Detecting Partisan Gerrymandering through Mathematical Analysis: A Case Study of South Carolina

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Abstract

Partisan gerrymandering, the process of drawing the district boundaries of election maps unfairly for a given party's political gain, is a practice that originated in the early 1800s and persists today. As partisan gerrymandering cases have met varying degrees of success in state and federal courts, now more than ever is there a need to define rigorous methods of assessing the partisan bias of election maps. In practice, two primary mathematical tools are utilized to quantify gerrymandering: measures of partisan bias and outlier analysis methods.

This thesis analyzes the 2011 and 2018 Congressional districting plans of Pennsylvania and uses the results to inform an analysis of the 2011 South Carolina Congressional and state legislative maps. The goals of this study are to (1) provide a state-specific analysis of South Carolina and (2) contribute to the current body of literature by increasing transparency in gerrymandering detection procedures, investigating how outlier analysis methods perform for gerrymandered versus fair maps and how various partisan bias measures available perform compared to one another. Results of this study indicate that, using the $\sqrt{\varepsilon}$ test as defined by Chikina et al. in the 2017 case *League of Women Voters v. Commonwealth of Pennsylvania*, relatively unbiased maps indicate clear levels of non-significance compared to their biased counterparts. Certain measures of partisan bias, such as *median-mean* and the *geometric bias measure* defined under a variable partisan swing assumption, are also found to perform more consistently as indicators of gerrymandering than other metrics. In addition, this study finds that there is insufficient evidence of partisan gerrymandering in the Congressional, state Senate, and state House maps of South Carolina drawn during the 2011 redistricting cycle.

Keywords: *election maps*, *gerrymandering*, *redistricting*, *partisan bias*, *outlier analysis*, *Markov chains*, *statistics*, *Pennsylvania*, *South Carolina*

Dedication

This thesis is dedicated to my family. Mom, Dad, Anthony, and Hannah, your love and support throughout grad school and beyond will always mean more than you know.

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Lastly, I would be remiss to not mention the constant support of my colleagues and my students who offered encouragement and inspiration throughout the duration of this thesis. Research takes a village. I am grateful for mine.

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Chapter 1

Introduction

In March of 1812, the *Boston Gazette* featured an artist's rendition of an unusual-looking voting district in the northeastern corner of Massachusetts, accompanied by the caption "*The Gerry-mander:* A new species of *Monster*, which appeared in *Essex South District* in January last" [18]. Then-governor of Massachusetts Elbridge Gerry had signed a bill authorizing the redrawing of the voting districts in the state and, intent on securing representation in the state senate, members of Gerry's political party carefully drew district boundaries to include their constituents [39]. Snaking around the outer edge of Essex county, the resulting district was said to resemble a salamander, most likely referring to the mythological monster from classical folklore [30]. It was then that the term *gerrymandering* was born, and within a year its usage had extended to include all instances of drawing voting districts unfairly for political gain.

Despite being centuries old, gerrymandering is no less prevalent today than it was in the nineteenth century. In recent years, the dialogue about gerrymandering has focused particularly on *partisan* gerrymandering, in which district boundary lines are drawn to advantage one political party over another, much like the Massachusetts district that drew so much attention in 1812. Like all forms of gerrymandering, partisan gerrymandering is believed to undermine democracy and dilute the power of the individual vote. It is thus worthwhile to answer the question of how unfair maps can be quantified and use these standards to assess the maps in one's own state.

While much work has been done on measures of partisan fairness, both in their development and application, detailed analyses of election maps are not available for every state, and those that do exist often consist of only one or two measures. Though this is a start, comparing a selection of *multiple* bias measures is beneficial to gain a more comprehensive understanding of an election map, as bias measures are not necessarily standardized across settings. In addition, analyses for the election map of a single state are often discussed in isolation, leaving limited information available that explicitly compares fair maps to gerrymandered ones. This thesis contributes to the growing body of literature by providing a thorough, comparative mathematical analysis of the Pennsylvania Congressional maps from 2011[1] and 2018, which serve as examples of a gerrymandered map and a fair map, respectively, using multiple partisan bias measures. These results are then used to inform an analysis of several South Carolina election maps.

(a) A map of the 1812 Essex South District.

(b) The cartoon mocking its bizarre shape.

Figure 1.1: The districting that coined the term *gerrymandering* as it appeared in the *Boston Gazette*. Map reproductions courtesy of the Norman B. Leventhal Map & Education Center at the Boston Public Library [47].

The goal of this paper is twofold: (1) to provide a state-specific analysis of South Carolina election maps that does not exist elsewhere to evaluate whether there is significant evidence of partisan bias, and (2) to increase transparency in how partisan fairness measures and other assessments of election maps can be applied and understood within the context of detecting gerrymandering. This thesis aims to increase transparency in gerrymandering detection procedures by exploring how measures of partisan bias and outlier analysis methods perform across a variety of conditions, and

[1]The 2011 Congressional map for Pennsylvania refers to that which was drawn during the redistricting process immediately following the 2010 Census.

in doing so, answers the following questions:

- How do methods compare for gerrymandered maps versus fair maps?
- How do methods compare to one another on a given election map?

The first chapter of this work provides an introduction to the basic principles of redistricting and describes the current state of partisan gerrymandering in the United States. Chapter 2 then summarizes the current body of literature on mathematical approaches to detecting gerrymandered election maps. A discussion of the data used in this analysis is presented in Chapter 3, and the methods for and results of the analysis are detailed in Chapter 4. The concluding chapter contextualizes the results of the study, discusses limitations and implications, and provides suggested areas for future work.

1.1 Principles of Redistricting

Gerrymandering occurs during the process of *redistricting*, in which politicians draw voting district boundaries for the election of federal senators and representatives according to the provisions set forth in the United States Constitution [13]. While each state is allotted two senators, representatives are apportioned to states every ten years based on the population as accounted for in the most recent U.S. Census [11, 12]. The number of U.S. Representatives was permanently fixed at 435 by the Reapportionment Act of 1929, leaving a set number of seats to fluctuate between states with shifts in the country's population [7]. Similarly, each state has its own number of representatives and senators that serve in the state legislature. For both federal and state elections, each seat in a given state is represented by a single voting district called a *single-member district*. The method of drawing these districts varies from state to state; for example, six states utilize independent commissions that expressly limit direct participation in the redistricting process by elected officials [37]. However, in the majority of states, the state legislature is charged with redrawing district boundary lines, a power granted by Article I, Section 4 of the U.S. Constitution, subject to the regulation of Congress. It has been observed throughout history that state legislatures controlled by a majority party are often likely to engage in gerrymandering when it suits their party's interest, and thus the primary focus of this paper is on the latter scenario in which the state legislature is in charge of the redistricting process.

