Graph Theory Exercises 12

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Exercise 1

If the graph G has no two-connected blocks then it has no cycle which means that its components are either an isolated vertex or a tree. If G has no edge then all of its components are isolated vertices and so it has a 1-colouring. Else, as trees have a 2-colouring the graph also has a 2-colouring.

Exercise 2

First colour each block using at most $k = \max_{i \in [r]} \chi(B_i)$ colours using a different function for each block B_i that we will call $c_i : B_i \to [k]$. At each cutvertex where there is a disagreement between the colouring of an arbitrary number of different functions we will resolve the problem in the following way:

1. Select a block, lets call its vertices B_x and call x the colouring it gave to the cutvertex c.

2. Select a block whose colouring conflicts with B_x, let's call it B_y and call y the colouring it gave to the cutvertex.

3. Give B_y a new colouring function

$$c'_y : B_y \to [k]$$
$$c'_y(v) = c_y(v) + x - y \bmod k.$$

   This function is one to one so its still a valid colouring, furthermore we now have resolved the conflict on the cutvertex as $c'_y(c) = y + x - y = x$.

4. Go back to step 2 until there is no conflict at that cutvertex.

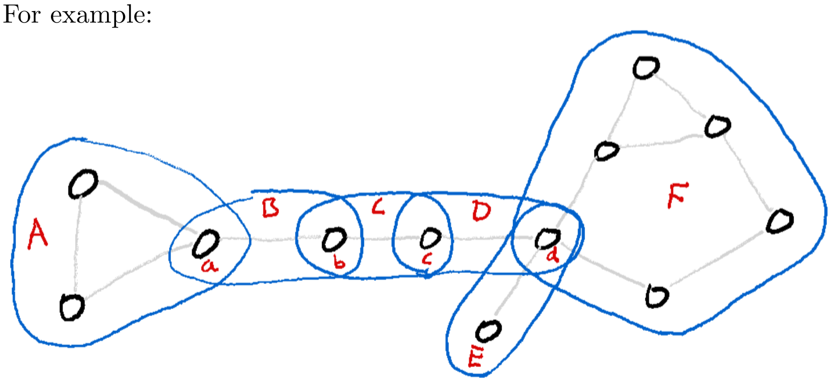
Using that method to solve conflict and knowing that a block graph is a forest we can solve all the colouring conflicts of the graph in a simple way:

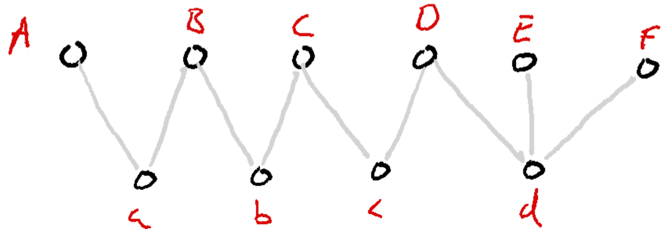
1. Pick a component of the block graph of G, if its a single block there is no problem thus we only consider the case where it is a tree.

2. Pick a leaf of the tree, it will be a block because a cutvertex in a block graph is adjacent to two trees.

3. Solve conflicts between blocks by keeping intact the colouring function of the block closest to the leaf.

For example:





In that block graph you would select a leaf A, E or F, let us say we pick A. Then you will change the coloring function at B in order to solve the possible conflict on cutvertex a, and by doing that B now agrees with A on a. You repeat the process with the block C, because C is farther away from the leaf A you change C's coloring function, not B's. You repeat this process until every possible conflict of coloring on cutvertex is solved. This process is guaranteed to finish as there are no cycles on the block graph.

Exercise 3

In this exercise, we use some *spectral* methods for deriving results about the chromatic number. We rely on the following lemma, which can be proved using the technique of Rayleigh-quotients:

Lemma 1. *Let G be a graph and H an induced subgraph of G, and let their adjacency matrices be A_G and A_H respectively. Then*

$$\lambda_{\min}(A_G) \leq \lambda_{\min}(A_H) \leq \lambda_{\max}(A_H) \leq \lambda_{\max}(A_G),$$

and

$$\delta(G) \leq \lambda_{\max}(A_G) \leq \Delta(G),$$

where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of G.

Use the above lemma to prove the below theorem:

Theorem 1 (Wilf, 1967)**.** *For any graph G with adjacency matrix A_G, we have*

$$\chi(G) \leq \lambda_{\max}(A_G) + 1.$$

Hint: Let H be a minimal induced subgraph of G with $\chi(H) = \chi(G)$. Can you relate the minimum degree of H to the chromatic number of G?

Solution from earlier lecture notes: Among all induced subgraphs of G there exists a minimal subgraph H (w.r.t inclusion) with $\chi(H) = \chi(G)$. Let v be a vertex of H. Then $H - \{v\}$ admits a $\chi(G) - 1$- colouring, and if $d_H(v) < \chi(G) - 1$, then this colouring could be extended to a $\chi(G) - 1$-colouring of H, contradicting our choice of H. Hence, the minimum degree in H is at least $\chi(G) - 1$. Denote by A_H the adjacency matrix of H. Then,

$$\chi(G) \leq \delta(H) + 1 \leq \lambda_{max}(A_H) + 1 \leq \lambda_{max}(A_G) + 1$$

where we used the inequalities from Lemma 1.

Exercise 4 (Extra)

Assume you have to separate English alphabet letters into boxes such that no two consecutive letters end up in the same box. What is the minimum number of boxes you need for this task?[1]

Solution: If we transform the problem into a graph where each letter is a vertex and vertices represent consecutive edges, we will have a graph with vertices of degree 1 (the letters A and Z) and 2 (other letters). By Brook's theorem, this graph has $\chi(G) = 2$ and the graph is not complete nor has odd cycles. Hence, it is enough with two boxes only.

Exercise 5 (Extra)

Prove that if a graph has at most two cycles of odd length then it can be coloured with 3 colours. *Hint*: a bipartite graph has $\chi(G) = 2$.

Solution: (Based on University of Victoria course notes in Discrete and Combinatorial Mathematics).

We must consider three distinct cases.

Case 1: Suppose G has no odd cycles. Then G is bipartite with $\chi(G) = 2$, so certainly we can colour G with three colours.

Case 2: Suppose that G contains exactly one odd cycle, C. Then certainly $\chi(G) > 2$ as this graph is not bipartite. Consider removing an arbitrary vertex

[1] Source: `https://shorturl.at/bsIVY`

$u \in C$ from G – this would create a graph with no odd cycles. So $G - u$ is 2-colourable. Adding u back to G would only require one additional colour, so G is 3-colourable.

Case 3: Suppose G contains exactly two odd cycles, C_1 and C_2, we now consider two subcases:

Case 3a: Suppose that C_1 and C_2 share a common vertex u. Consider $G-u$, which now has no odd cycles, since removing a vertex from a cycle breaks the cycle. Thus, $G - u$ is bipartite and 2-colourable. Adding back u will require at most one additional colour, so G is 3-colourable.

Case 3b: Suppose C_1 and C_2 share no common vertices. If every vertex in C_1 is adjacent to every vertex in C_2 then there would be another odd cycle in G which is impossible by assumption. Thus, there exists two vertices, $u \in C_1$ and $v \in C_2$, such that $u \sim v \notin E(G)$. Consider obtaining the graph $G - u - v$, this will break both cycles in G, making $G - u - v$ a graph free of odd cycles, and hence 2-colourable. As u and v are not adjacent in G colouring them will require at most one additional colour. Thus, G is 3-colourable.

This completes the proof.