, and in such a way, that for any cluster B where we embedded vertices from $N(R)$ (and possibly reserved space), we still have at least some unused and unreserved vertices, i.e., $|B \setminus (U \cup W \cup \tilde{W})| \geq \eta r/8 \cdot |B|$.

Now we shall proceed with the 'moreover' part, i.e., the embedding the left-over of the trees $K_j \in F - R$. Let u, v in cluster B be the images of $K_j \cap N(R)$ (alternatively there is only one such image). Set $W_j = \emptyset$ (and thus remove from W a set of vertices of the size $|K_j \cap F_1 \setminus N(R)|$). Similarly as above, we find $D \in V(\mathbf{H})$ such that

$$\deg(u, D \setminus (U \cup W \cup \tilde{W} \cup \tilde{U}) \geq |K_j \cap F_2| + \eta r/8 \cdot |D|,$$

and similarly

$$\deg(v, D \setminus (U \cup W \cup \tilde{W} \cup \tilde{U}) \geq |K_j \cap F_2| + \eta r/8 \cdot |D|.$$

As we have $|B \setminus (U \cup W \cup \tilde{W} \cup \tilde{U})| \geq |K_j \cap F_1 \setminus N(R)| + r\eta/8 \cdot |B|$ and $r\eta/8 > 3\varepsilon$, we may use Lemma 5 with $T_{L5} := K_j$, $R_{L5} := K_j \cap N(R)$, $X'_{L5} := B \setminus (U \cup W \cup \tilde{W} \cup \tilde{U})$, $Y'_{L5} := D \setminus (U \cup W \cup \tilde{U} \cup \tilde{W})$, $\alpha_{L5} := \frac{32\varepsilon}{r\eta}$, $\varepsilon_{L5} := \varepsilon$, and $d_{L5} := d$ to extend φ to the whole K_j with $F_1 \cap K_j \subseteq B$ and $F_2 \cap K_j \subseteq D$ and add the used vertices to $\tilde{W}$. Observe that after the embedding of K_j, we still have in each cluster B and D at least $r\eta/8 \cdot |B|$ and $r\eta/8 \cdot |D|$ vertices, respectively, outside U, W, $\tilde{U}$, and $\tilde{W}$. We continue until every $K \in F - R$ is embedded. □

7. Proof of Proposition 11

Given $\delta, q, d > 0$ and $\tilde{r} \leq r' \leq 1/2$ set

$$\varepsilon := \min\left\{\varepsilon_{P18}(\frac{q\delta}{20}, d, r'), \varepsilon_{P19}(\frac{q\delta}{20}, d, r'), \left(\frac{\delta q}{3}\right)^2, d/17\right\},$$

$$\beta := \min\left\{\beta_{P18}(\frac{q\delta}{20}, r', \varepsilon, \tilde{N}_{max}), \beta_{P19}(\frac{q\delta}{20}, r', \varepsilon, \tilde{N}_{max}), \delta r'/8\right\},$$

$$n_0 := \frac{200}{\delta q r' \beta}.$$

We gradually construct an injective homomorphism φ of T into H. To this end we consider the four introduced cases.

In each case, we start by embedding the vertices of W_A and W_B to ultratypical vertices of A and B, respectively. This can be done by applying Lemma 5 with X'_{L5} and Y'_{L5} being the sets of ultratypical vertices of A and B, respectively, T_{L5} being any tree with colour classes W_A and W_B such that $T[W_A \cup W_B]$ is a subgraph of T_{L5}, $\alpha_{L5} = 5\varepsilon$ and $R_{L5} = \emptyset$. Note that the assumptions of Lemma 5 are satisfied, since the pair (A, B) has density at least $d - \varepsilon > 15\varepsilon$, by Lemma 4 at least $1 - \sqrt{\varepsilon} > 4/5$ of vertices of A or B, respectively, are ultratypical, and moreover $|W_A| < \varepsilon|A|$, $|W_B| < \varepsilon|B|$ by definition of fine partition.

We embed the rest of the tree T using different strategy for each case. In what follows, we use indexes 1 and 2 to denote that the structure is a substructure of T_1 or T_2, respectively.

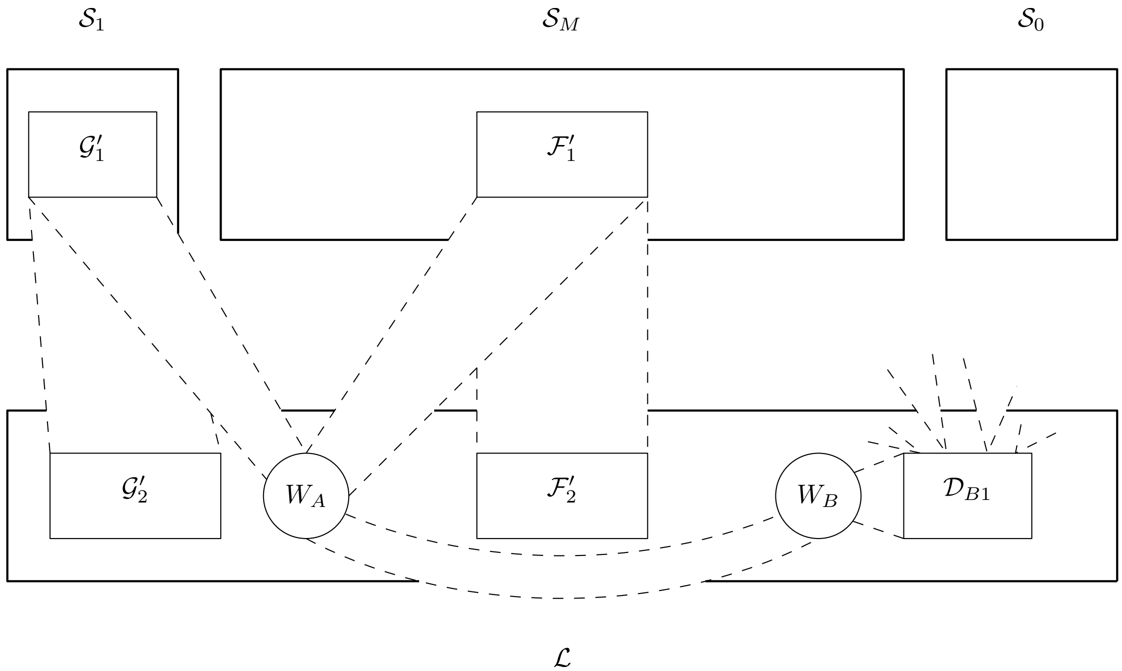

FIGURE 7.1. The embedding configuration in the case A. After inserting the vertices of W_A, W_B in the ultratypical vertices of clusters A and B we use Proposition 18 to embed $\mathcal{F}'$ in the matching connecting $\mathcal{S}_M := \mathcal{S} \cap M$ and $\mathcal{L} \cap M$. Then we invoke Proposition 19 to embed $\mathcal{G}'$ using the vertices in $\mathcal{S}_1$. Finally, we again invoke Proposition 19 to embed $\mathcal{D}_B$. Note that in this case, as well as in all of the subsequent cases, it may be the case that $B \in \mathcal{S}_1$.

When using Propositions 18 and 19, we shall always use (here we use the index $_P$ to indicate the parameter of the propositions) $d_P := d$, $r_P := r'$, $\tilde{N}_{max,P} := \tilde{N}_{max}$, $n_P := n$, $H_P := H$, $\mathbf{H}_P := \mathbf{H}$, and R_P will be either W_A or W_B depending whether we embed part of $\mathcal{D}_A$, or $\mathcal{D}_B$, respectively. In some cases, we shall use Proposition 19 several times. To avoid confusion, we shall use upper indices in parenthesis, e.g., $U^{(1)}_{P19}$, to indicate to which application of the proposition we refer. We will write $\mathcal{D}_{B1}$ as a shortcut for $\mathcal{D}_B \cap V(T_1)$ and $\mathcal{D}_{B2} := \mathcal{D}_B \cap V(T_2)$ and $\mathcal{D}_{A1}$ as a shortcut for $\mathcal{D}_A \cap V(T_2)$ (sic) and $\mathcal{D}_{A2} := \mathcal{D}_A \cap V(T_1)$. Thus, neighbours of W_A or W_B are in $\mathcal{D}_{A1}$ or $\mathcal{D}_{A2}$, respectively.

Case A. In this case we assume that there are two adjacent clusters A and B in H such that $\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M) \geq \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{D}_{A2}| + \delta k$ and $\overline{\deg}(B, \mathcal{L}) \geq (\tilde{r} + \delta)k$.

