# Some stochastic models in ecology

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Joint work with

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MATHÉMATIQUES APPLIQUÉES - INFORMATIQUE

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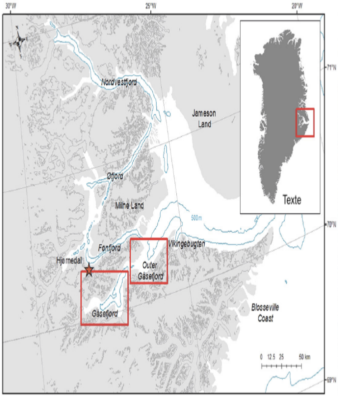
# Outline of the talk

- **1. Climate change in Greenland**
  - Effect on endemic whales
  - Design of experimental data
- **2. Sound data**
  - Point process
  - Mediation effect of depth
- **3. Spatial data**
  - Hypoelliptic diffusion
  - Numerical scheme and statistical inference

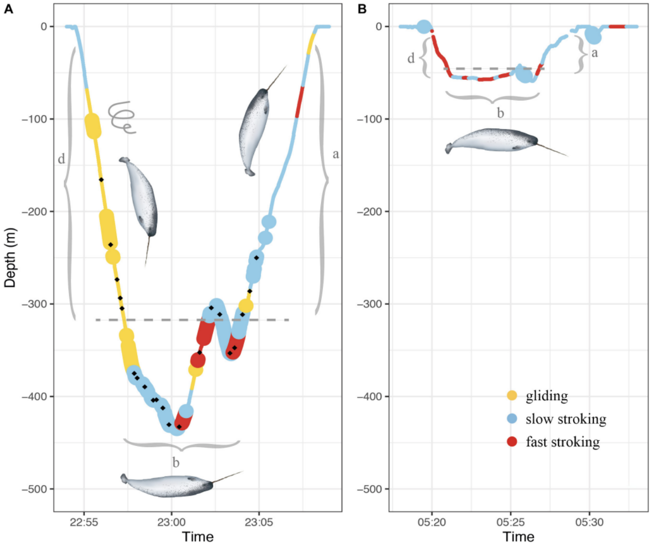


# 1. Climate change: a danger for the narwhals?

- Ice melt
- Decrease of sea ice coverage
- Increase of anthropogenic activities, of mining activities

- Reactions of narwhals to noise and human presence?

# Narwhal dive




Foraging V dive (A) and Non-foraging U dive (B), Buzzes (black dots)

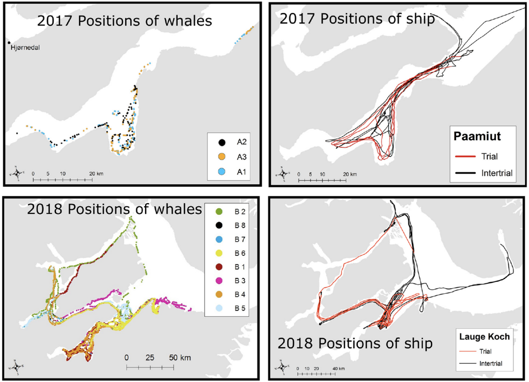
# Experimental design

- 16 animals
- Instrumented with satellite tags and Acousonde acoustic-behavioral recorders

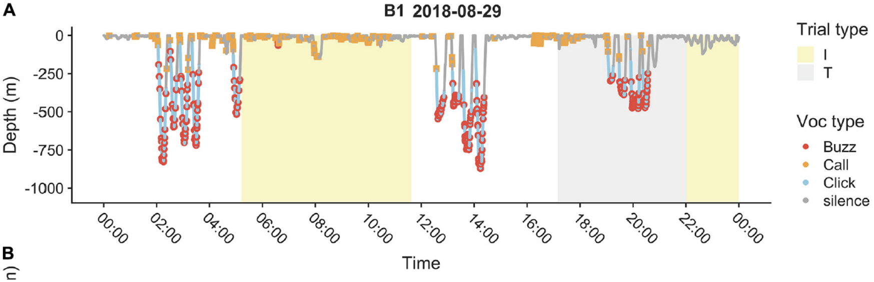
- Exposition to airgun pulses and vessel sounds
  - Pre exposure
  - Trials: airgun and ship-noise
  - Intertrial: only ship-noise
  - Post exposure

# Recorded data

- Data recorded each second
- Sound (click/buzz)
- Depth
- Position of whales
- Position of ship
- Stroke
- Accelerometer