State mapmakers must abide by several guidelines at the federal and state level when redistricting. The Apportionment Act of 1842, the first of many pieces of apportionment legislation that were subsequently enacted, requires geographic, single-member districts as opposed to the at-large electoral regions that were used previous to the passing of the Act [1]. The landmark U.S. Supreme Court case *Baker v. Carr*, in which the Court held that redistricting qualifies as a justiciable question, opened the door for the Court to consider legal disputes over redistricting and paved the way for new legislation regarding redistricting procedures [2]. The series of court cases following *Baker* emphasized the legal principle of "one person, one vote," which required that electoral districts be apportioned according to population. In 1964, it was ruled in *Wesberry v. Sanders* that Congressional districts must be nearly equal in population, and it shortly followed in *Reynolds v. Sims* that state legislative districts must also possess this quality [8, 16]. The Voting Rights Act of 1965 also had a significant impact on the redistricting process, providing racial protections by requiring that voting districts must not unduly disadvantage minority voters [15].

Many states have established additional requirements regarding the shape, size, and nature of voting districts, oftentimes explicitly stated in their state constitutions [58]. One such state is South Carolina, whose most recent redistricting guidelines were released in 2011 by the South Carolina House and Senate Judiciary Committees [26, 45]. In addition to reinforcing federally mandated principles, several guidelines are presented that provide a relatively thorough summary of requirements appearing in other states' redistricting laws.

- *Population.* While *Wesberry v. Sanders* and *Reynolds v. Sims* mandate that districts have approximately equal populations, South Carolina laws further go on to say that Congressional districts must achieve strict equality whenever possible. For the population estimates from the 2010 Census, this equates to an ideal district population of 660,766 people. Legislative districts must also be drawn to achieve roughly equipopulous regions, but the constraints are somewhat relaxed. The guidelines by the Senate Judiciary Committee require that Senate district populations must be within a plus-or-minus 5% deviation from 100,551 persons for any given district. The House Judiciary Committee guidelines, on the other hand, place a stricter bound on this deviation, requiring that each House district should be drawn within a $\pm 2.5\%$ deviation from 37,301 individuals.

- *Contiguity.* Both Congressional and legislative districts must be made up of contiguous geographic areas. In other words, a single district may not be comprised of two disconnected regions that do not share a boundary. Though not required by any federal law, all fifty states require some form of contiguity in their redistricting guidelines [37, 58]. Both sets of South Carolina guidelines allow for contiguity by water, but have differing precedents for how district areas touching at a point are regarded. The Senate guidelines consider point-to-point contiguity acceptable in certain cases, while the House guidelines state that regions connected at a single point are not considered contiguous.

- *Compactness.* Unless being adjusted to meet the aforementioned federal and state principles, districts should be "compact in form." Compactness is a somewhat intuitive notion that limits how spread out and "bizarre" the boundary for a district is. In general, districts that are closer in shape to a circle are considered to be more compact, while sprawling, twisting districts like Elbridge Gerry's salamander are not. Although the South Carolina House redistricting guidelines explicitly state that compactness will not be judged by mathematical measures, several such measures for compactness exist. One of the most widely used is the Polsby-Popper metric, which compares the area of a district to the area of a circle with a circumference equal to district's perimeter [44]. In legal documentation, however, the notion of compactness may be rather subjective and vaguely defined only in terms of whether or not a map's districts look strange.

- *Communities of Interest.* Whenever possible, districts should preserve known communities of interest, which may be defined several ways. South Carolina defines such communities as those with similar political beliefs, voting behavior, geographic location, and economic, social, cultural, and historic influences. County boundaries, municipal boundaries, and precinct lines are also considered. Though not every district follows precinct or county lines, efforts are made to avoid cutting across such boundaries unless necessary to meet other redistricting requirements, such as population equality or compliance with the Voting Rights Act.

- *Incumbency Protection.* South Carolina law allows mapmakers to make reasonable efforts to ensure that incumbent legislators remain in their current districts and are not forced to run against other incumbent members of the state. This consideration is expressly prohibited in some states and is not without controversy.

The redistricting process relies on the assumption that state legislatures will draw district boundaries to create fair districts representative of their respective populations, and the associated redistricting laws aim to keep state legislatures honest so as to not violate this assumption. However, these laws still leave room for district boundaries to be manipulated in order to influence election outcomes, which can undermine the democratic process and dilute the votes of constituents.

1.2 How Gerrymandering Occurs

The main idea behind gerrymandering is straightforward: if one can draw districts that include certain constituents and exclude others, then one can increase the likelihood of obtaining more seats in the state legislature or federal House of Representatives for a given party.

In practice, gerrymandering takes two primary forms. Consider a community of interest that historically votes overwhelmingly for a particular party. The first tactic used to gerrymander, *cracking*, occurs when districts are drawn that spread members of the community out between several districts. Cracking dilutes constituents' voting power across multiple districts and constructs the map so the community's political party does not comprise a majority in any of the districts of which it is a member. The second tactic used in a gerrymander is *packing*, in which districts are drawn to cluster communities of interest into a single district or just a few districts. While the community's political party may overwhelmingly win the district or districts into which they are packed, they would win comparatively fewer seats overall than what might be considered their fair share.

Packing and cracking, used in conjunction with one another, sometimes result in violations of the compactness principle, but with the advances of map-making technology, it is entirely possible to draw districts that are in reality packed and cracked, but appear to be fairly reasonable shapes. There can be more to a voting map than meets the eye.

1.3 Gerrymandering Today

For much of its history, gerrymandering has taken the form of *racial* gerrymandering in an effort to dilute the voices of minority voters. The Voting Rights Act of 1965 was an important step in U.S. history in making redistricting a fairer process that creates election maps that are appropriate representations of their constituents. One example of the impact of the Voting Rights Act on redistricting is the provision in Section 5 that, until 2013, subjected certain jurisdictions to pass federal preclearance by the U.S Attorney General or Federal District Court for D.C. when implementing changes that affect voting, including the redrawing of election maps. Section 4(b) includes a "coverage formula" that outlines the criteria that trigger the preclearance provision in Section 5; these criteria, which South Carolina meets, include states that have historically enacted racially discriminatory voting requirements[2]. Preliminary research has estimated that preclearance requirements have had a positive effect on increasing minority voting participation and Congressional representation [17, 48].

