A Generalized Sphere Theorem for Positively Curved Combinatorial 3-Manifolds

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Abstract

The generalized sphere theorem of Grove and Shiohama is an important result in the differential geometry of positively curved manifolds. It shows that if M is a Riemannian manifold whose sectional curvature is everywhere greater than a positive constant and M has diameter greater than half the maximum allowed by the Bonnet-Myers theorem, then M must be homeomorphic to a sphere. In this paper, we present a novel discrete version of this theorem which applies to positively curved *combinatorial* 3-manifolds. That is, those with at most five tetrahedra incident at each edge, or equivalently, those with an angle deficit at each edge in the standard piecewise-linear metric. Such spaces have been studied previously and found to satisfy a beautiful discrete version of the Bonnet-Myers theorem. Here, we use a complete census of the positively curved 3-manifolds completed by Lutz and Sullivan to prove a corresponding *discrete* generalized sphere theorem.

Keywords: discrete differential geometry; generalized sphere theorem; piecewise-linear geometry; positively curved combinatorial manifold

1 Introduction

Differential geometry is of central importance not only to geometers but also topologists, physicists and, increasingly, many of those interested in applied topics such as finite element analysis and computer graphics. One significant offshoot in this area is *discrete differential geometry* (DDG) which seeks discrete analogues to the classical theorems and concepts from differential geometry. Since many computational treatments of differential geometry involve discretizing shapes, this subject has particular relevance for those with applied interests, see [9] for examples. Other recent DDG work with a more pure-mathematics flavor can be found in [1, 2, 3, 4, 5, 6, 7, 16].

A particularly important goal in classical differential geometry is to elucidate the relationship between the curvature of a Riemannian (or semi-Riemannian) space and its topology. The classical results in this area are numerous, elegant, and have inspired an enormous amount of subsequent research. In this paper, we will present a discrete analogue of the "generalized" sphere theorem of Grove and Shiohama [10].

Theorem 1 (Grove-Shiohama)**.** *Let M be a complete, connected, n-dimensional Riemannian manifold with sectional curvature $K \geq \delta > 0$ and diameter greater than $\frac{\pi}{2\sqrt{\delta}}$. Then, M is homeomorphic to a sphere.*

Note that this bound is sharp, since the real projective space $\mathbb{RP}^n$ admits a metric with uniform sectional curvature $K = 1$ and diameter $\pi/2$. We should also note that the diameter bound in Theorem 1 is exactly half the maximum diameter allowed by the Bonnet-Myers theorem:

Theorem 2 (Bonnet-Myers)**.** *Let M be a complete, connected, n-dimensional Riemannian manifold with sectional curvature $K \geq \delta > 0$. Then the diameter of M is at most $\frac{\pi}{\sqrt{\delta}}$.*

The main result of this paper, Theorem 9 is a discrete version of Theorem 1 and is proved by brute-force checking of a combinatorial 3-manifold census created by Lutz and Sullivan [11]. Before precisely stating this result we need a few preliminaries. See Theorem 8 for the corresponding discrete version of Theorem 2.

2 Preliminaries

The discrete version of Theorem 1 given in this paper will apply to positively curved combinatorial 3-manifolds. A **combinatorial 3-manifold** M is an abstract simplicial complex in which the link of each vertex is a 2-sphere. We call such a space **positively curved** if at most five tetrahedra are incident along each edge. Why this terminology? If we endow M with the standard piecewise-linear (PL) metric in which all edges have unit-length, this condition is equivalent to requiring an angle deficit along each edge. In classical differential geometry an angle deficit is intimately related to positive curvature.

A very natural discrete definition of distance in an abstract simplicial complex uses **edge-paths**. That is, paths entirely contained in the 1-skeleton of M. Let $\mathcal{P}_1$ denote the set of all edge-paths on M.

Definition 3. The **edge-distance** between two vertices v and w in an abstract simplicial complex M is the minimum length (as a PL-path in the standard PL-metric) of an edge-path from v to w. We denote this quantity $d_1(v, w)$. The **edge-diameter** of M, written as $diam_1(M)$, is the maximum of $d_1(v, w)$ over all pairs of vertices v and w in M.

Note that the length of an edge-path is simply the number of edges it traverses.

A discrete version of the Bonnet-Myers theorem was previously proved in [18]. It applies to positively curved combinatorial 3-manifolds and gives a

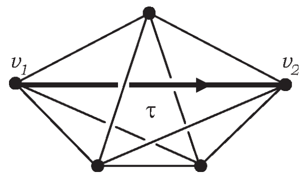


Figure 1: A *hop* from vertex v_1 to vertex v_2

diameter bound in terms of the edge-diameter.