We start by embedding the vertices of W_A and W_B to ultratypical vertices of clusters A and B, respectively. We then further partition the rest of T and embed it in the following three steps which we describe in detail later. We partition the trees from $\mathcal{D}_A$ in two sets – $\mathcal{F}$ and $\mathcal{G}$ and define $\mathcal{F}'$ and $\mathcal{G}'$ as sets of subtrees of $\mathcal{F}$ and $\mathcal{G}$, respectively, with leaves in $\mathcal{D}_{A1}$ removed. We denote $\mathcal{F} \cap \mathcal{D}_{Ai}$ and $\mathcal{G} \cap \mathcal{D}_{Ai}$ by $\mathcal{F}_i$ and $\mathcal{G}_i$ respectively, for $i = 1, 2$. Analogously, we define $\mathcal{F}_i'$ and $\mathcal{G}_i'$ for $i = 1, 2$.

In the first step, we embed $\mathcal{F}'$ into the edges of the matching $\mathbf{M}$ using Proposition 18 and we embed $\mathcal{G}'$ through $\mathcal{S}_1$ vertices using Proposition 19 (i.e., $\varphi(\mathcal{G}_1') \subseteq \mathcal{S}_1$ and $\varphi(\mathcal{G}_2') \subseteq \mathcal{L}$).

In the second step, we embed the trees from $\mathcal{D}_{B1}$ using again Proposition 19. To this end we again use the bound on the degree of the cluster B – specifically, as $\overline{\deg}(B, \mathcal{L}) \geq \tilde{r}k + \delta k = |\mathcal{D}_{A2} \cup \mathcal{D}_{B1}| + \delta k$, the cluster B has enough neighbours for embedding $\mathcal{D}_{B1}$, even though $\mathcal{D}_{A2}$ is already embedded.

In the third step, we embed $\mathcal{F} \setminus \mathcal{F}'$ and $\mathcal{G} \setminus \mathcal{G}'$ greedily. The structure of the embedded tree is sketched in Figure 7.1.

(1) In this step we embed the trees from the anchored forest $\mathcal{D}_A$ except of several leaves, ensuring that the neighbours of those left-out leaves are mapped to ultratypical vertices in $\mathcal{L}$-clusters. We split the anchored βk-forest $\mathcal{D}_A$ into two disjoint forests $\mathcal{F}$ and $\mathcal{G}$ in the following way. Let $\mathcal{F}$ be

a maximal subset of trees of $\mathcal{D}_A$ such that

$$|\mathcal{F}_2| \leq \frac{r'}{1-r'}\operatorname{d\overline{eg}}(A,\mathcal{S}_M) - \frac{r'}{1-r'}\delta k/2, \tag{7.1}$$

and we choose it as an empty set if the size of the expression is less than zero.

This means that if $\mathcal{G}$ is non-empty then

$$|\mathcal{F}_2| \geq \frac{r'}{1-r'}\operatorname{d\overline{eg}}(A,\mathcal{S}_M) - \frac{r'}{1-r'}\delta k/2 - \beta k, \tag{7.2}$$

otherwise we could move a suitable tree from $\mathcal{G}$ to $\mathcal{F}$ while retaining the condition imposed on $\mathcal{F}$. By deleting the leaves of trees in $\mathcal{D}_A$ that are contained in $\mathcal{F}_1 \cup \mathcal{G}_1$ we get forests $\mathcal{F}'$ and $\mathcal{G}'$. For each tree $K \in \mathcal{F}' \cup \mathcal{G}'$ we have $|K \cap (\mathcal{F}'_1 \cup \mathcal{G}'_1)| \leq |K \cap (\mathcal{F}'_2 \cup \mathcal{G}'_2)|$, because each vertex from $K \cap (\mathcal{F}'_1 \cup \mathcal{G}'_1)$ has at least one child in $(\mathcal{F}'_2 \cup \mathcal{G}'_2)$. Specifically, $|\mathcal{G}'_1| \leq |\mathcal{G}'_2|$.

Now we apply Proposition 18 to our anchored forest $F_{P18} := \mathcal{F}'$ if it is non-empty. Set $U_{P18} := \varphi(W_A \cup W_B)$, $\eta_{P18} := q\delta/4$, $\mathbf{M}_{P18} := \mathbf{M}$, and $A_{P18} := A$. From Definition 8 we know that $|U_{P18}| = |W_A \cup W_B| \leq 12k/(\beta k) = 12/\beta$.

To apply the proposition it suffices to verify that the degree of A in $\mathcal{S}_M$ is sufficiently large, as by definition of $\mathcal{F}'$ we know that for each $K \in \mathcal{D}_A$ we have $|K \cap \mathcal{F}'_1| \leq |K \cap \mathcal{F}'_2|$. We have

$$\begin{aligned}
\operatorname{d\overline{eg}}(A,\mathcal{S}_M) &\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \delta k/2 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U_{P18}| + \delta k/4 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}'_2| + \sum_{C\subseteq\mathcal{S}\,:\,CD\in\mathbf{M}} \max\{|U_{P18}\cap C|, \frac{1-r'}{r'}|U_{P18}\cap D|\} \; + \eta_{P18}n \ ,
\end{aligned}$$

where the first inequality is due to the definition of $\mathcal{F}$ (bound 7.1) and the second one and third one are due to the facts that $\delta k/4 \geq \frac{1-r'}{r'}|U_{P18}|$ (from the choice of n_0) and $\delta k/4 \geq \eta_{P18}n$ (from the choice of η_{P18}).

If $\mathcal{G}$ is non-empty and, thus, the bound 7.2 holds, we proceed by embedding $\mathcal{G}'$.

We apply Proposition 19 (Configuration 2) to the anchored forest $\mathcal{G}'$ and $\mathcal{B}^{(1)}_{P19} := \mathcal{S}_1$. As we know that $N_{\mathbf{H}}(\mathcal{S}_1)$ is disjoint from $\mathcal{S}_M$, there is no need to include $\varphi(\mathcal{F}_1) \subseteq \mathcal{S}_M$ in the forbidden set U that ensures the injectiveness of φ. Thus, we set $U^{(1)}_{P19} := \varphi(\mathcal{F}_2 \cup W_A \cup W_B)$.

Also note that $\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19} \subseteq \varphi(W_A \cup W_B)$, because $\varphi(\mathcal{F}_2) \in \mathcal{L}$ (we could actually replace $W_A \cup W_B$ by W_B). Let $\eta^{(1)}_{P19} := \delta q/4$, and $A^{(1)}_{P19} := A$. Now we verify the first condition from Proposition 19. For the degree of the cluster A in $\mathcal{S}_1$ we have

$$\begin{aligned}
\operatorname{d\overline{eg}}(A,\mathcal{S}_1) &= \operatorname{d\overline{eg}}(A,\mathcal{S}_1\cup\mathcal{S}_M) - \operatorname{d\overline{eg}}(A,\mathcal{S}_M) \\
\boxed{\text{assumption of this configuration}} \quad &\geq \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{D}_{A2}| + \delta k - \operatorname{d\overline{eg}}(A,\mathcal{S}_M) \\
\boxed{\tilde{r} \leq r',\text{ bound (7.2)}} \quad &\geq \frac{1-r'}{r'}|\mathcal{F}_2\cup\mathcal{G}_2| + \delta k - \frac{1-r'}{r'}|\mathcal{F}_2| - \delta k/2 - \frac{1-r'}{r'}\beta k \\
\boxed{\text{bounding error terms}} \quad &\geq \frac{1-r'}{r'}|\mathcal{G}_2| + 3\delta k/8 \\
\boxed{|\mathcal{G}_2| \geq |\mathcal{G}'_2| \geq |\mathcal{G}'_1|} \quad &\geq |\mathcal{G}'_2| + 3\delta k/8 \\
&\geq |\mathcal{G}'_1| + |\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19}| + \eta^{(1)}_{P19}n \ ,
\end{aligned}$$

where we at first used the lower bound on the degree of A in $\mathcal{S}_1 \cup \mathcal{S}_M$, then the lower bound on the size of $\mathcal{F}_2$, after bounding the error terms we used the fact that $|\mathcal{G}'_2| \geq |\mathcal{G}'_1|$ and then we again bounded the errors terms by using the facts that $|\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19}| \le |W_A \cup W_B| \le \delta k/8$ and $\eta^{(1)}_{P19} n \le \delta k/4$.

Further we verify that for each cluster $C \in \mathcal{S}_1$ we have

$$\begin{aligned} \overline{\deg}(C, V(\mathbf{H}) \setminus \mathcal{B}^{(1)}_{P19}) = \overline{\deg}(C, \mathcal{L}) &\ge \tilde{r}k + \delta k \\ \boxed{\text{bound on the skew of } T} \quad &\ge |\mathcal{D}_{A2}| + |\varphi(W_A \cup W_B)| + \eta^{(1)}_{P19} n \\ &= |\mathcal{F}_2| + |\mathcal{G}_2| + |\varphi(W_A \cup W_B)| + \eta^{(1)}_{P19} n \\ &\ge |\mathcal{G}'_2| + |U^{(1)}_{P19}| + \eta^{(1)}_{P19} n \ , \end{aligned}$$

Thus we can extend φ to $\mathcal{G}$. Note that $\varphi(\mathcal{G}_2) \subseteq \mathcal{L}$.