Even with the provisions in the Voting Rights Act that still offer legal backing to contest racially discriminatory political districts, a more difficult-to-detect form of gerrymandering may creep into the map-making room. Increasingly, the focus of the legal landscape has shifted to partisan gerrymandering, and several accusations of election maps being drawn to disadvantage a given political party based on previous voting patterns have surfaced in state and federal courts. With increasing partisan polarization in the last decade, it is perhaps of little surprise that both Republican and Democratic legislatures have been accused of bringing partisan bias to the redistricting process in an attempt to gain seats. For example, the Wisconsin legislature was accused of a Republican gerrymander in *Gill v. Whitford*, while a case of Democratic gerrymandering arose in Maryland in *Benisek v. Lamone* [3, 4].

For several years, the U.S. Supreme Court remained undecided on how to address cases that presented claims of partisan gerrymandering and consistently redirected such cases back to lower courts. One challenge facing the courts was the amount of abstractness that can quickly arise in claims of partisan discrimination. This challenge began to prompt the question of how one can *objectively* analyze whether the voters of a given party are unfairly disadvantaged by a districting plan. When Pennsylvania brought claims of partisan gerrymandering to the courts for the first time in the 2004 case *Vieth v. Jubelirer*, the case was struck down on the basis that there is no Constitutional provision for courts to restrict election maps from being drawn with partisan intent. Still, Justice Anthony Kennedy acknowledged that "comprehensive and neutral principles for drawing electoral boundaries" did not yet exist, but these concrete principles could one day be discovered [14].

[2] It is of note that while the coverage formula held for the redistricting process through 2013, it was struck down as unconstitutional in the 5-4 case Supreme Court case *Shelby County v. Holder*, with the majority reasoning that the criteria are outdated and thus violate the principles of equal state sovereignty and federalism. While the Court did not strike down the preclearance requirement in Section 5, unless a new coverage formula is written, the provision in Section 5 is unenforceable [10].

This ruling was a source of hope for U.S. citizens involved in the fight against gerrymandering, and as a result of Justice Kennedy's statement, mathematicians, political scientists, and other interested parties have set out to explore what such standards might look like. Though elusive at first, several rigorous methods for evaluating the partisan bias of maps have since been developed, taking into account both existing redistricting principles and political demographics of voters. Cases in which these methods are utilized have often been stalled in the federal court, but some have met success at the state level, resulting in the drawing of fairer election maps.

Arguably one of the most successful partisan gerrymandering lawsuits is the *League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania* case (hereafter referred to as *LWV v. PA*), which was introduced to the Pennsylvania Supreme Court in 2017 and resolved the following year. Due to the combined efforts of mathematician Wes Pegden from Carnegie Mellon, political scientist Jowei Chen from University of Michigan, and political scientist Christopher Warshaw from George Washington University, several rigorous methods based in mathematical theory (described in Chapter 2) were utilized to demonstrate that the 2011 Pennsylvania Congressional map was drawn with partisan intent, resulting in extreme partisan bias towards Republicans and thus diluting the voting power of Democrats. The results of their analysis were so overwhelming, the plaintiffs argued, that they could *only* be the result of carefully drawn districts and could not have occurred simply due to the voter makeup of the state. The state court ruled in favor of the League of Women Voters, holding that the districting plan violated Article I, Section 5 of the Pennsylvania State Constitution, which requires that the state hold "free and fair elections." As a result, the districting plan was redrawn in time for the 2018 midterm elections [6].

In efforts to bring cases like *LWV v. PA* to the Federal Supreme Court, Justice Kennedy was considered to be a swing vote who might help along such efforts if clear and identifiable standards were produced. However, after the retirement of Judge Kennedy and the appointment of new Supreme Court Justice Brett Kavanaugh, the political landscape of the court shifted. A combined partisan gerrymandering case entitled *Rucho v. Common Cause*, which considered the claims made by plaintiffs in North Carolina and Maryland of a Republican and a Democratic gerrymander, respectively, was introduced in March 2019. Three months later, the Supreme Court released a 5-4 decision to remain uninvolved in partisan gerrymandering cases, stating that such claims present political questions beyond the reach of the federal courts. Chief Justice John Roberts clarified that the conclusion of the case neither "condone[d] excessive partisan gerrymandering" nor "condemn[ed] complaints about districting to echo into a void," but rather that the majority felt there are other methods of recourse available that are more appropriate than the intervention of federal courts [9]. Thus, partisan gerrymandering cases remain an issue for state supreme courts.

Fortunately, the fight to end partisan gerrymandering does not appear to be a lost cause. The premise of free and fair elections found in the Pennsylvania state constitution is also echoed in the constitutions of twenty-seven other states [57]. This could be used as a substantial argument against the practice of partisan gerrymandering. In addition, bills that would establish independent commissions to draw election maps every ten years have been proposed in many states; in the *Rucho* court opinion, Chief Justice Roberts even mentions examples of such commissions as a reasonable potential alternative to bring partisan gerrymandering claims to federal courts. Though not a foolproof solution, the passing of such bills would take the redistricting process out of the hands of legislators who may possess partisan intent and put it into the hands of a potentially more balanced group of individuals committed to upholding fair democracy rather than furthering political gain. Such commissions are already in place in six states, though their makeup varies [37]. In South Carolina, bills were introduced in 2017 to both the House and Senate with bipartisan support to form such a redistricting commission.

So despite the stepping back of the U.S. Supreme Court, hope still exists at the state level for partisan gerrymandering cases. However, the question of how to rigorously determine whether a map is biased towards a given political party and thus demonstrate a need for its redrawing remains.

