Assignment 2

Submitted by : VADHRI VENKATA RATNAM 19.03.2023

1 Which of the following vectors represent valid quantum states? why or why not??

A valid quantum state will have sum of mod square of probability amplitudes equal to 1. Inner product of the vector with itself is 1.

$\langle\psi|\psi\rangle = 1$

(a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = 0.0 + 1.1 = 1$

(b) $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} = (0.5).(0.5) + (0.5).(0.5) = 0.25 + 0.25 = 0.5$

(c) $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} = 0.0 + 1.1 + 1.1 + 0.0 = 2$

(d) $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} = 1/\sqrt{2}.1/\sqrt{2} + 0.0 + 0.0 + 1/\sqrt{2}.1/\sqrt{2} = 0.5 + 0.5 = 1$

(e) $\begin{bmatrix} 0 \\ 1/\sqrt{2} \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} \end{bmatrix} = 0.0 + 1/\sqrt{2}.1/\sqrt{2} = 0.5$

(f) $\begin{bmatrix} 0.3/\sqrt{2} \\ 0.7/\sqrt{2} \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} 0.3/\sqrt{2} \\ 0.7/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0.3/\sqrt{2} & 0.7/\sqrt{2} \end{bmatrix} = 0.09/2 + 0.49/2 = 0.29$

(g) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\langle\psi|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} = 0.0 + 0.0 + 1.1 + 0.0 = 1$$

(a), (d) and (g) are valid quantum states.

2. What is the expected output from the series of Stern-Gerlach Machines

$I : SG_z - > I/2(|0\rangle)), I/2(dropped)$

$I/2(|0\rangle) : SG_x - > I/4(|+\rangle), I/4(|-\rangle, dropped)$

(a) i = x

$I/4|+\rangle) : SG_x = I/4(|+\rangle)$

(b) i = y

$I/4|+\rangle) : SG_y = I/8(|i\rangle), I/8(|-i\rangle)$

(c) i = z

$I/4|+\rangle) : SG_z = I/8(|0\rangle), I/8(|1\rangle)$

3. Suppose A and B are commuting Hermitian operators. Prove that: exp(A) exp(B) = exp(A + B)

$exp^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} \ldots.$

$exp^B = 1 + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} \ldots.$

$A.\, exp^A.\, exp^B = (1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} \ldots.)(1 + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} \ldots.)$

$= 1 + (A + B) + \frac{(A^2 + B^2 + 2AB)}{2!} + \frac{(A^3 + B^3 + 3AB^2 + 3A^2B)}{3!} \ldots$

$B.\, exp^{(A+B)} = 1 + (A + B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} \ldots$

$= 1 + (A + B) + \frac{(A^2 + B^2 + AB + BA)}{2!} + \frac{(A^3 + B^3 + AB^2 + ABA + BA^2 + BAB + B^2A)}{3!} \ldots$

Since A and B commute, $AB = BA$

$= 1 + (A + B) + \frac{(A^2 + B^2 + 2AB)}{2!} + \frac{(A^3 + B^3 + 3AB^2 + 3BA^2)}{3!} \ldots$

Hence, $exp^A exp^B = exp^{(A+B)}$

4. In a three-dimensional vector space spanned by the orthonormal basis {|1〉,|2〉,|3〉}, the linear operators A and B has the following action on the basis vectors :

$A|1\rangle = |1\rangle, B|1\rangle = |1\rangle$

$A|2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle), A|3\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$

$B|2\rangle = \frac{\sqrt{3}}{2}|2\rangle + \frac{1}{2}|3\rangle, B|3\rangle = \frac{-1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle$

Is it possible to find a common set of eigenvectors for A and B ? Justify your answer.

If there is a common set of eigenvectors between A and B, then A and B would commute.

$AB = BA$

(a) $A|1\rangle = |1\rangle, B|1\rangle = |1\rangle$

$A|1\rangle = |1\rangle$

Multiply B on both sides.

$BA|1\rangle = B|1\rangle$

since $B|1\rangle = |1\rangle$

$$BA|1\rangle = |1\rangle$$

$B|1\rangle = |1\rangle$

Multiply A on both sides.

$AB|1\rangle = A|1\rangle$

since $A|1\rangle = |1\rangle$

$$AB|1\rangle = |1\rangle$$

Hence, $AB|1\rangle = BA|1\rangle => AB = BA$

(b) $A|2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \quad B|2\rangle = \frac{\sqrt{3}}{2}|2\rangle + \frac{1}{2}|3\rangle$

$A|2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$

Multiply B on both sides.