Theorem 4 (Trout [18]). *For any connected positively curved combinatorial 3-manifold M we have $diam_1(M) \leq 5$. Moreover, this bound is sharp.*

Interestingly, while the statement of Theorem 4 uses edge-diameter, its proof relies on expanding the set of paths under consideration to include those which contain not only edges, but also other types of PL-paths between vertices. In this work, this enlarged set of paths will be crucial to obtaining a discrete result fully analogous to the classical generalized sphere theorem.

3 Hops and Jumps

The first new type of path is called a *hop*.

Definition 5 (Hops). Suppose τ is a 2-simplex in M and v_1 and v_2 are vertices in M such that $v_1 * \tau$ and $v_2 * \tau$ are 3-simplices in M. The PL-path from v_1 through the barycenter of τ and ending on v_2 will be called a **hop** from v_1 to v_2. See Figure 1 for an illustration.

The other new type of path we call a *jump*.

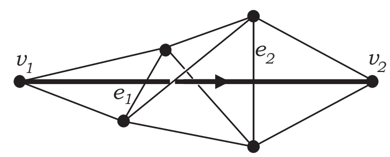


Figure 2: A *jump* from vertex v_1 to vertex v_2

Definition 6 (Jumps)**.** Suppose there are edges e_1 and e_2 and vertices v_1 and v_2 in M so that $e_1 * e_2$ is a 3-simplex in M and $v_1 * e_1$ and $v_2 * e_2$ are 2-simplices in M. We call the PL-path from v_1 through the barycenters of e_1 and e_2 and ending on v_2 a **jump** from v_1 to v_2. For an illustration, see Figure 2.

Just as for edges in an edge-path, the length of each hop and jump will be its length as a PL-path in the standard PL metric. Some Euclidean geometry tells us these lengths are

$$H = \frac{2}{3}\sqrt{2} \tag{1}$$

and

$$J = \frac{1}{2}\sqrt{2} + \sqrt{3} \tag{2}$$

respectively. We let $d(v, w)$ denote the **distance between vertices** v **and** w obtained by minimizing over all paths containing edges as well as hops and jumps. Similarly, we let $diam(M)$ denote the **diameter of** M defined in terms of the distance function d. Finally, let $\mathcal{P}$ denote the set of all the paths containing edges, hops or jumps.

Why are these paths relevant to this paper? In the classical setting, the generalized sphere theorem requires a manifold to have diameter more than half the maximum allowed by the Bonnet-Myers theorem, and this diameter bound cannot be improved. If we naively imitate the classical results in the discrete setting, from Theorem 4 we would want to assume the bound $diam_1(M) > \frac{5}{2}$. Right away we see a discrepancy between the discrete and classical results since this bound cannot possibly be sharp. Worse, the smallest weakening that could be sharp is $diam_1(M) \geq 3$ and this bound does not work. As we will see, there are positively curved combinatorial 3-manifolds M with $diam_1(M) = 3$ but which are *not* homeomorphic to the 3-sphere. However, if we instead use the finer-grained measure of diameter, $diam(M)$, the problem disappears and the discrete results mirror the classical ones exactly. That is, we can prove a discrete version of the generalized sphere theorem with a diameter bound exactly half the maximum allowed by the corresponding discrete Bonnet-Myers theorem.

What is the discrete Bonnet-Myers theorem corresponding to Theorem 4 but which uses $diam(M)$? From the proof of Theorem 4 in [18], we have the following bound on the possible distances which occur in a positively curved 3-manifold.

Lemma 7 (Trout [18]). *If v and w are vertices in a connected positively curved combinatorial 3-manifold M then $d(v, w) \in \{0, 1, H, 2, J, 3, 2H, 4, 2J\}$.*

Note that we have listed the possible distances in increasing numerical order. This result implies the desired discrete Bonnet-Myers type theorem.

Theorem 8 (Trout [18]). *For any connected positively curved combinatorial*

3-manifold M we have $diam(M) \leq 2J$.

Note, it is shown in [18] that this diameter bound is sharp.

4 Main Results

We are now able to state our main result.

