# *Grading of January 5 2024 exam: Rubrics, statistics, reflection · 1MA170*

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This file contains information about how the exam was graded – how the points were assigned to each question – and also statistics on the outcome of the exam. The full data of the scores is available in an anonymized form in the GitHub repository of the course.

## *General observations*

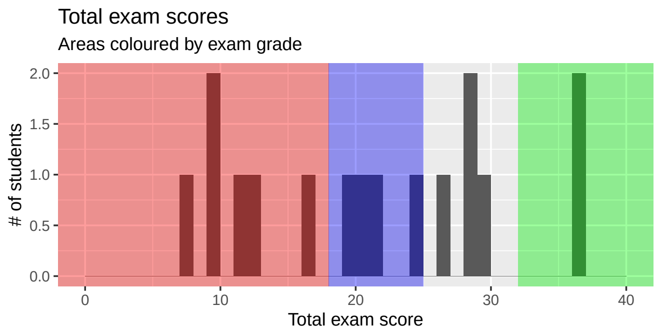


Figure 1: Plot of the total scores gotten by each student on the exam.

The overall distribution of exam scores is illustrated in Figure 1. A total of sixteen students wrote the exam, of whom ten passed it. The mean and median score of all students was a touch over 21,[1] but as can be seen in the figure, and as is usual on most exams, there was a clear gap of a few students who failed with very low scores, with the rest passing – only one student ended up just barely failing. The mean and median score among *passing* students were 27.35 and 27.375 respectively.

[1] Median 21.125, mean 21.34.

In Figure 2, one can see a correlation plot of correlations between scores on the different questions and with the total score. As expected, most questions correlate with each other – doing well on one question tends to indicate that you understood the course well overall.

The one noticeable exception is question nine, which actually has a *negative* correlation with the overall score. The reason for this is presumably that the question was hard, and each student only had to answer eight out of the ten questions. So the best-performing students chose this question as one of the two to not answer, and thus get a zero on. Why the lower-scoring students chose to answer this one might have two plausible explanations:

1. They already had at least two of the other questions which they did

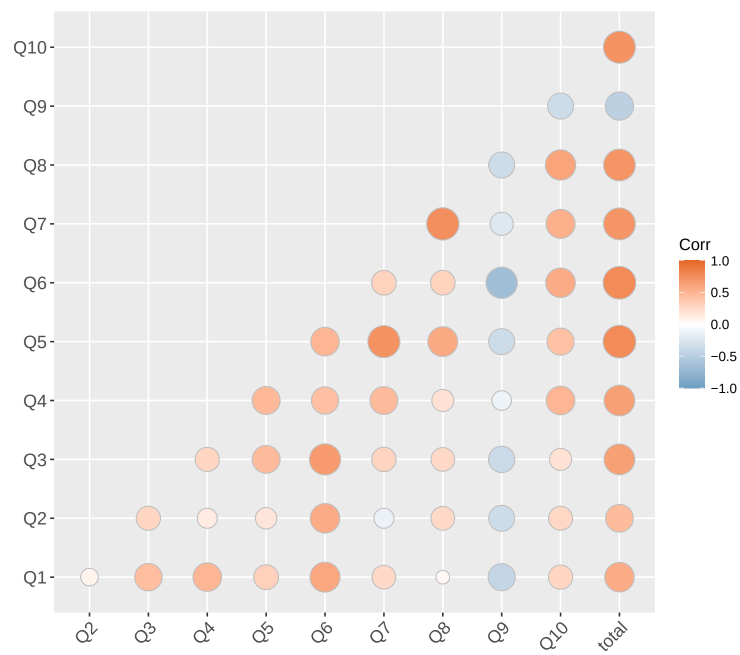


Figure 2: Correlation matrix for scores on each question, and on the exam as a whole.

not recall the answer for, and so they lost nothing by making an attempt at question nine and getting some score for it.

