Solution - 3

Quantum Computation and Machine Learning

April 2023

Module 3: Quantum Circuits

1. : Reference.

2. The actions of the controlled–U transformation, **C**, on a two–qubit state are given by:

$$\mathbf{C}\,|0\rangle_1\,|\psi\rangle_2 \rightarrow |0\rangle_1\,|\psi\rangle_2$$
$$\mathbf{C}\,|1\rangle_1\,|\psi\rangle_2 \rightarrow |1\rangle_1\,U\,|\psi\rangle_2$$

where $U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix}$

As you can see, the general single-qubit unitary transformation is applied on the 2nd qubit only when the 1st qubit is $|1\rangle$. So, the matrix for this two-qubit controlled–U transformation, **C** is given by the following 4×4 matrix:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha \end{bmatrix}$$

3. Here, we will run through the problem of teleportation of an entangled quantum state using 2 Bell pairs shared between Alice and Bob. Suppose that Alice has two qubits entangled state given as:

$$|\psi\rangle_{12} = \alpha\,|00\rangle + \beta\,|11\rangle$$

and Alice and Bob share two-sets of following Bell-state:

$$|\psi\rangle_{35} = |\psi\rangle_{46} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

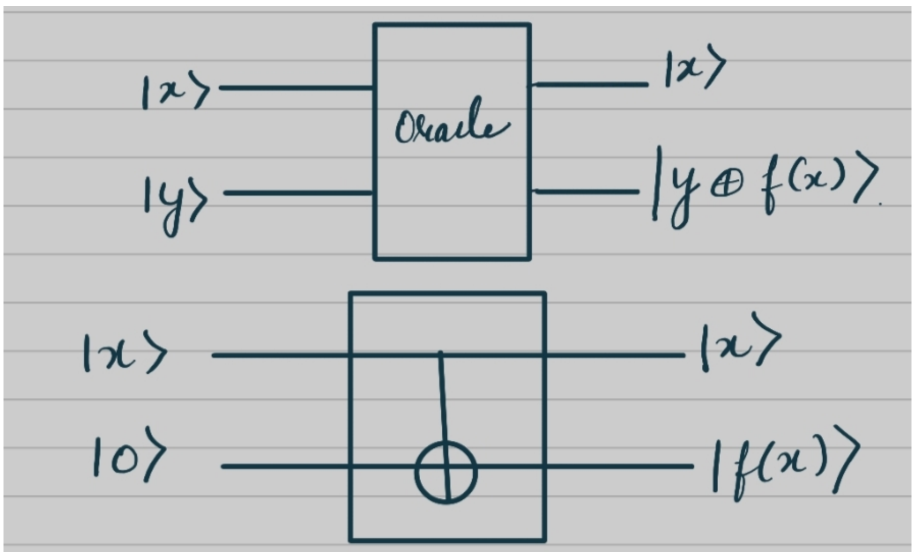
Notice that Alice has four qubits (1,2,3,4) and Bob has two (5,6). Now the combined six-qubit state will be

$$|\Psi\rangle_{12\,35\,46} = |\psi\rangle_{12} \otimes |\psi\rangle_{35} \otimes |\psi\rangle_{46} = \alpha\,|00\rangle + \beta\,|11\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

According to the problem here, we have to apply teleportation protocol 2-times: one on the qubits $1, 3, 5$ and another on the qubit $2, 4, 6$.

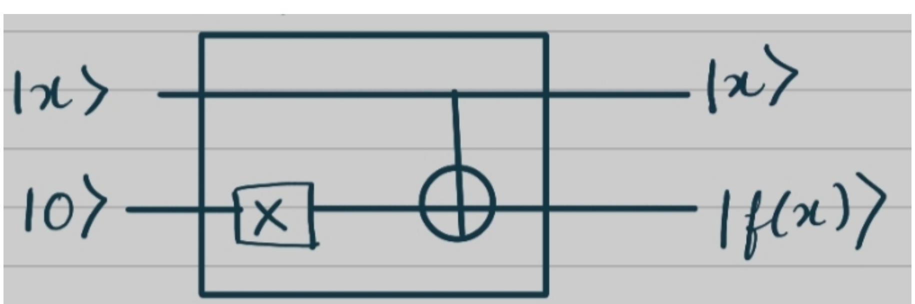
Click here and observe the states of qubit 1,2(with Alice prior to the measurement operation done by Alice as once measurement is done the state will collapse.) and qubit 5,6 (with bob) you will find that the qubits that bob have represents the same state that is the bell pair Alice wanted to transfer i.e., $\alpha |00\rangle + \beta |11\rangle$