(2) In this step we embed the trees from $\mathcal{D}_B$ using Configuration 1 from Proposition 19. The appropriate set $U^{(2)}_{P19}$ guaranteeing the injectiveness of φ consists of $\varphi(\mathcal{F}'_1 \cup \mathcal{F}'_2 \cup \mathcal{G}'_1 \cup \mathcal{G}'_2 \cup W_A \cup W_B)$. We set $\mathcal{B}^{(2)}_{P19} := \mathcal{L}$, $\eta^{(2)}_{P19} := \delta q/2$ and $A^{(2)}_{P19} := B$. First we verify the first condition of the proposition. We have

$$\begin{aligned} \overline{\deg}(B, \mathcal{L}) &\ge \tilde{r}k + \delta k \\ \boxed{\text{bound on the skew of } T} \quad &= |\mathcal{F}_2 \cup \mathcal{G}_2 \cup \mathcal{D}_{B1}| + \delta k \\ \boxed{\text{bounding error terms}} \quad &\ge |\varphi(\mathcal{F}_2 \cup \mathcal{G}_2 \cup W_A \cup W_B)| + |\mathcal{D}_{B1}| + \delta k/2 \\ &\ge |\bigcup \mathcal{B}^{(2)}_{P19} \cap U^{(2)}_{P19}| + |\mathcal{D}_{B1}| + \eta^{(2)}_{P19} n \ , \end{aligned}$$

We immediately use the 'moreover' part of the proposition with $\tilde{U}^{(2)}_{P19} = \emptyset$ and verify that for each L-cluster C we have

$$\begin{aligned} \overline{\deg}(C) &\ge k + \delta k \\ &\ge |\mathcal{D}_{B1}| + |\mathcal{D}_{B2}| + |\mathcal{D}_A| + |W_A \cup W_B| + \delta k \\ &\ge |\mathcal{D}_{B1}| + |\mathcal{D}_{B2}| + |U^{(2)}_{P19}| + \eta^{(2)}_{P19} n \ , \end{aligned}$$

where we use mainly the fact that $|\mathcal{D}_A \cup \mathcal{D}_B \cup W_A \cup W_B| = k$.

(3) We have defined an injective homomorphism φ of the whole tree T except of its leaves from $\mathcal{F}_1 \setminus \mathcal{F}'_1$ and $\mathcal{G}_1 \setminus \mathcal{G}'_1$. We know that their neighbours are embedded in ultratypical vertices of L-clusters. By Proposition 7, such vertices have degree at least $k + \delta k - 2\sqrt{\varepsilon} n/r' \ge k$ as $\delta q > 2\sqrt{\varepsilon}/r'$. Thus we can greedily extend φ to the whole tree T.

Case B. In this case we assume that $\tilde{r}|\mathcal{D}_{A1}| \ge (1-\tilde{r})|\mathcal{D}_{A2}|$ and that there are two adjacent clusters A, B such that $\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \ge (1+\delta)k$ and $\overline{\deg}(B, \mathcal{L}) \ge (\tilde{r} + \delta)k$. The embedding procedure is roughly similar to the one from Case A. However, for embedding $\mathcal{D}_A$ we now also use $\mathcal{L}$.

We start by embedding certain part of the anchored forest $\mathcal{D}_A$ using the matching $\mathbf{M}$ and the set $\mathcal{S}_1$ similarly to the Case A. Then, we proceed by reserving $|\mathcal{D}_{B1}|$ vertices that will later help us to embed the anchored trees from $\mathcal{D}_B$. In the third part we embed the rest of the forest $\mathcal{D}_A$ using the high degree vertices in $\mathcal{L}$, and then proceed by embedding $\mathcal{D}_B$ using the reserved vertices. Finally, we argue that we can embed several leftover leaves of the tree as in the previous case.

(1) Analogously to the preceding case we split the anchored forest $\mathcal{D}_A$ into three disjoint sets $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ in the following way.

Let $K_1, K_2, \dots$ be the trees of $\mathcal{D}_A$ sorted according to their skew, i.e., according to the ratio $|K_i \cap V(T_2)|/|K_i \cap V(T_1)|$ in descending order. We define $\mathcal{F}$ as the union $K_1 \cup \dots \cup K_j$, where j is taken to be maximal such that

$$|\mathcal{F}_2| = \sum_{i=1}^{j} |K_i \cap V(T_1)| \le \frac{r'}{1-r'} \overline{\deg}(A, \mathcal{S}_M) - \frac{r'}{1-r'} \delta k/3. \tag{7.3}$$

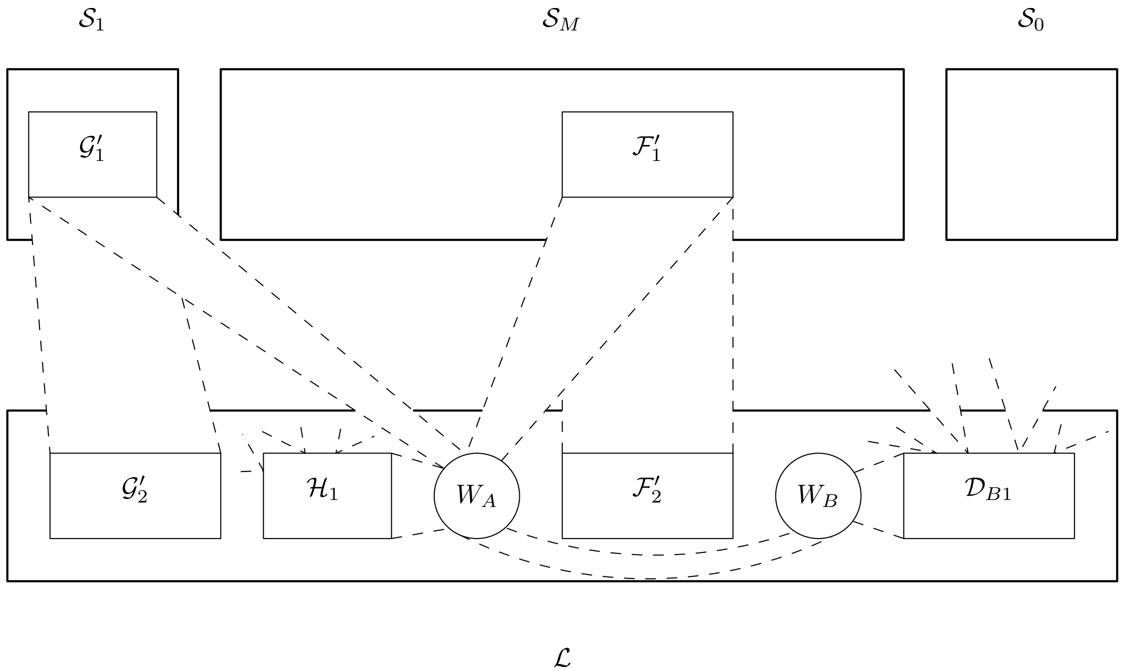

FIGURE 7.2. The embedding configuration in the case B. After inserting the vertices of W_A, W_B in the ultratypical vertices of clusters A and B we use Proposition 18 to embed $\mathcal{F}'$ in the matching $\mathbf{M}$. Then we invoke Proposition 19 to embed $\mathcal{G}'$ using the vertices in $\mathcal{S}_1$. Then we reserve suitable vertices in the neighbourhood of the cluster B in $\mathcal{L}$ that will later serve for embedding of $\mathcal{D}_{B1}$ using Proposition 19. Then we embed $\mathcal{H}_1$ using the same proposition and finally we embed $\mathcal{D}_B$ through the reserved vertices.

If the right hand side is less than zero, define $\mathcal{F}$ as the empty set. Then we similarly define $\mathcal{G}$ as the union of trees $K_{j+1}, \dots, K_{j'}$ where j' is maximal such that

$$|\mathcal{G}_2| = \sum_{i=j+1}^{j'} |K_i \cap V(T_1)| \le \frac{r'}{1-r'}\mathrm{d\overline{eg}}(A, \mathcal{S}_1) - \frac{r'}{1-r'}\delta k/3. \tag{7.4}$$

Finally we set $\mathcal{H} = \mathcal{D}_A \setminus (\mathcal{F} \cup \mathcal{G})$.

As before, we have

$$|\mathcal{F}_2| \ge \frac{r'}{1-r'}\mathrm{d\overline{eg}}(A, \mathcal{S}_M) - \frac{r'}{1-r'}\delta k/3 - \beta k\ , \tag{7.5}$$

if $\mathcal{F} \neq \mathcal{D}_A$ and

$$|\mathcal{G}_2| \ge \frac{r'}{1-r'}\mathrm{d\overline{eg}}(A, \mathcal{S}_1) - \frac{r'}{1-r'}\delta k/3 - \beta k\ , \tag{7.6}$$

if $\mathcal{F} \cup \mathcal{G} \neq \mathcal{D}_A$. Additionally, we also have

$$\tilde{r}|\mathcal{F}_1 \cup \mathcal{G}_1| \ge (1-\tilde{r})|\mathcal{F}_2 \cup \mathcal{G}_2|\ , \tag{7.7}$$

because of the assumption $\tilde{r}|\mathcal{D}_{A1}| \ge (1-\tilde{r})|\mathcal{D}_{A2}|$ and the fact that in $\mathcal{F} \cup \mathcal{G}$ there are the anchored trees with biggest skew.

