Some stochastic and statistical models for marine ecology

Adeline Leclercq Samson (Grenoble, France)

Joint work with

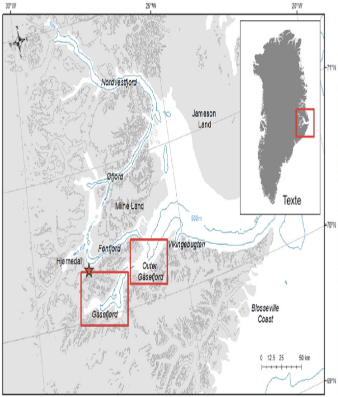
E. Buckwar (Linz), S. Ditlevsen (Copenhagen), M.P. Heide-Jørgensen (Copenhagen),
A. Melnykova (Avignon) M. Tamborrino (Warwick),
O. Tervo (Copenhagen), I. Tubikanec (Linz)

LABORATOIRE JEAN KUNTZMANN
MATHÉMATIQUES APPLIQUÉES - INFORMATIQUE

Université Grenoble Alpes

Outline of the talk

- **1. Climate change in Greenland**
  - Effect on endemic whales
  - Design of experimental data



- **2. Sound data**
  - Point process
  - Mediation effect of depth

- **3. Dive data**
  - Duration process
  - Competitive risks

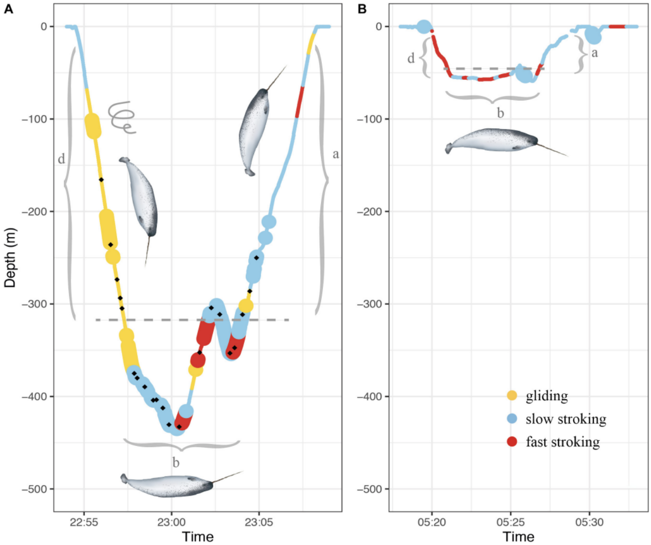
- **4. Spatial data**
  - Hypoelliptic diffusion
  - Numerical scheme and statistical inference

1. Climate change: a danger for the narwhals?

- Ice melt
- Decrease of sea ice coverage
- Increase of anthropogenic activities, of mining activities

- Reactions of narwhals to noise and human presence?

Narwhal dive



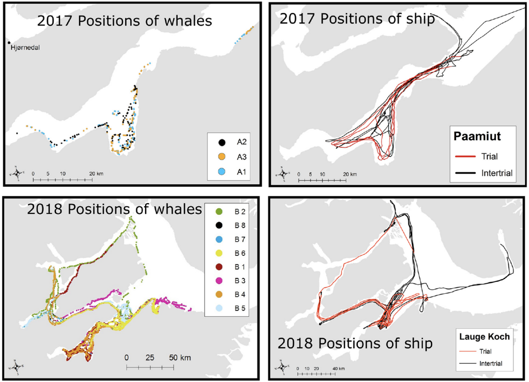
Foraging V dive (A) and Non-foraging U dive (B), Buzzes (black dots)

Experimental design

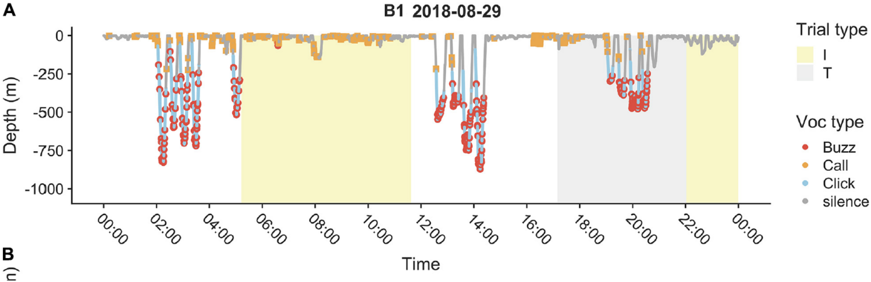
- 16 animals
- Instrumented with satellite tags and Acousonde acoustic-behavioral recorders
- Exposition to airgun pulses and vessel sounds
  - Pre exposure
  - Trials: airgun and ship-noise
  - Intertrial: only ship-noise
  - Post exposure

Recorded data

- Data recorded each second
- Sound (click/buzz)
- Depth
- Position of whales
- Position of ship
- Stroke
- Accelerometer



2. Sound data



A
B1 2018-08-29
0
-250
-500
-750
-1000
Depth (m)
00:00
02:00
04:00
06:00
08:00
10:00
12:00
14:00
16:00
18:00
20:00
22:00
00:00
Time
Trial type
I
T
Voc type
Buzz
Call
Click
silence
B
n)