4. (i) $f(0) = 0$ and $f(1) = 1$



Oracle for this case will be given by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

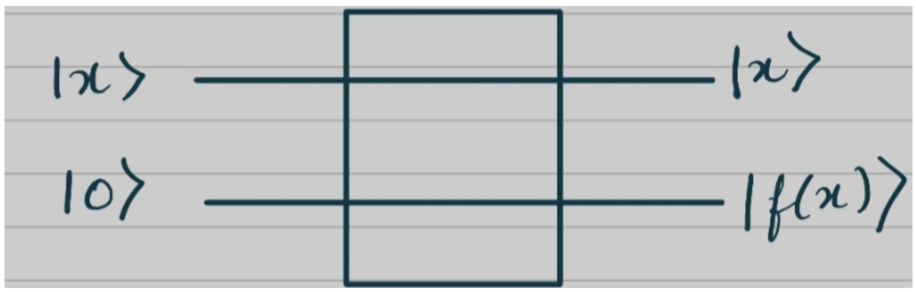
(2) $f(0) = 1$ and $f(1) = 0$



Oracle for this case will be given by $(I \otimes X).CNOT$ i.e.,

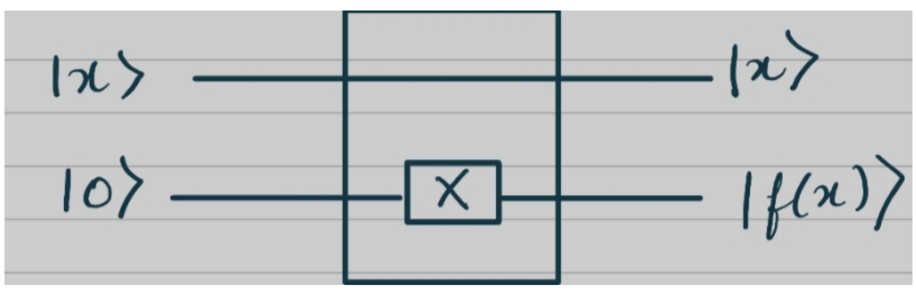
$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

(3) $f(0) = 0$ and $f(1) = 0$



Oracle for this case will be given by $I_{(4\times4)}$ i.e., $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

(4) $f(0) = 1$ and $f(1) = 1$



Oracle for this case will be given by $(I \otimes X)$ i.e., $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$

The oracle(U) can be written as,
$U = e^{-iH} = \sum \lambda_i |\psi_i\rangle \langle\psi_i|$ i.e., in terms of the Eigenvalues(λ_i) and corresponding Eigenvectors $|\psi_i\rangle$ where H is the Hamiltonian.

(i) Oracle for this case will be given by, $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ The Eigenvalues for the above matrix are, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1\ and\ \lambda_4 = -1$ with corresponding Eigenvector say $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\ and\ |\psi_4\rangle$, U can be rewritten as,

$$U = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| - |\psi_4\rangle\langle\psi_4| =$$
$$e^0 |\psi_1\rangle\langle\psi_1| + e^0 |\psi_2\rangle\langle\psi_2| + e^0 |\psi_3\rangle\langle\psi_3| + e^{-\pi} |\psi_4\rangle\langle\psi_4|$$
$$\implies e^{-iH} = e^0 |\psi_1\rangle\langle\psi_1| + e^0 |\psi_2\rangle\langle\psi_2| + e^0 |\psi_3\rangle\langle\psi_3| + e^{-\pi} |\psi_4\rangle\langle\psi_4|$$
$$\implies H = 0 |\psi_1\rangle\langle\psi_1| + 0 |\psi_2\rangle\langle\psi_2| + 0 |\psi_3\rangle\langle\psi_3| + \pi |\psi_4\rangle\langle\psi_4|$$
$$\implies H = \pi |\psi_4\rangle\langle\psi_4|$$