2. Some may have misunderstood the question and not realized its difficulty, believing their incorrect solutions to actually be correct. Several students mistakenly believed that all directed acyclic graphs must be trees, and gave an answer that would have been roughly correct under that false assumption.

## *Question 1 (5p)*

What does it mean for a graph to be $k$-connected? State a correct definition.

We proved a characterisation of two-connected graphs as being exactly the graphs that can be constructed by a certain process starting from a cycle graph. State the theorem, and give a proof of it.

## *Question 2 (5p)*

Define the adjacency matrix $A$, incidence matrix $D$, and Laplacian matrix $Q$ of a graph. Prove that $Q = DD^t$.

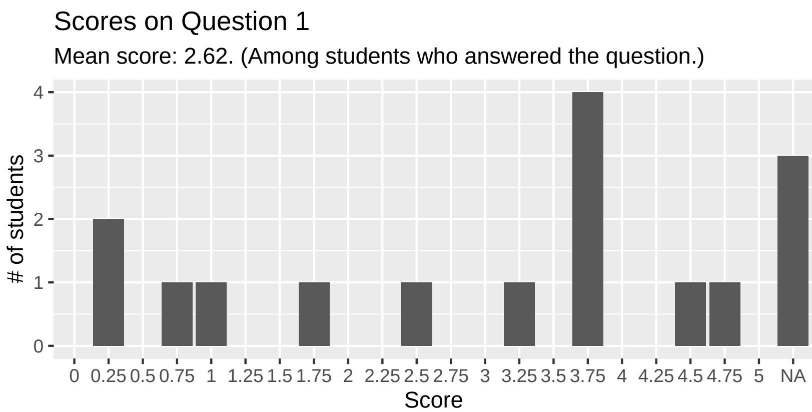


Figure 3: Histogram of student scores on question one.

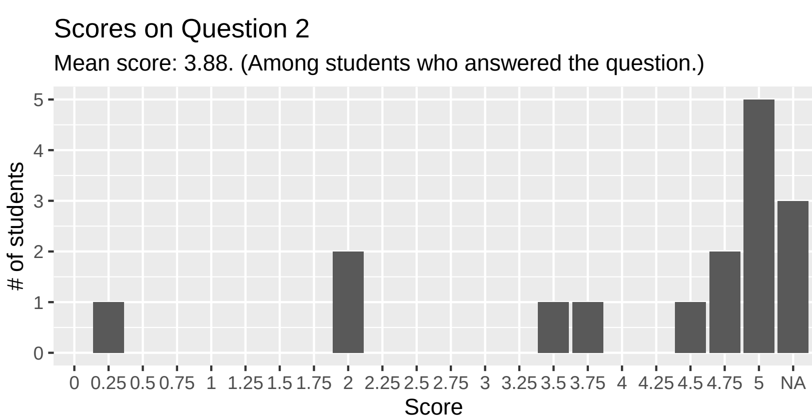


Figure 4: Histogram of student scores on question two.

## *Question 3 (5p)*

Recall that Hall's marriage theorem gives a condition for the existence of a matching of one side of a bipartite graph into the other in terms of a condition on the size of sets $Q \subseteq A$ and their neighbour sets $N(Q) \subseteq B$.

Give a precise statement of the theorem, and then give a proof. (The proof we gave in the lecture was a clever application of the max-flow min-cut theorem, where we turned our bipartite graph into a flow network. If you choose to give this proof, please also draw a figure of the construction.)

