Assignment 4

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In [1]:
```
from numpy import pi
from qiskit import *
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit, Aer, assemble
from qiskit.visualization import plot_histogram, plot_bloch_vector, plot_state_qsphere
from math import sqrt, pi
import numpy as np
```

In [2]:
```
%%html
<style>
    img {float:left}
</style>
```

1 Design a network consisting of conditional gate and two hadamard gates

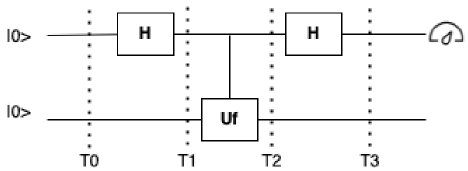
There are the following three gates for construction.

- A conditional U_f gate with transformation function $|\phi\rangle = (-1)^y|\phi\rangle$
- Two Hadamard Gates

The objective is to find the vaue of y added by the U_f gate.

Multiple Qubits

Consider the following circuit.



$T0 = |00\rangle$

$T1 = |0\rangle \otimes H|0\rangle$

$= \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + |1\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

$T2 = U_f(T1)$

$= U_f\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

Conditional activation of U_f => activates when Qubit 1 is 1

$= \frac{1}{\sqrt{2}}(|00\rangle + U_f|01\rangle)$

$= \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + U_f|1\rangle)$

$= \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + (-1)^y|1\rangle)$

If y = 0 (unknown bit inside the U_f gate)

$T2 = \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + |1\rangle) = |0+\rangle$

Apply hadamard on Qubit 1.

$T3 = |0\rangle H|+\rangle = |00\rangle$

Measurement on Qubit 1 gives 0

If y = 1 (unknown bit inside the U_f gate)

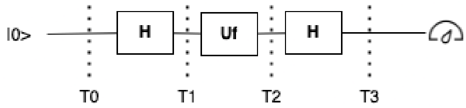
$T2 = \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle - |1\rangle) = |0-\rangle$

Apply hadamard on Qubit 1.

$T3 = |0\rangle H|-\rangle = |01\rangle$

Measurement on Qubit 1 gives 1

Single Qubit



$T0 = |0\rangle$

$T1 = H|0\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$T2 = U_f(T1)$

$= U_f \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Conditional activation of U_f => activates when Qubit 1 is 1

$= \frac{1}{\sqrt{2}}(|0\rangle + U_f|1\rangle)$

$= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle)$

If y = 0 (unknown bit inside the U_f gate)

$T2 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

Apply hadamard on Qubit 1.

$T3 = H|+\rangle = |0\rangle$

Measurement on Qubit 1 gives 0

If y = 1 (unknown bit inside the U_f gate)

$T2 = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

Apply hadamard on Qubit 1.

$T3 = H|-\rangle = |1\rangle$

Measurement on Qubit 1 gives 1

2. Which single operator application will yield value of x.

Hardamard Gate

3. Proove the following

$H^{\otimes n}|x\rangle_n = (-1)^{x.y}|y\rangle_n$

$x.y = x_1 y_1 \oplus x_2 y_2 \oplus \ldots x_n y_n$

Method 1

Single hadamard gate is reprsented as below.

$H|x\rangle = \Sigma_y (-1)^{x \cdot y}|y\rangle$

For all values of x => 0,1,2...n

$H^{\otimes n}|x\rangle_n = \Sigma_{y_1,y_2,y_3..y_n}(-1)^{x1.y1 \oplus x_2 \cdot y_2 .. \oplus x_n \cdot y_n}|y_1\rangle y_2\rangle \ldots |y_n\rangle$

$= \Sigma_y (-1)^{x \cdot y}|y\rangle_n$

Hence proved that

$H^{\otimes n}|x\rangle_n = (-1)^{x.y}|y\rangle_n$

Method 2

$H|x\rangle = \Sigma_y(-1)^{x.y}|y\rangle$

$H^{\otimes n}|x\rangle_n = H^{\otimes n}|x_1\rangle|x_2\rangle..|x_n\rangle = H|x_1\rangle H|x_2\rangle..H|x_n\rangle$

$= \Sigma_{y_1}(-1)^{x_1.y_1}|y_1\rangle \otimes \Sigma_{y_2}(-1)^{x_2.y_2}|y_2\rangle \ldots \Sigma_{y_n}(-1)^{x_n.y_n}|y_n\rangle$

$= \Sigma_{y_1,y_2,y_3..y_n}(-1)^{x1.y1\oplus x_2.y_2..\oplus x_n.y_n}|y_1\rangle y_2\rangle \ldots |y_n\rangle$

$= \Sigma_y(-1)^{x.y}|y\rangle_n$

Hence proved that

$H^{\otimes n}|x\rangle_n = (-1)^{x.y}|y\rangle_n$

4. Probability of detection of a balanced function.

The probability of having a balanced function or const function is $\frac{1}{2}$

The minimum number of queries to make so that we have a 100% accurate answer is $2^{N/2} + 1$

The probablity of error happens in the case where we have a balanced function and all the $2^{N/2}$ queries returned the same value (0 or 1).