$BA|2\rangle = \frac{1}{\sqrt{2}}(B|2\rangle - B|3\rangle)$

$= \frac{1}{\sqrt{2}}\left[\left[\frac{\sqrt{3}}{2}|2\rangle + \frac{1}{2}|3\rangle\right] - \left[\frac{-1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle\right]\right]$

$= \frac{1}{\sqrt{2}}\left[\frac{\sqrt{3}+1}{2}|2\rangle + \frac{1-\sqrt{3}}{2}|3\rangle\right]$

$$= \frac{1}{2\sqrt{2}}\left[(\sqrt{3}+1)|2\rangle + (1-\sqrt{3})|3\rangle\right]$$

$B|2\rangle = \frac{\sqrt{3}}{2}|2\rangle + \frac{1}{2}|3\rangle$

Multiply A on both sides.

$AB|2\rangle = \frac{\sqrt{3}}{2}A|2\rangle + \frac{1}{2}A|3\rangle$

$= \frac{\sqrt{3}}{2}\left[\frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)\right] + \frac{1}{2}\left[\frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)\right]$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sqrt{3}+1}{2}|2\rangle + \frac{1-\sqrt{3}}{2}|3\rangle\right]$$

Hence, $BA|2\rangle = AB|2\rangle => AB = BA$

(b) $A|3\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \quad B|3\rangle = \frac{-1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle$

$A|3\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$

Multiply B on both sides.

$BA|3\rangle = \frac{1}{\sqrt{2}}(B|2\rangle + B|3\rangle)$

$= \frac{1}{\sqrt{2}}(B|2\rangle + B|3\rangle)$

$= \frac{1}{\sqrt{2}}\left[\left[\frac{\sqrt{3}}{2}|2\rangle + \frac{1}{2}|3\rangle\right] + \left[\frac{-1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle\right]\right]$

$$= \frac{1}{2\sqrt{2}}\left[(\sqrt{3}-1)|2\rangle + (1+\sqrt{3})|3\rangle\right]$$

$B|3\rangle = \frac{-1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle$

Multiply A on both sides.

$AB|3\rangle = \left[\frac{-1}{2}A|2\rangle + \frac{\sqrt{3}}{2}A|3\rangle\right]$

$= \left[\frac{-1}{2}\left[\frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)\right] + \left[\frac{\sqrt{3}}{2}\left[\frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)\right]\right]\right]$

$$= \frac{1}{2\sqrt{2}} \left[\left[(-|2\rangle + |3\rangle) \right] + \left[\sqrt{3} \left[(|2\rangle + |3\rangle) \right] \right] \right]$$

$$= \frac{1}{2\sqrt{2}} \left[(\sqrt{3} - 1)|2\rangle + (1 + \sqrt{3})|3\rangle \right]$$

Hence, $BA = AB$

5. Average value of the observable X1Z2

$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$\langle\psi| = \frac{\langle 00| + \langle 11|}{\sqrt{2}}$

$\langle L\rangle = \langle\psi|L|\psi\rangle$

Observable $L = X_1 Z_2$

$\langle X_1 Z_2\rangle = \langle\psi|X_1 Z_2|\psi\rangle$

$= \quad \left\langle \frac{\langle 00| + \langle 11|}{\sqrt{2}} \middle| X_1 Z_2 \middle| \frac{|00\rangle + |11\rangle}{\sqrt{2}} \right\rangle$

$= \quad \left\langle \frac{\langle 00| + \langle 11|}{\sqrt{2}} \middle| \frac{X_1|0\rangle Z_2|0\rangle + X_1|1\rangle Z_2|1\rangle}{\sqrt{2}} \right\rangle$

$Substitute, X|0\rangle = |1\rangle, X|1\rangle = |0\rangle; Z|0\rangle = |0\rangle, Z|1\rangle = -1.|1\rangle$

$= \quad \left\langle \frac{\langle 00| + \langle 11|}{\sqrt{2}} \middle| \frac{|10\rangle - |01\rangle}{\sqrt{2}} \right\rangle = 0$

6. Show that the bell states form an orthonormal basis.

In order for the vectors to form an orthonormal basis, we need to proove the following.

- Vectors are linearly independent.
- Vectors are unit vectors
- Vectors are orthogonal.

The following are the computational basis for the bell states.

$$|\psi_+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} |\psi_-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$|\phi_+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} |\phi_-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

(a) Linear independence.

Method 1

If the colomns of states put together form a unitary matrix, its columns form an basis.

