How to create superposition of quantum states?

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Qubit

- The superposition of states of classical bits is used to perform computations in quantum informatics
  $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$
- Examples of systems that can be used as a qubit:
  spin $\frac{1}{2}$;
  photon polarization;
  two-level atom.

Qubit

- If we have a system of two qubits and the first one is described by the space $\mathcal{H}_A$ and the second one is described by the space $\mathcal{H}_B$ then the whole system is described by the space $\mathcal{H}_A \otimes \mathcal{H}_B$
- A qubit $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ that describes a system of two qubits $|\psi_a\rangle_A \in \mathcal{H}_A$ and $|\psi_b\rangle_B \in \mathcal{H}_B$ can be describe in such way $|\psi\rangle_{AB} = |\psi_a\rangle_A \otimes |\psi_b\rangle_B = |\psi_a \psi_b\rangle$

Density matrix

- Set of states $\{|\psi_i\rangle\}$ and set of probabilities $\{p_i\}$ is introduced a density matrix
$$\rho = \Sigma p_i |\psi_i\rangle \langle\psi_i|$$
- The density matrix for a particular state $|\psi\rangle$ looks as follows $\rho_\psi = |\psi\rangle \langle\psi|$
- Density matrix ρ for the state of n several systems $\{\rho_i\}$ is equal $\rho = \rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$

Quantum Gates

- Quantum gates are used to implement quantum algorithms

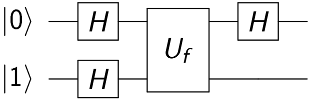


Figure: Deutsch's algorithm

- The quantum gate operator is a unitary operator: $H^\dagger H = \mathbb{I}$

Quantum gates

- X, Y, Z

  $X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

  $Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

  $Z|\psi\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

- H - Hadamard operator:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- Control Not Gate: $|\psi\rangle \to (|0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X)|\psi\rangle$
  $|00\rangle \to |00\rangle$
  $|01\rangle \to |01\rangle$
  $|10\rangle \to |11\rangle$
  $|11\rangle \to |10\rangle$

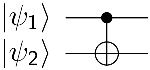


Figure: CNOT

- Creation of superposition from the general state:

$|\psi\rangle \rightarrow \frac{|\psi\rangle+|\psi^{\perp}\rangle}{\sqrt{2}} = |\Psi\rangle$, in general $\breve{H}\,|\psi\rangle_a\,|Q\rangle_{b\ldots} = \frac{|\psi\rangle+|\psi^{\perp}\rangle}{\sqrt{2}}_a\,|Q_{\psi}\rangle_{\ldots}$

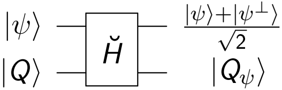


Figure: U-Hadamard

Universal Hadamard Gate

Nonlinearity of the U-Hadamard operator

$$\hat{G}|\psi\rangle_a|Q\rangle_b = \hat{G}(\alpha|0\rangle_a + \beta|1\rangle_a)|Q\rangle_b =$$
$$\alpha\hat{G}|0\rangle_a|Q\rangle_b + \beta\hat{G}|1\rangle_a|Q\rangle_b = \alpha\hat{G}|1\rangle_a|Q_0\rangle_b + \beta\hat{G}|0\rangle_a|Q_1\rangle_b$$

This operator is not linear, which means it is not unitary and cannot be realized as a quantum gate

Bures Fidelity

If we cannot create an exact state σ, we can create a state ρ as close to the required state as possible to implement the algorithm.

Bures Fidelity

$F = Tr(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}) \in (0,\ 1)$ or for pure state $\sigma = |\Psi\rangle\langle\Psi| \Rightarrow$
$F = \langle\Psi|\rho|\Psi\rangle$

Universal Cloner and Universal NOT Gates

- $|\psi\rangle \rightarrow \frac{|\psi\rangle+|\psi^{\perp}\rangle}{\sqrt{2}}$
- Universal Cloner $|\psi\phi\rangle \rightarrow |\psi\psi\rangle = U_c\,|\psi\phi\rangle$

Nonlinearity of U-Cloner

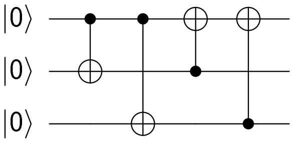
$$|\psi\phi\rangle = \alpha\,|0\phi\rangle + \beta\,|1\phi\rangle \rightarrow \alpha\,|00\rangle + \beta\,|11\rangle \neq |\psi\psi\rangle$$

