Determination of unit vectors in the direction of forces from the knot to the pulleys

In order to calculate the tensions in the ropes, we need to find the vectors from the knot to the points where the ropes from the knot make contact with the pulleys. Using one of the pulleys as an example, this note describes how to find these vectors from the measurements we have made.

Figure 1 shows a diagram for one of the pulleys. The points shown in the diagram are described in Table 1.

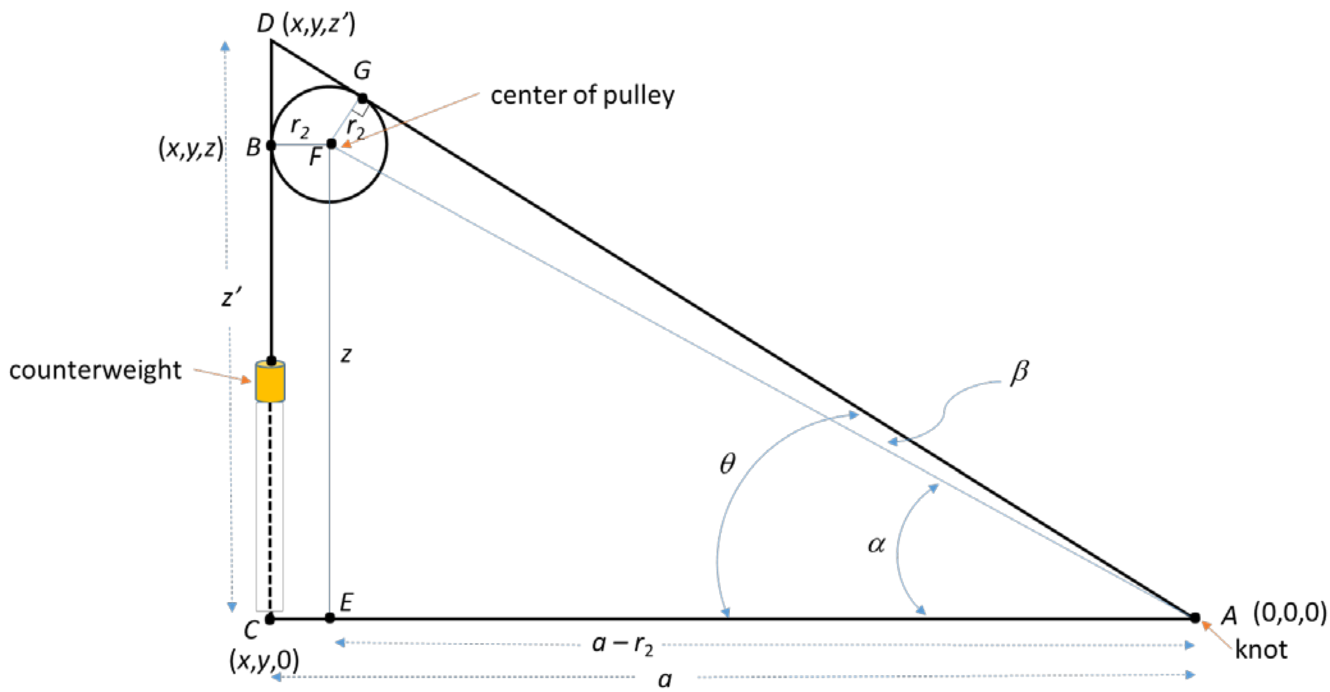

Figure 1. Diagram for one of the pulleys

Point	Coordinates	Notes
A	(0,0,0)	Location of knot, which is used as the origin of the coordinate system
B	(x,y,z)	**Measured values** for (x, y, z) where z is the height of the center of the pulley
C	(x,y,0)	**Measured values** for x & y at position of rope having the counterweight at the end
D	(x,y,z')	Point along the line AG projected to the known x & y coordinates of point B
E	(-,-,0)	Point directly below pulley center at height $z = 0$ (x & y not given)
F	(-,-,z)	Center of the pulley. (z is the measured height, x & y not given)
G	(-,-,-)	Point where rope from knot passes over the pulley. (Coordinates not given)

Table 1: Description of points shown in Figure 1.

We have measurements for the coordinates of the knot center (point *A*), the *x* and *y* coordinates of the rope with the counterweight (graduated cylinder) hanging vertically from the pulley, the height *z* of the pulley center, and the (outer) diameter of the pulley, r_2. We can translate the coordinate system such that the knot defines the origin by subtracting the measured knot center coordinates from all of the measured coordinates. What we wish to find is the unit vector along the direction of the rope from the knot to the pulley.

If the diameter of the pulley is small compared to the distance from the knot to the pulley, we could use the vector pointing from *A* to our measured coordinates for *B*. This is not a bad approximation, however, a more accurate value can be obtained by finding the actual direction of the rope from the knot to the point where the rope first makes contact with the pulley. This can be accomplished using geometry to find the coordinates of point *D* on the diagram. The *x* and *y* coordinates for point *D* are the same as the measured values for point *C*. The *z* coordinate (denoted as *z'* on the diagram) can be determined by projecting the line between the knot and rope contact point on the pulley ($\overline{AG}$) to the *x* and *y* coordinates of point *D*. This requires finding the distances and angles shown in Figure 1.

The procedure is as follows: find the angle α in the right triangle defined by the points *EAF*; find the angle β in right triangle *GAF*; sum these to find angle θ in right triangle *CAD*. Using the distance between points *A* and *C*, and the angle θ, find projected height, *z'*, for the point *D*.

Angle α

The distance from point *A* to *C* is found from the measured values for point *C*. $\overline{AC} \equiv a = \sqrt{x^2 + y^2}$. The distance from point A to E is $\overline{AE} = a - r_2$ (i.e., subtract the known radius of the pulley from the distance from *A* to *C* as shown in Figure 1). Since the triangle *EAF* is a right triangle,

$$\tan(\alpha) = \frac{z}{a - r_2} \quad \Rightarrow \quad \alpha = \tan^{-1}\left(\frac{z}{a - r_2}\right),$$

where arc tangent is denoted $\tan^{-1}$.

Angle β

The angle at vertex *G* in triangle *GAF* is a right angle since the rope is tangent to the pulley at the point of contact. The hypotenuse of triangle *GAF* is the same as that of triangle *EAF*, namely the distance between points *F* and *A*: $\overline{AF} = \sqrt{(a - r_2)^2 + z^2}$. From triangle *GAF* we see that:

$$\sin(\beta) = \frac{r_2}{AF} = \frac{r_2}{\sqrt{(a - r_2)^2 + z^2}} \quad \Rightarrow \quad \beta = \sin^{-1}\left(\frac{r_2}{\sqrt{(a - r_2)^2 + z^2}}\right)$$

Coordinates of projected point *D*

From triangle *CAD*, we see that $\tan(\theta) = z'/a$. Since $\theta = \alpha + \beta$, $z' = a\tan(\alpha + \beta)$. Thus, the coordinates of the projected point *D* are (x, y, z') where

$$z' = a\tan(\alpha + \beta),\ a = \sqrt{x^2 + y^2},\ \alpha = \tan^{-1}\left(\frac{z}{a - r_2}\right), \text{and } \beta = \sin^{-1}\left(\frac{r_2}{\sqrt{(a - r_2)^2 + z^2}}\right)$$

Unit vector

Since the knot (point *A*) is at the origin, the unit vector in the direction of the tension in the rope is:

$$\hat{u} = \frac{x\hat{i} + y\hat{j} + z'\hat{k}}{\sqrt{x^2 + y^2 + z'^2}}.$$