$P(Error) = P(0)_1.P(0)_2 \ldots P(0)_K$

$= \frac{1}{2}.\frac{1}{2}\ldots\ldots\frac{1}{2}ktimes$

$= \frac{1}{2^k}$

Probability of error is inversely proportional to the number of tries (k).

Hence the larger the number of tries, the more the probability of detection of balanced function.

5. Probability of getting y != a is 0.

The final state of the function is below.

Final state = $\sum_{x,y\in\{0,1\}_n}(-1)^{f(x)+x.y}|y\rangle$

For a specific y, the probablity is the square of the term below

$P(Y = y) = \left[\sum_{x\in\{0,1\}_n}(-1)^{x.a+x.y}\right]^2$

$= \left[\sum_{x\in\{0,1\}_n}(-1)^{x.(a\oplus y)}\right]^2$

$= \left[\sum_{x\in\{0,1\}_n}(-1)^{x.(a\oplus y)}\right]^2$

$P(Y = y) = 1$ if $a == y$ else 0

6. Execute Bernstein-Vazirani for f(x) = a.x with a = 100101

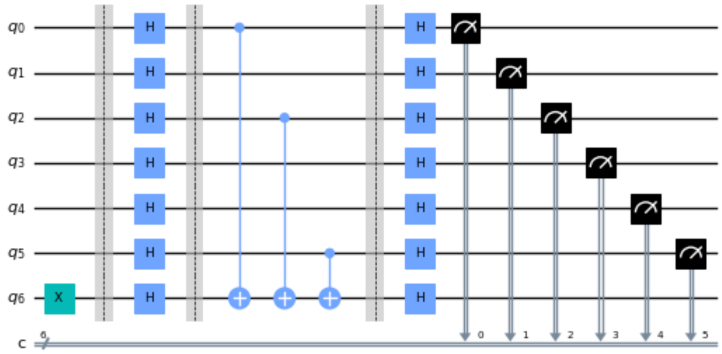
In [3]:
```
qreg_q = QuantumRegister(7, 'q')
creg_c = ClassicalRegister(6, 'c')
circuit = QuantumCircuit(qreg_q, creg_c)

circuit.x(qreg_q[6])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4], qreg_q[5], qreg_q[6])
circuit.h(qreg_q[0])
circuit.h(qreg_q[1])
circuit.h(qreg_q[2])
circuit.h(qreg_q[3])
circuit.h(qreg_q[4])
circuit.h(qreg_q[5])
circuit.h(qreg_q[6])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4], qreg_q[5], qreg_q[6])
circuit.cx(qreg_q[0], qreg_q[6])
circuit.cx(qreg_q[2], qreg_q[6])
circuit.cx(qreg_q[5], qreg_q[6])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4], qreg_q[5], qreg_q[6])
circuit.h(qreg_q[0])
circuit.h(qreg_q[1])
circuit.h(qreg_q[2])
circuit.h(qreg_q[3])
circuit.h(qreg_q[4])
circuit.h(qreg_q[5])
circuit.h(qreg_q[6])
```

```
circuit.measure(qreg_q[0], creg_c[0])
circuit.measure(qreg_q[1], creg_c[1])
circuit.measure(qreg_q[2], creg_c[2])
circuit.measure(qreg_q[3], creg_c[3])
circuit.measure(qreg_q[4], creg_c[4])
circuit.measure(qreg_q[5], creg_c[5])

circuit.draw(output='mpl', scale=0.5)
```

Out[3]:



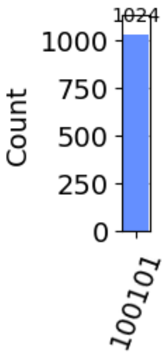
In [4]:

```
sim = Aer.get_backend('aer_simulator')

job = execute(circuit, sim)
result = job.result()

counts = result.get_counts()
plot_histogram(counts, figsize=(0.25,2))
```

Out[4]:



7. Step by step: Bernstien vaziarni

Method 1

we start with s = 0110

Step 1. Start with Hadamard gates applied to 4 Qubits. 5th Qubit used for gate calc.

