Lecture 16: Edge-colourings and Ramsey theory · 1MA020

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Continuing our previous work on vertex colourings, we now consider the related notion of edge colourings. Then we discuss the basic notions of Ramsey theory.

Edge-colourings

We already saw the definition of an edge-colouring in the exercise session, but let us restate it here as well:

Definition 1. Let $G = (V, E)$ be a graph. A *proper*[2] *k-edge-colouring* is a function $c : E \to [k]$ such that no two edges which are incident to each other (i.e. share an endpoint) are assigned the same colour. If we do not have this restriction on incident edges, we call it just an (improper) edge colouring.

[2] Unlike for vertex colourings, we will actually be interested in improper edge colourings more often than proper ones, so we choose the opposite convention of including the word proper and omitting the word improper for them.

The *edge-chromatic number* of G, denoted $\chi_1(G)$,[3] is the smallest integer k such that G has a proper k-edge-colouring.

[3] This is sometimes also called the *chromatic index* of G. The 1 in the notation indicates that edges are one-dimensional – if we ever need to refer to both the chromatic number and the edge-chromatic number at the same time, we may thus denote the chromatic number by $\chi_0(G)$, since vertices are zero-dimensional. In some texts the edge-chromatic number is denoted by $\chi'(G)$, but χ and χ' look way too similar in LaTeX and on a blackboard, so let us avoid that notation.

Remark 2. We notice immediately that for a proper edge-colouring, the colour classes are all matchings, just like how for a vertex colouring the colour classes are all independent sets. In fact, of course, a proper edge-colouring is a vertex colouring of the line graph, so this is just an instance of the correspondence between matchings on G and independent sets in $L(G)$.

We can also see that if v is a vertex of maximum degree, its incident edges must all have different colours, so we must have that $\chi_1(G) \geq \Delta$.[4] In fact this bound is attained for bipartite graphs.

[4] We could also phrase this as that these Δ edges form a clique in the line graph, so the chromatic number of the line graph must be at least Δ, and $\chi(L(G)) = \chi_1(G)$.

Theorem 3 (König's line-colouring theorem, 1916)**.** *Every bipartite graph G with maximum degree Δ has edge-chromatic number $\chi_1(G) = \Delta$.*

Proof. We prove the result by induction on the number $m = |E|$ of edges. The base case of $m = 0$ is trivial.

So, let $G = (V, E)$ be bipartite, with m edges and maximum degree Δ. Pick an arbitrary edge $v \sim w$, and let G' be the graph obtained from G by deleting this edge.

Now G' is a bipartite graph on fewer edges, and so by the induction hypothesis it can be properly edge-coloured with $\Delta(G') \leq \Delta$ colours, so we pick such a colouring.

Each of the vertices v and w are incident to at most $\Delta - 1$ edges in G', so there must exist a colour i not used by any edge incident to v, and likewise a colour j not incident to w. If $i = j$ we can just colour the edge $v \sim w$ in this colour, so assume that $i \neq j$.

The rest of the argument essentially proceeds by considering the Kempe chains of the colouring, and doing a Kempe change. Of course, we defined Kempe chains only for *vertex* colourings, so we need to consider edge-induced subgraphs instead of induced subgraphs.

So consider the graph $G\langle i, j\rangle$ – the edge-induced subgraph of G consisting only of the edges coloured with i or with j. If we could show that v and w are in different connected components of this graph, then we could swap the two colours i and j in the component containing w, enabling us to colour the edge $v \sim w$ in colour i.

So suppose for contradiction that there were a path P from v to w in $G\langle i, j\rangle$. Then, since v is not incident to any edge of colour i by assumption, the path P has to start with an edge coloured j. Likewise, w is not incident to any edge coloured j, so P has to end with an edge coloured i.

So if we just look at the sequence of colours of edges in P, it has to look like

$$j\,i\,j\,i \ldots j\,i\,j\,i$$

and so in particular we see that it must be of even length, since all the is occur in even-numbered positions and the sequence ends on an i.