We define $\mathcal{F}'$ and $\mathcal{G}'$ as in the previous case. We have $|K \cap \mathcal{F}'_1| \le |K \cap \mathcal{F}'_2|$ for each $K \in \mathcal{F}'$ and $|\mathcal{G}'_1| \le |\mathcal{G}'_2|$.

If $\mathcal{F}'$ is non-empty we apply Proposition 18 to embed the anchored forest $F_{P18} := \mathcal{F}'$ in the same way as in the previous case. Set $U_{P18} = \varphi(W_A \cup W_B)$, $\eta_{P18} = \delta q/4$, $r_{P18} := r'$, $\mathbf{M}_{P18} := \mathbf{M}$, and $A_{P18} := A$. Similarly to the previous case we verify that

$$\begin{aligned}
\overline{\deg}(A, \mathcal{S}_M) &\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \delta k/3 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U_{P18}| + \delta k/4 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}'_2| + \sum_{C\subseteq \mathcal{S}\,:\,CD\in\mathbf{M}} \max\{|U_{P18}\cap C|, \frac{1-r'}{r'}|U_{P18}\cap D|\} \; + \eta_{P18}n \ .
\end{aligned}$$

If $\mathcal{G}$ is non-empty we proceed by embedding $\mathcal{G}'$. This is also done in an analogous way to the preceding case.

We apply Proposition 19 (Configuration 2) to the anchored forest $F^{(1)}_{P19} := \mathcal{G}'$ and set $\mathcal{B}^{(1)}_{P19} := \mathcal{S}_1$. By the properties of a skew LKS graph, the set $N_{\mathbf{H}}(\mathcal{S}_1) \cup \mathcal{S}_1$ is disjoint from $\mathcal{S}_M \supseteq \varphi(\mathcal{F}'_1)$, thus for ensuring injectiveness of φ it suffices to set $U^{(1)}_{P19} := \varphi(\mathcal{F}'_2 \cup W_A \cup W_B)$ and then we also have $\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19} \subseteq W_A \cup W_B$. Set $\eta^{(1)}_{P19} := \delta q/4$, and $A^{(1)}_{P19} := A$.

Now we verify the first condition from the proposition. For the degree of the cluster A in $\mathcal{S}_1$ we have

$$\begin{aligned}
\overline{\deg}(A, \mathcal{S}_1) &\geq \frac{1-r'}{r'}|\mathcal{G}_2| + \delta k/3 \\
&\geq |\mathcal{G}'_1| + |\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19}| + \eta^{(1)}_{P19}n \ ,
\end{aligned}$$

where we use the definition of $\mathcal{G}$, the fact that $|\mathcal{G}_2| \geq |\mathcal{G}'_1|$ and the fact that $|\bigcup \mathcal{B}^{(1)}_{P19} \cap U^{(1)}_{P19}| \leq \delta q/12$.

Further, we verify that for each cluster $C \in \mathcal{S}_1$ we have

$$\begin{aligned}
\overline{\deg}(C, V(\mathbf{H}) \setminus \mathcal{B}^{(1)}_{P19}) = \overline{\deg}(C, \mathcal{L}) &\geq \tilde{r}k + \delta k \\
&\geq |\mathcal{D}_{A2}| + |\varphi(W_A \cup W_B)| + \delta k/2 \\
&\geq |\mathcal{F}_2| + |\mathcal{G}_2| + |\varphi(W_A \cup W_B)| + \delta k/2 \\
&\geq |\mathcal{G}'_2| + |U^{(1)}_{P19}| + \eta^{(1)}_{P19}n \ ,
\end{aligned}$$

where we used the facts that $|\mathcal{D}_{A2}| \leq \tilde{r}k$ and bounded the error terms in the usual manner.

(2) In this step we reserve suitable vertices for embedding $\mathcal{D}_B$ and use Proposition 19, Configuration 1, to this end.

We apply the proposition the anchored forest $F^{(2)}_{P19} := \mathcal{D}_B$, $A^{(2)}_{P19} := B$, the set $U^{(2)}_{P19} := \varphi(W_A \cup W_B \cup \mathcal{F}_2 \cup \mathcal{G}_2)$, and $\mathcal{B}^{(2)}_{P19} := \mathcal{L}$. Take $\eta^{(2)}_{P19} := q\delta/20$. We start by verifying the first condition:

$$\begin{aligned}
\overline{\deg}(B, \mathcal{L}) &\geq \tilde{r}k + \delta k \\
&\geq |\mathcal{F}_2 \cup \mathcal{G}_2 \cup \mathcal{H}_2 \cup \mathcal{D}_{B1}| + \delta k \\
&\geq |\mathcal{F}_2 \cup \mathcal{G}_2 \cup \mathcal{D}_{B1}| + \delta k \\
&\geq |\mathcal{D}_{B1}| + |\varphi(W_A \cup W_B \cup \mathcal{F}_2 \cup \mathcal{G}_2)| + \delta k/2 \\
&\geq |\mathcal{D}_{B1}| + |U^{(2)}_{P19}| + \eta^{(2)}_{P19}n \ ,
\end{aligned}$$

where we use the upper bound on the smaller colour class of T and then we bound the error terms as usual. This gives us an embedding of $N(W_B) \cap \mathcal{D}_B$ as well as the reservation set W that will help us later for embedding $\mathcal{D}_{B1}$.

Before finishing the embedding of $\mathcal{D}_B$ by invoking the 'moreover' part of Proposition 19, Configuration 1, we shall embed the anchored forest $\mathcal{H}$, which will define the set $\tilde{U}^{(2)}_{P19} := \varphi(\mathcal{H})$.

(3) We proceed with embedding of $F^{(3)}_{P19} := \mathcal{H}$, using a third time Proposition 19, Configuration 1. Let $U' = \varphi(N(W_B) \cap \mathcal{D}_B) \cup W$, $|U'| = |\mathcal{D}_{B1}|$ and set $U^{(3)}_{P19} := \varphi(W_A \cup W_B \cup \mathcal{F} \cup \mathcal{G}) \cup U'$. Thus $U^{(3)}_{P19} \cap \mathcal{L} \subseteq \varphi(W_A \cup W_B \cup \mathcal{F}'_2 \cup \mathcal{G}'_2) \cup U'$. Further set $\mathcal{B}^{(3)}_{P19} := \mathcal{L}$, $\eta^{(3)}_{P19} := \delta q/4 \geq \eta^{(2)}_{P19}$, and $A^{(3)}_{P19} := A$. We verify the first condition of the proposition:

$$\begin{aligned}
\overline{\deg}(A,\mathcal{L}) &\geq k + \delta k - \overline{\deg}(A,\mathcal{S}_M) - \overline{\deg}(A,\mathcal{S}_1)\\
\boxed{\text{bounds (7.5) and (7.6)}}\quad &\geq k + \delta k - \Big(\frac{1-r'}{r'}|\mathcal{F}_2| + \delta k/3 + \frac{1-r'}{r'}\beta k\Big) - \Big(\frac{1-r'}{r'}|\mathcal{G}_2| + \delta k/3 + \frac{1-r'}{r'}\beta k\Big)\\
\boxed{\text{bounding error terms}}\quad &\geq k - \frac{1-r'}{r'}(|\mathcal{F}_2| + |\mathcal{G}_2|) + \delta k/4\\
\boxed{\tilde{r} \leq r'}\quad &\geq k - \frac{1-\tilde{r}}{\tilde{r}}(|\mathcal{F}_2| + |\mathcal{G}_2|) + \delta k/4\\
\boxed{\text{bound (7.7)}}\quad &\geq k - (|\mathcal{F}_1| + |\mathcal{G}_1|) + \delta k/4\\
\boxed{T \text{ is of size } k}\quad &\geq |\mathcal{F}_2| + |\mathcal{G}_2| + |\mathcal{H}| + |\mathcal{D}_B| + |W_A \cup W_B| + \delta k/4\\
&\geq |\mathcal{H}_1| + |\varphi(W_A \cup W_B \cup \mathcal{F}'_2 \cup \mathcal{G}'_2)| + |\mathcal{D}_{B1}| + \delta k/4\\
&\geq |\mathcal{H}_1| + |U^{(3)}_{P19} \cap \bigcup \mathcal{B}^{(3)}_{P19}| + \eta^{(3)}_{P19} n\ ,
\end{aligned}$$

where we at first used our bounds on $|\mathcal{F}_2|$ and $|\mathcal{G}_2|$. Then we used the inequality $\tilde{r}|\mathcal{F}_1 \cup \mathcal{G}_1| \geq (1-\tilde{r})|\mathcal{F}_2 \cup \mathcal{G}_2|$, we followed by interpreting k as the size of T and used trivial bounds on error term throughout the computation.