Theorem 9. *Let M be a connected, positively curved, combinatorial 3-manifold. If M has $diam(M) > J$ then it is homeomorphic to a 3-sphere. Moreover, this bound is sharp and equal to half the maximum diameter which occurs for any such M.*

The proof of this theorem relies on the work of Lutz, Sullivan and Sulanke in [15, 12, 17] who created a complete census of positively curved combinatorial 3-manifolds along with a list of their topological types. See [13] and [14]. This census contains 4787 manifolds, with the following topologies represented: the 3-sphere S^3, the real projective space $\mathbb{RP}^3$, the lens spaces $L(3,1)$ and $L(4,1)$, and the cube space S^3/Q. Note that all of these manifolds are homeomorphic to either a 3-sphere or to the quotient space of a 3-sphere under the action of a finite group.

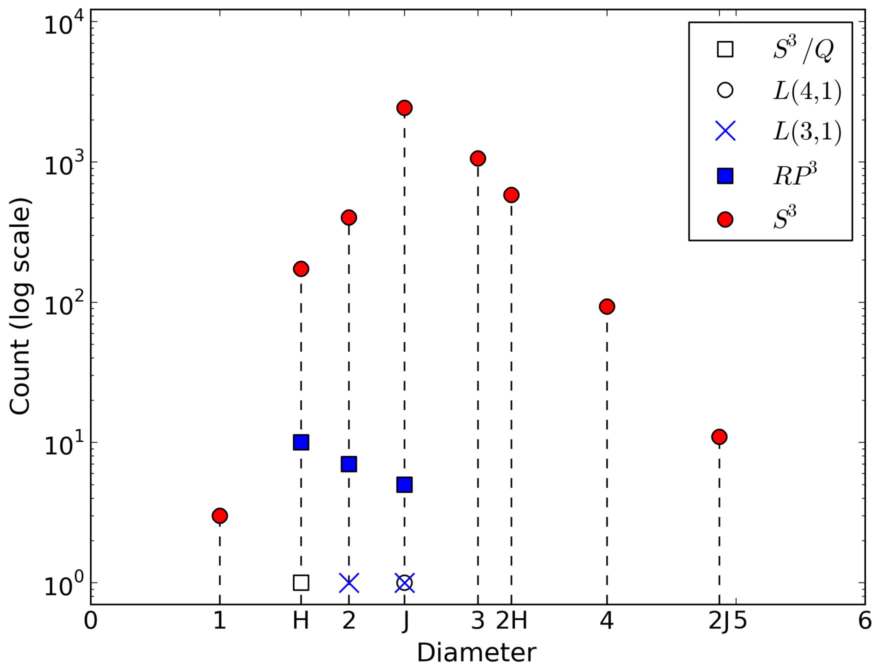
Python programs were written to examine each manifold in the census and compute its diameter and edge-diameter. Code is available online [8] along with a complete table of results for each manifold. See Figure 3 for a plot of the number of manifolds by topological type and diameter. As claimed, every manifold with diameter greater than J is a 3-sphere. The corresponding data using the coarser-grained edge-diameter can be found in Figure 4. Note the existence of 3-manifolds with edge-diameter greater than half the maximum allowed edge-diameter and which are *not* homeomorphic to the 3-sphere.

When only edge-paths are considered, as expected we obtain the following result from the census:

Theorem 10. *If M is a connected, positively curved, combinatorial 3-manifold satisfying the bound $diam_1(M) > 3$ then M is homeomorphic to the 3-sphere. Moreover, this bound is sharp.*