Sound analysis: definition

T_{ij}: jth sound emitted by animal i
$\lambda_i(t)$: Rate of initiating a sound at time t for whale i
$X(t)$: Exposure level of seismic activity
$D(t)$: Depth
$Seismic(t)$: Seismic activity. Equals 0 if no airgun pulses, equals 1 if airgun pulses

We define the exposure X by

$$\text{if } Seismic(t) = 0, \quad X(t) = 0$$

$$\text{if } Seismic(t) = 1, \quad X(t) = 1/distance\ to\ ship$$

Sound stochastic model

For each animal i,

- Counting process $N_i = \sum_{j \geq 1} 1_{t_{ij} \leq t}$ associated with $(T_{ij})_j$
- Intensity process adapted to a filtration $\mathcal{F}_t$

$$\lambda_i(t|\mathcal{F}_t) \quad = \quad \lim_{h \to 0} \mathbb{E}\left(\frac{N_i(t+h) - N_i(t)}{h} |\mathcal{F}_t\right)$$

Stochastic model on $\lambda_i(t|\mathcal{F}_t)$ should include

- Autoregressive memory
- Covariates that depend on time $X(t), D(t)$, with non-linear effect
- Individual random effect b_i
- Mediation analysis

Hypothesis: mediation effect of Depth

Depth is an intermediate variable

- Exposure causes animal to dive less
- Less dives make animals produce less sounds

Question

- Does the sound production rate decrease when adjusting on depth?

Direct and indirect effects

- Direct effect on sound production not caused by depth: at a fixed depth, whale produces less sound under exposure
- Indirect effect on sound production caused by depth: whale dives less deep

Formalism of mediation

Definitions

- S_i is the presence of exposure or not ($= Seismic_i$)
- $D_i(s)$ is the mediator value when $S_i = s$
- $Y_i(s, d)$ is the observation would the exposure be s and the mediator d

The observation is

$$Y_i = Y_i(S_i, D_i(S_i))$$

Effect definitions

Total Effect	$\tau_i = Y_i(1, D_i(1)) - Y_i(0, D_i(0))$
Causal Mediation Effect	$\delta_i(s) = Y_i(s, D_i(1)) - Y_i(s, D_i(0))$
Direct Effect	$\xi_i(s) = Y_i(1, D_i(s)) - Y_i(0, D_i(s))$

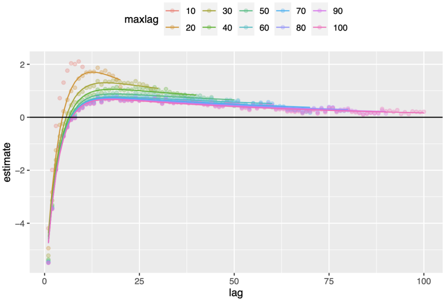
Estimation with a regression model

Poisson model with memory

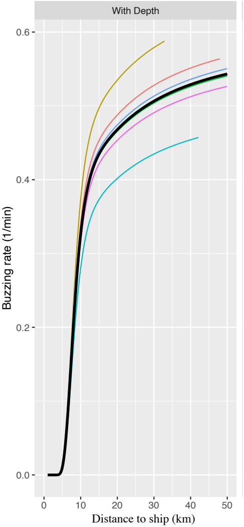
$$\begin{aligned}\log(\lambda_i(t)) \quad &= \quad \beta_0 + \beta_1 X(t) + \beta_2 P(t) + spline(D(t)) + b_i \\ &\quad\ \ \alpha_1 AR_1(t) + \ldots + \alpha_p AR_p(t)\end{aligned}$$

Estimation of the memory kernel

- From individual data before exposure
- No prior assumption on the shape
- Several lags p compared with BIC



Estimated buzzing rate



Mediation proportion = Mediation effect / Total effect

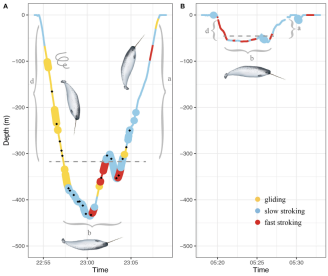
Seismic activity	0.343
Presence of ship	0.461

Still to be done

- Estimation of the mediation effect by bootstrap
- Hawkes process instead of the Poisson model
  - Non-parametric intensity
  - Random effects
  - Mediation

3. Depth data

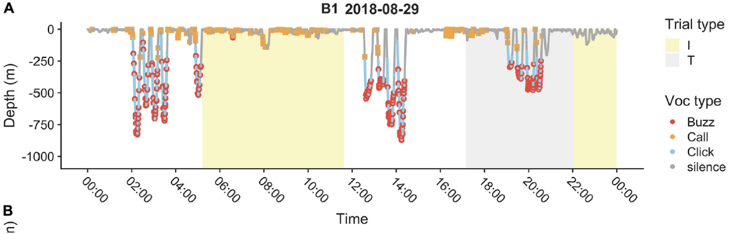
Deep dive Medium dive



Do animals

1. dive less under exposure?

2. dive less deep under exposure?



Dive less?