Therefore, the path P together with the edge $v \sim w$ forms a cycle in G of odd length. This, however, is impossible, since a graph is bipartite if and only if it contains no cycles of odd length.[5] □

[5] This, unfortunately, is not a statement we have had occasion to prove in the course. Fortunately, it makes for a very nice exercise.

Exercise 1. Prove this characterization of bipartite graphs.

So we have seen that the obvious bound is sharp for the bipartite graphs. What other values can the edge-chromatic number take? It turns out that the range of values is far more restricted than for vertex colourings.

Theorem 4 (Vizing, 1964)**.** *For any graph G of maximum degree Δ, it holds that $\chi_1(G) \in \{\Delta, \Delta + 1\}$. A graph such that $\chi_1(G) = \Delta$ is said to be of* class one*, and a graph such that $\chi_1(G) = \Delta + 1$ is said to be of* class two.[6]

[6] It turns out that there is no known classification of which graphs are of which class, but some partial results are known.

Exercise 2. For which values of n is K_n of class one and for which of class two?

To see this result, it suffices to see the following lemma:

Lemma 5. *Let G be a graph, v a vertex of G, and $k \geq \Delta$ an integer. Suppose $G[V \setminus \{v\}]$ has a proper k-edge-colouring such that for every neighbour w of v except possibly one, there are at least two colours not used by any edge incident to w, and the possible exception has one colour available. Then G is k-edge-colourable.*

Let us start by seeing how the theorem follows from the lemma, and then proving the lemma.

Proof of Theorem 4 from Lemma 5. The lower bound $\chi_1(G) \geq \Delta$ is clear, so what we need to show is the upper bound. So suppose for contradiction that there were a counterexample to this, and let G be an edge-minimum[7] such counterexample. That is, G is such that there exists a single edge $e = v \sim w$ such that the graph $G \setminus e$ is $\Delta + 1$-edge-colourable.

[7] That is, G is a graph which is a counterexample, and no other counterexample has fewer edges than G.

So pick a $\Delta + 1$-edge-colouring of $G \setminus e$, and notice that this gives us a $\Delta + 1$-edge-colouring of $G[V \setminus v]$ as well. So let us check that this colouring of $G[V \setminus v]$ satisfies the conditions of our lemma.

Every vertex always has at least one colour available, since we have $\Delta + 1$ colours and only at most Δ incident edges. Removing v will also remove the edges between v and its neighbours in $G \setminus e$, so all neighbours other than w in G will have one more colour freed up, namely the one used by their edge with v. So by our lemma there is a $\Delta + 1$-edge-colouring of G. □

Proof of Lemma 5. fix the induction hypothesis - needs to somehow be an induction in both k and Delta?

We prove the result by induction in k. The base case is not entirely trivial.[8]

[8] It is, however, simple enough that we can have it be an exercise instead of doing it in the lecture:

Exercise 3. Prove the base case. What value of k do we actually start at?

Pick a k-edge-colouring of $G' = G[V \setminus v]$, and for each $i \in [k]$, let X_i be the set of neighbours of v where colour i is available, that is, the neighbours which are not incident to any edge coloured with colour i. Let us assume that we have chosen the colouring c which minimizes $\sum_{i=1}^{k} |X_i|^2$ – that is, we have chosen it so that all the parts are of as equal sizes as possible.

We lose no generality in assuming that there is in fact an exceptional neighbour w at which only one colour is available, and that every other neighbour has exactly two colours available – if some neighbour has too many colours available, we can just add a leaf to it and use up one colour with the edge to that leaf. If we can edge-colour this larger graph, we can certainly colour the smaller one we started with, so no generality was lost.