We immediately use the second part of the proposition with $\tilde{U}^{(3)}_{P19} = \emptyset$. We verify that for each $C \in \mathcal{L}$ we have

$$\begin{aligned}
\overline{\deg}(C) &\geq k + \delta k\\
&= |\mathcal{F} \cup \mathcal{G} \cup \mathcal{H} \cup \mathcal{D}_B \cup W_A \cup W_B| + \delta k\\
&\geq |\mathcal{H}_1| + |\mathcal{H}_2| + |U^{(3)}_{P19} \cup \tilde{U}^{(3)}_{P19}| + \eta^{(3)}_{P19} n\ .
\end{aligned}$$

Thus, we can extend φ to $\mathcal{H}$. Note that $|C \setminus (U \cup U' \cup \tilde{U})| \geq r'\eta^{(3)}_{P19}|C|/8$ for each cluster C with $C \cap \varphi(\mathcal{H})$.

(4) Now, we finish up the embedding of $\mathcal{D}_B$, using the 'moreover' part of the second application of Proposition 19. The first condition of the proposition is satisfied, as for each $C \in \mathcal{L}$ we have

$$\begin{aligned}
\overline{\deg}(C) &\geq k + \delta k\\
&= |\mathcal{D}_A \cup \mathcal{D}_B \cup W_A \cup W_B| + \delta k\\
&\geq |\mathcal{D}_{B1}| + |\mathcal{D}_{B2}| + |\varphi(W_A \cup W_B \cup \mathcal{F}' \cup \mathcal{G}' \cup \mathcal{H})| + \eta^{(2)}_{P19} n\\
&\geq |\mathcal{D}_{B1}| + |\mathcal{D}_{B2}| + |U^{(2)}_{P19} \cup \tilde{U}^{(2)}_{P19}| + \eta^{(2)}_{P19} n\ .
\end{aligned}$$

The second condition is that for each cluster C with $C \cap \varphi(\mathcal{H})$ we have $|C \setminus (U \cup U' \cup \tilde{U})| \geq r'\eta^{(2)}_{P19}|C|/8$. This is satisfied as $\eta^{(3)}_{P19} \geq \eta^{(2)}_{P19}$ and by the property of the embedding of $\mathcal{H}$, guaranteed by the third application of Proposition 19.

(5) We have defined an injective homomorphism φ on the whole tree T except of its leaves from $\mathcal{F}_1 \setminus \mathcal{F}'_1$ and $\mathcal{G}_1 \setminus \mathcal{G}'_1$. As we know that their neighbours are embedded in ultratypical vertices of L-clusters, we can greedily extend the embedding to the whole tree T, as in Case A.

Case C. In this case we assume that $\tilde{r}|\mathcal{D}_{A1}| \leq (1-\tilde{r})|\mathcal{D}_{A2}|$ and that there are adjacent clusters A and B such that $\overline{\deg}(A, \mathcal{S}_1 \cup \mathcal{S}_M \cup \mathcal{L}) \geq (1+\delta)k$ and $\overline{\deg}(B,\mathcal{L}) \geq |\mathcal{D}_{B2}| + \delta k = |\mathcal{D}_{B1}| + \delta k$. The embedding procedure is very similar to the one from the preceding case, the difference being in the order in which we embed the parts of T in the host graph.

We start by reserving vertices for the embedding of the anchored forest $\mathcal{D}_B$ using Proposition 19. Then we embed parts of $\mathcal{D}_A$ using the matching $\mathbf{M}$ and $\mathcal{S}_1$ as in the previous cases. We have to be

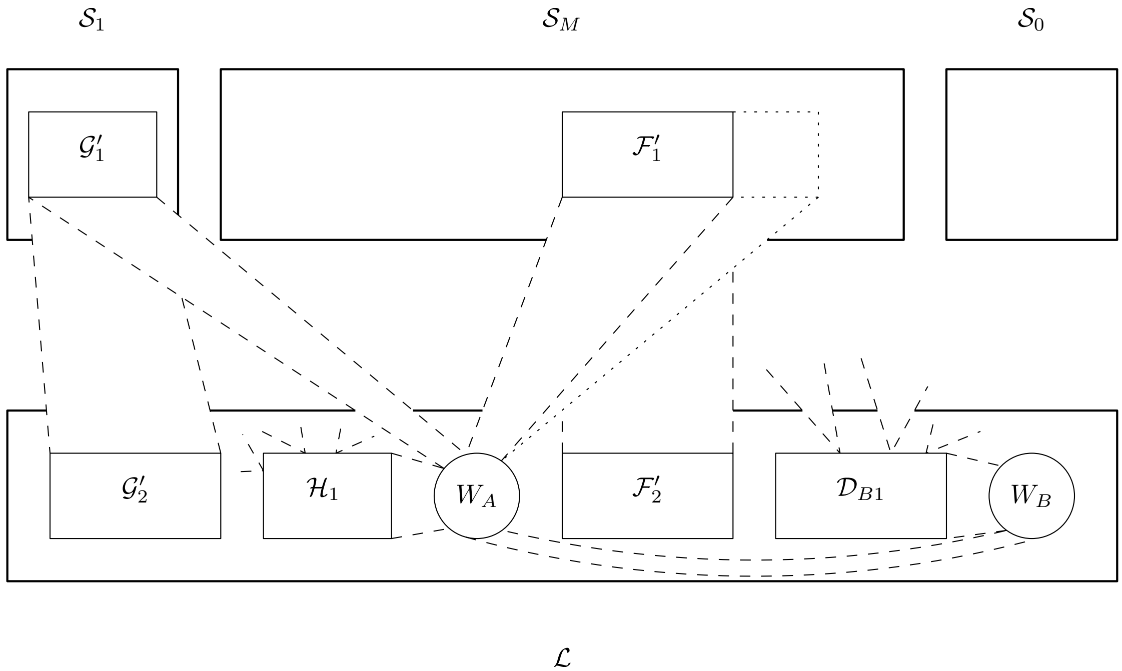

FIGURE 7.3. The embedding configuration in the case C. The configuration is very similar to the preceding one from case B. However, in this case we start by embedding $\mathcal{D}_B$ in the neighbourhood of the cluster B. The figure suggests that because of the vertices reserved for $\mathcal{D}_{B1}$ we must be more careful in the application of Proposition 18 and add those vertices in the forbidden set U_{P18}.

more careful, though, as the vertices reserved for $\mathcal{D}_B$ can cover substantial part of $\mathbf{M}$. We finish by embedding the rest of $\mathcal{D}_A$ through high degree L-clusters using Proposition 19.

(1) We start by reserving vertices for embedding the anchored forest $\mathcal{D}_B$ such that $\mathcal{D}_{B1} := \mathcal{D}_B \cap V(T_1)$ will be embedded in the neighbourhood of the cluster B. Set $\mathcal{B}^{(1)}_{P19} := \mathcal{L}$ and $U^{(1)}_{P19} := \varphi(W_A \cup W_B)$. Set $\eta^{(1)}_{P19} := q\delta/20$, and $A^{(1)}_{P19} := B$. We apply Proposition 19, Configuration 1, to reserve vertices in $\mathcal{L}$ that will later serve for embedding of $\mathcal{D}_B$. We verify that the first condition of the proposition is satisfied. Indeed:

$$\begin{aligned}\overline{\deg}(B, \mathcal{L}) &\geq |\mathcal{D}_{B1}| + \delta k \\ &\geq |\mathcal{D}_{B1}| + |\varphi(W_A \cup W_B)| + \eta^{(1)}_{P19} n\ ,\end{aligned}$$

where we used the standard error estimation.

This gives us embedding of $N(W_B) \cap \mathcal{D}_B$ as well as a reserved set W. We set $U' = \varphi(N(W_B) \cap \mathcal{D}_B) \cup W$, $|U'| = |\mathcal{D}_{B1}|$. After embedding the whole T except of several of its leaf neighbours, we will invoke the second part of the proposition with $\tilde{U}^{(1)} = \varphi(\mathcal{F}' \cup \mathcal{G}' \cup \mathcal{H})$ where $\mathcal{F}' \cup \mathcal{G}' \cup \mathcal{H} \subseteq \mathcal{D}_A$. Note that if we set $\tilde{U}^{(1)}$ to such value, we will satisfy the first condition needed for the actual embedding of $\mathcal{D}_B$, because for any cluster $C \in \mathcal{L}$ we have

$$\begin{aligned}\overline{\deg}(C) &\geq k + \delta k \\ &= |\mathcal{D}_A \cup \mathcal{D}_B \cup W_A \cup W_B| + \delta k \\ &\geq |\mathcal{D}_{B1}| + |\mathcal{D}_{B2}| + |U^{(1)}_{P19} \cup \tilde{U}^{(1)}| + \eta^{(1)}_{P19} n\ .\end{aligned}$$

To satisfy the second condition we will ensure that for all subsequent applications of Propositions 18 and 19 we choose the value η being greater than $\eta^{(1)}_{P19}$.