- Time at surface since the last dive for animal i, stay j: T_{ij}
- Hazard function:

$$\lambda(t) = \lim_{dt \to 0} \frac{P(t \leq T < t + dt | T \geq t)}{dt}$$

Frailty model = Cox model with random effects

$$\lambda_i(t) = \lambda_0(t) b_i \exp(\beta_1 X(t) + \beta_2 P(t))$$

- $\lambda_0(t)$ common baseline hazard over time
- Time-varying covariates X
- b_i unobserved random risk factor shared by all the events from animal i

$$b_i \sim \Gamma(a, b)$$

Dive less deep?

Two types of dive: Medium or a Deep Dive

T_{ijM}	Time at surface before the next Medium dive
T_{ijD}	Time at surface before the next Deep dive
$Z_{ij} = M$ or D	Type of dive
$T_{ij} = \min(T_{ijM}, T_{ijD})$	Observations

Competitive events models [Gill and Andersen, 2006]

Cause-specific hazard function of dive M

- rate at which a Medium dive is occurring at presence of the other type of dive

$$\lambda_M(t) = \lim_{dt \to 0} \frac{P(t \leq T < t + dt, Z = M | T \geq t)}{dt}$$

Two different Cox models with random individual effects

$$\lambda_M(t) = \lambda_{0M}(t) b_{iM} e^{\beta_{1M} X(t) + \beta_{2M} P(t)}$$

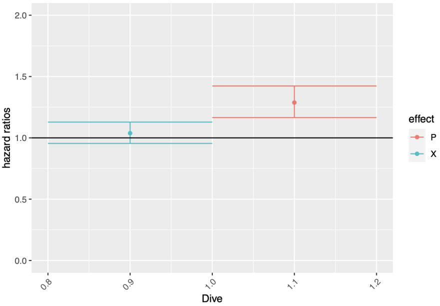
and

$$\lambda_D(t) = \lambda_{0D}(t) b_{iD} e^{\beta_{1D} X(t) + \beta_{2D} P(t)}$$

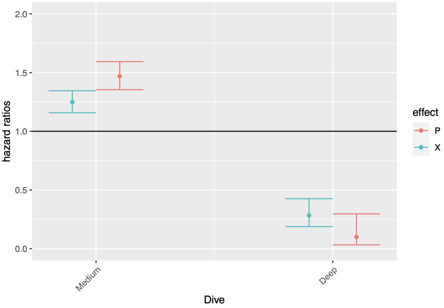
- λ_{0M} and λ_{0D} are the baseline hazard functions
- β quantifies the effect of exposure, expressed as Hazard Ratio

Effect of exposure

Time at Surface



Time to Deep/Medium dive



4. Spatial position data

Do animal adapt their position due to

- anthropogenic activities?
- warming of ocean temperatures?

Stochastic model of animal movement [Blackwell 1997, Brillinger et al 2002, Preisler et al 2004, Hooten et al 2010, Michelot et al 2019, Gloaguen et al 2018]

- X_t position at time t
- SDE on the position X_t

$$dX_t = \mu(X_t, S_t, \theta)dt + \Sigma(X_t, t)dW_t$$

- $\mu(x, S, \theta)$ models the direction preference depending on position x and spatial covariates S

Effect of spatial maps

- **Reflected SDE** [Hanks et al 2017)]
  - Drift should include reflection term to constrain the animal in space
- **Spatial covariates** [Brillinger et al 2002, Gloaguen et al 2018]

$$\mu(x, S, \theta) = \sum_{k=1}^{L} \gamma_k H'(S_k(x), \theta)$$

  - H potential function, H' gradient of the potential
  - S_k different spatial maps (temperature, depth, ice coverage, etc)
  - γ_k weights of the mixture

- **SDE on the velocity V_t** [Hamiltonian models]

$$
\begin{aligned}
dX_t &= V_t dt \\
dV_t &= \mu(V_t, S_t, \theta)dt + \Sigma(V_t, t)dW_t
\end{aligned}
$$

  - Hypoelliptic system

- **Modeling Depth**

$$
\begin{aligned}
dX_t &= \mu_X(X_t, S_t, D_t, \theta)dt + \Sigma(X_t, t)dW_t^1 \\
dD_t &= \mu_D(X_t, S_t, D_t, \theta)dt + \sigma_D dW_t^2
\end{aligned}
$$

  - Noise could be possibly degenerated ($\sigma_D = 0$)

Parametric estimation for hypoelliptic systems

$$\begin{aligned} dX_{1t} &= b_1(X_{1t}, X_{2t}, \mu)dt \\ dX_{2t} &= b_2(X_{1t}, X_{2t}, \mu)dt + \sigma dW_t \end{aligned}$$

Observations

- Data: discrete observations $X_{10:n} = (X_{10}, \ldots, X_{1n})$ at times $t_0 = 0 < t_1 = \Delta < \ldots < t_n = n\Delta$
- Hidden coordinate (X_{2t})

Difficult because

- Hypoellipticity
- No explicit transition density of the SDE
- Hidden coordinate X_2

Different strategies

Based on simulation

- Exact simulation
- Approximated simulation with a numerical scheme
- Estimation methods based on simulation
  - ABC
  - Monte Carlo, Importance Sampling

