4.4 -

Eq 9 - $q(x_t | x_{t-1}) = \mathcal{N}(x_t; \sqrt{\alpha_t}\, x_{t-1}, (1-\alpha_t) I)$, $\alpha_T > 0$

$\mu = \mu + \sigma \varepsilon$ → Mean $\sigma = \sqrt{1-\alpha_t}$ → Standard deviation

So, we can write $x_t = \sqrt{\alpha_t}\, x_{t-1} + \sqrt{1-\alpha_t}\, \varepsilon_t$

At $t=1$ $x_1 = \sqrt{\alpha_1}\, x_0 + \sqrt{1-\alpha_1}\, \varepsilon_1$

At $t=2$ $x_2 = \sqrt{\alpha_2}\, x_1 + \sqrt{1-\alpha_2}\, \varepsilon_2$

$$= \sqrt{\alpha_2}\,(\sqrt{\alpha_1}\, x_0 + \sqrt{1-\alpha_1}\, \varepsilon_1) + \sqrt{1-\alpha_2}\, \varepsilon_2$$

$$= \sqrt{\alpha_2 \alpha_1}\, x_0 + \sqrt{\alpha_2 (1-\alpha_1)}\, \varepsilon_1 + \sqrt{1-\alpha_2}\, \varepsilon_2$$

And keeps on going till t.

So, $x_t = \sqrt{\alpha_t \alpha_{t-1}}\, x_{t-2} + \sqrt{\alpha_t (1-\alpha_{t-1})}\, \varepsilon_{t-1} + \sqrt{1-\alpha_t}\, \varepsilon_t$

$$= \sqrt{\alpha_t}\,(\sqrt{\alpha_{t-1}}\, x_{t-2} + \sqrt{(1-\alpha_{t-1})}\, \varepsilon_{t-1}) + \sqrt{1-\alpha_t}\, \varepsilon_t$$

Let $\bar{\alpha}_t = \prod_{i=1}^{t} \alpha_i$ So,

$$x_t = \sqrt{\bar{\alpha}_t}\, x_0 + \sqrt{(1-\bar{\alpha}_t)}\, \varepsilon_t \quad \text{where } \varepsilon_t \sim \mathcal{N}(0, I)$$

Hence Eq 9 can be written as

$$q(x_t | x_0) = \mathcal{N}(x_t; \sqrt{\bar{\alpha}_t}\, x_0, (1-\bar{\alpha}_t) I)$$

where $\bar{\alpha}_t = \prod_{i=1}^{t} \alpha_i$

Hence Proved.

Eq 9. = Eq 11.