Assignment 1

Submission : 23.02

Team: VADHRI VENKATA RATNAM

---

1. Inner product of A, B

Both A and B are reprsented as col matrices, hence we calcuate $A^T. B$

$1.1 + 2.2 + 3.3 = 1 + 4 + 9 = 14$

Ans : D

---

2. Compute $A. A^T$

$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$A. A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1.1 + 0.0 & 1.2 + 0.(-1) \\ 2.1 + (-1).0 & 2.2 + (-1).(-1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Ans : B

3. Eigenvalues and Eigenvectors

Method to be used

Step 1. Calculate λ values using the characteristic equation. det($A - \lambda I$) = 0

Step 2. Subsitute each λ value into $AX = \lambda X$ to get each eigen vector.

1. $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Step 1 : Find eigenvalues

$det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$

$det(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$

$\lambda^2 - 1 = 0; \lambda^2 = 1; \lambda = \pm 1$

Step 2 : Find eigenvectors

For $\lambda = 1;$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} x2 \\ x1 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

All vectors that satisfy $x1 = x2$ can be an eigenvector.

Example : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Normalized $1/\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -1;$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = -1. \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} x2 \\ x1 \end{bmatrix} = \begin{bmatrix} -x1 \\ -x2 \end{bmatrix}$

All vectors that satisfy $x1 = -x2$ can be an eignvector.

Example : $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Normalized $1/\sqrt{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\text{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \lambda = \pm 1 \text{ eigenvectors} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Step 1 : Find eigenvalues

$det(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$

$det(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0$

$(1 - \lambda)(-1 - \lambda) = 0$

$(1 - \lambda^2) = 0; \lambda^2 = 1; \lambda = \pm 1;$

Step 2 : Find eigenvectors

For $\lambda = 1;$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} x1 \\ -x2 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

All vectors to satisfy above has any x1 and x2 = 0 (since $x2 == -x2$)

Example : $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Normalized $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = -1;$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = (-1). \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} x1 \\ -x2 \end{bmatrix} = \begin{bmatrix} -x1 \\ -x2 \end{bmatrix}$

All vectors to satisfy above has x1 = 0 (since $x1 == -x1$) and any x2

Example : $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Normalized $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\mathrm{X} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \lambda = \pm 1 \text{ eigenvectors} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. $X = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Step 1 : Find eigenvalues

$det(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$

$det(\begin{bmatrix} \lambda & -i \\ i & -\lambda \end{bmatrix})$ = 0

$-\lambda^2 - i.(-i) = 0$

$(\lambda^2 - 1) = 0; \lambda^2 = 1; \lambda = \pm 1;$

Step 2 : Find eigenvectors

For $\lambda = 1;$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} -i.x2 \\ i.x1 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

All vectors to satisfy above $x2 = ix1$ and $x1 = -ix2$

Example : $\begin{bmatrix} 1 \\ i \end{bmatrix}$ Normalized $1/\sqrt{2} \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = -1;$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \end{bmatrix} = (-1). \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

$\begin{bmatrix} -i.x2 \\ i.x1 \end{bmatrix} = \begin{bmatrix} -x1 \\ -x2 \end{bmatrix}$

All vectors to satisfy above $x2 = -ix1$ and $x1 = ix2$

Example : $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ Normalized $1/\sqrt{2} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\mathrm{X} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \lambda = \pm 1 \text{ eigenvectors} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

4. Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$$

(a) Check if A is hermitian

$A = A^{\dagger} = [A^T]^*$

$$[A^T] = \left[\begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}^T\right]^* = \begin{bmatrix} 2 & -i & 1 \\ i & 2 & -i \\ 1 & i & 2 \end{bmatrix}^* = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} = A$$

(b) Eigenvalues of A

$$A = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}; det(A - \lambda I) = 0$$

$$det(\begin{bmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{bmatrix}) = 0$$

$$(2-\lambda).\,det(\begin{bmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{bmatrix}) - i.\,det(\begin{bmatrix} -i & i \\ 1 & 2-\lambda \end{bmatrix}) + 1.det(\begin{bmatrix} -i & 2-\lambda \\ 1 & -i \end{bmatrix}) = 0$$