Based on approximation of the transition density

- What is a "good" approximation? Depends on the properties of the numerical scheme
- Estimation methods based on approximation
  - Contrast estimator [Thieullen Samson 2012, Ditlevsen Samson 2019, Melnykova 2020]
  - MCMC
  - EM algorithm [Beskos et al 2005, Gloaguen et al 2018, Ditlevsen, Samson 2014, 2019]

Numerical approximation schemes

Matricial notations: $X_t = (X_{1t}, X_{2t}), b(X) = (b_1(X), b_2(X))$

$$dX_t = b(X_t, \mu) + \Sigma dW_t$$

- Expected properties of a numerical scheme
  - Locally Lipschitz conditions on b
  - Exact moments up to a certain order
  - Mean-square convergence of order p for a step size Δ

$$\max_{t_i} \left(\mathbb{E} \left(\|X(t_i) - \tilde{X}(t_i)\|^2 \right) \right)^{1/2} \leq c\Delta^p$$

  - Preservation of structural properties: hypoellipticity, ergodicity, amplitudes, frequences, phases of oscillations

Some numerical scheme

Discretization with Euler-Maruyama

$$X_{i+1} = X_i + \Delta b(X_i) + \sqrt{\Delta}\Sigma\eta_i, \quad \eta_i \sim_{iid} \mathcal{N}(0, I)$$

- Does not preserve hypoellipticity

$$Var(X_{i+1}|X_i) = \sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}$$

- Not mean-square convergent

Local linearization

[Ozaki 1989, Biscay et al. 1996, Jimenez et al 2015, Melnykova 2020]

Approximate the solution by the solution of an autonomous linear SDE
On each interval $[i\Delta, (i+1)\Delta[$,

$$d\tilde{X}_t = \left(b(\tilde{X}_i) + \partial_x b(\tilde{X}_i)(\tilde{X}_t - \tilde{X}_i) + \frac{1}{2}\sigma^2 \partial_c^2 b(\tilde{X}_i)(t - i\Delta) \right) + \Sigma dW_t$$

- Linear SDE $\rightarrow$ explicit solution
- 1-step hypoellipticity

$$Cov(\tilde{X}_{i+1}|\tilde{X}_i) = \sigma^2 \begin{pmatrix} \frac{\Delta^3}{3}\partial_c b_1^2 & \frac{\Delta^2}{2}\partial_c b_1 \\ \frac{\Delta^2}{2}\partial_c b_1 & \Delta \end{pmatrix}$$

- Mean-square convergent of order 1
- Does not preserve other structural properties (ergodicity, oscillations)

Numerical splitting scheme

[Buckwar, Samson, Tamborrino, Tubikanec, 2021; Ditlevsen, Pilipovic, Samson, 2022]

Introduction of two subsystems.

1. **Subsystem a:** Linear SDE with exact solution

$$dX_t^a = AX_t^a dt \; + \; \Sigma dW_t$$

2. **Subsystem b:** Non-linear (decoupled) ODE with (exact) solution

$$dX_t^b = N(X_t^b)dt$$

Numerical splitting schemes with time step Δ

- Lie-Trotter

$$\hat{\mathbf{X}}^{LT} = X_\Delta^a \circ X_\Delta^b$$

- Strang

$$\hat{\mathbf{X}}^{S} = X_{\Delta/2}^b \circ X_\Delta^a \circ X_{\Delta/2}^b$$

Properties of splitting schemes

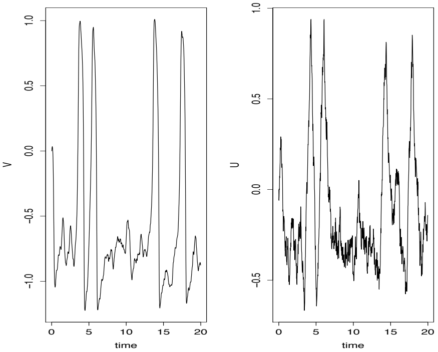
- Exact first moment up to Δ^3, covariance matrix up to Δ^3
- Mean-square convergence with order 1
- Preservation of noise structure, 1-step hypoellitpticity
- Preservation of Lyapounov structure
- Geometric ergodicity
- Transition density of the scheme highly non-linear -¿ difficult to exploit?

Example on the Fitzhugh-Nagumo model

[Lindner et al 1999, Gerstner and Kistler, 2002, Lindner et al 2004, Berglund and Gentz, 2006]

$$\begin{aligned} dV_t &= \frac{1}{\varepsilon}(V_t - V_t^3 - C_t - s)dt, \\ dC_t &= (\gamma V_t - C_t + \beta)\, dt + \sigma dW_t, \end{aligned}$$

- ε time scale separation
- s stimulus input
- β position of the fixed point
- γ duration of excitation



Splitting scheme on the FHN model

1. **Subsystem a:** Linear SDE

$$dX_t = \begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ \gamma & -1 \end{pmatrix} X_t dt \; + \; \Sigma dW_t$$