# 2. Sound data


A
B1 2018-08-29
0
-250
-500
-750
-1000
Depth (m)
00:00
02:00
04:00
06:00
08:00
10:00
12:00
14:00
16:00
18:00
20:00
22:00
00:00
Time
Trial type
I
T
Voc type
Buzz
Call
Click
silence
B
n)

# Sound analysis: definition

**Point process**

- $T_j$, the time of the $j$th sound emitted by the animal
- $T_0 = 0 < T_1 < T_2 < \dots$
- $T_n \to \infty$ when $n \to \infty$ (no accumulation)

**Count process**

- $N(t)$ the number of sounds emitted in the interval $]0; t]$
- $(N(t))_{t \in \mathbb{R}^+}$ is a stochastic process at continuous time
  - $N(0) = 0$
  - $N(b) - N(a)$ is the number of sounds emitted in the interval $]a; b]$

# Poisson process

A **Poisson process** is a count process $(N(t))_{t\in\mathbb{R}^+}$ such that

- $N(0) = 0$
- For all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ the variables $N(t_{j+1}) - N(t_j)$ are independent
- For all $(s, t) \in \mathbb{R}_+^2$, $N(s+t) - N(s)$ follows a Poisson distribution of parameter $\lambda t$:
$$\mathbb{P}(N(s+t) - N(s) = k) = e^{\lambda t}\frac{(\lambda t)^k}{k!}$$

Thus

- $\mathbb{P}(N(t) = k) = e^{\lambda t}\frac{(\lambda t)^k}{k!}$
- $\mathbb{E}(N(t)) = \lambda t$

# Non homogeneous Poisson process

Let $\lambda(t)$ an integrable function. $(N(t))_{t\in\mathbb{R}^+}$ is a **nonhomogeneous Poisson process** with **rate** or **intensity** $\lambda(t)$ if

- $N(0) = 0$
- For all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ the variables $N(t_{j+1}) - N(t_j)$ are independent
- For all $(t, s) \in \mathbb{R}^2_+$, $N(s+t) - N(s)$ follows a Poisson distribution of parameter $\int_t^{t+s} \lambda(u)du$:

$$N(s+t) - N(s) \sim Poisson\left(\int_t^{t+s} \lambda(u)du\right)$$

Thus

- $\mathbb{P}(N(t+\Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + o(\Delta)$
- $\mathbb{P}(N(t+\Delta) - N(t) = 1) = \lambda(t)\Delta + o(\Delta)$
- $\mathbb{P}(N(t+\Delta) - N(t) \geq 2) = o(\Delta)$

If the time interval $[t, t+\Delta[$ is small:

- the probability of a single event is approximately $\lambda(t)\Delta$
- the probability of more than 1 event is negligible

Simplification of the process:

- $Y(t) := N(t+\Delta) - N(t)$ takes value $0$ or $1$
- Bernoulli distribution with $\mathbb{E}(Y(t)) = \mathbb{P}(Y(t) = 1) = \lambda(t)\Delta$

# Inclusion of covariates

To ensure the positivity of the intensity, we consider the log link:

$$\log \lambda(t, Z(t)) = \gamma_0 + \gamma_1 D(t) + \gamma_2 X(t) = \gamma^t Z(t)$$

with
- $D(t)$ : Depth
- $X(t)$ : Exposure level of seismic activity = 1/distance to ship
- $Z(t) = (1, D(t), X(t))$ vector of covariates

**Estimation by maximum likelihood** for iid sample $Y_1, \ldots, Y_n$:

$$L(\gamma | Y, Z) = \prod_{j=1}^{n} e^{-\lambda(t_j, Z_j)} \frac{\lambda^{Y_j}}{Y_j!}$$

Maximisation of the log-likelihood

$$\ell(\gamma | Y, Z) = \sum_{j=1}^{n} (Y_j \gamma^t Z_j - e^{\gamma^t Z_j})$$

# To include variability between individuals (random effects)

Covariance between observations of a given individual:

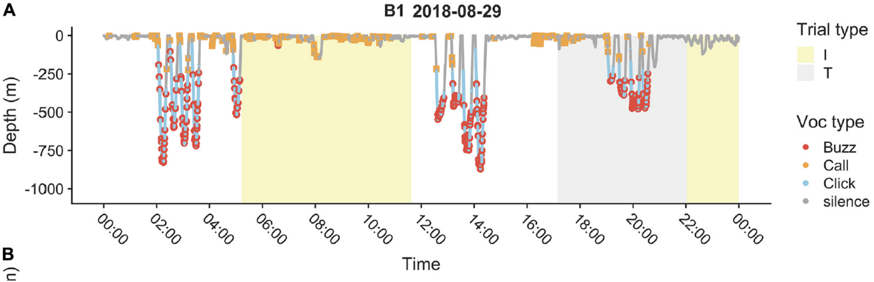
$$\log \lambda(t|b_i, Z(t)) = \gamma_0 + b_i + \gamma_1 D(t) + \gamma_2 X(t) = b_i + \gamma^t Z(t)$$

with
- $b_i$ random effect of individual $i$
- $b_i \sim \mathcal{N}(0, \omega^2)$

**Estimation by maximum likelihood** for iid sample of animal $i$: $Y_{i1}, \ldots, Y_{in_i}$

$$L(\gamma|Y, Z) = \int p(Y_i|Z_i, b_i, \gamma) p(b_i, \omega^2) db_i$$

# Non independence of observations




- When buzzes start to be emitted, a whole sequence of buzzes is emitted
- Dependence of the emission between observations of a given animal

$\rightarrow$ Intensity should depend on the past and thus be a random variable

# Poisson model with random intensity function

**Non homogeneous Poisson model with random intensity function**

- $\mathcal{F}_t$ denotes the filtration of the process up to time $t$
- $\lambda(t)$ is a random variable whose value is determined by $\mathcal{F}_t$ such that
  - $\mathbb{P}(N(t+\Delta) - N(t) = 1 | \mathcal{F}_t) = \lambda(t)\Delta + o(\Delta)$
  - $\mathbb{P}(N(t+\Delta) - N(t) \geq 2 | \mathcal{F}_t) = o(\Delta)$

**Example: Hawkes process**

$$\lambda(t) = \lambda + \sum_{j=1}^{N(t)} M_j e^{-\alpha(t-T_j)}$$

**To ensure a positive intensity**:

$$\log \lambda(t) = \lambda + \sum_{j=1}^{N(t)} M_j e^{-\alpha(t-T_j)}$$

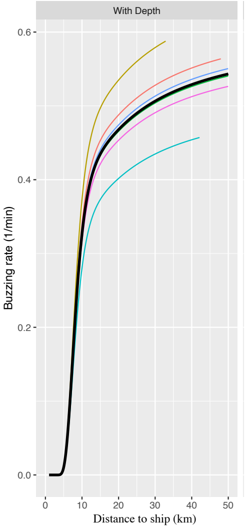
**With covariates and random effects**:

$$\log \lambda(t) = \lambda + b_i + \sum_{j=1}^{N(t)} M_i e^{-\alpha(t-T_{ij})} + \gamma_1 D_i(t) + \gamma_2 X_i(t)$$

with
- $D(t)$ : Depth
- $X(t)$ : Exposure level of seismic activity $= 1/$distance to ship

**Estimation**: two steps to estimate the auto-regressive memory and then the other effects

# Estimated buzzing rate




- Variability between individuals
- When ship is close, no buzz
- Buzzing rate without exposure $= 0.55/\mathrm{min}$

# Still to be done

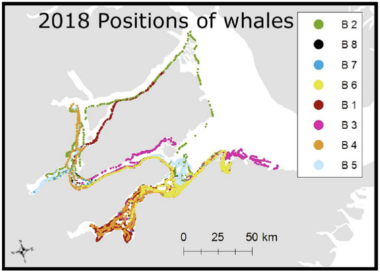
- Estimation of the mediation effect of Depth:
  - Exposure causes animal to dive less
  - Less dives make animals produce less sounds

  - What is the direct of Exposure and its indirect effect through Depth?
  - Is Depth a mediation factor?
- Hawkes process instead of the Poisson model
  - Non-parametric intensity
  - Random effects
  - Mediation

# 3. Spatial position data

Do animal adapt their position due to

- anthropogenic activities?
- warming of ocean temperatures?