(2) We now proceed by embedding the anchored forest $\mathcal{D}_A$ analogously to the previous case. We split the forest $\mathcal{D}_A$ into three (possibly empty) forests $\mathcal{F}, \mathcal{G}, \mathcal{H}$ in such a way that $\mathcal{F}$ is maximal

with

$$|\mathcal{F}_2| \leq \frac{r'}{1-r'}\overline{\deg}(A,\mathcal{S}_M) - |U'| - \frac{r'}{1-r'}\delta k/3 \;, \tag{7.8}$$

or $\mathcal{F}$ is empty if the value of right hand side is smaller then zero. Moreover, if $\mathcal{F} \neq \mathcal{D}_A$, we have

$$|\mathcal{F}_2| \geq \frac{r'}{1-r'}\overline{\deg}(A,\mathcal{S}_M) - |U'| - \frac{r'}{1-r'}\delta k/3 - \beta k \;. \tag{7.9}$$

Then we similarly define $\mathcal{G}$ to be maximal such that

$$|\mathcal{G}_2| \leq \frac{r'}{1-r'}\overline{\deg}(A,\mathcal{S}_1) - \frac{r'}{1-r'}\delta k/3 \;, \tag{7.10}$$

or $\mathcal{G}$ is empty if the value of right hand side is smaller then zero. Moreover, if $\mathcal{F}\cup\mathcal{G} \neq \mathcal{D}_A$, we have

$$|\mathcal{G}_2| \geq \frac{r'}{1-r'}\overline{\deg}(A,\mathcal{S}_1) - \frac{r'}{1-r'}\delta k/3 - \beta k \;. \tag{7.11}$$

We have $\mathcal{H} := \mathcal{D}_A \setminus (\mathcal{F}\cup\mathcal{G})$ and, as in the previous case, $\mathcal{F}\cup\mathcal{G}$ consist of the trees with big skew, so if $\mathcal{D}_{A2}$ is non-empty we have:

$$\frac{1-\tilde{r}}{\tilde{r}} \geq \frac{|\mathcal{D}_{A1}|}{|\mathcal{D}_{A2}|} \geq \frac{|\mathcal{H}_1|}{|\mathcal{H}_2|} \;. \tag{7.12}$$

We define $\mathcal{F}'$ and $\mathcal{G}'$ as usual. We use Proposition 18 to embed the forest $F_{P18} := \mathcal{F}'$ as in the previous cases. Set $U_{P18} := \varphi(W_A \cup W_B) \cup U'$, $\eta_{P18} := \delta q/4$, $\mathbf{M}_{P18} := \mathbf{M}$, and $A_{P18} := A$. We verify that

$$\begin{aligned}
\overline{\deg}(A,\mathcal{S}_M) &\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U'| + \delta k/3 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U_{P18}| + \delta k/4 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}_2'| + \sum_{C\subseteq\mathcal{S}\,:\,CD\in\mathbf{M}} \max\{|U_{P18}\cap C|, \frac{1-r'}{r'}|U_{P18}\cap D|\} \;+\eta_{P18}n \;,
\end{aligned}$$

where we used the fact that $|U_{P18}| = |\varphi(W_A\cup W_B)| + |U'| \leq |U'| + \delta k/12$.

If $\mathcal{G}$ is non-empty, we proceed by embedding $\mathcal{G}'$. As in the preceding cases, we apply Proposition 19, Configuration 2, to $F^{(2)}_{P19} := \mathcal{G}'$ and set $\mathcal{B}^{(2)}_{P19} := \mathcal{S}_1$. As we know that $N_{\mathbf{H}}(\mathcal{S}_1)\cup\mathcal{S}_1$ is disjoint from $\bigcup\mathcal{S}_M \supseteq \varphi(\mathcal{F}_1')$, for ensuring the injectiveness of φ it suffices to set $U^{(2)}_{P19} := \varphi(\mathcal{F}_2'\cup W_A\cup W_B)\cup U'$. Because $\varphi(\mathcal{F}_2')\cup U' \subseteq \mathcal{L}$, we have $\bigcup\mathcal{B}^{(2)}_{P19}\cap U^{(2)}_{P19} \subseteq \varphi(W_A\cup W_B)$. Set $\eta^{(2)}_{P19} := \delta q/4$, and $A^{(2)}_{P19} := A$. We start by verifying the first condition from the proposition. We have

$$\begin{aligned}
\overline{\deg}(A,\mathcal{S}_1) &\geq \frac{1-r'}{r'}|\mathcal{G}_2| + \delta k/3 \\
&\geq |\mathcal{G}_1'| + |\bigcup\mathcal{B}^{(2)}_{P19}\cap U^{(2)}_{P19}| + \eta^{(2)}_{P19}n \;,
\end{aligned}$$

where we use the definition of $\mathcal{G}$, the fact that $|\mathcal{G}_2| \geq |\mathcal{G}_1'|$ and the fact that $|\bigcup\mathcal{B}^{(2)}_{P19}\cap U^{(2)}_{P19}| \leq 12/\beta$.

Further we verify that for each cluster $C\in\mathcal{S}_1$ we have

$$\begin{aligned}
\overline{\deg}(C, V(\mathbf{H})\setminus\bigcup\mathcal{B}^{(2)}_{P19}) = \overline{\deg}(C,\mathcal{L}) &\geq \tilde{r}k + \delta k \\
\boxed{\text{bound on skew of } T} \quad &\geq |\mathcal{D}_{A2}| + |\mathcal{D}_{B2}| + \delta k \\
&\geq (|\mathcal{F}_2| + |\mathcal{G}_2|) + |\mathcal{D}_{B1}| + |\varphi(W_A\cup W_B)| + \delta k/2 \\
&\geq |\mathcal{G}_2'| + |\varphi(\mathcal{F}_2'\cup W_A\cup W_B)| + |U'| + \delta k/2 \\
&\geq |\mathcal{G}_2'| + |U^{(2)}_{P19}| + \eta n \;,
\end{aligned}$$

where we started by using the bound on the skew of T, i.e., $|\mathcal{D}_{A2}|+|\mathcal{D}_{B2}|\leq\tilde{r}k$, then bounded the error terms and rearranged suitable terms.

(3) Now we apply Proposition 19, the first part, to embed the forest $F^{(3)}_{P19}:=\mathcal{H}$. Set $\mathcal{B}^{(3)}_{P19}:=\mathcal{L}$ and $U^{(3)}_{P19}:=\varphi(W_A\cup W_B\cup\mathcal{F}'\cup\mathcal{G}')\cup U'$, thus $U^{(3)}_{P19}\cap\mathcal{L}\subseteq\varphi(W_A\cup W_B\cup\mathcal{F}'_2\cup\mathcal{G}'_2)\cup U'$. Set $\eta^{(3)}_{P19}:=\delta q/8$, and $A^{(3)}_{P19}:=A$. We start by verifying the first condition:

$$\begin{aligned}
\mathrm{d}\overline{\mathrm{eg}}(A,\mathcal{L}) &\geq k+\delta k-\mathrm{d}\overline{\mathrm{eg}}(A,\mathcal{S}_M)-\mathrm{d}\overline{\mathrm{eg}}(A,\mathcal{S}_1)\\
\boxed{\text{bounds (7.9) and (7.11)}}\quad &\geq k+\delta k-\Big(\frac{1-r'}{r'}|\mathcal{F}_2|+\delta k/3+\frac{1-r'}{r'}|\mathcal{D}_{B1}|+\frac{1-r'}{r'}\beta k\Big)\\
&\quad-\Big(\frac{1-r'}{r'}|\mathcal{G}_2|+\delta k/3+\frac{1-r'}{r'}\beta k\Big)\\
\boxed{\text{bounding error terms}}\quad &\geq k-\frac{1-r'}{r'}(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
\boxed{\tilde{r}\leq r'}\quad &\geq k-\frac{1-\tilde{r}}{\tilde{r}}(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
&= k-\frac{1}{\tilde{r}}(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
\boxed{\text{bound on skew of } T}\quad &\geq k-\frac{1}{\tilde{r}}(\tilde{r}k-|\mathcal{H}_2|)+(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
&= \frac{1}{\tilde{r}}|\mathcal{H}_2|+(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
&\geq \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{H}_2|+(|\mathcal{F}_2\cup\mathcal{G}_2\cup\mathcal{D}_{B1}|)+\delta k/4\\
\boxed{\text{bound (7.12)}}\quad &\geq |\mathcal{H}_1|+|\varphi(W_A\cup W_B\cup\mathcal{F}'_2\cup\mathcal{G}'_2)|+|\mathcal{D}_{B1}|+\eta^{(3)}_{P19}n\\
&\geq |\mathcal{H}_1|+|U^{(3)}_{P19}\cap\bigcup\mathcal{B}^{(3)}_{P19}|+\eta^{(3)}_{P19}n\ ,
\end{aligned}$$