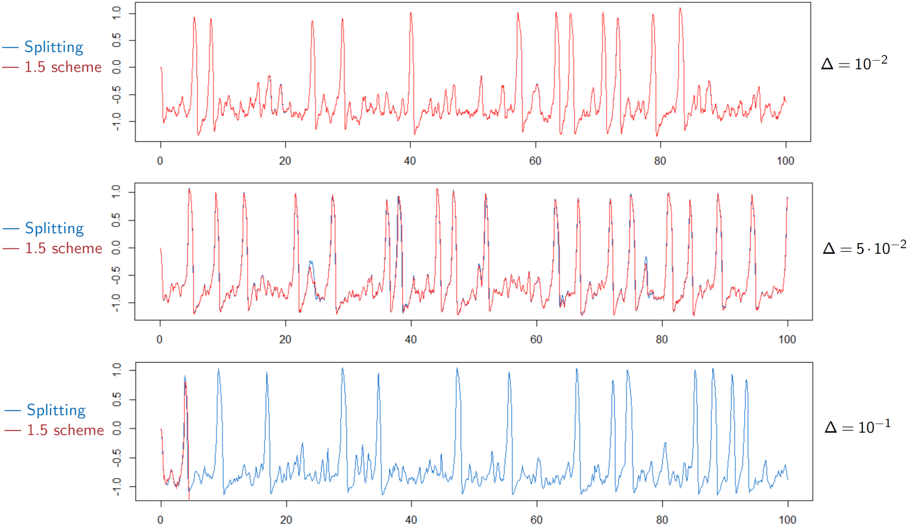
2. **Subsystem b:** Non-linear ODE

$$dX_t = \begin{pmatrix} \frac{1}{\varepsilon}(X_{1t} - X_{1t}^3) \\ \beta \end{pmatrix} dt$$

Explicit solution for both systems

Comparison of Splitting and Order 1.5 Strong Taylor Scheme

[Buckwar, Samson, Tamborrino, Tubikanec, 2021]



Estimation for hypoelliptic SDE

Case of complete observations

Constrats based on numerical approximations [Genon-Catolot, Jacod, 1993; Kessler 1996, Ditlevsen, Samson, 2019, Melnykova 2020]

$$\hat{\mu} = \arg\min_{\mu\in\Theta} \left(\sum_{i=1}^{n-1} \left(X_{i+1} - X_i - \Delta \tilde{b}_\mu(X_i)\right)' \Gamma^{-1} \left(X_{i+1} - X_i - \Delta \tilde{b}_\mu(X_i)\right) + \sum_{i=1}^{n-1} \log\det\Gamma \right)$$

Rate of convergence specific to hypoelliptic systems

- $\hat{\mu}_1$ asymptotically normal at rate $\sqrt{n/\Delta}$
- $\hat{\mu}_2$ asymptotically normal at rate $\sqrt{n\Delta}$

What about partial observations?

- **Filtering** X_{2t}
  - Particle filter for the partial observations
  - Stochastic Approximation EM algorithm coupled to Particle filter [Ditlevsen, Samson, 2019]
  - Exact simulation and EM [Gloaguen et al 2019]
- **Optimal control**
  - Control of W_t
  - Contrast based on Euler discretization [Clairon, Samson, 2021]
- **Approximate Bayesian Computation (ABC)**
  - Likelihood free method
  - Approximate numerical scheme for the simulations [Buckwar, Tamborrino, Tubikanec, 2020]

Approximate Bayesian Computation (ABC)

Posterior distribution

$$\underbrace{\pi(\theta|X)}_{\textit{posterior}} \propto \underbrace{\pi(X|\theta)}_{\substack{\textit{likelihood}\\(\textit{intractable})}} \underbrace{\pi(\theta)}_{\textit{prior}}$$

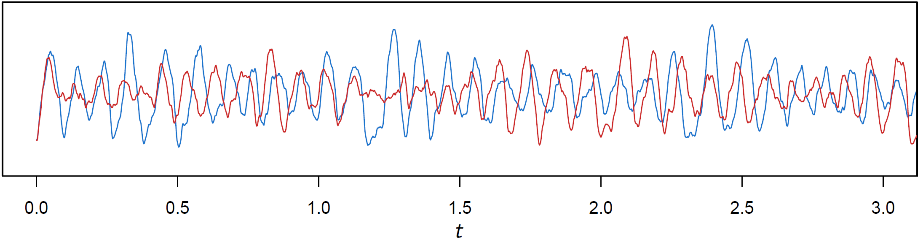
Approximate Bayesian Computation

1. Simulate $\theta^k \sim \pi(\theta)$ for $k = 1, \ldots, K$
2. Generate pseudo-data X_θ from the SDE model for each θ^k
3. Introduce summaries $s(X)$ and $s(X_\theta)$ of the data
4. Approximate the posterior, for a 'small' η

$$\pi(\theta|X) \approx \pi_{d,\eta,s}(\theta|X) = \pi(\ \theta \mid d(s(X), s(X_\theta)) < \eta\ )$$

Choice of the summaries in ABC [Buckwar, Tamborrino, Tubikanec, 2020]

How to account for the variability in the data for identical θ?