$=> (2-\lambda).\,((2-\lambda)^2 - (i.-i)) - i.\,((-i)(2-\lambda) - i.1) + 1.((-i)(-i) - (2-\lambda).1) = 0$

$=> (2-\lambda).\,((2-\lambda)^2 - 1) - i.\,((-i)(2-\lambda+1)) + 1.(-1-(2-\lambda)) = 0$

$=> (2-\lambda).\,((2-\lambda)^2 - 1) - 1.(3-\lambda) + 1.(-3+\lambda)) = 0$

$=> (2-\lambda).\,((2-\lambda)^2 - 1) - 3 + \lambda - 3 + \lambda = 0$

$=> (2-\lambda).\,((2-\lambda)^2 - 1) - 6 + 2\lambda = 0$

$=> (2-\lambda).\,(3 - 4\lambda + \lambda^2) - 6 + 2\lambda = 0$

$=> 6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - 6 + 2\lambda = 0$

$=> -\lambda^3 + 6\lambda^2 - 9\lambda = 0$

$=> \lambda.\,(-\lambda^2 + 6\lambda - 9) = 0$

$=> \lambda.\,(\lambda - 3)^2 = 0$

$\lambda = 3 \text{ or } \lambda = 0$

(c) Normalized eigenvectors of A and check orthoganality

$Ax = \lambda x$

For $\lambda = 3$

$$\begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \lambda . \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = 3 . \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

The above gives us the following equations.

$-x1 + ix2 + x3 = 0$

$-ix1 - x2 + ix3 = 0$

$x1 - ix2 - x3 = 0$

$x3 = x1 - ix2$

.if $x3 = 0$, then $x1 = ix2$ and if $x2 = 1, x1 = i$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Normalized eignenvector, length = $\sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

.if $x2 = 0$, then $x2 = x3$ and if $x1 = 1, x2 = 1$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

.if $x1 = 0$, then $x3 = -ix2; => x2 = 1; x3 = -i$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

Normalized eignenvector, length = $\sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

$$1\sqrt{2}. \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \quad 1\sqrt{2}. \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad 1\sqrt{2}. \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

For $\lambda = 0$

$$\begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \lambda. \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The following equations are derived from above.

$2x1 + ix2 + x3 = 0 \ldots (1)$

$-ix1 + 2x2 + ix3 = 0$

$x1 - ix2 + 2x3 = 0$

From equation (1) above, $x3 = -2x1 - ix2$, if we subsitute that in the equation (2) above.

$-ix1 + 2x2 + i(-2x1 + ix2) = 0 \; x2 = ix1$

$x1 = 1, x2 = i, x3 = -1$

Normalized eignenvector, length = $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$1\sqrt{3}. \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}$$

Eigenvectors are NOT orthogonal to each other.

(d) Determinant and Trace of A

$$A = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$$

$$2.det(\begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}) - i.det(\begin{bmatrix} -i & i \\ 1 & 2 \end{bmatrix}) + 1.det(\begin{bmatrix} -i & 2 \\ 1 & -i \end{bmatrix})$$

$2.3 - 3 - 3 = 6 - 6 = 0$

Determinant of matrix is 0

$tr(A) = 2 + 2 + 2 = 6$

Trace of matrix is 6

5. Find matrix representation

$A\left|\uparrow\right\rangle = \left|\downarrow\right\rangle$ and $A\left|\downarrow\right\rangle = \left|\uparrow\right\rangle$

$\left|\uparrow\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left|\downarrow\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$Let A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$

$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$=> \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\alpha_1 = 0; \alpha_3 = 1;$

$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$=> \begin{pmatrix} \alpha_2 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha_2 = 1; \alpha_4 = 0;$

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6. Matrix representations

$A = |\alpha\rangle \langle\alpha| + |\beta\rangle \langle\beta|$

(a) $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$=> \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$

$=> \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$=> \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $\alpha = 1/\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\beta = 1/\sqrt{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$=> \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$=> \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$$=> \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) $\alpha = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ $\beta = \begin{bmatrix} sin\theta \\ -cos\theta \end{bmatrix}$