1.4 Identifying a Gerrymander

While violating state and federal districting requirements is enough to give a reasonable suspicion that a map may be gerrymandered, it turns out that identifying a gerrymander with clear standards is a bit more complicated. For example, consider the principle of compactness. One can look at an election map and quickly identify districts that appear to be unusual with respect to the others, and the compactness of the districting plan can be measured using one of the traditional metrics, such as Polsby-Popper. But just how "unusual" does a districting have to be to be considered an extreme enough instance of gerrymandering by the state courts? In addition to the already complicated task of finding a rigorous enough metric to assess partisan bias, sometimes bias occurs naturally due to the political geography of a state. For example, Democratic voters tend to cluster in urban areas, diluting their voting power among more spread-out rural districts that tend to vote Republican [40]. How can one discern whether the bias of a map is due to a malicious manipulation of voting district boundaries or simply the natural bias that arises due to a state's voter makeup?

We thus turn to mathematics in such an endeavor. To evaluate whether an election map was *intentionally* drawn with the purpose of diluting a party's votes, one can consider looking at a large sample of possible maps that could be drawn for a given state, taking into account voter preferences, populations, county lines, and other factors. The map in question can then be compared against the others to evaluate how extreme it is with respect to the sample, thus assessing the likelihood of a gerrymander. This method has been utilized to evaluate several election maps in the last several years, including the Congressional districting plan that was redrawn as a result of *LWV v. PA*, and may be considered a reasonable way to approach the challenge of partisan gerrymandering.

In light of the aforementioned redistricting principles in Section 1.1, the next chapter presents a review of the current body of literature on evaluating districting plans, summarizing proposed measures of partisan bias and methods of outlier analysis that have been used in both academic literature and legal cases in the United States. A selection of the measures and methods discussed is then used in the following chapters to analyze the 2011 Congressional, State House, and State Senate election maps of South Carolina[3].

[3] As with the district maps drawn for Pennsylvania, the "2011 maps" for South Carolina refer collectively to the Congressional and legislative maps drawn during the redistricting cycle following the 2010 Census.

Chapter 2

Literature Review

As mathematicians and political scientists alike have set out in search of rigorous ways to detect partisan gerrymanders, the body of literature regarding possible methods has grown substantially, particularly over the last twenty years. This chapter reviews multiple methods that have been used to assess election maps for partisan bias in existing literature, summarizing their advantages and disadvantages, conditions under which they break down, and their usefulness in detecting gerrymandered maps as discussed by the academic community.

It is of note that, in the literature discussed, a two-party system of Democrats and Republicans is assumed for simplicity. For most states, in which the proportion of votes cast for third-party candidates is small, this is a reasonable assumption. It is often helpful in practice to define a reference party in the definition of a metric; in the literature reviewed, this has been observed to typically be the party that is supposed to be disadvantaged by a suspected gerrymander. For purposes of this paper, when it is helpful to utilize a reference party, we allow the Democratic party to serve as this party of interest. For example, if discussing an average district vote share of 0.45, we are describing a scenario in which Democratic candidates received an average of 45% of the votes cast in each district. Also note that the variables used in this chapter may be adjusted from the original works of literature in which they are defined for consistency throughout this document.

Katz et al. are credited for their significant work in bringing together much of the existing literature on partisan fairness in their 2019 paper for *American Political Science Review* [33]. In addition to drawing from other sources, the first section of this chapter emphasizes many pieces of their work most closely related to the goals of this thesis.

2.1 The Seats-Votes Curve

The first step in conducting a thorough analysis of an election map's "fairness," which is often used as a rather arbitrary designation, is to formally define a standard by which fairness will be evaluated. This leads to the identification of several metrics that adhere to this standard and quantify the degree of partisan bias in a given map. This chapter begins with a discussion of a theoretical graph that lies at the heart of many such bias metrics and a formal definition of what is meant by partisan bias.

2.1.1 Definition

In Katz et al.'s evaluation of partisan fairness measures, they define a state to have the following three underlying components [33]:

- A *populace* $\mathbb{P}$, which represents the set of all individuals living in a state, as well as their voting behavior;
- An *electoral system* $\mathbb{E}$, which describes the system by which individuals in the populace have their votes turned into seats and includes the district boundary lines that are of interest when discussing an election map; and
- All *measured exogeneous influences* X on voter behavior, which includes factors such as demographic variables, geographic location, and whether or not there are incumbents in a particular election.

For a state with L voting districts as defined by $\mathbb{E}$, the random variables $v_1, \ldots, v_L$ represent the district-level proportions of reference party votes in each district and follow a joint probability distribution with density $p(v_1, \ldots, v_L|X)$. Recalling that the function of a random variable is itself a random variable, $v_1, \ldots, v_L$ can be used to define two new random variables that represent two quantities observed in any given statewide election: the *statewide average district vote*, formally defined as

$$V = V(v_1, \ldots, v_L) = \frac{1}{L}\sum_{d=1}^{L} v_d, \tag{2.1}$$

and the *statewide seat proportion*, stated as

$$S = S(v_1, \ldots, v_L) = \frac{1}{L} \sum_{d=1}^{L} \mathbb{1}(v_d > 0.5), \tag{2.2}$$

where $\mathbb{1}$ is the indicator function. As Katz et al. point out, the electoral system $\mathbb{E}$ is crucial in the understanding of these two random variables, as the effect of redistricting reflected in $\mathbb{E}$ dictates how the boundary lines are drawn and can result in two sets of district-level votes that have the same average district vote share V but different values of the seat share S.

These random variables provide the basis for the *seats-votes function*, denoted $S(V)$ and defined as the expected value of the statewide seat proportion for a given statewide vote V:

$$S(V) = E_p\left[S|X, V\right] = \int_0^1 \cdots \int_0^1 S(v_1, \ldots, v_L) \cdot p(v_1, \ldots, v_L|V, X) dv_1 \cdots dv_L. \tag{2.3}$$