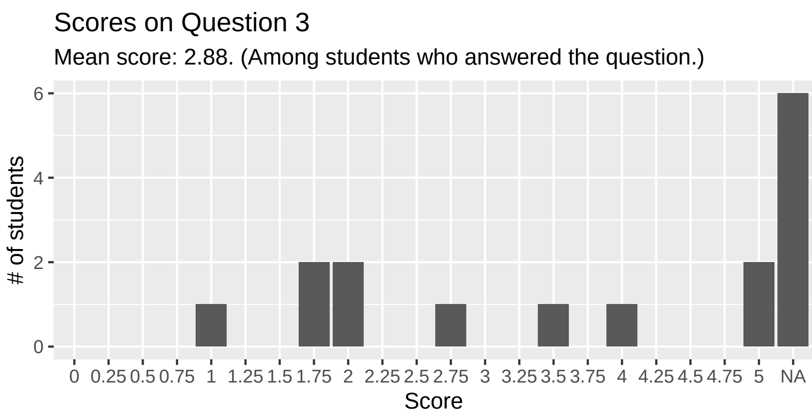


Figure 5: Histogram of student scores on question three.

## *Question 4 (5p)*

**Part a:** Prove that if you remove $k$ edges from a tree $T$, (Where the tree has more than $k$ vertices, so there are indeed $k$ edges to remove.) the resulting graph is a forest of $k+1$ trees.

**Part b:** Pick one of Kruskal's or Prim's algorithms. State it and prove its correctness.

## *Question 5 (5p)*

What does it mean for a graph to be planar? What is the planar dual of a planar graph? Give definitions, and include a correct figure illustrating the definition.

Prove the following result from the course: (Hint: Consider a spanning tree of $G$ and its complement in $G^*$.)

**Theorem 1** (Euler's formula). *Let $G = (V, E)$ be a connected planar graph, and let $f$ be the number of faces for some planar embedding of $G$. Then*

$$|V| - |E| + f = 2,$$

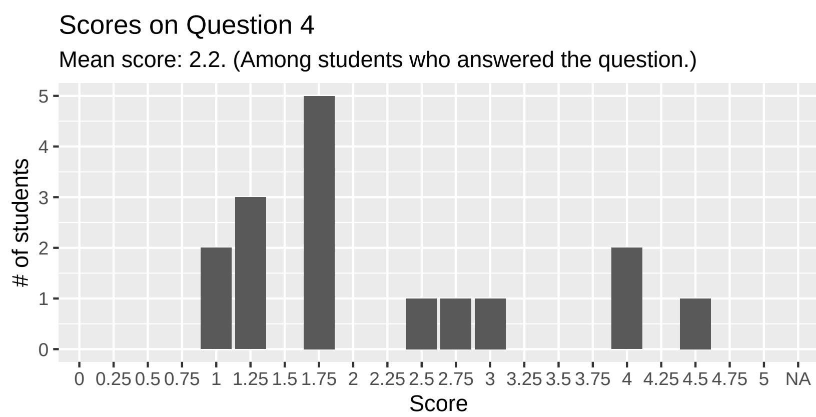


Figure 6: Histogram of student scores on question four.

*and so in particular any two planar embeddings have the same number of faces.*

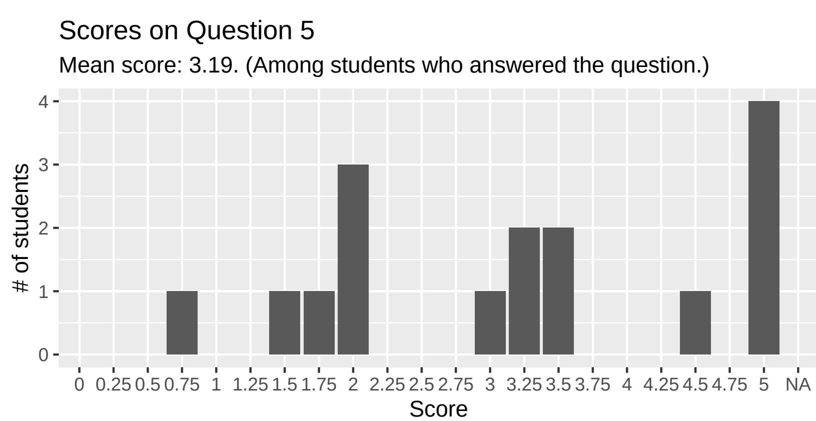


Figure 7: Histogram of student scores on question five.