$\Longrightarrow$ **Transform the data from time to frequency domain**

Spectral Density (Periodogram)

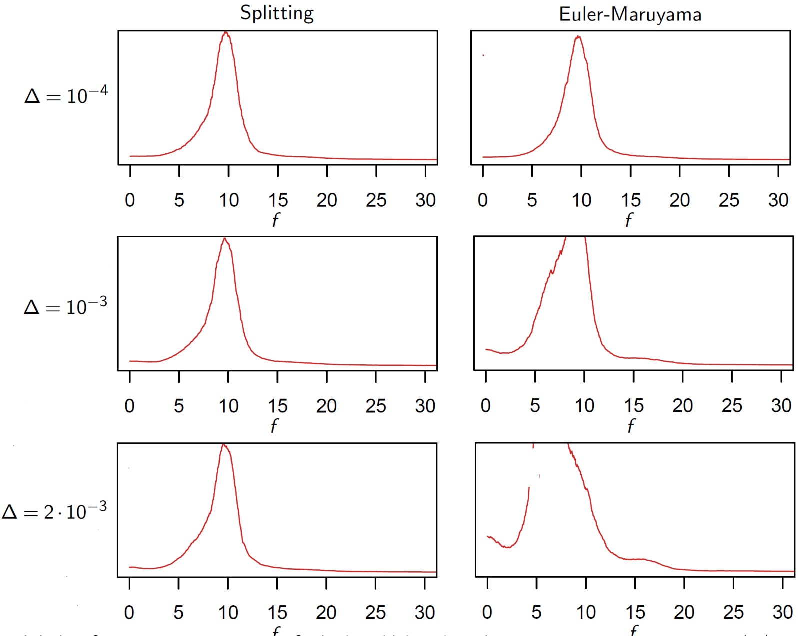
- Stationary stochastic process: $\mathbf{X}_\theta = (X_\theta(t))_{t\geq 0}$
- Autocovariance function: $Cov(X_\theta(t), X_\theta(s)) = r_\theta(\tau = t-s)$

$$S_{\mathbf{X}_\theta}(\omega = 2\pi f) = \int_{-\infty}^{\infty} r_\theta(\tau) e^{-i\omega\tau} \, d\tau, \quad \omega \in [-\pi, \pi]$$

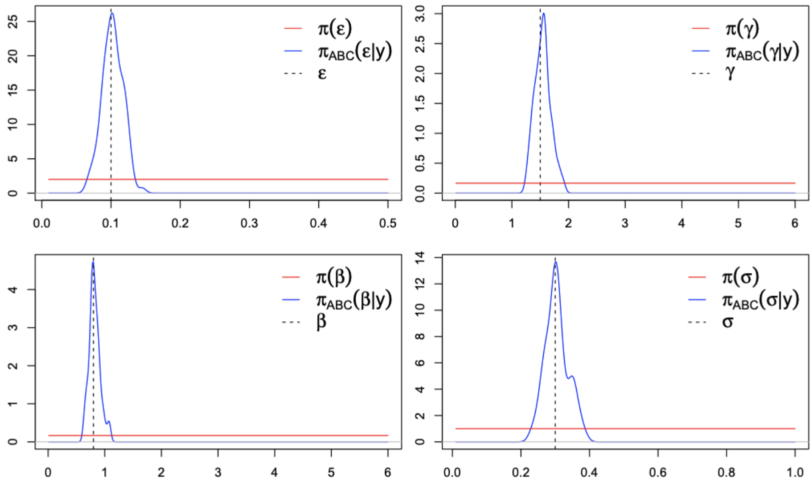
- Periodogram

$$s(X) = \hat{S}_X(\omega) = \frac{1}{n} \left| \sum_{j=1}^{n} X_j e^{-i\omega j} \right|^2$$

The Splitting Scheme preserves the Periodogram



ABC for the FHN Model



Perspectives

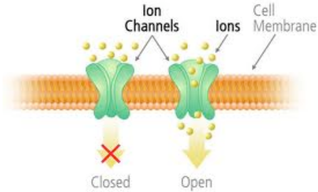
- Narwhals data are very rich
  - Time-dependent variables
  - Multi-dimensional analysis
  - Mediation effect
- Hypoelliptic SDE for movement/spatial data
  - Splitting schemes are promising
  - Adaptation to drift with gradient of potential

MERCI !

Stochastic models for intracellular neuronal data

Stochastic Morris-Lecar neuronal model [Wainrib, Thieullen, Pakdaman, 2012]
Weak law of large number, Langevin approximation

- Calcium, potassium, leakage ionic currents
- g_{Ca}, g_K, g_L maximal conductances
- V_{Ca}, V_K, V_L reversal potential
- I input current
- X_{2t} proportion of opened potassium channels
- Functions α and β: opening and closing rates



$$
\begin{aligned}
\frac{dX_{1t}}{dt} &= -g_{Ca} m_\infty(X_{1t})(X_{1t} - V_{C_a}) - g_K X_{2t}(X_{1t} - V_K) - g_L(X_{1t} - V_L) + I \\
\frac{dX_{2t}}{dt} &= \alpha(X_{1t})(1 - X_{2t}) - \beta(X_{1t}) X_{2t} + \sigma(V_t, X_{2t}) dW_t
\end{aligned}
$$

where (W_t) is a Brownian motion

3. Particle filter approach

$$\begin{aligned} dX_{1t} &= b_1(X_{1t}, X_{2t}, \mu)dt \\ dX_{2t} &= b_2(X_{1t}, X_{2t}, \mu)dt + \sigma_2 dB_{2t} \end{aligned}$$

- Previous trick where C_i is replaced by $\frac{V_{i+1}-V_i}{\Delta}$ not available
- We want to filter C_i and compute $\pi_{n,\theta} f = \mathbb{E}\left(f(X_{2n})|X_{10:n};\theta\right)$
  - Kalman filter when SDE is linear and Gaussian
  - **Particle filter/Sequential Monte Carlo** (SMC)
    [Del Moral et al, 2001; Doucet et al, 2001; Chopin, 2004; ...]