$=> \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \end{bmatrix} + \begin{bmatrix} sin\theta \\ cos\theta \end{bmatrix} \begin{bmatrix} sin\theta & -cos\theta \end{bmatrix}$

$=> \begin{bmatrix} cos^2\theta & sin\theta.cos\theta \\ sin\theta.cos\theta & sin^2\theta \end{bmatrix} + \begin{bmatrix} sin^2\theta & -cos\theta.sin\theta \\ cos\theta.sin\theta & cos^2\theta \end{bmatrix}$

$=> \begin{bmatrix} cos^2\theta + sin^2\theta & 0 \\ 0 & cos^2\theta + sin^2\theta \end{bmatrix}$

$$=> \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Tensor product

$\psi = 1/\sqrt{2}.\,(|0\rangle + |1\rangle)$

(a) $\psi^{\otimes}2$

$\psi^{\otimes}2 = \psi \otimes \psi = 1/\sqrt{2}.\,(|0\rangle + |1\rangle) \otimes 1/\sqrt{2}.\,(|0\rangle + |1\rangle)$

$= 1/2(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$

$= 1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle$

(a) $\psi^{\otimes}3$

$\psi^{\otimes}3 = \psi \otimes \psi \otimes \psi = 1/\sqrt{2}.\,(|0\rangle + |1\rangle) \otimes 1/\sqrt{2}.\,(|0\rangle + |1\rangle) \otimes 1/\sqrt{2}.\,(|0\rangle + |1\rangle)$

$= 1/(2\sqrt{2})\,[\,(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle\,]$

$= 1/(2\sqrt{2})\,[\,(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes (|0\rangle + |1\rangle)\,]$

$= 1/(2\sqrt{2})\,[\,(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)\,]$

---

8. Outer product

$$a = \begin{bmatrix} 3i \\ 2-i \\ 4 \end{bmatrix} \; b = \begin{bmatrix} 2 \\ i \\ 2+3i \end{bmatrix}$$

$$= a \otimes b = \begin{bmatrix} 2.3i & i.3i & (2+3i).3i \\ 2.(2-i) & i.\,(2-i) & (2+3i).\,(2-i) \\ 2.4 & 4.i & 4.(2+3i) \end{bmatrix}$$

$$= \begin{bmatrix} 6i & -3 & 6i-9 \\ 4-2i & 2i+1 & 6i+7 \\ 8 & 4i & 8+12i \end{bmatrix}$$

9. Unitary operator.

$A = i.\,(I - U).\,(I + U^\dagger)$

If U is unitary operator, then $U.U^\dagger = U^\dagger.U = I$

If A is hermitian than $A = A^\dagger$

$A^\dagger = \left[\, i.\,(I - U).\,(I + U^\dagger)\,\right]^\dagger$

$= (-i).\left[\,(I - U).\,(I + U^\dagger)\,\right]^\dagger$

$= (-i).\left[\,(I - U)^\dagger.\,(I + U^\dagger)^\dagger\,\right]$

$= (-i).\left[\,(I^\dagger - U^\dagger)(I^\dagger + U^\dagger)^\dagger)\,\right]$

$= (-i).\left[\,(I - U^\dagger)(I + U)\,\right]$

$= (-i).\left[\,(I^2 + U.I - I.U^\dagger - U^\dagger U)\,\right]$

$= (-i).\left[\,(I + U - U^\dagger - I)\,\right]$

$$A^\dagger = i.\left[\,(U^\dagger - U)\,\right]$$

$A = \left[\, i.\,(I - U).\,(I + U^\dagger)\,\right]$

$= i.\,[\,I^2 + I.U^\dagger - U.I - U.U\dagger\,]$

$= i.\,[\,I + U^\dagger - U - I\,]$

$$A = i.\,[\,U^\dagger - U\,]$$

$$A = A^\dagger$$

10. Check if operators are equal.

$AB = 1/2[A, B] + 1/2\{A, B\}?$

$1/2[A, B] + 1/2\{A, B\} = 1/2(AB - BA + AB + BA)$

$= 1/2.(2AB)$

$= AB$

```
In [ ]:
```