## *Question 6 (5p)*

What does it mean for a graph to be Eulerian? State and prove Euler's theorem giving a criterion for a graph to be Eulerian.

## *Question 7 (5p)*

Recall from the lectures that the *minimum bisection problem* asks for a partition of the vertices of a graph into two equally-sized sets (So, as we state it, it only applies to graphs with an even number of vertices.) such that the number of edges between them is minimal.

We proved an upper bound on how many edges you could be forced to include in such a bisection using the probabilistic method, where

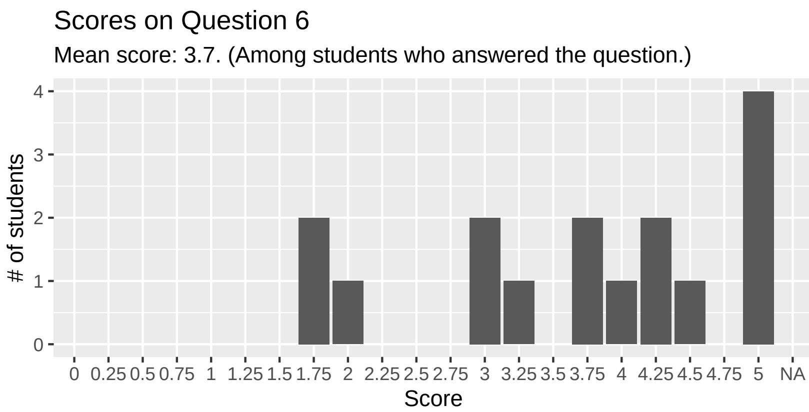


Figure 8: Histogram of student scores on question six.

we started by picking a matching on the complement graph. State and prove this result.

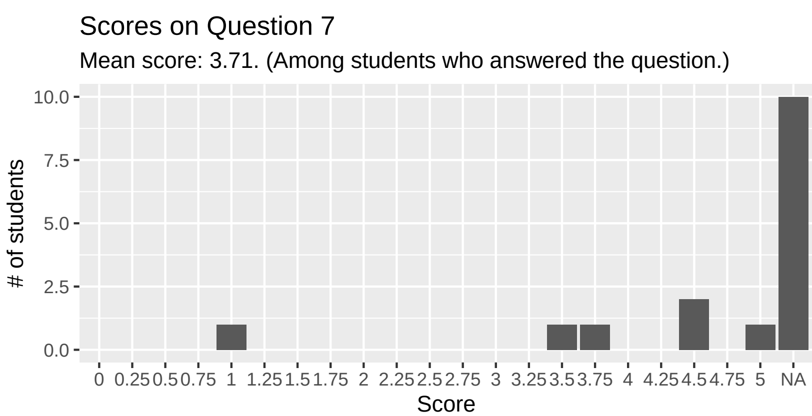


Figure 9: Histogram of student scores on question seven.

## *Question 8 (5p)*

Given a graph $H$ and an integer $n$, define the extremal function $\mathrm{ex}(n; H)$ of $H$.

Turáns theorem says that

$$\mathrm{ex}(n; K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Prove this. (We gave two proofs in the course, and there are many more pretty proofs floating around. Any correct proof is of course a correct answer to the question. The first proof we gave in the course used the Caro-Wei result on independent sets, and then applied the Cauchy-Schwarz inequality

$$|\langle a, b \rangle| \leq \|a\| \, \|b\|$$

to a clever choice of $a$ and $b$.)