Particle filter/Sequential Monte Carlo

- Iterative algorithm
- Simulation of K particles $X^k_{20:n}$ and computation of weights $w_n(X^k_{20:n})$
- Empirical measure $\Psi^K_{n;\theta} = \sum_{k=1}^K w_n(X^k_{20:n})\mathbf{1}_{X^k_{20:n}}$

At time $j = 1, \ldots, n$, $\forall\, k = 1, \ldots, K$:

1. simulation of $C_j^{(k)} \sim q(\cdot|V_j, C_{j-1}^{(k)}; \theta)$
2. calculation of weights

$$w\left(C_{0:j}^{(k)}\right) = \frac{p(V_{0:j}, C_{0:j}^{(k)}|\theta)}{p(V_{0:j-1}, C_{0:j-1}^{(k)}\theta)q(C_j^{(k)}|V_j, C_{j-1}^{(k)}; \theta)}$$

Particle filter for hypoelliptic SDE

- **Degenerate Hidden Markov Model**:
  - (V_i, C_i) Markovian but not (C_i)
  - Set $X_i = (X_{1i}, X_{2i})$, with Markov kernel $p(dX_{1i}, dX_{2i}|X_{1,i-1}, X_{2i-1})$
  - $V_i = X_i^{(1)}$ with transition kernel $\mathbb{1}_{\{V=X^{(1)}\}}$ (zero almost everywhere)
- **Computation of the weights** [Ditlevsen, Samson, 2014, 2019]
  - Elliptic SDE: Euler-Maruyama approximation
  - Hypoelliptic SDE: 1.5 order scheme approximation

Exact conditional expectation: $\pi_{n,\theta} f \quad = \mathbb{E}\left(f(X_{20:n}) | X_{10:n}; \theta\right)$

Approximate conditional expectation: $\pi_{\Delta,n,\theta} f = \mathbb{E}_{\Delta}\left(f(X_{20:n}) | X_{10:n}; \theta\right)$

Particle filter approximation: $\Psi^K_{n,\theta} f \quad = \sum_{k=1}^K f(X^k_{20:n}) w_{n,\theta}(X^k_{20:n})$

Particle filter convergence for elliptic case [Ditlevsen, Samson, 2014]

For any $\varepsilon > 0$, for any bounded Borel function f

$$\mathbb{P}\left(\left|\Psi^K_{n,\theta} f - \pi_{\Delta,n,\theta} f\right| \geq \varepsilon\right) \leq C_1 \exp\left(-K \frac{\varepsilon^2}{C_2 ||f||^2}\right)$$

If $\Delta \leq 1/\sqrt{K}$,

$$\mathbb{P}\left(\left|\Psi^K_{n,\theta} f - \pi_{n,\theta} f\right| \geq \varepsilon\right) \leq C_3 \exp\left(-K \frac{\varepsilon^2}{C_4 ||f||^2}\right)$$

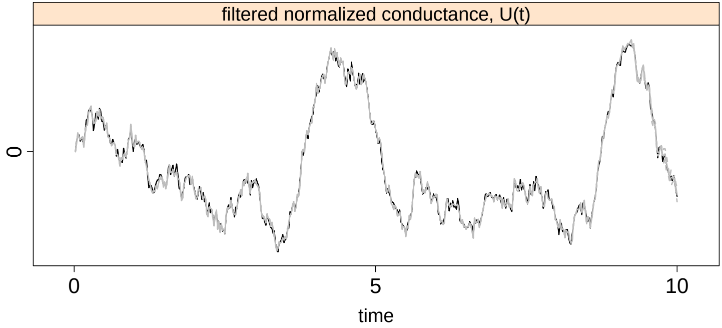
Tools: exponential inequalities (generalization of Del Moral and Jacod's proof)

Hypoelliptic case

Don't work with Ito-Taylor scheme
Splitting scheme with $\Delta^2 \leq 1/\sqrt{K}$?

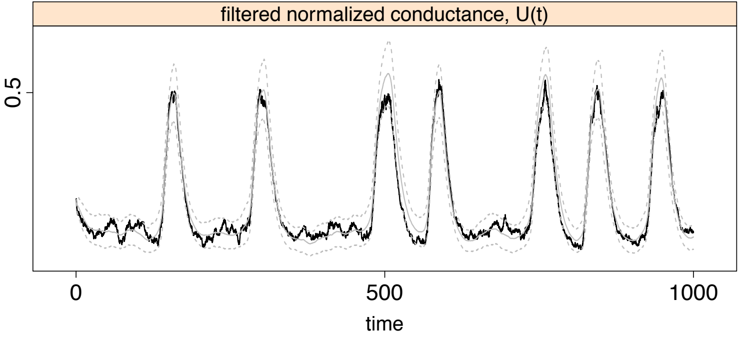
Filtering with the hypoelliptic neuronal FitzHugh-Nagumo