$|\psi_0\rangle = H^{\otimes 4}|0000\rangle$

$= \frac{1}{\sqrt{2^n}}\left[\,\Sigma_{x\in\{0,1\}^4}|x\rangle\,\right]$

Step 2. Apply U_f gate for the Qubits.

$= \frac{1}{\sqrt{2^4}}\left[\,\Sigma_{x\in\{0,1\}^4}(-1)^{f(x)}|x\rangle\,\right]$

Step 3. Apply Hadamard gate.

$= H^{\otimes 4}\frac{1}{\sqrt{2^4}}\left[\,\Sigma_{x\in\{0,1\}^4}(-1)^{f(x)}|x\rangle\,\right]$

$= \frac{1}{2^4}\left[\,\Sigma_{x\in\{0,1\}^4}(-1)^{f(x)}\left[\,\Sigma_{y\in\{0,1\}^4}(-1)^{x.y}|y\rangle\,\right]\,\right]$

$= \frac{1}{2^4}\left[\,\Sigma_{x\in\{0,1\}^4}(-1)^{f(x)+x.y}|y\rangle\,\right]$

$= \frac{1}{2^4}\left[\,\Sigma_{x\in\{0,1\}^4}(-1)^{s.x+x.y}|y\rangle\,\right]$

$= |0110\rangle$

Method 2

Step 1 : Init (0,1,2,3,4) qubits

$|\psi\rangle_0 = |10000\rangle$

Step 2: Apply Hadamard gates to all bits

$|\psi\rangle_1 = |-++++\rangle$

Step 3: Apply CNOT gates as per secret message (0110) inside U_f

Due to phase kickback, $CNOT|+-\rangle$ becomes $CNOT|--\rangle$

$CNOT(3,0)|\psi\rangle_2 = |-++-+\rangle$

$CNOT(2,0)|\psi\rangle_3 = |-+--+\rangle$

Step 4 : Apply Hadamard gates again.

$|\psi\rangle_4 = H|-+--+\rangle$

$= |10110\rangle$

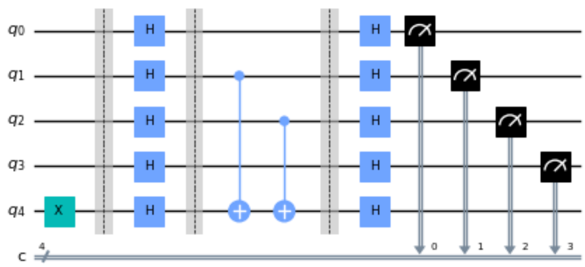
If we drop the 0th bit and measure 1,2,3,4 they are the secret key.

In [5]:
```
qreg_q = QuantumRegister(5, 'q')
creg_c = ClassicalRegister(4, 'c')
circuit = QuantumCircuit(qreg_q, creg_c)

circuit.x(qreg_q[4])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4])
circuit.h(qreg_q[0])
circuit.h(qreg_q[1])
circuit.h(qreg_q[2])
circuit.h(qreg_q[3])
circuit.h(qreg_q[4])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4])
circuit.cx(qreg_q[1], qreg_q[4])
circuit.cx(qreg_q[2], qreg_q[4])
circuit.barrier(qreg_q[0], qreg_q[1], qreg_q[2], qreg_q[3], qreg_q[4])
circuit.h(qreg_q[0])
circuit.h(qreg_q[1])
circuit.h(qreg_q[2])
circuit.h(qreg_q[3])
circuit.h(qreg_q[4])
circuit.measure(qreg_q[0], creg_c[0])
circuit.measure(qreg_q[1], creg_c[1])
circuit.measure(qreg_q[2], creg_c[2])
circuit.measure(qreg_q[3], creg_c[3])

circuit.draw(output='mpl', scale=0.5)
```

Out[5]:

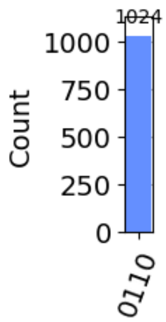


In [6]:
```
sim = Aer.get_backend('aer_simulator')

job = execute(circuit, sim)
result = job.result()

counts = result.get_counts()
plot_histogram(counts,figsize=(0.25,2))
```

Out[6]:



8. Find new quantum state after applying QFT

(i) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$QFT|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\end{bmatrix}$

(ii) $|\psi\rangle = \alpha|10\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\end{bmatrix} \otimes \begin{bmatrix}1\\ 0\end{bmatrix}$

$QFT|\psi\rangle = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = |+-\rangle$

(iii) $|\psi\rangle = \frac{1}{\sqrt{2}}[\,|10\rangle + |01\rangle\,]$

$QFT|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1\\ 0\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\ i-1\\ 0\\ -i-1\end{bmatrix}$

9. Rotation gates in QFT.

The following is the circuit that can achieve quantum addition. If we need to add x to y, we can add that in the freq domain too.

