

# Exam in Graph Theory, 5 January 2024 · 1MA170

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There are a total of ten questions on this exam. You are to **pick eight of them to answer.**<sup>2</sup>

Good luck!



## Question 1 (5p)

What does it mean for a graph to be  $k$ -connected? State a correct definition.

We proved a characterisation of two-connected graphs as being exactly the graphs that can be constructed by a certain process starting from a cycle graph. State the theorem, and give a proof of it.

## Question 2 (5p)

Define the adjacency matrix  $A$ , incidence matrix  $D$ , and Laplacian matrix  $Q$  of a graph. Prove that  $Q = DD^t$ .

## Question 3 (5p)

Recall that Hall's marriage theorem gives a condition for the existence of a matching of one side of a bipartite graph into the other in terms of a condition on the size of sets  $Q \subseteq A$  and their neighbour sets  $N(Q) \subseteq B$ .

Give a precise statement of the theorem, and then give a proof.<sup>3</sup>

## Question 4 (5p)

**Part a:** Prove that if you remove  $k$  edges from a tree  $T$ ,<sup>4</sup> the resulting graph is a forest of  $k + 1$  trees.

**Part b:** Pick one of Kruskal's or Prim's algorithms. State it and prove its correctness.

## Question 5 (5p)

What does it mean for a graph to be planar? What is the planar dual of a planar graph? Give definitions, and include a correct figure illustrating the definition.

<sup>1</sup> I will visit the exam hall at some point during the exam to answer any questions. If you need to reach me outside that time, I am available by email at [vilhelm.agdur@math.uu.se](mailto:vilhelm.agdur@math.uu.se) or by phone at 072-373 32 90.

<sup>2</sup> If you answer more than eight questions, your exam total score will be the sum of your eight *lowest* scores, so there is absolutely **no** benefit to answering more than eight questions.

<sup>3</sup> The proof we gave in the lecture was a clever application of the max-flow min-cut theorem, where we turned our bipartite graph into a flow network. If you choose to give this proof, please also draw a figure of the construction.

<sup>4</sup> Where the tree has more than  $k$  vertices, so there are indeed  $k$  edges to remove.

Prove the following result from the course:<sup>5</sup>

**Theorem 1** (Euler's formula). *Let  $G = (V, E)$  be a connected planar graph, and let  $f$  be the number of faces for some planar embedding of  $G$ . Then*

$$|V| - |E| + f = 2,$$

*and so in particular any two planar embeddings have the same number of faces.*

<sup>5</sup> Hint: Consider a spanning tree of  $G$  and its complement in  $G^*$ .

### Question 6 (5p)

What does it mean for a graph to be Eulerian? State and prove Euler's theorem giving a criterion for a graph to be Eulerian.

### Question 7 (5p)

Recall from the lectures that the *minimum bisection problem* asks for a partition of the vertices of a graph into two equally-sized sets<sup>6</sup> such that the number of edges between them is minimal.

We proved an upper bound on how many edges you could be forced to include in such a bisection using the probabilistic method, where we started by picking a matching on the complement graph. State and prove this result.

<sup>6</sup> So, as we state it, it only applies to graphs with an even number of vertices.

### Question 8 (5p)

Given a graph  $H$  and an integer  $n$ , define the extremal function  $\text{ex}(n; H)$  of  $H$ .

Turán's theorem says that

$$\text{ex}(n; K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Prove this.<sup>7</sup>

<sup>7</sup> We gave two proofs in the course, and there are many more pretty proofs floating around. Any correct proof is of course a correct answer to the question.

The first proof we gave in the course used the Caro-Wei result on independent sets, and then applied the Cauchy-Schwarz inequality

$$|\langle a, b \rangle| \leq \|a\| \|b\|$$

to a clever choice of  $a$  and  $b$ .

<sup>8</sup> That is, not just a brute-force method, but something you'd actually use in practice.

### Question 9 (5p)

A *directed acyclic graph* (often called just a DAG) is a directed graph that contains no cycles. Devise a reasonable<sup>8</sup> algorithm for determining when such a graph is Hamiltonian. For directed graphs, being Hamiltonian means having a *directed* Hamiltonian path.

### Question 10 (5p)

For each of these statements, determine if it is true or false and give a proof or disproof:<sup>9</sup>

<sup>9</sup> Each statement gives one point, except for c) and d), which give half a point each.

- a) For every  $k \geq 3$ , there exists a  $k$ -regular Hamiltonian graph.
- b) For every  $k \geq 1$ , there exists a  $k$ -regular planar graph.
- c) For every  $k \geq 1$ , there exists a multigraph with exactly  $k$  spanning trees.
- d) For every  $k \geq 1$ , there exists a graph with exactly  $k$  spanning trees.
- e) There exists a 6-connected planar graph.
- f) There exists a tree in which every vertex has degree 2.