$U.U^\dagger = I$

The colums of bell states can be formed as below.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$U.U^\dagger = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = I$$

Method 2

If the bell states are dependent, there will exist a,b,c,d (atleast one not zero) that satisfies below.

$$\frac{1}{\sqrt{2}}\left[a\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + c\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right] = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}a+b\\c+d\\c-d\\a-b\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

The only solution for the 4 equations above is $a = 0; b = 0; c = 0; d = 0$

(b) Unit vectors

$$\langle\psi_+|\psi_+\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = 2/2 = 1$$

$$\langle\psi_-|\psi_-\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = 2/2 = 1$$

$$\langle\phi_+|\phi_+\rangle = \frac{1}{2}\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = 2/2 = 1$$

$$\langle\phi_-|\phi_-\rangle = \frac{1}{2}\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = 2/2 = 1$$

(b) Orthogonal to each other

$$\langle\psi_+|\psi_-\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = 0/2 = 0$$

$$\langle\psi_+|\phi_+\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = 0/2 = 0$$

$$\langle\psi_-|\phi_+\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = 0/2 = 0$$

$$\langle\phi_-|\phi_+\rangle = \frac{1}{2}\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = 0/2 = 0$$

7. Multi-qubit joint measument

Consider a state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

If we measure the state above in $|+\rangle and|-\rangle$ basis,

What are the outcomes of joint measurements ?

Method 1

$\langle +|0\rangle = \frac{1}{\sqrt{2}}$

$\langle +|1\rangle = \frac{1}{\sqrt{2}}$

$\langle -|0\rangle = \frac{1}{\sqrt{2}}$

$\langle -|1\rangle = \frac{-1}{\sqrt{2}}$

If we measure all qu-bits in the $|+\rangle$ and $|-\rangle$ basis, the following are the outcomes of the measurement.

$P(+++) = |\langle +++|\psi\rangle|^2 = \frac{1}{4}$

$P(++-) = |\langle ++-|\psi\rangle|^2 = 0$

$P(+--) = |\langle +--|\psi\rangle|^2 = \frac{1}{4}$

$P(+-+) = |\langle +-+|\psi\rangle|^2 = 0$

$P(---) = |\langle ---|\psi\rangle|^2 = 0$

$P(-++) = |\langle -++|\psi\rangle|^2 = 0$

$P(--+) = |\langle --+|\psi\rangle|^2 = \frac{1}{4}$

$P(-+-) = |\langle -+-|\psi\rangle|^2 = \frac{1}{4}$

Method 2

$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

Convert the state into the $|+\rangle$ and $|-\rangle$ basis, by substituting below.

$|0\rangle| = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|1\rangle| = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&= \frac{1}{\sqrt{2}}\Big[\,\frac{1}{\sqrt{2}}(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle)\Big] \\
&= \frac{1}{\sqrt{2}}\Big[\,\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\Big] \\
&= \frac{1}{4}[\,(|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle) + (|+\rangle - |-\rangle) \otimes (|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)\,] \\
&= \frac{1}{4}[\,(|+++\rangle + |++-\rangle + |+-+\rangle + |+--\rangle + |-++\rangle + |-+-\rangle + |--+\rangle + |---\rangle) + (|+++\rangle - |++-\rangle - | \\
&= \frac{1}{4}[\,(2|+++\rangle + 2|+--\rangle + 2|-+-\rangle + 2|--+\rangle)\,] \\
&= \frac{1}{2}[\,(|+++\rangle + |+--\rangle + |-+-\rangle + |--+\rangle)\,]
\end{aligned}
$$

Using the born rule on the state above, the following probabilities exist,

$P(+++) = |\frac{1}{2}|^2 = \frac{1}{4}$

$P(+--) = |\frac{1}{2}|^2 = \frac{1}{4}$

$P(-+-) = |\frac{1}{2}|^2 = \frac{1}{4}$

$P(--+) = |\frac{1}{2}|^2 = \frac{1}{4}$

The rest of the statts can be infrerred to be 0.

$P(++-) = |0|^2 = 0$

$P(+-+) = |0|^2 = 0$

$P(---) = |0|^2 = 0$

$P(-++) = |0|^2 = 0$

what are the joint measuements and probabilities for +, i, -i states.

For sake of clarity in expression we will represent i and j for i and -i states respectively.

$\langle +|0\rangle = \frac{1}{\sqrt{2}}$

$\langle +|1\rangle = \frac{1}{\sqrt{2}}$

$\langle -|0\rangle = \frac{1}{\sqrt{2}}$

$\langle -|1\rangle = \frac{-1}{\sqrt{2}}$

$\langle i|0\rangle = \frac{i}{\sqrt{2}}$

$\langle i|1\rangle = \frac{i}{\sqrt{2}}$

$\langle j|0\rangle = \frac{i}{\sqrt{2}}$

$\langle j|1\rangle = \frac{-i}{\sqrt{2}}$

The following are the joint measurements and their probabilities.

