4.1. Prove that Eq 7 is same as eq 8.

Eq 7: $$\log p(x) \geq \mathbb{E}_{q(x_{1:T}|x_0)}\left[\log \frac{p_\theta(x_{0:T})}{q(x_{1:T}|x_0)}\right] \quad \text{— (a)}$$

For $p_\theta(x_{0:T})$ in Eq 7:

$$\rightarrow \quad p_\theta(x_{0:T}) = p(x_T) \times \prod_{t=1}^{T} p_\theta(x_{t-1}|x_t) = p_\theta(x_0|x_1)\, p_\theta(x_1|x_2) \ldots\ldots \ldots p_\theta(x_{T-1}|x_T)\, p(x_T)$$

So, $p_\theta(x_{0:T})$ can be written as:

$$\log(p_\theta(x_{0:T})) = \log p_\theta(x_0|x_1) + \sum_{t=1}^{T-1} \log p_\theta(x_t|x_{t+1}) + \log p(x_T)$$

$$= \log\left[(p_\theta(x_0|x_1))\left(\sum_{t=1}^{T-1} p_\theta(x_t|x_{t+1})\right)(p(x_T))\right]$$

Substituting this eq^n in (a)

$$\log p(x) \geq \mathbb{E}_{q(x_{1:T}|x_0)}\left[\log \frac{p_\theta(x_0|x_1)\sum_{t=1}^{T-1} p_\theta(x_t|x_{t+1})\, p(x_T)}{q(x_{1:T}|x_0)}\right]$$

Expanding the terms:-

$$\log p(x) \geq \mathbb{E}_{q(x_{1:T}|x_0)}[\log p_\theta(x_0|x_1)] + \sum_{t=1}^{T-1} \mathbb{E}_{q(x_{1:T}|x_0)}[\log p_\theta(x_t|x_{t+1})]$$
$$+ \underbrace{\cancel{\mathbb{E}_{q(x_{1:T}|x_0)}[\log p(x_T)]}}_{\hookrightarrow \text{Constant.}} - \mathbb{E}_{q(x_{1:T}|x_0)}[\log q(x_{1:T}|x_0)]$$

Here, $\mathbb{E}_{q(x_{1:T}|x_0)}[\log p(x_T)]$ is a fixed distribution. So, this is constant.

Also, Rearranging the terms.

$$\boxed{\log p(x) \geq \mathbb{E}_{q(x_{1:T}|x_0)}[\log p_\theta(x_0|x_1)] + C - \sum_{t=1}^{T-1} \mathbb{E}_{q(x_{t-1},x_t,x_{t+1}|x_0)}\left[\log \frac{q(x_t|x_{t-1})}{p_\theta(x_t|x_{t+1})}\right]}$$

(a) = Eq 8 (H.P.)