In order for this function to be coherent and single-valued, an assumption that Katz et al. refer to as the Stable Electoral System Assumption is made. This assumptions states that the joint probability density p is defined independently of any one set of observed district vote proportions $v_1^O, \ldots, v_L^O$ that occur in a given election, which allows the *seats-votes curve* to then be defined as the set of all possible values of $S(V)$ for the given $\mathbb{P}$, $\mathbb{E}$, and X, denoted

$$\mathcal{S} = \left\{S(V) \ : \ V \in [0, 1]\right\}. \tag{2.4}$$

2.1.2 Partisan Symmetry and Bias

The seats-votes relationship gives rise to the notion of *partisan symmetry*, which King and Browning characterize as an absence of partisan bias [35]. Katz et al. formally define partisan symmetry in terms of the seats-votes curve, achieved if an electoral system $\mathbb{E}$ satisfies

$$S(V) = 1 - S(1 - V) \text{ for all } V \in [0, 1]. \tag{2.5}$$

The intuition behind the partisan symmetry standard is that neither party should have an advantage over the other when it comes to turning votes into seats. In other words, in a symmetric electoral system, we would expect an average district vote of V to win a proportion of S seats, regardless of which party achieved the average district vote of V. Partisan symmetry reasonably formalizes the idea of partisan "fairness," which Nagle states should be "a primary outcome of redistricting" [41]. McDonald and Best emphasize that symmetry serves as a good "reality-based benchmark," as it does not require one to choose between competing political ideologies to determine what is considered fair [40]. Even where not explicitly stated, these sentiments are echoed by many authors in the literature regarding the evaluation of election maps, and thus partisan symmetry is assumed as the standard by which to evaluate the fairness of a given map in this paper.

When partisan bias is present, asymmetry is introduced into the seats-votes relationship [35]. In general, this bias can be viewed as a deviation from partisan symmetry [29]. Katz et al. can again be credited with formally defining *partisan bias* in terms of the seats-votes curve as a random variable that captures the deviation from partisan symmetry, stated as

$$\beta(V) = \frac{S(V) - [1 - S(1 - V)]}{2}. \tag{2.6}$$

Values of this random variable can be thought of as the distance from *each* party to symmetry, given a certain statewide average district vote V. In practice, $\beta(V)$ represents the proportion of seats that would have to be taken away from the reference party and given to the opposing party for the system to be symmetric with respect to partisanship, with positive values of $\beta(V)$ indicating an advantage for Democrats and negative values of $\beta(V)$ indicating an advantage for Republicans. For example, if the bias of a particular map with a statewide average district vote of $V = 0.48$ is $\beta(0.48) = -0.12$, this indicates that 12% of the seats in the legislature would need to be shifted from Republicans to Democrats in order to achieve partisan symmetry.

Partisan bias is sometimes summarized at a point; for instance, we could summarize the bias of a map when each party receives an average of 50% of the votes in each district, $\beta(0.5)$. This summary is only useful, however, if it represents the map's bias for *all* empirically likely values of V. If this is not the case, a summary measure could lead one to believe that an election map is fair when it is actually biased at another point that could occur in a real election. In other words, even if $\beta(0.5) = 0$, it does not imply that the districting plan is unbiased for all reasonable statewide vote proportions; it could be biased at $\beta(0.45)$, for instance, where $V = 0.45$ is a statewide vote proportion that could possibly occur under the given plan. It may be desirable, then, to estimate the bias random variable $\beta(V)$ at all possible values of V using the seats-votes curve.

2.1.3 Estimating the Seats-Votes Curve

The seats-votes curve is purely theoretical at this point, given that the joint density of $v_1, ..., v_L$ is unknown and quite challenging to estimate. Only a small sample of election results may be observed under a given electoral system; for example, only five Congressional races occur under a particular districting before the next census is taken and the maps are redrawn. As a result, the joint distribution of district vote proportions is not easily observed through voting data, so the seats-votes curve must be estimated using other methods.

Method 1: Functional Forms.

One method of estimating the seats-votes curve is to specify a class of parametric functional forms that model the seats-votes relationship and estimate the parameters of a chosen form using observed election data. For example, Tufte expounds on both a linear model, used as early as the 1950s by Robert Dahl, and a more statistically "graceful" logit model [52]. Each functional form includes parameters related to two distinct features of an electoral system: (1) its partisan bias and (2) its degree of *electoral responsiveness*, also referred to as *competitiveness* in several texts, which captures how quickly a party's proportion of seats $S(V)$ changes with shifts in the statewide vote V. King and Browning credit Tufte as probably the first to include bias and competitiveness in one equation but as independent parameters, a strength also cited by Nagle [35, 41].

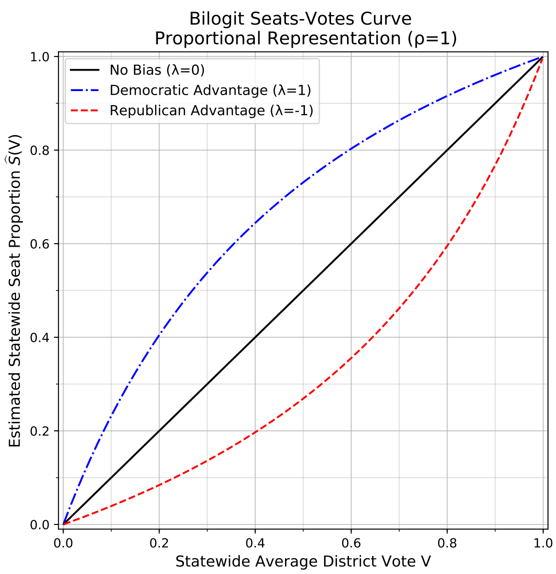
Another functional form introduced by King and Browning that has been increasingly used by scholars such as Nagle is the *bilogit* model [35]. Stated in terms of the seats-votes curve, this bilogit form can be written as

$$S(V) = \frac{1}{1 + \exp\left\{-\lambda - \rho \ln\left(\frac{V}{1-V}\right)\right\}}, \tag{2.7}$$

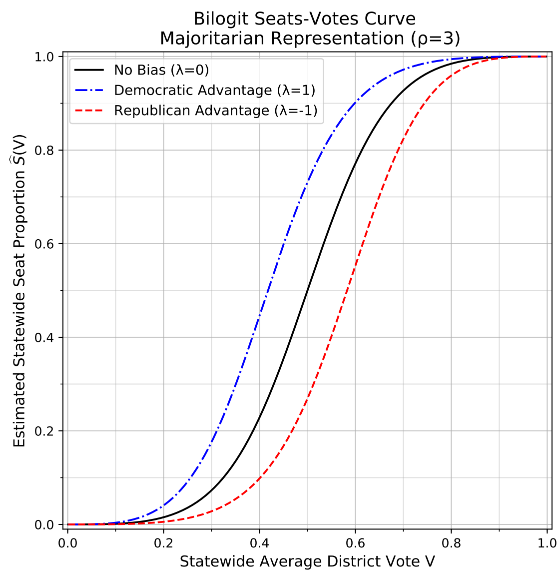
where λ is the bias parameter and ρ is the competitiveness or responsiveness parameter. Though not identical, the bias parameter can be interpreted in a similar manner to $\beta(V)$ defined in Equation 2.6, with $\lambda < 0$ representing a Republican advantage, $\lambda > 0$ representing a Democratic advantage, and $\lambda = 0$ corresponding to an unbiased system[1]. The competitiveness parameter, on the other hand,