(ii) Oracle for this case will be given by $(I \otimes X).CNOT$, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The Eigenvalues for the above matrix are, $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1 \, and \, \lambda_4 = 1$ with corresponding Eigenvector say $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \, and \, |\psi_4\rangle$, U can be rewritten as,

$$U = -1 |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4| =$$
$$e^{-i\pi} |\psi_1\rangle\langle\psi_1| + e^0 |\psi_2\rangle\langle\psi_2| + e^0 |\psi_3\rangle\langle\psi_3| + e^0 |\psi_4\rangle\langle\psi_4|$$
$$\implies e^{-iH} = e^{-i\pi} |\psi_1\rangle\langle\psi_1| + e^0 |\psi_2\rangle\langle\psi_2| + e^0 |\psi_3\rangle\langle\psi_3| + e^{-0} |\psi_4\rangle\langle\psi_4|$$
$$\implies H = \pi |\psi_1\rangle\langle\psi_1| + 0 |\psi_2\rangle\langle\psi_2| + 0 |\psi_3\rangle\langle\psi_3| + 0 |\psi_4\rangle\langle\psi_4|$$
$$\implies H = \pi |\psi_1\rangle\langle\psi_1|$$

Note This $|\psi_i\rangle$ is different from the above part as the U matrix has change for this case. Similarly we can find H for the other cases also,
(iii) H = 0
(iv) H = $\pi |\psi_2\rangle\langle\psi_2| + \pi |\psi_4\rangle\langle\psi_4|$

5. Setting b = 0 we obtain from the map

$$|a\rangle \otimes |0\rangle \to |a\rangle \otimes |a\rangle$$

since a $\otimes$ 0 = a. Thus we have cloned a bit

6. Such a matrix does not exist. This can be seen as follows:-

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right) \otimes \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right)$$

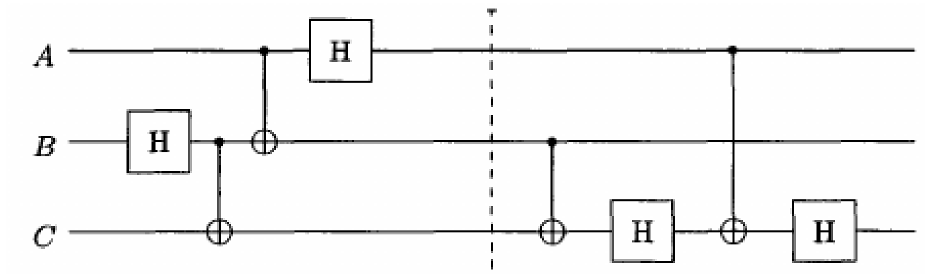
$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

On the other hand, we have:-

$$\mathbf{U} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \mathbf{U} \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$
$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

where we used the linearity of $\mathbf{U}$. Comparing these two equations we find a contradiction. This is the no cloning theorem

7. The input state of the circuit given in the question is $|\psi\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C$, where $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ and $|a|^2 + |b|^2 = 1$.