Now we can see by an easy double-counting argument that that the sum over all colours of the number of neighbours it is available in equals the sum over all neighbours of the number of colours it has available, and so

$$\sum_{i=1}^{k} |X_i| = 2d_v - 1.$$

Now we claim that for every pair i, j of colours,

$$\left||X_i| - |X_j|\right| \leq 2.$$

Suppose for contradiction that this were not the case for some pair i, j, so that $|X_i| > |X_j| + 2$, and consider the edge-induced subgraph

$G'\langle i,j\rangle$ of edges coloured i or j.[9]

This is where we really see why edge-colourings are more restricted than vertex colourings – a Kempe chain[10] can look like any connected two-colourable graph, but this $G'\langle i,j\rangle$ has to have maximum degree two, and so every connected component has to be a cycle or a path.[11]

If it is a cycle, then at no vertex in the cycle will either colour i or j be available, and so in particular this cycle will be disjoint from X_i and X_j. So if a component contains a vertex from X_i, that component has to be a path, and in fact the vertex from X_i has to be one of the endpoints.

There clearly has to be at least one component of $G'\langle i,j\rangle$ which contains more vertices from X_i than from X_j – and so in particular this component has to be a path with one or both endpoints in X_i and neither in X_j. So if we do a Kempe change on this component, swapping the two colours i and j, this will decrease $|X_i|^2 + |X_j|^2$, contradicting our assumption of minimality.

So $\left||X_i| - |X_j|\right| \leq 2$ for all i and j, so all the values $|X_i|$ lie in an interval of integers of length three, which in particular implies that there must be some i such that $|X_i| = 1$. If there were not, there are two cases:

1. There is some j such that $|X_j| = 0$. Then for every k, $|X_k| \in \{0, 2\}$. Then, however, the sum of their sizes would be even, but we know that $\sum_{i=1}^{k} |X_i| = 2d_v - 1$, which is odd.
2. The least $|X_j|$ is at least two. Then, however,

$$2d_v - 1 = \sum_{i=1}^{k} |X_i| \geq 2k$$

and so $d_v > k \geq \Delta$, which is a contradiction.

So suppose we have such an i, and say $X_i = \{u\}$. So let

$$G'' = G \setminus (v \sim u \cup c^{-1}(i)),$$

that is, G'' is the graph gotten by deleting the edge $u \sim v$ and every edge coloured i. So $G''[V \setminus v]$ is $(k-1)$-edge-coloured, and The remaining idea is that this $G''[V \setminus v]$ will satisfy the induction hypothesis, so we can extend to a (k-1)-colouring of all of G'', and then we can add all the edges coloured i to the graph and colour the one bad edge u sim v with colour i.

□

[9] That is, the edge-induced subgraph $G'\langle c^{-1}(\{i,j\})\rangle$.

[10] That is, a connected component of the induced subgraph of vertices coloured in two colours by a vertex colouring.

[11] Where the path is allowed to be of length one, that is, be just a single isolated vertex.

Ramsey theory

Let us now dip our toes into the field of Ramsey theory. It is a large research area with many questions to be posed, and we will give it a shamefully short treatment. Let us start by stating the second most elementary definition in the area.[12]

[12] In the exercises, we saw *the* most basic definition. Let's make it mildly more interesting here.

Definition 6. For any integers r and k, the *Ramsey number* $R(r,k)$ is the smallest integer n such that every edge-colouring of K_n using only the two colours red and blue contains a red K_r or a blue K_k.[13] Equivalently, it is the smallest integer n such that any graph on n vertices contains either an independent set of size k or has a clique of size r.

[13] By "containing a monochromatic K_r" we mean that for some colour i, the edge-induced subgraph $K_n\langle c^{-1}(i)\rangle$ contains an r-clique.

Remark 7. Why are the two things we stated equivalent? Given any graph $G = (V,E)$ on n vertices, we can edge-colour the complete graph K_n by colouring the edge $i \sim j$ red if $i \sim j \in E$, and blue otherwise. Then a red clique in the colouring is a clique in G, and a blue clique is an independent set in G.