$P(+ii) = |\frac{-1}{2}|^2 = \frac{1}{4}$

$P(+ij) = |0|^2 = 0$

$P(+ji) = |0|^2 = 0$

$P(+jj) = |\frac{-1}{2}|^2 = \frac{1}{4}$

$P(-ii) = |\frac{-1}{2}|^2 = \frac{1}{4}$

$P(-ij) = |0|^2 = 0$

$P(-ji) = |0|^2 = 0$

$P(-jj) = |\frac{-1}{2}|^2 = \frac{1}{4}$

what is the expectation value for $X_1X_2X_3$

Pauli X-gate is a bit-flip operation

$$E = \left\langle \frac{1}{\sqrt{2}}(\langle 000|+\rangle 111|)\middle| X_1X_2X_3 \middle| \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right\rangle$$

$$= \frac{1}{2}\langle(\langle 000| + \langle 111|)|(X_1|0\rangle X_2|0\rangle X_3|0\rangle + X_1|1\rangle X_2|0\rangle X_3|0\rangle)\rangle$$

$$= \frac{1}{2}\langle(\langle 000| + \langle 111|)|(|111\rangle + |000\rangle)\rangle$$

$$= 1$$

8. Transformation matrix

Find U

We are trying to find a matrix, U that satisfies the following equations.

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \quad |\psi_-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \; |\phi_-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

$U|00\rangle = |\phi_+\rangle$

$U|01\rangle = |\phi_-\rangle$

$U|10\rangle = |\psi_+\rangle$

$U|11\rangle = |\psi_-\rangle$

After calculating each row of 4x4 U matrix and comparing with two-qubit state in col vector form, we get the following.

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Find H

$U = e^{iH}$

Assuming $\theta = 1$

$U = I.cos(1) + iH.sin(1)$

$$H = \frac{-i}{sin(1)}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{bmatrix} - I.cos(1)$$

$$H = \frac{-i}{sin(1)}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{bmatrix} - \begin{bmatrix} cos(1) & 0 & 0 & 0 \\ 0 & cos(1) & 0 & 0 \\ 0 & 0 & cos(1) & 0 \\ 0 & 0 & 0 & cos(1) \end{bmatrix}$$

$$= \frac{-i}{sin(1)}\begin{bmatrix} \frac{1}{\sqrt{2}} - cos(1) & \frac{1}{\sqrt{2}} & 0 & 0 \\ \\ 0 & -cos(1) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ 0 & 0 & \frac{1}{\sqrt{2}} - cos(1) & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & -cos(1) \end{bmatrix}$$

9. Calculate eigenvectors and eigenvalues of density matrix ρ.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Density matrix $\rho = |\psi\rangle\langle\psi|$

$\rho = (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|)$

$$= \alpha\alpha^*|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0| + \beta\beta^*|1\rangle\langle 1| = \begin{bmatrix}\alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^*\end{bmatrix}$$

Calculate eigen values and eigen vectors of ρ

Using the characeristic equation.

$det(\rho - \lambda I)$ = 0

$$det\left[\begin{bmatrix}\alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^*\end{bmatrix} - \begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}\right] = 0$$

$$det\begin{bmatrix}\alpha\alpha^* - \lambda & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* - \lambda\end{bmatrix} = 0$$

$$(\alpha\alpha^* - \lambda)(\beta\beta^* - \lambda) - \alpha\beta^*\beta\alpha^* = 0$$

$$\lambda^2 - \lambda(\alpha\alpha^* + \beta^*\beta) = 0$$

Two eigen values are $\lambda = 0 \quad \lambda = \alpha\alpha^* + \beta^*\beta)$

Eigen vectors calculation

$$\begin{bmatrix}\alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^*\end{bmatrix}\begin{bmatrix}x1 \\ x2\end{bmatrix} = (\alpha\alpha^* + \beta^*\beta)\begin{bmatrix}x1 \\ x2\end{bmatrix}$$

Solving the equation above we get the following relation between x1 and x2

$$\alpha x2 = \beta x1$$

If $x1 == 1, x2 == \frac{\beta}{\alpha}$ and if $x2 == 1, x1 == \frac{\alpha}{\beta}$

Eigen vectors => $\begin{bmatrix}1 \\ \frac{\beta}{\alpha}\end{bmatrix}$ and $\begin{bmatrix}\frac{\alpha}{\beta} \\ 1\end{bmatrix}$

10. Is the expression a density operator.

Note: Assuming that the second half of the expression in the question, has an $\otimes$ rather than multiplication.

$(|0\rangle \otimes |0\rangle) \otimes (\langle 0| \otimes \langle 0|)$ is 4x4 matrix.

$$\rho = \frac{1}{4}(1-\epsilon)\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} + \epsilon\begin{bmatrix}1 \\ 0 \\ 0 \\ 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1-\epsilon & 0 & 0 \\ 0 & 0 & 1-\epsilon & 0 \\ 0 & 0 & 0 & 1-\epsilon\end{bmatrix}$$

For a valid density operator, trace = 1 and $tr(\rho) = \frac{4-3\epsilon}{4}$

Hence, the density operator is not valid one.

```
In [ ]:
```