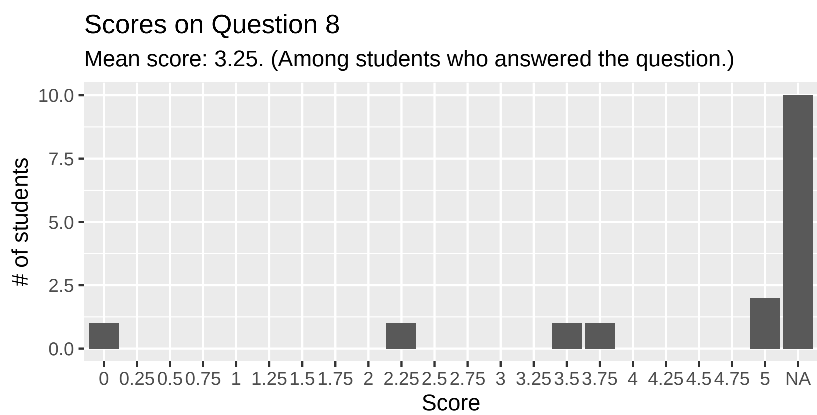


Figure 10: Histogram of student scores on question eight.

## *Question 9 (5p)*

A *directed acyclic graph* (often called just a *DAG*) is a directed graph that contains no cycles. Devise a reasonable (That is, not just a brute-force method, but something you'd actually use in practice.) algorithm for determining when such a graph is Hamiltonian. For directed graphs, being Hamiltonian means having a *directed* Hamiltonian path.

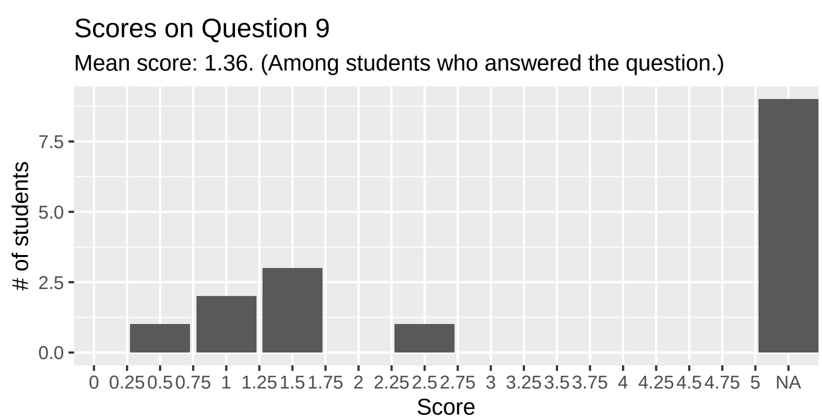


Figure 11: Histogram of student scores on question nine.

## *Question 10 (5p)*

For each of these statements, determine if it is true or false and give a proof or disproof:

a) For every $k \geq 3$, there exists a $k$-regular Hamiltonian graph.

b) For every $k \geq 1$, there exists a $k$-regular planar graph.

c) For every $k \geq 1$, there exists a multigraph with exactly $k$ spanning trees.

d) For every $k \geq 1$, there exists a graph with exactly $k$ spanning trees.

e) There exists a 6-connected planar graph.

f) There exists a tree in which every vertex has degree 2.

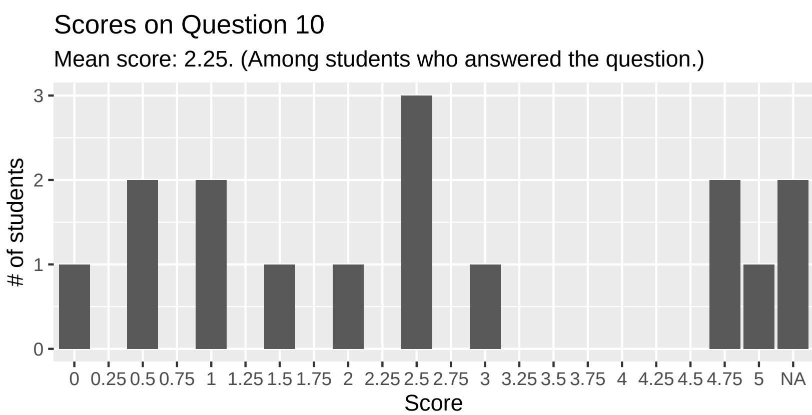


Figure 12: Histogram of student scores on question ten.