- Universal NOT Gate
  $\alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle \rightarrow |\psi^{\perp}\rangle = \beta^*\,|0\rangle - \alpha^*\,|1\rangle$

Nonlinearity of U-NOT

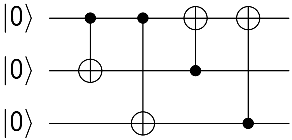
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \alpha\,|1\rangle + \beta\,|0\rangle \neq |\psi^{\perp}\rangle$$

U-Cloner and U-NOT Machine



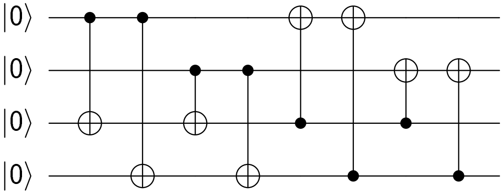
- $|\psi\rangle_a |\Xi_{00}\rangle_{bc} = |\psi\rangle \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) \rightarrow |\psi\rangle_a |\Xi\rangle_{bc}$
- $|\psi\rangle_a |\Xi_{0+}\rangle_{bc} = |\psi\rangle \left(\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)\right) \rightarrow |\psi\rangle_b |\Xi\rangle_{ac}$
- $|\Xi\rangle_{bc} = c_0 |\Xi_{00}\rangle_{bc} + c_1 |\Xi_{0x}\rangle_{bc}$

U-Cloner and U-NOT Machine



- $\rho_1^{out} = (c_0^2 + c_0 c_1) |\psi\rangle \langle\psi| + \frac{1}{2} c_1^2 \mathbb{I}$
  $\rho_2^{out} = (c_0^1 + c_0 c_1) |\psi\rangle \langle\psi| + \frac{1}{2} c_0^2 \mathbb{I}$
  $\rho_3^{out} = c_0 c_1 |\psi^*\rangle \langle\psi^*| + \frac{1}{2}(1 - c_0 c_1)\mathbb{I}$
- $c_0 = c_1 = \frac{1}{\sqrt{3}} \Rightarrow \rho_1^{out} = \rho_2^{out} = \frac{5}{6} |\psi\rangle \langle\psi| + \frac{1}{6} |\psi^\perp\rangle \langle\psi^\perp|$
  $F_{|\psi\rangle} = \frac{5}{6}$
- $(iY)\rho_3^{out}(iY)^\dagger = \frac{1}{3} |\psi\rangle \langle\psi| + \frac{2}{3} |\psi^\perp\rangle \langle\psi^\perp|$
  $F_{|\psi^\perp\rangle} = \frac{2}{3}$

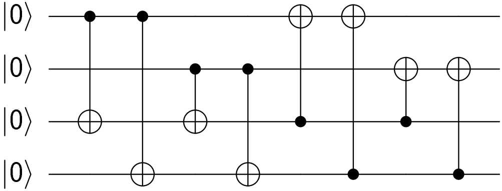
Idea of U-Hadamard



$$\rho_2^{out}{}_a \otimes (iY)\rho_3^{out}(iY)_b^\dagger \otimes \check{\rho}_{cd} \to \rho_b^{goal}\tilde{\rho}_{acd}$$

Idea of U-Hadamard

By varying the constants C_i, $i = \{1, 2, 3, 4\}$, the maximum value of F can be obtained



$$\check{\rho} = C_1 |00\rangle\langle 00| + C_2 |01\rangle\langle 00| + C_3 |00\rangle\langle 01| + C_4 |11\rangle\langle 11|$$
$$\rho^{goal} = A|\psi\rangle\langle\psi| + B|\psi\rangle\langle\psi^{\perp}| + C|\psi^{\perp}\rangle\langle\psi| + D|\psi^{\perp}\rangle\langle\psi^{\perp}|$$
$$F = \frac{\langle\psi| + \langle\psi^{\perp}|}{\sqrt{2}} \rho^{goal} \frac{|\psi\rangle + |\psi^{\perp}\rangle}{\sqrt{2}} = \frac{1}{2}(1 + C + D)$$ *To be continued...*