# Stochastic model of animal movement

[Blackwell 1997, Brillinger et al 2002, Preisler et al 2004, Hooten et al 2010, Michelot et al 2019, Gloaguen et al 2018]

**Stochastic Differential Equation** (SDE) to describe the dynamic of the position:

- $X_t$ position at time $t$
- SDE on the position $X_t$

$$dX_t = b(X_t, \theta)dt + \Sigma(X_t, t)dW_t$$

- The drift $b(x, \theta)$ models the direction preference depending on position $x$
- The diffusion coefficient $\Sigma(X_t, t)$ models the variability around the mean

# A first SDE

**Ornstein-Uhlenbeck process**

$$dX_t = -\beta(X_t - \mu)dt + \Sigma dW_t$$

**Solution obtained by the Ito lemma** applied to the function $f(X_t, t) = X_t e^{\beta t}$

$$\begin{aligned} df(X_t, t) &= \beta X_t e^{\beta t} dt + e^{\beta t} dX_t \\ &= e^{\beta t} \beta \mu dt + \Sigma e^{\beta t} dW_t \end{aligned}$$

Integration from 0 to t

$$X_t e^{\beta t} = X_0 + \int_0^t e^{\beta s} \beta \mu ds + \int_0^t \Sigma e^{\beta s} dW_s$$

Thus

$$X_t = X_0 e^{-\beta t} + \mu(1 - e^{-\beta t}) + \int_0^t \Sigma e^{\beta(s-t)} dW_s$$

The moments of the Ornstein-Uhlenbeck process are

- $\mathbb{E}(X_t) = X_0 e^{-\beta t} + \mu(1 - e^{-\beta t})$
- $Cov(X_s, X_t) = \frac{\sigma^2}{2\beta} e^{-\beta(s+t)}(e^{2\beta(s \wedge t)} - 1)$
- $Var(X_t) = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$

# Parameter estimation

## Transition density

$$X_{t+\Delta}|X_t \sim \mathcal{N}\left(X_t e^{-\Delta} + \mu(1-e^{-\beta\Delta}), \frac{\sigma^2}{2\beta}(1-e^{-2\beta\Delta})\right)$$

## Likelihood

$$L(\beta,\mu,\sigma^2) = \prod_{j=1}^{n} p(X_{t_{j+1}}|X_{t_j})$$

## Maximum Likelihood Estimator

$$\widehat{e^{-\beta\Delta}} = \frac{\sum_j X_j X_{j-1}/n - \sum_i X_i/n \sum_i X_{i-1}/n}{\sum_i X_{i-1}^2/n - (\sum_i X_{i-1}/n)^2}$$

$$\hat{\mu} = \frac{\sum_j (X_j - \widehat{e^{-\beta\Delta}} X_{j-1})}{1-\widehat{e^{-\beta\Delta}}}$$

# More complex SDE for movement ecology

- **SDE on the velocity** $V_t \in \mathbb{R}^2$ [Hamiltonian models]

$$
\begin{aligned}
dX_t &= V_t dt \\
dV_t &= b(V_t, S_t, \theta)dt + \Sigma(V_t, t)dW_t
\end{aligned}
$$

with $S_t$ different spatial maps (temperature, depth, ice coverage, etc)

  - Hypoelliptic system

- **Modeling Depth**

$$
\begin{aligned}
dX_t &= b_X(X_t, S_t, D_t, \theta)dt + \Sigma(X_t, t)dW_t^1 \\
dD_t &= b_D(X_t, S_t, D_t, \theta)dt + \sigma_D dW_t^2
\end{aligned}
$$

  - Noise could be possibly degenerated ($\sigma_D = 0$)

# Parametric estimation for multidimensionnal systems

Matricial notations $X_t \in \mathbb{R}^d$:

$$dX_t = b(X_t, \theta) + \Sigma dW_t$$

**Difficult** because

- Data: discrete observations $X_{10:n} = (X_{10}, \ldots, X_{1n})$ at times $t_0 = 0 < t_1 = \Delta < \ldots < t_n = n\Delta$
- No explicit transition density of the SDE except if $b$ is linear
- Hypoellipticity: $\Sigma$ is degenerated but $X_t$ has a smooth density (noise propagates to $\mathbb{R}^d$)

# Different strategies of inference

**Based on simulation**

- Exact simulation
- Approximated simulation with a numerical scheme

- Estimation methods based on simulation
  - ABC
  - Monte Carlo, Importance Sampling

**Based on approximation of the transition density**

- What is a "good" approximation? Depends on the properties of the numerical scheme

- Estimation methods based on approximation
  - Contrast estimator [Thieullen Samson 2012, Ditlevsen Samson 2019, Melnykova 2020]
  - MCMC
  - EM algorithm [Beskos et al 2005, Gloaguen et al 2018, Ditlevsen, Samson 2014, 2019]

