Frictional Force in Pulleys

The tension in the rope attached to the counterweight (graduated cylinder with water) pulling vertically down on the pulley and the tension in the rope from the pulley to the knot are not equal in magnitude due to friction in the pulley. The pulley consists of a sheave (grooved wheel over which the rope passes) and a fixed axle about which the sheave can rotate. At equilibrium, the sheave is stationary. In our simple model for friction, we imagine the friction as occurring at a single point where the sheave and axle are in contact. Depending on which way the sheave is trying to turn, friction will act in the direction opposing that motion.

Consider one of the pulleys as shown in Figure 1. In order for the sheave to not be moving (i.e., to be in equilibrium), the net force on the sheave must be zero*:

$$\mathbf{T} + \mathbf{W} + \mathbf{R} = 0 \quad \Rightarrow \quad \mathbf{R} = -\left(\mathbf{T} + \mathbf{W}\right) \tag{1}$$

where **T** is the tension in the rope from the pulley to the knot, **W** is the tension in the rope attached to the counterweight, and **R** is the force at the point where the sheave is in contact with the fixed axle (point ***A*** in Figure 1). This contact force can be decomposed into two components; one tangential to the sheave-axle interface and one perpendicular to the interface (the "normal" component). The resulting force is the vector sum of these two components:

$$\mathbf{R} = \mathbf{f} + \mathbf{N} \tag{2}$$

The tangential component, **f**, is the friction force that opposes the potential rotation of the sheave.

The standard model of friction is that the magnitude of the friction force is proportional to the force pressing the objects (sheave and axle) together, i.e., the normal force. This relationship is expressed by:

$$f = \mu N \tag{3}$$

where the proportionality constant, μ, is defined as the coefficient of static friction. (There is a similar relationship for sliding friction, with a different proportionality constant, however, we are not concerned with that here since we are only considering the equilibrium situation.)

We can find the magnitude of the friction force by considering the sum of the moments about the center of the axle (point ***O*** in Figure 1), which must be zero for the sheave to be in equilibrium:

$$Wr_2 + fr_1 - Tr_2 = 0 \tag{4}$$

where r_1 and r_2 are the radial distances to the inner and outer surfaces of the sheave respectively. Only magnitudes enter since all of the forces are tangential to these surfaces. The signs for the terms are determined by the right-hand rule. Note that there is no moment due to the normal component of the contact force since its line of action is through the center of the axle.

Solving for f in Eq. (4):

$$f = (T - W)\frac{r_2}{r_1}. \tag{5}$$

If this quantity is positive, the tension is greater than the counterweight and friction is resisting the rotation of the pulley toward the knot; if it is negative, the counterweight is greater than the tension and the pulley is trying to rotate toward the counterweight. In either case, the magnitude of the friction is positive and is always resisting the impending motion. To make this explicit we will write the magnitude of the friction as:

$$f = |T - W|\frac{r_2}{r_1} \tag{6}$$

Having an expression for the magnitude of the friction, a relationship between the normal force and the friction, and knowing the vectors **T** and **W**, we are now in a position to find the coefficient of static friction for the pulley. Since the contact force has been decomposed into two orthogonal components, the friction force and normal force, these form a right triangle as shown in Figure 2. Thus,

$$f^2 + N^2 = R^2 = |\mathbf{T} + \mathbf{W}|^2. \tag{7}$$

Using the Eq. (3) relating the friction to the normal force in Eq. (7),

$$f^2 + \frac{f^2}{\mu^2} = |\mathbf{T} + \mathbf{W}|^2 \quad \Rightarrow \quad 1 + \frac{1}{\mu^2} = \frac{|\mathbf{T} + \mathbf{W}|^2}{f^2} \Rightarrow \quad \frac{1}{\mu^2} = \frac{|\mathbf{T} + \mathbf{W}|^2 - f^2}{f^2}. \tag{8}$$

Solving for the coefficient of friction and using Eq. (6) for the magnitude of the friction:

$$\boxed{\mu = \frac{f}{\sqrt{|\mathbf{T} + \mathbf{W}|^2 - f^2}} \quad \text{where} \quad f = |T - W|\frac{r_2}{r_1}}$$

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*Note: bold characters represent vectors, non-bold, italicized characters represent scalars.

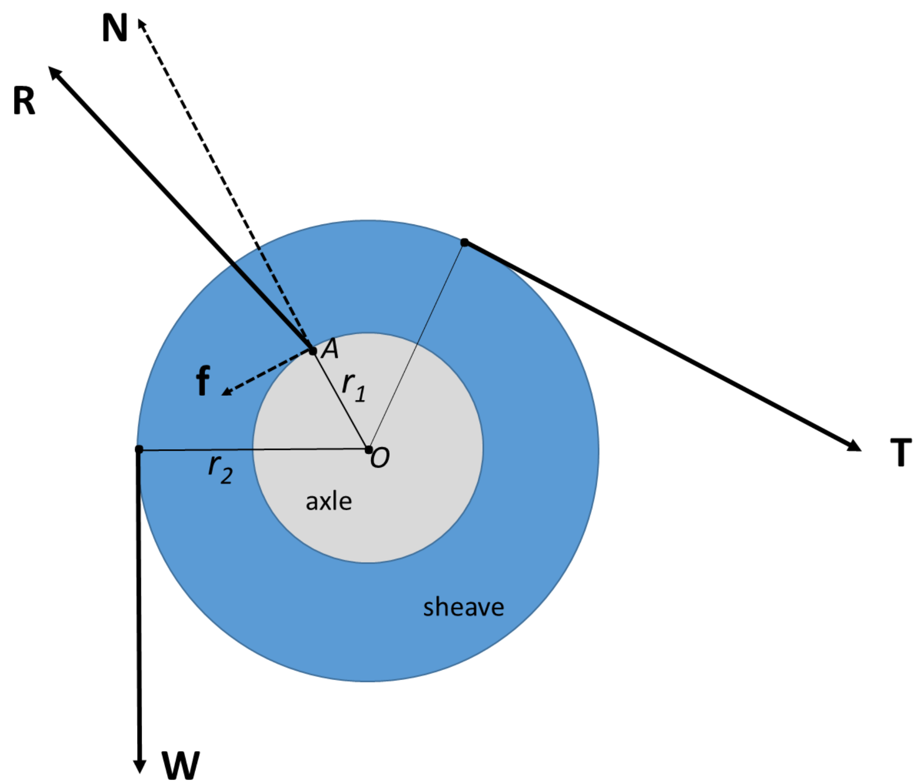

Figure 1. Forces on pulley. The contact force **R** is shown with its decomposition into tangential (**f**) and normal (**N**) components. In this figure, the tension **T** is greater than the counterweight **W**. The friction force adds to the counterweight to prevent the sheave from rotating.

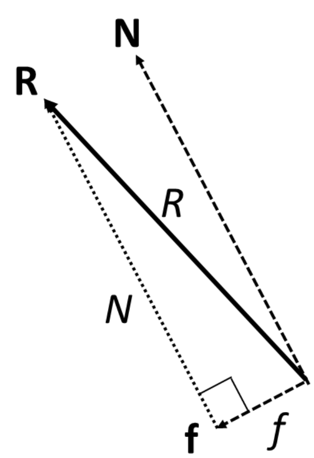

Figure 2. Right triangle formed by normal and tangential components of contact force.