Now according to circuit, we first apply H (Hadamard) gate to qubit-B; so, the state at this point will be:

$$|\psi\rangle_A \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + |1\rangle\,\big)_B \otimes |0\rangle_C$$

Next, we have a C-NOT gate with qubit-B as a control-qubit and qubit-C as target; with the application of this C-NOT gate, we have our state in the following state:

$$|\psi\rangle_A \otimes \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)_{BC}$$

Now, with another C-NOT gate the joint state of our system will be: (here qubit-A as a control-qubit and qubit-B as target)

$$\frac{1}{\sqrt{2}}\big(a\,|000\rangle + a\,|011\rangle + b\,|110\rangle + b\,|101\rangle\,\big)_{ABC}$$

$$\Big\downarrow \text{ H on qubit-A}$$

$$\frac{1}{2}\big(a\,|000\rangle + a\,|100\rangle + a\,|011\rangle + a\,|111\rangle + b\,|010\rangle - b\,|110\rangle + b\,|001\rangle - b\,|101\rangle\,\big)_{ABC}$$

$$\Big\downarrow \text{ C-NOT: control:B and target:C}$$

$$\frac{1}{2}\big(a\,|000\rangle + a\,|100\rangle + a\,|010\rangle + a\,|110\rangle + b\,|011\rangle - b\,|111\rangle + b\,|001\rangle - b\,|101\rangle\,\big)_{ABC}$$

$$\Big\downarrow \text{ H on qubit-C}$$

$$\frac{1}{2\sqrt{2}}\big(a\,|000\rangle + a\,|001\rangle + a\,|100\rangle + a\,|101\rangle + a\,|010\rangle + a\,|011\rangle + a\,|110\rangle + a\,|111\rangle$$

$$+b\,|010\rangle - b\,|011\rangle - b\,|110\rangle + b\,|111\rangle + b\,|000\rangle - b\,|001\rangle - b\,|100\rangle + b\,|101\rangle\,\big)_{ABC}$$

$$\Big\downarrow \text{C-NOT: control:A and target:C}$$

$$\frac{1}{2\sqrt{2}}\big(a\,|000\rangle + a\,|001\rangle + a\,|101\rangle + a\,|100\rangle + a\,|010\rangle + a\,|011\rangle + a\,|111\rangle + a\,|110\rangle$$

$$+b\,|010\rangle - b\,|011\rangle - b\,|111\rangle + b\,|110\rangle + b\,|000\rangle - b\,|001\rangle - b\,|101\rangle + b\,|100\rangle\,\big)_{ABC}$$

$$\Big\downarrow \text{H on qubit-C}$$

$$\frac{1}{4}\big(a\,|000\rangle + a\,|001\rangle + a\,|000\rangle - a\,|001\rangle + a\,|100\rangle - a\,|101\rangle + a\,|100\rangle + a\,|101\rangle + a\,|010\rangle$$

$$+a\,|011\rangle + a\,|010\rangle - a\,|011\rangle + a\,|110\rangle - a\,|111\rangle + a\,|110\rangle - a\,|111\rangle$$

$$+b\,|010\rangle + b\,|011\rangle - b\,|010\rangle + b\,|011\rangle - b\,|110\rangle + b\,|111\rangle + b\,|110\rangle + b\,|111\rangle + b\,|000\rangle + b\,|001\rangle$$

$$-b\,|000\rangle + b\,|001\rangle - b\,|100\rangle + b\,|101\rangle + b\,|100\rangle + b\,|101\rangle\,\big)_{ABC}$$

Simplifying,

$$\frac{1}{4}\big(a\,|000\rangle + \cancel{a\,|001\rangle} + a\,|000\rangle - \cancel{a\,|001\rangle} + a\,|100\rangle - \cancel{a\,|101\rangle} + a\,|100\rangle + \cancel{a\,|101\rangle} + a\,|010\rangle$$

$$+\cancel{a\,|011\rangle} + a\,|010\rangle - \cancel{a\,|011\rangle} + a\,|110\rangle - \cancel{a\,|111\rangle} + a\,|110\rangle - \cancel{a\,|111\rangle}$$

$$+\cancel{b\,|010\rangle} + b\,|011\rangle - \cancel{b\,|010\rangle} + b\,|011\rangle - \cancel{b\,|110\rangle} + b\,|111\rangle + \cancel{b\,|110\rangle} + b\,|111\rangle + \cancel{b\,|000\rangle} + b\,|001\rangle$$

$$-\cancel{b\,|000\rangle} + b\,|001\rangle - \cancel{b\,|100\rangle} + b\,|101\rangle + \cancel{b\,|100\rangle} + b\,|101\rangle\,\big)_{ABC}$$