# Numerical approximation schemes

**Expected properties of a numerical scheme**

- Locally Lipschitz conditions on $b$
- Exact moments up to a certain order
- Mean-square convergence of order $p$ for a step size $\Delta$

$$\max_{t_i} \left( \mathbb{E} \left( \|X(t_i) - \tilde{X}(t_i)\|^2 \right) \right)^{1/2} \leq c\Delta^p$$

- Preservation of structural properties: hypoellipticity, ergodicity, amplitudes, frequences, phases of oscillations

# Some numerical schemes

**Discretization with Euler-Maruyama**

$$X_{i+1} = X_i + \Delta b(X_i) + \sqrt{\Delta}\Sigma\eta_i, \quad \eta_i \sim_{iid} \mathcal{N}(0, I)$$

- Not mean-square convergent
- Does not preserve ergodicity
- Does not preserve hypoellipticity

$$Var(X_{i+1}|X_i) = \sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}$$

## Local linearization

[Ozaki 1989, Biscay et al. 1996, Jimenez et al 2015, Melnykova 2020]

**Approximate the solution by the solution of an autonomous linear SDE**

On each interval $[i\Delta, (i+1)\Delta[$,

$$d\tilde{X}_t = \left( b(\tilde{X}_i) + \partial_x b(\tilde{X}_i)(\tilde{X}_t - \tilde{X}_i) + \frac{1}{2}\sigma^2 \partial_c^2 b(\tilde{X}_i)(t - i\Delta) \right) + \Sigma dW_t$$

- Linear SDE $\rightarrow$ explicit solution
- 1-step hypoellipticity

$$Cov(\tilde{X}_{i+1}|\tilde{X}_i) = \sigma^2 \begin{pmatrix} \frac{\Delta^3}{3}\partial_c b_1^2 & \frac{\Delta^2}{2}\partial_c b_1 \\ \frac{\Delta^2}{2}\partial_c b_1 & \Delta \end{pmatrix}$$

- Mean-square convergent of order 1
- Does not preserve other structural properties (ergodicity, oscillations)

# Numerical splitting scheme

[Buckwar, Samson, Tamborrino, Tubikanec, 2021; Ditlevsen, Pilipovic, Samson, 2022]

Introduction of two subsystems.

1. **Subsystem a:** Linear SDE with exact solution
$$dX_t^a = AX_t^a dt \; + \; \Sigma dW_t$$
2. **Subsystem b:** Non-linear (decoupled) ODE with (exact) solution
$$dX_t^b = N(X_t^b)dt$$

**Numerical splitting schemes** with time step $\Delta$

- Lie-Trotter
$$\hat{\mathbf{X}}^{LT} = X_\Delta^a \circ X_\Delta^b$$
- Strang
$$\hat{\mathbf{X}}^{S} = X_{\Delta/2}^b \circ X_\Delta^a \circ X_{\Delta/2}^b$$

## Properties of splitting schemes

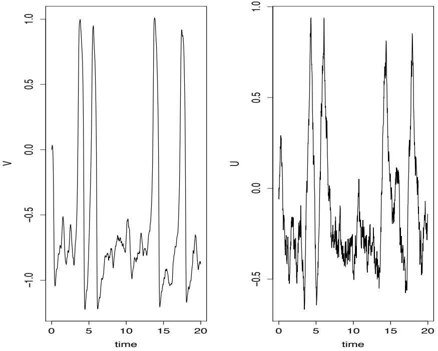
- Exact first moment up to $\Delta^3$, covariance matrix up to $\Delta^3$
- Mean-square convergence with order 1
- Preservation of noise structure, 1-step hypoellitpticity
- Preservation of Lyapounov structure
- Geometric ergodicity
- Transition density of the scheme highly non-linear -¿ difficult to exploit?

# Example on the Fitzhugh-Nagumo model

[Lindner et al 1999, Gerstner and Kistler, 2002, Lindner et al 2004, Berglund and Gentz, 2006]

$$
\begin{aligned}
dV_t &= \frac{1}{\varepsilon}(V_t - V_t^3 - C_t - s)dt, \\
dC_t &= (\gamma V_t - C_t + \beta)\, dt + \sigma dW_t,
\end{aligned}
$$

- $\varepsilon$ time scale separation
- $s$ stimulus input
- $\beta$ position of the fixed point
- $\gamma$ duration of excitation



# Splitting scheme on the FHN model

1. **Subsystem a:** Linear SDE

$$dX_t = \begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ \gamma & -1 \end{pmatrix} X_t dt \; + \; \Sigma dW_t$$