[1]The variable λ used for the partisan bias parameter in this thesis corresponds with King's usage. In his 1989 paper, he defines λ in this manner, noting its equivalence to $\ln(\beta)$ used in his 1987 paper with Browning, where β is a quantity unrelated to the partisan bias random variable $\beta(V)$ presented in Equation 2.6 [34, 35]. This λ is also equivalent to α_0 used by Katz et al. [33]. Note that Nagle defines λ in the opposite manner, with positive values representing a Republican advantage, which alters Equation 2.7 to instead be $\left(1 + \exp\left\{\lambda - \rho \ln\left(\frac{V}{1-V}\right)\right\}\right)^{-1}$ [41].

changes with the type of democratic representation that exists in an electoral system. An electoral system with perfect *proportional representation* would have a competitiveness parameter of $\rho = 1$; however, this is typically infeasible in a district-based electoral system, a fact that was cited in *Rucho v. Common Cause* [9]. A more reasonable range of plausible values consistent with what has been observed empirically in U.S. elections would be $1 < \rho < \infty$, which represents a *majoritarian* electoral system [33, 34]. For example, the bilogit model may use a parameter value of $\rho = 3$ to correspond with the *cube law* referenced by Tufte [52].



(a) Proportional Representation



(b) Majoritarian Representation

Figure 2.1: Two examples of theoretical seats-votes curves drawn according to the bilogit model.

In practice, the parameters λ and ρ can be estimated using data from an observed election outcome either by performing a direct fit to the data or by using the max-likelihood method of estimation [41]. King and Browning argue that, not only does the bilogit model hold advantages over the linear model in that it can model extreme values of V and $S(V)$, but it also is more realistic than the logit model presented in Tufte's work, allowing for the range of all possible degrees of partisan bias and forms of democratic representation [35].

A major limitation of using functional forms to estimate the seats-votes curve is the fact that the parameters must be estimated with very few available observations. In the case of Congressional and state legislative districts, the maximum amount of available data would be five elections occurring under the same districting plan, and this small sample results in a rather uncertain model. In addition, if the purpose of evaluating the bias of a map is to investigate whether it ought to be redrawn to be fair for future elections, waiting until five elections have occurred under the redistricting plan is an impractical solution. For this reason, Katz et al. view these functional forms as more appropriate for studying broad patterns in electoral systems than for the practical evaluation of individual electoral maps. This method does, however, provide a way of visually understanding the theoretical seats-votes curve and how it changes based on the bias and competitiveness of a given districting.

Method 2: Uniform Partisan Swing Assumption.

Because the election data available for a given districting plan is so limited, additional assumptions must be in place to estimate the seats-votes curve if not using functional forms. One such assumption is that of *uniform partisan swing* between elections. Katz et al. note that when shifts in the average district vote proportion V occur, they are typically observed in two ways: as an absolute average partisan swing affecting almost all districts in a state, which can be volatile and difficult to predict, and as the relative positions of district votes for a particular election, which tend to be highly stable over time and are valuable for evaluating districting plans. The assumption is formally stated as follows:

Uniform Partisan Swing Assumption. When the average district vote swings between elections under the same electoral system $\mathbb{E}$ from V to V', every district vote proportion moves uniformly by $\delta = V' - V$, so that $\{v_1, \ldots, v_L\}$ from one election becomes $\{v_1 + \delta, \ldots, v_L + \delta\}$ in the next, with elements truncated to $[0, 1]$ if necessary. [33]

This approach allows one to estimate the seats-votes curve using a single observed election under a given electoral system (i.e., a particular redistricting plan), along with its average district vote proportion $V^O = V(v_1^O, \ldots, v_L^O)$ and an estimate of the statewide seat proportion $S(V^O)$, which is denoted as

$$\widehat{S}(V^O) = \frac{1}{L} \sum_{d=1}^{L} \mathbb{1}(v_d^O > 0.5). \tag{2.8}$$

Fixed values of partisan swing δ can then be chosen to estimate the new statewide district vote and seat proportions for hypothetical elections under the same electoral system. A useful application of this assumption would be to choose multiple values for δ in order to estimate several points along the seats-votes curve. In practice, the swing values can each be chosen to be the partisan shift needed for a single seat to switch from one party to another. The algorithm for constructing the estimated curve in this manner can be summarized as follows:

1. Observe an election outcome $\{v_1^O, \ldots, v_L^O\}$ with statewide average district vote V^O and estimated statewide seat proportion $\widehat{S}(V^O)$. Plot $\left(V^O, \widehat{S}(V^O)\right)$ on the curve.

2. For each v_i^O, $i = 1, \ldots, L$:

   (a) Choose the minimum swing δ_i such that seat i switches parties[2].

   - If $v_i^O < 0.5$ (Republican seat), choose the minimum δ_i such that $v_i^O + \delta_i > 0.5$.
   - If $v_i^O > 0.5$ (Democrat seat), choose the minimum δ_i such that $v_i^O + \delta_i < 0.5$.

   (b) Apply the partisan swing δ_i uniformly to each district to obtain the new district vote proportions $\{u_1, \ldots, u_L\}$, where for $d = 1, \ldots L$,

$$u_d = \begin{cases} 0 & \text{if } v_d^O + \delta_i < 0, \\ 1 & \text{if } v_d^O + \delta_i > 1, \text{ and} \\ v_d^O + \delta_i & \text{otherwise.} \end{cases}$$

   (c) Calculate the new statewide average district vote $V_i' = \frac{1}{L} \sum_{d=1}^{L} u_d$.

   (d) Calculate the new estimated statewide seat proportion $\widehat{S}(V_i') = \frac{1}{L} \sum_{d=1}^{L} \mathbb{1}(u_d > 0.5)$.

   (e) Plot the point $\left(V_i', \widehat{S}(V_i')\right)$ on the estimated seats-votes curve.