Parameter θ fixed at the true value



Filtering with the hypoelliptic neuronal Morris-Lecar

Parameter θ fixed at the true value



Stochastic Approximation (SAEM) algorithm [Delyon, Lavielle, Moulines, 1999, Ditlevsen, Samson, 2014]

- *E Step*
  - *S Step*: simulation of $X_{20:n}^{(m)}$ under $p_\Delta(X_{20:n}|X_{10:n};\widehat{\theta}_m)$ with particle filter
  - *SA Step*: stochastic approximation of Q_{m+1}

$$Q_{m+1}(\theta) = (1-\alpha_m)Q_m(\theta) + \alpha_m \log p_\Delta(X_{10:n}, X_{20:n}^{(m)};\theta)$$

- *M Step*: $\widehat{\theta}_{m+1} = \arg\max_\theta Q_{m+1}(\theta)$

Convergence [Ditlevsen, Samson, 2014, 2018]

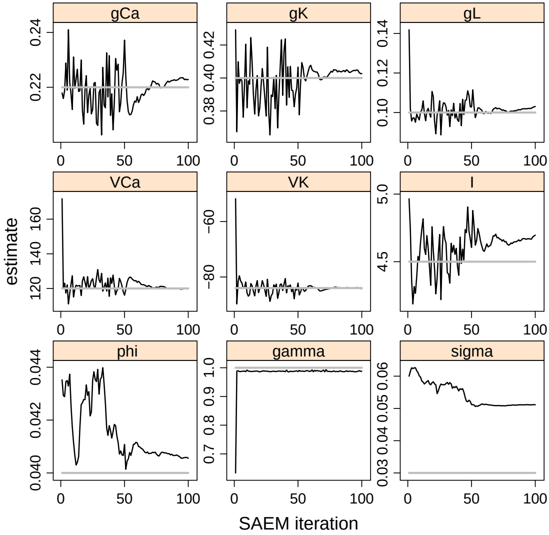
Assumptions

1. $\sum_m \alpha_m = \infty$, $\sum_m \alpha_m^2 < \infty$.
2. Number of particles $K(m) = \log(m^{1+\delta})$

$$\widehat{\theta}_m \xrightarrow[m\to\infty]{a.s.} \textit{(local) max of likelihood } p_\Delta(X_{10:n};\theta)$$

Tool: convergence of Robbins-Monroe scheme and inequality deviation for the particle filter

SAEM-SMC on hypoelliptic Morris-Lecar simulated data



- σ: not enough information in the membrane potential data

SAEM-SMC on hypoelliptic FHN simulated data

Estimation results obtained from 100 simulated data sets

		ε fixed	ε fixed	ε estimated
	True	1.5 contrast	Euler Contrast	1.5 contrast
ε	0.100	–	–	0.105 (0.010)
γ	1.500	1.523 (0.130)	1.499 (0.196)	1.592 (0.160)
β	0.800	0.821 (0.110)	0.779 (0.107)	0.866 (0.130)
σ	0.300	0.293 (0.008)	0.381 (0.038)	0.306 (0.020)

Optimal Control

[Clairon, Samson, work in progress]

Intuition

- Cost function

$$h(u, V_{0:n}, \theta) = \sum_{i=0}^{n} \left(V_i - \tilde{V}_{i,u,\theta} \right)^2 + \frac{1}{\Delta} \sum_{i=0}^{n} u_i^2$$

  where $V_{i,u,\theta}$ is the approximated solution similar to a 1.5 order scheme

$$V_{i+1,u,\theta} = V_{i,u,\theta} + \tilde{b}_1(V_{i,u,\theta}) + \tilde{\Sigma} u_i$$

  and $u = (u_0, \ldots, u_i, \ldots)$ mimics the Brownian process (B_t)

- Calculate the *best control* $\bar{u}$ that minimizes $h(u, V_{0:n}, \theta)$
- Estimate θ as

$$\hat{\theta} = \arg\min_{\theta} h(\bar{u}, V_{0:n}, \theta)$$

Optimal control problem

Main step: solve $\bar{u} = \arg\min_u h(u, V_{0:n}, \theta)$

- Linear-Quadratic problem thanks to the 1.5 order scheme (pseudo-linear formulation)
- Solving a finite difference equation (Riccati equation)
- Well-known problem and explicit solution

Introduction of weights ω

$$h_\omega(u, V_{0:n}, \theta) = \omega \sum_{i=0}^{n} \left(V_i - \tilde{V}_{i,u,\theta} \right)^2 + \frac{1}{\Delta} \sum_{i=0}^{n} u_i^2$$

- Allow to relax the problem of optimal control
- Optimal control problem to optimize the weights
- More stable numerical results

Comparison on FHN simulated data

Estimation results obtained from 100 simulated data sets

	True	SAEM-SMC	Optim. Control
ε	0.100	0.105 (0.01)	0.098 (0.001)
γ	1.500	1.59 (0.16)	1.58 (0.06)
β	0.800	0.87 (0.13)	0.87 (0.05)
σ	0.300	0.31 (0.02)	0.29 (0.001)