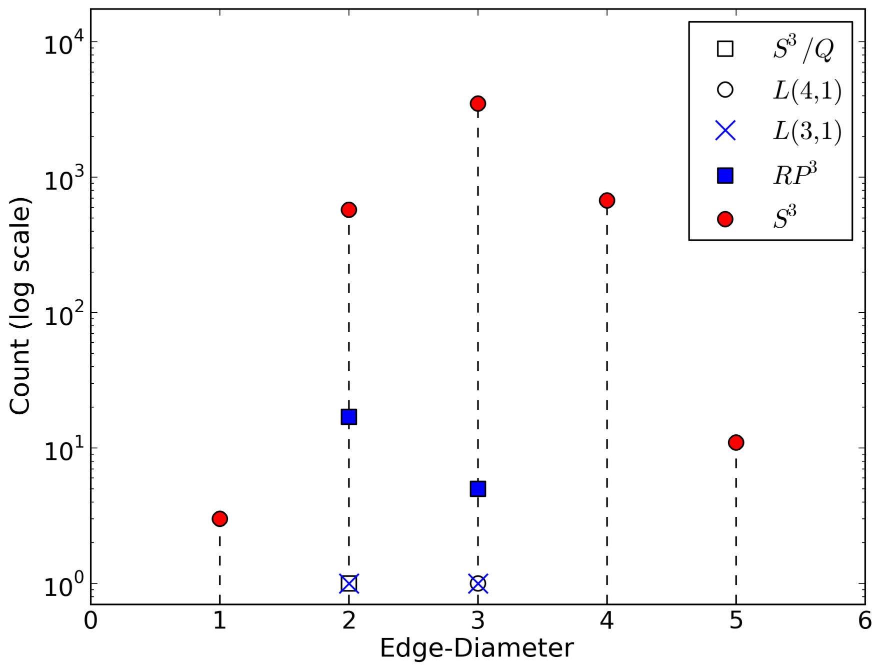
5 Discussion

We should mention another notable advantage of expanding the space of paths to include hops and jumps. In classical differential geometry, a geodesic segment is completely determined by its starting point, initial direction, and length. There is no "branching" of geodesics on a Riemannian manifold. Indeed, this property is of fundamental importance to constructing many arguments in this area. However in the discrete setting, if we use only edge-paths then this valuable characteristic does not hold. It is possible to have two minimal edge-paths with the same length and which begin by traversing the same edge, yet reach distinct destination vertices.



Diameter	S^3/Q	$L(4,1)$	$L(3,1)$	RP^3	S^3	Total
1					3	3
H	1			10	173	184
2			1	7	401	409
J		1	1	5	2438	2445
3					1060	1060
$2H$					582	582
4					93	93
$2J$					11	11
Total	1	1	2	22	4761	4787

Figure 3: Number of connected positively curved combinatorial 3-manifolds by topology and diameter, displayed graphically (above) and in table form (below). Note that all manifolds with diameter greater than half the maximum are 3-spheres. Results were computed using a 3-manifold census created by Lutz and Sullivan [12].



Edge-Diameter	S^3/Q	$L(4,1)$	$L(3,1)$	RP^3	S^3	Total
1					3	3
2	1		1	17	574	593
3		1	1	5	3498	3505
4					675	675
5					11	11
Total	1	1	2	22	4761	4787

Figure 4: Number of connected positively curved combinatorial 3-manifolds by topology and edge-diameter, displayed graphically (above) and in table form (below). Note that when diameters are measured in edges, there are manifolds with diameter greater than half the maximum which are *not* 3-spheres. Results were computed using a 3-manifold census created by Lutz and Sullivan [12].

To see this, consider a combinatorial 3-manifold M containing a minimal jump from vertex v_1 to vertex v_2 passing through edges e_1 and e_2. See Figure 2 for an illustration. (Note that such jumps exist since diameter J triangulations exist. See Figure 3.) Using the structure of a jump, we will construct edge-paths Q_1 and Q_2 starting at v_1. For both paths we begin with an edge from v_1 to some fixed vertex x in e_1. Then for Q_1 we continue along the edge from x to a vertex y_1 in e_2. For Q_2 we instead use the edge from x to the other vertex y_2 in e_2. The resulting paths Q_1 and Q_2 must be minimal among edge-paths. If some Q_i were not, then its endpoints v_1 and y_i would be connected by an edge, creating a two-edge path from v to w and contradicting the minimality of the jump. Thus Q_1 and Q_2 provide examples of minimal edge-paths which "branch" at the vertex x. The non-branching property of minimal paths is restored, at least for positively curved manifolds, if we include hops and jumps. In particular we have:

Theorem 11 (**Unique Extension**, Trout [18]). *If two minimal paths P_1 and P_2 in $\mathcal{P}$ pass through the same first two simplices of a positively curved 3-manifold M and P_1 and P_2 have the same length, then $P_1 = P_2$. This result is false if $\mathcal{P}$ is replaced by $\mathcal{P}_1$.*

See [18] for details.

It is not entirely surprising that discrete differential geometry on a PL-manifold M comes to resemble classical differential geometry more and more as the set of admissible PL-paths on M is expanded. Forcing ourselves to use only edges amounts to coarse-graining the underlying PL-geometry, and expanding the set of paths clearly reduces this coarse-graining. We could even include *all* the PL-paths in M and still have a “discrete” geometry in some sense, because our PL-metric is determined solely by the structure of M as a simplicial complex. However, since PL-paths admit infinitesimal deformations we would lose much of the “discrete” flavor to the theory. The successful use of hops and jumps in this paper and in [18] indicate that one can improve the behavior of discrete geometries by judicious inclusion of PL-paths in addition to edges. Investigating the possibilities for such inclusions and their benefits represents an exciting topic of further research.

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