The idea of a uniform partisan swing occurring in a state can be traced back to Butler in his analysis of the British General Election of 1950 [19]. Although variations in partisan shift between the elections of 1945 and 1950 did occur in some areas of the country, the partisan swing was rather consistent across regions overall. Butler attributed some of the variations to uncontested candidates in certain regions and others to the effect of votes cast for third-party candidates. However, he observed very few instances of deviations from the national average.

[2] In this theoretical context, exact ties are not considered; in existing literature, this assumption is either implied or stated explicitly, as it is it Katz et al. where the authors assume that each district has an odd number of voters [33]. Indeed, exact ties in district-level elections are certainly rare events, yet they are still plausible and must be considered when dealing with actual election data. An explanation of how ties are handled in this analysis is included in Subsection 4.2.1.

The reasonableness of the uniform partisan swing approach to estimating the seats-votes curve is furthered by Katz et al., who conducted a study on the empirical accuracy of predictions from the seats-votes curve under the uniform partisan swing assumption. A criticism offered by Nagle is that the assumption is unrealistic, as it assumes, for instance, that the same number of Democrats will shift in a district with few Democratic voters as will in a district with many Democratic voters, echoing King's supposition that the curve estimated this way is most sensible in the competitive region near $V = 0.5$ but less so for extreme outcomes [34, 41]. Nagle and King also emphasize that the assumption allows for scenarios in which one or more estimated district-level vote proportions could fall below 0 or above 1, though Katz et al. handle this in their formal statement of the assumption by simply truncating such values to be within the bounds. However, the study conducted by Katz et al. using a large sample of pairs of successive elections under the same districting plan would suggest that these criticisms need not necessarily be a concern. Upon analyzing the out-of-sample error rate for the statewide seat proportion estimates, they found that the predictions afforded by this estimated seats-votes curve were remarkably accurate and did not seem to break down for larger swings, implying that uniform partisan swing is relatively fixed for elections [33].

Another limitation of the uniform partisan swing assumption as stated by multiple authors is that the model is deterministic rather than stochastic, which would be more reasonable when working with real election data [33, 34]. Gelman and King propose such a model that they suggest provides "substantially improved ... estimates" for the seats-votes relationship in a two-party electoral system [29]. The uniform partisan swing model, however, is a reasonable first approximation.

Method 3: Variable Partisan Swing Assumption.

As an alternative to the uniform partisan swing assumption, Nagle suggests another reasonable method of approximating the seats-votes curve which we designate the *variable partisan swing assumption*. Rather than supposing that a statewide partisan shift equally affects all districts, this approach assumes that this shift has an equal *probability* of applying to any individual voter in any district, allowing for the degree of partisan swing to vary for each district [41]. Under this assumption, Nagle shows that we can expect a seat for district d currently won with a proportion of v_d votes to be lost if the average district vote decreases to $\frac{V}{2v_d}$. This leads to the algorithm for estimating points along this alternate seats-votes curve, which can be summarized as follows:

1. Observe an election outcome $\{v_1^O, \ldots, v_L^O\}$ with a statewide average district vote V^O.

2. For each v_d^O, $d = 1, \ldots, L$, find the new statewide average district vote V_d' that will cause that seat to be lost[3] to the opposite party:

   - If $v_d^O < 0.5$, the new statewide average district vote is $V_d' = 1 - \dfrac{(1 - V^O)}{2(1 - v_d^O)}$.
   - If $v_d^O > 0.5$, the new statewide average district vote is $V_d' = \dfrac{V^O}{2v_d^O}$.

3. Sort the $L+1$ statewide average district vote proportions $\{V^O, V_1', \ldots, V_L'\}$ in order from least to greatest.

4. Denote the ordered average district vote proportions as $V_{(i)}'$, $i = 0, \ldots, L$. For each i, plot $\left(V_{(i)}', \frac{i}{L}\right)$ on the estimated seats-votes curve.

It is of note that constructing the estimated seats-votes curve under the variable partisan swing assumption results in a curve that is very similar to that under the uniform partisan swing assumption in the $0.4 < V < 0.6$ region, but typically deviates in the tails of the curve. Further, the seats-votes curve estimated under this assumption does not have the issue of theoretical values of V falling outside the [0,1] range and does not rely on truncation. While slightly different in their implementation, both this assumption and the uniform partisan swing assumption provide valid approaches to the challenge of assessing fair maps, and considering seats-votes curves constructed under both assumptions adds to the rigor of this analysis.

The seats-votes curve lies at the heart of many partisan bias measures, as it provides a way of both defining and visualizing partisan symmetry, which we can equate with our intuitive notion of the "fairness" of an election map. Having an understanding of this curve and multiple methods of estimating it using election data lays the groundwork for calculating measures of partisan bias. Thus, the selection of bias measures used in the Chapter 4 analysis are considered under estimated seats-votes curves using both Methods 2 and 3.

[3] Again, note that ties are not considered in this algorithm. Rather, the assumptions described by Nagle suppose that a seat d held by Democrats would be lost to Republicans if the vote proportion in that district falls to $v_d^O = 0.5$, which implies an odd number of voters. Subsection 4.2.1 describes how ties may be handled in practice.

2.2 Measures of Partisan Bias

To analyze an election map, it is essential to have some way of measuring its degree of partisan bias (i.e., its degree of partisan asymmetry) through a robust and comprehensible metric. Throughout the literature available on the topic, several measures appear quite often and are widely accepted by scholars as reasonable approaches — some of which have been utilized for over a century — while others have been more recently proposed and lack widespread usage. Most approaches to assessing the likelihood of gerrymandering do have one thing in common: they almost always use more than one metric to assess the bias of a map. Referring to this tendency to conduct districting plan analyses using a variety of metrics, Nagle comments that while a single superior measure would be convenient, "complex social problems often do not yield to simple solutions" and that a comparative practice of evaluating a map using several bias measures is rigorous and consistent with approaches used to evaluate other features of election maps, such as compactness [41].

Again, for purposes of this thesis, partisan symmetry is used as the standard for the fairness of an election map. That being said, any reasonable measure of partisan bias indicating some degree of unfairness should correspond with nonzero values of the bias random variable $\beta(V)$, representing a deviation from the ideal symmetry we wish to see in the seats-votes relationship for a fair election system. Several such methods are discussed in this section, some of which use the estimated seats-votes curve directly, and some which use only values from an observed election. Measures that are commonly used to assess election maps but are not indicators of asymmetry are also briefly discussed.