We set $\tilde{U}^{(3)}_{P19}=\emptyset$ and immediately invoke the second part of proposition. We verify that for each $C\in\mathcal{L}$ we have

$$\begin{aligned}
\mathrm{d}\overline{\mathrm{eg}}(C) &\geq k+\delta k\\
&=|\mathcal{D}_A\cup\mathcal{D}_B\cup W_A\cup W_B|+\delta k\\
&\geq |\mathcal{H}_1|+|\mathcal{H}_2|+|\varphi(W_A\cup W_B\cup\mathcal{F}'\cup\mathcal{G}')\cup U'|+\eta^{(3)}_{P19}n\\
&\geq |\mathcal{H}_1|+|\mathcal{H}_2|+|U^{(3)}_{P19}\cup\tilde{U}^{(3)}_{P19}|+\eta^{(3)}_{P19}n\ .
\end{aligned}$$

Thus we can extend φ to $\mathcal{H}$. Moreover, note that after each application of Propositions 18 and 19 it was true that φ avoided at least $r'\eta^{(1)}_{P19}|C|/8$ vertices of each cluster C. Thus, we can extend φ to $\mathcal{D}_B$ as we promised in the first part of the analysis of this case.

(4) We have defined φ on the whole tree T except for $\mathcal{F}_1\setminus\mathcal{F}'_1$ and $\mathcal{G}_1\setminus\mathcal{G}'_1$. We can again extend φ to the whole T in the usual greedy manner.

Case D. In this case we assume the existence of two adjacent clusters A,B such that $\mathrm{d}\overline{\mathrm{eg}}(A,\mathcal{S}_M\cup\mathcal{L})\geq k+\delta k$ and $\mathrm{d}\overline{\mathrm{eg}}(B,\mathcal{L})\geq|\mathcal{D}_{B2}|+\delta k$. Moreover, we assume that $\tilde{r}|\mathcal{D}_{A1}|\geq(1-\tilde{r})|\mathcal{D}_{A2}|$ and $|\mathcal{D}_{B2}|\leq\frac{\tilde{r}}{1-\tilde{r}}rk$ and for each edge $(C,D)\subseteq\mathbf{M}$ either $\mathrm{d}\overline{\mathrm{eg}}(A,C)=0$ or $\mathrm{d}\overline{\mathrm{eg}}(A,D)=0$.

We proceed in the same way as in the previous case, although the analysis is different.

(1) We start by reserving vertices for embedding the anchored forest $F^{(1)}_{P19}:=\mathcal{D}_B=\mathcal{D}_{B1}\cup\mathcal{D}_{B2}$ such that $\mathcal{D}_{B1}$ will be embedded in the L-neighbourhood of the cluster B. This is done using Proposition 19 in the exactly same way as in the previous case. We get an embedding of $N(W_B)\cap\mathcal{D}_B$ and a set of reserved vertices W. We set $U'=\varphi(N(W_B)\cap\mathcal{D}_B)\cup W$, $|U'|=|\mathcal{D}_{B1}|$.

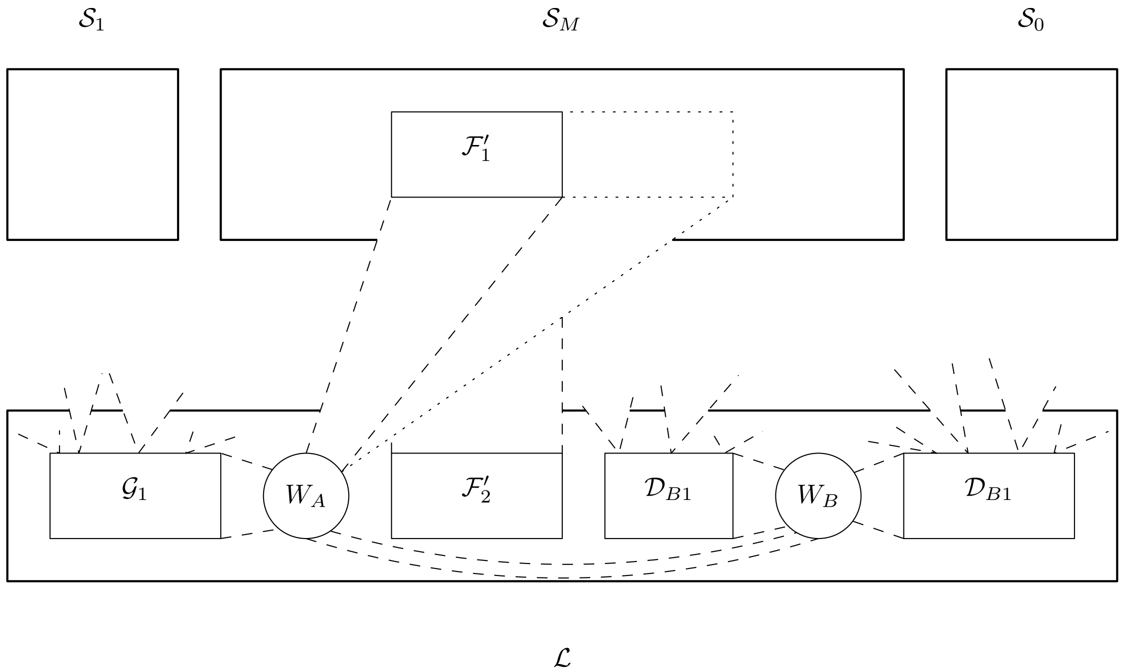

FIGURE 7.4. The embedding configuration in the case D. The order of operations is the same as in the preceding case, but the analysis is different. The figure suggests that as in the previous case we have to me more careful in the application of Proposition 18. The special condition on the neighbourhood of the cluster A plays the following role: we split the reserved vertices for $\mathcal{D}_{B1}$ into two parts – the vertices in the neighbourhood of A (the right rectangle on the figure) and those that are not neighbours of A (the left rectangle). Now the condition implies that the first type of vertices does not play a role in the embedding of $\mathcal{F}'$ using the matching, whilst the second type of vertices does not have to be considered in the embedding of $\mathcal{G}$ through the $\mathcal{L}$-neighbourhood of A.

We will also invoke the 'moreover' part Proposition 19 after embedding the rest of T and then we set $\tilde{U}^{(1)}_{P19} = \varphi(\mathcal{F}' \cup \mathcal{G})$ for $\mathcal{F}' \cup \mathcal{G} \subseteq \mathcal{D}_A$. We have to ensure that for subsequent applications of Propositions 18 and 19 we have $\eta \geq \eta^{(1)}_{P19} = q\delta/20$.

Moreover, we split the set $U' \subseteq \mathcal{L}$ in two sets U'_1 and U'_2 such that U'_1 contains the vertices from U' contained in clusters C such that $C \in N_{\mathbf{H}}(A)$ (we define $N_{\mathbf{H}}(A)$ as the set of clusters C with $\overline{\deg}(A, C) > 0$) and $U'_2 := U' \setminus U'_1$. Note that our assumption on the neighbourhood of cluster A states that if we have $(C, D) \subseteq \mathbf{M}$ with $D \cap U'_1 \neq \emptyset$, we have then $\overline{\deg}(A, C) = 0$.

(2) We continue by embedding the anchored forest $\mathcal{D}_A$ analogously to previous cases. Partition $\mathcal{D}_A = \mathcal{F} \cup \mathcal{G}$, ordering the components by decreasing order f their skew, in such a way that $\mathcal{F}$ is maximal with

$$|\mathcal{F}_2| \leq \frac{r'}{1-r'}\overline{\deg}(A, \mathcal{S}_M) - |U'_2| - \frac{r'}{1-r'}\delta k/2 \ , \tag{7.13}$$

or $\mathcal{F}$ is empty if the right hand side is smaller than zero. We define $\mathcal{F}'$ as usual. If $\mathcal{F} \neq \mathcal{D}_A$, we have

$$|\mathcal{F}_2| \geq \frac{r'}{1-r'}\overline{\deg}(A, \mathcal{S}_M) - |U'_2| - \frac{r'}{1-r'}\delta k/2 - \beta k \ . \tag{7.14}$$

Moreover, $\mathcal{F}$ is chosen so that it contains the trees with maximal skew, thus if it is non-empty we have

$$\frac{|\mathcal{F}_1|}{|\mathcal{F}_2|} \geq \frac{|\mathcal{F}_1 \cup \mathcal{G}_1|}{|\mathcal{F}_2 \cup \mathcal{G}_2|} \geq \frac{1-\tilde{r}}{\tilde{r}} \ . \tag{7.15}$$

