Tension in the ropes from the knot to the pulleys

A diagram of the rope-pulley system is shown in Figure 1. The weights and tensions are described in Table 1.

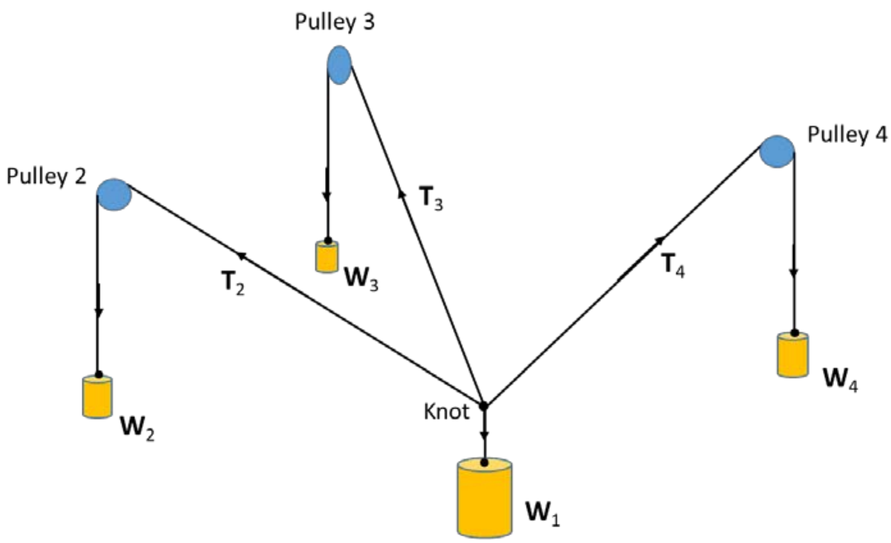

Figure 1: Rope-pulley system

Point	Description
W_1	Weight of object hanging from the knot
W_2	Counterweight attached to the end of rope 2 hanging over pulley 2
T_2	Tension in rope 2 from knot to pulley 2
W_3	Counterweight attached to the end of rope 3 hanging over pulley 3
T_3	Tension in rope 3 from knot to pulley 3
W_4	Counterweight attached to the end of rope 4 hanging over pulley 4
T_4	Tension in rope 4 from knot to pulley 4

Table 1: Description of symbols shown in Figure 1.

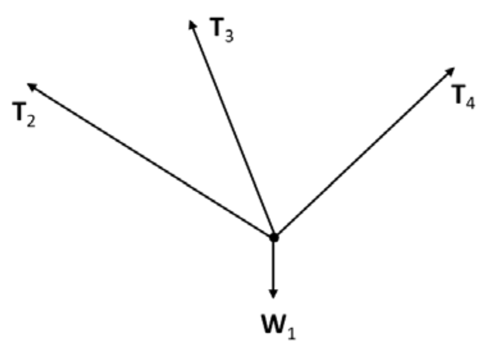

Figure 2: Free-body diagram of system

For the system to be in equilibrium, the sum of the forces at the knot must be zero. From the free-body diagram shown in Figure 2:

$$\mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 + \mathbf{W}_1 = 0 \tag{1}$$

Each of the tensions can be represented by the product of the magnitude of the force and a unit vector pointing in the direction of the force.

$$\begin{aligned}
\mathbf{W}_1 &= -W_1\mathbf{k} && \text{since the force of gravity is in the } \textit{-z} \text{ direction.} \\
\mathbf{T}_2 &= T_2\mathbf{u}_2 && \text{where} \quad \mathbf{u}_2 = u_{2x}\mathbf{i} + u_{2y}\mathbf{j} + u_{2z}\mathbf{k} \\
\mathbf{T}_3 &= T_3\mathbf{u}_3 && \text{where} \quad \mathbf{u}_3 = u_{3x}\mathbf{i} + u_{3y}\mathbf{j} + u_{3z}\mathbf{k} \\
\mathbf{T}_4 &= T_4\mathbf{u}_4 && \text{where} \quad \mathbf{u}_4 = u_{4x}\mathbf{i} + u_{4y}\mathbf{j} + u_{4z}\mathbf{k}
\end{aligned} \tag{2}$$

Combining Eqs. (1) and (2)

$$\begin{aligned}
0 = {} & T_2u_{2x}\mathbf{i} + T_2u_{2y}\mathbf{j} + T_2u_{2z}\mathbf{k} \\
& + T_3u_{3x}\mathbf{i} + T_3u_{3y}\mathbf{j} + T_3u_{3z}\mathbf{k} \\
& + T_4u_{4x}\mathbf{i} + T_4u_{4y}\mathbf{j} + T_4u_{4z}\mathbf{k} \\
& - W_1\mathbf{k}
\end{aligned} \tag{3}$$

Each component of a vector equation must satisfy this equality, so the Eq. (3) must be true for the **i**, **j**, and **k** components separately.

$$\begin{aligned}
0 &= T_2u_{2x} + T_3u_{3x} + T_4u_{4x} \\
0 &= T_2u_{2y} + T_3u_{3y} + T_4u_{4y} \\
0 &= T_2u_{2z} + T_3u_{3z} + T_4u_{4z} - W_1
\end{aligned} \tag{4}$$

Rearranging Eq. (4) to put it in a more useful form:

$$\begin{aligned}
u_{2x}T_2 + u_{3x}T_3 + u_{4x}T_4 &= 0 \\
u_{2y}T_2 + u_{3y}T_3 + u_{4y}T_4 &= 0 \\
u_{2z}T_2 + u_{3z}T_3 + u_{4z}T_4 &= W_1
\end{aligned} \tag{5}$$

This is a set of three simultaneous equations which can be written in matrix notation as:

$$\begin{pmatrix} u_{2x} & u_{3x} & u_{4x} \\ u_{2y} & u_{3y} & u_{4y} \\ u_{2z} & u_{3z} & u_{4z} \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ W_1 \end{pmatrix} \tag{6}$$

It is more compact to write this in vector-matrix notation: **Ax = b** where

$$\mathbf{A} = \begin{pmatrix} u_{2x} & u_{3x} & u_{4x} \\ u_{2y} & u_{3y} & u_{4y} \\ u_{2z} & u_{3z} & u_{4z} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ W_1 \end{pmatrix} \tag{7}$$

We want to find the tensions, T_2, T_3, and T_4. We know how to find the unit vectors (direction of the forces) from measurements (as explained in another note), so we know the matrix **A**. We also know the weight attached to the knot, W_1, which gives us the vector **b**.

A matrix equation like this can be solved by finding the inverse of the matrix **A** (denoted $\mathbf{A}^{-1}$**):**

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{8}$$

Thus, knowing the weight attached to the knot and the unit vectors for the directions of the forces, we can find the vector **x**, whose components are the tensions in the ropes.