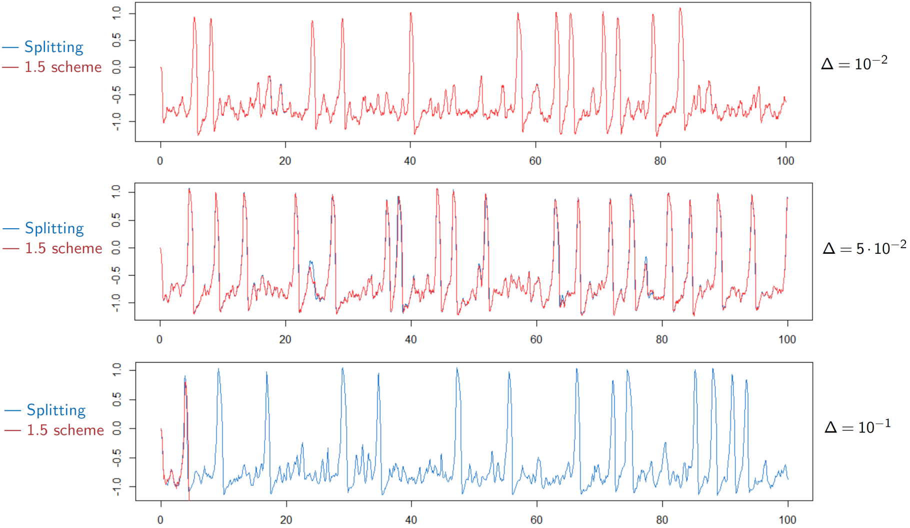
2. **Subsystem b:** Non-linear ODE

$$dX_t = \begin{pmatrix} \frac{1}{\varepsilon}(X_{1t} - X_{1t}^3) \\ \beta \end{pmatrix} dt$$

Explicit solution for both systems

# Comparison of Splitting and Order 1.5 Strong Taylor Scheme

[Buckwar, Samson, Tamborrino, Tubikanec, 2021]



# Estimation for elliptic SDE

$$dX_t = b(X_t, \theta)dt + \Sigma dW_t, \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

**Minimum contrast estimator** [Genon-Catolot, Jacod, 1993; Kessler 1996; Pilipovic, Samson, Ditlevsen, 2022]

Set $\Gamma = \Sigma'\Sigma$.

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \left( \sum_{i=1}^{n-1} (X_{i+1} - X_i - \Delta b(X_i, \theta))' \, \Gamma^{-1} \, (X_{i+1} - X_i - \Delta b(X_i, \theta)) + \sum_{i=1}^{n-1} \log \det \Gamma \right)$$

**Rate of convergence** specific to hypoelliptic systems

- $\hat{\theta}$ asymptotically normal at rate $\sqrt{n\Delta}$
- $\hat{\Gamma}$ asymptotically normal at rate $\sqrt{n}$

# Estimation for hypoelliptic SDE

**Constrats based on numerical approximations** [Genon-Catolot, Jacod, 1993; Kessler 1996, Ditlevsen, Samson, 2019, Melnykova 2020]

$$\hat{\theta} = \arg\min_{\mu \in \Theta} \left( \sum_{i=1}^{n-1} \left( X_{i+1} - X_i - \Delta \tilde{b}(X_i, \theta) \right)' \Gamma^{-1} \left( X_{i+1} - X_i - \Delta \tilde{b}(X_i, \theta) \right) + \sum_{i=1}^{n-1} \log \det \Gamma \right)$$

**Rate of convergence** specific to hypoelliptic systems

- $\hat{\theta}_1$ asymptotically normal at rate $\sqrt{n/\Delta}$
- $\hat{\theta}_2$ asymptotically normal at rate $\sqrt{n\Delta}$
- $\hat{\Gamma}$ asymptotically normal at rate $\sqrt{n}$

# Still to be done

- First numerical applications of contrast estimator based on splitting scheme are promising
- Extension to SDE with spatial covariates
  - numerical schemes?
  - asymptotic results of estimators?
- Modeling Depth together with the position to get a 3D system
  - where should we put noise?
  - can we test the presence of noise?

# Perspectives

- Narwhals data are very rich
  - Time-dependent variables
  - Multi-dimensional analysis
  - Mediation effect
- Hypoelliptic SDE for movement/spatial data
  - Splitting schemes are promising
  - Adaptation to drift with gradient of potential

MERCI !