Now we use Proposition 18 to embed $F_{P18} := \mathcal{F}'$. Set $U_{P18} := \varphi(W_A \cup W_B) \cup U'_2$ and $\mathbf{M}_{P18}$ be only those matching pairs $(C, D), C \subseteq \mathcal{S}$ such that $\overline{\deg}(A, C) > 0$. Observe that U'_1 is disjoint from $\bigcup V(\mathbf{M}_{P18})$. Set $\eta_{P18} := \delta q/3$, and $A_{P18} := A$. As in the previous cases we easily verify that

$$\begin{aligned}
\overline{\deg}(A, \mathcal{S}_M) &\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U'_2| + \delta k/2 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U_{P18}| + \delta k/3 \\
&\geq \frac{1-r'}{r'}|\mathcal{F}'_2| + \sum_{C\subseteq\mathcal{S}\,:\,CD\in\mathbf{M}} \max\{|U_{P18} \cap C|, \frac{1-r'}{r'}|U_{P18} \cap D|\} \ + \eta n.
\end{aligned}$$

Thus we can extend φ to $\mathcal{F}'$. Note that $\mathcal{F}'_2$ is embedded in L-clusters that are not in the neighbourhood of A. Indeed, from our assumption on the cluster A we have $\overline{\deg}(A, D) = 0$ for any edge $CD \in \mathbf{M}_{P18}$, with $C \subseteq \mathcal{S}$.

(3) We now apply Proposition 19, first part, to embed $F^{(2)}_{P19} := \mathcal{G}$ if it is non-empty. Set $\mathcal{B}^{(2)}_{P19} := \mathcal{L} \cap N_{\mathbf{H}}(A)$ and $U^{(2)}_{P19} := \varphi(W_A \cup W_B \cup \mathcal{F}) \cup U'$. Note that $U^{(2)}_{P19} \cap \bigcup \mathcal{B}^{(2)}_{P19} \subseteq \varphi(W_A \cup W_B) \cup U'_1$, as we know that neither U'_2, nor $\varphi(\mathcal{F}'_2)$ is in $N_{\mathbf{H}}(A)$ and $\varphi(\mathcal{F}'_1) \cap \mathcal{L} = \emptyset$. Set $\eta^{(2)}_{P19} := q\delta/4$, and $A^{(2)}_{P19} := A$.

We verify the first condition of the proposition:

$$\begin{aligned}
\overline{\deg}(A, \mathcal{L}) &\geq k + \delta k - \overline{\deg}(A, \mathcal{S}_M) \\
\boxed{\text{bound (7.14)}} \quad &\geq k + \delta k - (\frac{1-r'}{r'}|\mathcal{F}_2| + \frac{1-r'}{r'}|U'_2| + \delta k/2 + \frac{1-r'}{r'}\beta k) \\
\boxed{\text{definition of } U'} \quad &\geq k + \delta k - \frac{1-r'}{r'}|\mathcal{F}_2| - \frac{1-r'}{r'}(|\mathcal{D}_{B1}| - |U'_1|) - \delta k/2 - \frac{1-r'}{r'}\beta k \\
\boxed{\text{bounding error terms \& } \tilde{r} \leq r'} \quad &\geq k - \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{F}_2| - \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{D}_{B1}| + \frac{1-r'}{r'}|U'_1| + \delta k/3 \\
\boxed{\text{bound (7.15)}} \quad &\geq k - |\mathcal{F}_1| - \frac{1-\tilde{r}}{\tilde{r}}|\mathcal{D}_{B1}| + |U'_1| + \delta k/3 \\
\boxed{\text{assumed bound on } \mathcal{D}_{B1}} \quad &\geq k - |\mathcal{F}_1| - \frac{1-\tilde{r}}{\tilde{r}}\frac{\tilde{r}}{1-\tilde{r}}\tilde{r}k + |U'_1| + \delta k/3 \\
&= (1-\tilde{r})k - |\mathcal{F}_1| + |U'_1| + \delta k/3 \\
\boxed{\text{bound on the skew of } T} \quad &\geq |\mathcal{D}_{A1}| - |\mathcal{F}_1| + |U'_1| + \delta k/3 \\
&\geq |\mathcal{G}_1| + |U'_1| + \delta k/3 \\
&\geq |\mathcal{G}_1| + |U^{(2)}_{P19} \cap \bigcup \mathcal{B}^{(2)}_{P19}| + \eta^{(2)}_{P19} n\ .
\end{aligned}$$

We set $\tilde{U}^{(2)}_{P19} := \emptyset$ and immediately apply the second part of the proposition. We verify that for each $C \in \mathcal{L}$ we have

$$\begin{aligned}
\overline{\deg}(C) &\geq k + \delta k \\
&= |\mathcal{D}_A \cup \mathcal{D}_B \cup W_A \cup W_B| + \delta k \\
&\geq |\mathcal{G}_1| + |\mathcal{G}_2| + |\varphi(W_A \cup W_B \cup \mathcal{F}')| + |\mathcal{D}_{B1}| + \eta^{(2)}_{P19} n \\
&\geq |\mathcal{G}_1| + |\mathcal{G}_2| + |U^{(2)}_{P19} \cup \tilde{U}^{(2)}_{P19}| + \eta^{(2)}_{P19} n\ .
\end{aligned}$$

Thus, we can extend φ to $\mathcal{G}$. Moreover, after each operation it was true that φ avoided at least $r'\eta^{(1)}_{P19}|C|/8$ vertices of each cluster C. Thus, we can extend φ to $\mathcal{D}_B$ using the 'moreover' part of Proposition 19.

(4) We again extend the embedding of T greedily to $\mathcal{F}_1 \setminus \mathcal{F}_1'$ as usual.

8. Conclusion

In this last section we show a straightforward application of our result and then consult the possibilities of further research in this area.

Ramsey numbers for trees. The Ramsey number $R(G_1, \dots, G_m)$ is the least number such that any complete graph on $R(G_1, \dots, G_m)$ vertices with its edges coloured with m colours contains a monochromatic copy of G_i in colour i for some $1 \leq i \leq m$. It is not difficult to see that, if true, both the Loebl–Komlós–Sós conjecture and the Erdős–Sós conjecture would imply that for any pair of trees T_1, T_2 on $k+1$ and $l+1$ vertices, respectively, it holds that $R(T_1, T_2) \leq k + l$. This was shown to be asymptotically true in [PS12] and even finer asymptotic bound was obtained for $T_1 = T_2$ in [HLT02].

Our Conjecture 1 generalises this consequence for trees of given skew.

Suppose that we have trees $T_1, \dots, T_m$ such that the size of the i-th tree is $k_i + 1$ and the size of one of its colour class is at most $(k_i + 1)/m$. Then, assuming the validity of Conjecture 1, we deduce $R(T_1, \dots, T_m) \leq 2 + \sum_{i=1}^{m} (k_i - 1)$. Indeed, by the pigeonhole principle, for every vertex v there exists a colour i such that v is incident with at least k_i edges of colour i. Moreover, there exists a colour c such that at least $1/m$ of the vertices are incident with at least k_c edges of this colour. Thus, the subgraph formed by the edges of colour c satisfies the conditions of Conjecture 1. Using Theorem 1, we prove this consequence to be asymptotically true.

Corollary 20. *For trees $T_1, \dots, T_m$ with $|T_i| = k_i$ and such that one colour class of T_i has size at most k_i/m for $1 \leq i \leq m$ we have*

$$R(T_1, \dots, T_m) \leq \sum_{i=1}^{m} k_i + o\left(\sum_{i=1}^{m} k_i\right).$$

This generalises the asymptotic bound from [PS12] and can be shown in a very similar manner.

Note however that, if true, the Erdős–Sós conjecture implies the same bound but without the additional restriction on the skew of the trees.

Possible direction of research. We believe that, similarly as in [Coo09, HP16, Zha11], one could use Simonovits' stability method to prove that Conjecture 1 is true for dense graphs. Furthermore, by using techniques exposed in [HKP+17a, HKP+17b, HKP+17c, HKP+17d], one can probably prove that Conjecture 1 is asymptotically true even in the setting of sparse graphs.

Considering the structure of the graph witnessing the tightness of Conjecture 1 given in Section 1, it might seem feasible to strengthen the conjecture by replacing the condition on the size of the smaller colour class by the same condition on the size of the complement of a maximal independent set. However, this is not possible; a complete bipartite graph $K_{(k-1)/2,k}$ does not contain a bistar $B_{(k-1)/2,(k-1)/2}$ (that is, two stars with $(k-1)/2$ leaves with their centres joined by an edge) for $k \geq 7$ odd, even though almost $1/3$ of vertices of $K_{(k-1)/2,k}$ have degree at least k and the size of the complement of a maximal independent set in $B_{(k-1)/2,(k-1)/2}$ is 2, i.e., its relative size with respect to the whole bistar is very small, in particular at most $1/4$.

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