Of course, these definitions contain the hidden assumption that these numbers actually exist. A priori there might be some r and k such that there exist arbitrarily large graphs G with $\alpha(G) < k$ and $\omega(G) < r$. So let us prove that these numbers are indeed finite.

Lemma 8. *For all m and n, it holds that*

$$R(m,n) \leq R(m-1,n) + R(m,n-1).$$

Proof. To keep the notation manageable, let us let $s = R(m-1,n)$ and $t = R(m,n-1)$. Consider some edge-colouring of K_{s+t}, and pick an arbitrary vertex v. By the pigeonhole principle, this v has either s incident red edges or t incident blue edges.[14]

[14] If it did not, the total number of vertices in the K_{s+t} would be upper bounded by

$$1 + (s-1) + (t-1) = s + t - 1$$

which is a contradiction.

So assume without loss of generality that it has s neighbours by red edges,[15] and let S be this set of neighbours. Then the induced subgraph of just these neighbours is a K_s, which thus must contain either a blue K_n or a red K_{m-1}. In the first case, we are done, and in the second case, the red K_{m-1} together with v forms a red K_m, and we are again done. □

[15] The argument if it instead had t neighbours by blue edges is entirely the same.

Corollary 9. *For all m and n, $R(m,n)$ is finite, and in particular,*

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

Proof. It is easy to see that $R(m,1) = R(1,m) = 1$, so the finiteness is immediate by induction. To see the more explicit bound, we observe that it is nearly as immediate that $R(m,2) = m$ and $R(2,n) = n$, and then do a standard proof by induction, which we omit.[16] □

[16] It is just a version of the very basic "prove this formula with binomial coefficients" proofs that one sees when first introduced to induction, there are no fancy ideas here. Do work out the details if you like.

Having seen the upper bound, let us now also establish a lower bound on this quantity. To see a lower bound, it suffices to find a single edge-colouring of a K_n which contains no monochromatic K_k. This time, we will get to use the probabilistic method – the idea is just to consider a random colouring, and show that the probability that it has a monochromatic clique is less than one.[17]

Theorem 10 (Erdős, 1947). *If* $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, *then* $R(k,k) > n$. *In particular,* $R(k,k) \geq \left\lfloor 2^{k/2} \right\rfloor$ *for all* $k \geq 3$.[18]

Proof. As advertised, we consider a random edge-colouring of a K_n – we just assign each edge red or blue with equal probability.

So, for each set $S \in \binom{[n]}{k}$, let R_S be the event that $K_n[S]$ is monochrome red, and B_S the event that it is monochrome blue. Clearly $\mathbb{P}(R_S) = \mathbb{P}(B_S) = 2^{-\binom{k}{2}}$.

Then the event that the colouring has a monochrome K_k is just

$$\bigcup_{S \in \binom{[n]}{k}} R_S \cup B_S,$$

and so we can use a union bound to see that

$$\mathbb{P}\left(\bigcup_{S \in \binom{[n]}{k}} R_S \cup B_S\right) \leq \sum_{S \in \binom{[n]}{k}} \mathbb{P}(R_S) + \mathbb{P}(B_S) = \binom{n}{k} 2^{1-\binom{k}{2}}$$

which by assumption is less than one, and so there must exist an outcome in which the event does not occur, which will be our example. □

[17] This is one of the early proofs by Erdős using the probabilistic method, that really made its power clear.

[18] We only prove the first statement – the second follows just by seeing that

$$\binom{\lfloor 2^{k/2} \rfloor}{k} 2^{1-\binom{k}{2}} < 1,$$

which is not a calculation we want to have to do.

Exercises

Exercise 4. Suppose G is some graph of maximum degree Δ, and let D be the set of vertices of degree Δ. Use Lemma 5 to show the result by Fournier that if $G[D]$, the subgraph induced by the vertices of maximum degree, is a forest, then G is of class one.