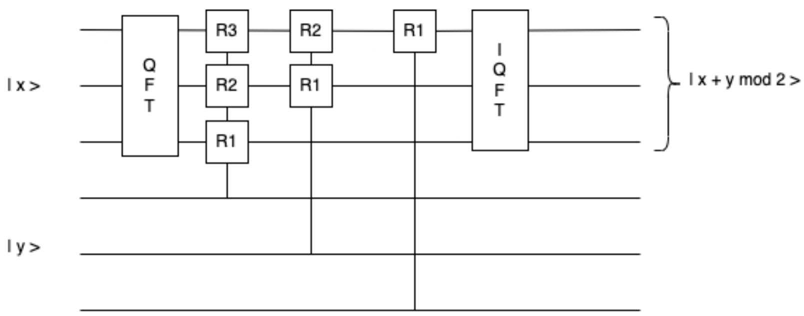
$|x\rangle = \frac{1}{\sqrt{2^n}}\sum_p e^{2\pi i\frac{xp}{2^n}}|p\rangle$

Add y to the phase ..

$|x\rangle = \frac{1}{\sqrt{2^n}}\sum_p e^{2\pi i\frac{x(p+y)}{2^n}}|p\rangle$

Apply QFT^{-1} ...

$|x+y \quad \%2^n\rangle$



Y can take values raised to power of 2, with less gates.

No of operations : $y^{(2^n-1)}$

10. First stage of QPE

N-bit QPE

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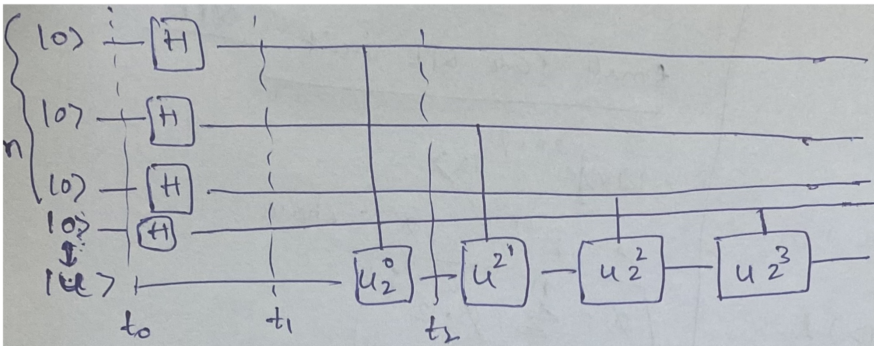

$t_0 = |0\rangle^{\otimes n} |u\rangle$

$t_1 = |+\rangle^{\otimes n} |u\rangle$

$\phi = 0.\phi_1\phi_2\phi_3 \cdot$

$t_2 = \frac{1}{\sqrt{2^n}}(|0\rangle + e^{2\pi i \phi}|1\rangle) \otimes (|0\rangle + |1\rangle) \cdots (|0\rangle + |1\rangle) \otimes |u\rangle$

$2\phi = \phi_1 . \phi_2$

$t_3 = \frac{1}{\sqrt{2^n}}\left[(|0\rangle + e^{2\pi i \phi}) \otimes (|0\rangle + e^{2\pi i (2\phi)}|1\rangle) \otimes \cdots \otimes (|0\rangle + |1\rangle) \otimes |u\rangle\right]$

$= \frac{1}{\sqrt{2^n}}\left[(|0\rangle + e^{2\pi i \phi}|1\rangle) \otimes (|0\rangle + e^{2\pi i (\phi_2\phi_3 \cdots)}|1\rangle) \otimes \cdots \otimes (|0\rangle + |1\rangle) \otimes |u\rangle\right]$

$t_n = \frac{1}{\sqrt{2^n}}\left[(|0\rangle + e^{2\pi i (0.\phi_1\phi_2\phi_3 \cdots)}|1\rangle) \otimes (|0\rangle + e^{2\pi i (\phi_2\phi_3 \cdots \phi_n)}|1\rangle) \cdots (|0\rangle + e^{2\pi i \phi_n}|1\rangle)\right.$

$= \frac{1}{\sqrt{2^n}}\left[(|0\rangle + e^{2\pi i 0.\phi_n}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.\phi_{n-1}\phi_n}) \otimes \cdots\right) |u\rangle.$

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