$\Rightarrow$

$$\frac{1}{2}\Bigg(\;|00\rangle_{AB}\otimes(a\,|0\rangle + b\,|1\rangle)_C + |01\rangle_{AB}\otimes(a\,|0\rangle + b\,|1\rangle)_C + |10\rangle_{AB}\otimes(a\,|0\rangle + b\,|1\rangle)_C + |11\rangle_{AB}\otimes(a\,|0\rangle + b\,|1\rangle)_C\Bigg)$$

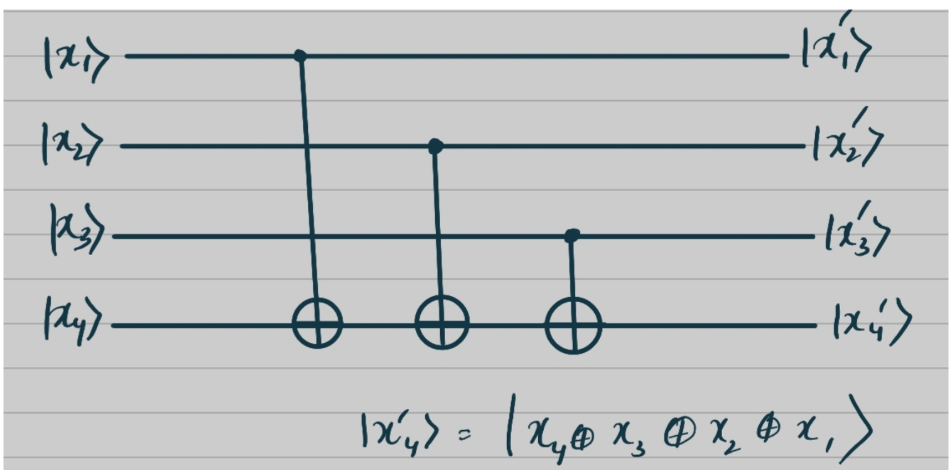
$$= \frac{1}{2}\Bigg(\;|00\rangle_{AB}\otimes|\psi\rangle_C + |01\rangle_{AB}\otimes|\psi\rangle_C + |10\rangle_{AB}\otimes|\psi\rangle_C + |11\rangle_{AB}\otimes|\psi\rangle_C\Bigg)$$

So, at the end of the circuit we get our state $|\psi\rangle$ on the qubit-C, which means this circuit implements the teleportation protocol.

8. The truth table and the Circuit for the same is given below and yes,this can be reversible which can be verified easily.

truth table:

x_1	x_2	x_3	x_4	x'_1	x'_2	x'_3	x'_4
0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	1
0	1	0	0	0	1	0	1
1	1	0	0	1	1	0	0
0	0	1	0	0	0	1	1
1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0
1	1	1	0	1	1	1	1
0	0	0	1	0	0	0	1
1	0	0	1	1	0	0	0
0	1	0	1	0	1	0	0
1	1	0	1	1	1	0	1
0	0	1	1	0	0	1	0
1	0	1	1	1	0	1	1
0	1	1	1	0	1	1	1
1	1	1	1	1	1	1	0



9. (i) We know that for any Unitary Matrix U, $U^{\dagger} = U^{-1}$ For the Toffoli Gate the T Operator is given as:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

It can be easily check that $T^\dagger = T^{-1} = T$ (ii) A NOT gate can be constructed from a Toffoli gate by setting the three input bits to $\{a, 1, 1\}$ that will give $\{a, 1, (a, 1) \oplus 1\}$
(iii) AND gate (a,b) can be constructed from a Toffoli gate by setting the three input bits to $\{a, b, 0\}$ that will give $\{a, b, (a, b) \oplus 0\}$
(iv) OR gate (a,b) can be implemented as Toffoli(1,b,Toffoli(a,b,a)).

10. the given state is a valid state as $\langle\psi|\psi\rangle = 1$. Circuit representation for the state is given below,