2.2.1 Bias Summary from the Seats-Votes Curve

The advantage of estimating the entire seats-votes curve by utilizing one of the methods described in Subsection 2.1.3 is that it allows us to easily obtain a point estimate of $\beta(V)$ for any statewide vote proportion V. Specifically, what we call the *bias point estimate* can be found using

$$\hat{\beta}(V) = \frac{\widehat{S}(V) - \left[1 - \widehat{S}(1-V)\right]}{2} \tag{2.9}$$

by simply reading the values of $\widehat{S}(V)$ and $\widehat{S}(1-V)$ off the constructed curve. The seats-votes curve allows us to observe how this deviation from partisan symmetry changes with shifts in V, which is notable because the direction of partisan bias can switch between parties for different average district vote proportions [33]. This is a more robust approach than utilizing a single summary measure of $\hat{\beta}(V)$ at a particular V, helping mitigate the potential problem of judging a map to be fair based solely on the summary as presented in Subsection 2.1.2. However, summary measures may still be used as an initial indicator of bias; for example, it may be of interest to summarize the bias of a particular election map at the observed statewide average district vote using the bias point estimate $\hat{\beta}(V^O)$. Zero values may not tell the whole story, but nonzero values can be clear indicators of bias for observed election outcomes.

One limitation of using this measure is that, for moderate values of $\hat{\beta}(V)$ that are closer to zero, the bias point estimate may be more meaningful for states with larger numbers of districts. Recall that it represents the (possibly negative) proportion of seats that would have to be taken from the reference party and given to the opposition party for the system to be symmetric. If we consider a map with an observed statewide average vote of 0.48, a bias point estimate value of $\hat{\beta}(0.48) = -0.12$ will hold stronger implications for an election map with 18 districts as opposed to a map with five. In the former, this measure of $\hat{\beta}(V)$ indicates that $0.12(18) = 2.16 \approx 2$ seats would have been won by Democrats in a symmetric election, while in the latter, the measure indicates that $0.12(5) = 0.6$ seats would have been won by Democrats, which does not represent an entire seat. In this regard, the election map with five districts might be judged to be relatively fair, while the one with more districts might be considered biased. Still, keeping its limitations in mind, the bias point estimate can provide a good initial metric to characterize the fairness of an election map.

2.2.2 Median-Mean

The *median-mean* metric (also called *symmetry vote bias* by McDonald and Best) is a metric with origins traced as early as the nineteenth century and with applications to gerrymandering found as early as the mid 1950s [19, 25, 40]. It is a straightforward, vote-denominated calculation of bias found by subtracting the observed average district vote proportion V^O from the median district vote share M:

$$\mathrm{MM} = M - V^O. \tag{2.10}$$

The median-mean metric characterizes the fairness of an election map by summarizing the skewness of the district vote distribution, with nonzero values corresponding to some degree of asymmetry in the seats-votes relationship. When the Democratic party is the reference party, positive values of MM represent an advantage for Democrats and negative quantities represent an advantage for Republicans. Chikina et al. provide an intuition as to the interpretation of the metric: if the median $M = 0.5$, this means that Democrats are winning half the seats, and if the mean V^O is much smaller than the median (thus making MM > 0), it means they are winning half the seats despite having a small minority of the total votes [22].

The median-mean measure is a popular metric for assessing districting bias due to its computational simplicity and easy-to-understand interpretation, described by McDonald and Best as a "manageable standard" and "leading indicator of potential unfairness" [40]. Its usefulness also stems from the fact that it is computed using two observable quantities from the election map rather than quantities obtained from an estimated seats-votes curve, a criticism appropriately made by Stephanopoulos and McGhee [50]. Katz et al. prove that MM is indeed a reliable measure of partisan symmetry in light of the theoretical seats-votes curve, demonstrating that MM $= 0$ if and only if $\hat{\beta}(0.5) = 0$, where $\hat{\beta}(0.5)$ is the bias point estimate at $V = 0.5$ from the estimated seats-votes curve [33]. The limitation of median-mean, however, is that it is a useful *indicator* for whether the true bias summary at $V = 0.5$ is zero, but *not* a general measure of partisan bias for the entire electoral system. Recall that a summary bias measure of zero does not imply that the map is fair for all V; thus, a median-mean value of zero could overlook bias occurring at another statewide average district vote that may be observed in a real election under the given districting plan.

Despite its limitations, median-mean is a useful measure of bias; although zero values do not necessarily imply that a map is truly unbiased, nonzero values point to asymmetry. The metric has been used in several analyses of districting plans, including that of the 2011 Pennsylvania Congressional Map made by Chikina et al. and presented by Pegden in *LWV v. PA* [22, 43].

2.2.3 A Geometric Measure of Bias

Nagle offers an alternative approach to quantifying partisan bias utilizing a measure that he denotes B_G, referred to in this paper as *geometric partisan bias.* While an intuitive definition of partisan symmetry is established in Subsection 2.1.2 — namely, if one party receives a proportion of seats $S(V)$ with an average district vote proportion of V, then so should the opposite party — Nagle's approach relates this to graphical symmetry. In a symmetric seats-votes curve, if a point exists on its graph at $(0.5 + x, 0.5 + y)$, then there should be another point on the curve at $(0.5 - x, 0.5 - y)$, a quality referred to as *inversion symmetry* about the (0.5, 0.5) midpoint of the seats-votes graphical space [41]. A calculation of geometric partisan bias is found by inverting the estimated seats-votes curve about this midpoint and then finding the area in between the original curve and the inverted one. The algorithm can be summarized in a few steps:

1. Graph the estimated seats-votes curve $\widehat{S}(V)$ using a selected method[4] from Subsection 2.1.3.
2. Graph the inverted seats-votes curve $\widehat{S}^{-1}(V)$ by mapping each point $\left(V,\ \widehat{S}(V)\right)$ on the original curve to a point $\left(1-V,\ 1-\widehat{S}(V)\right)$ on the new curve.
3. Calculate the area between $\widehat{S}(V)$ and $\widehat{S}^{-1}(V)$:

$$B_G = \int_0^1 \left|\widehat{S}(v) - \widehat{S}^{-1}(v)\right| dv. \tag{2.11}$$

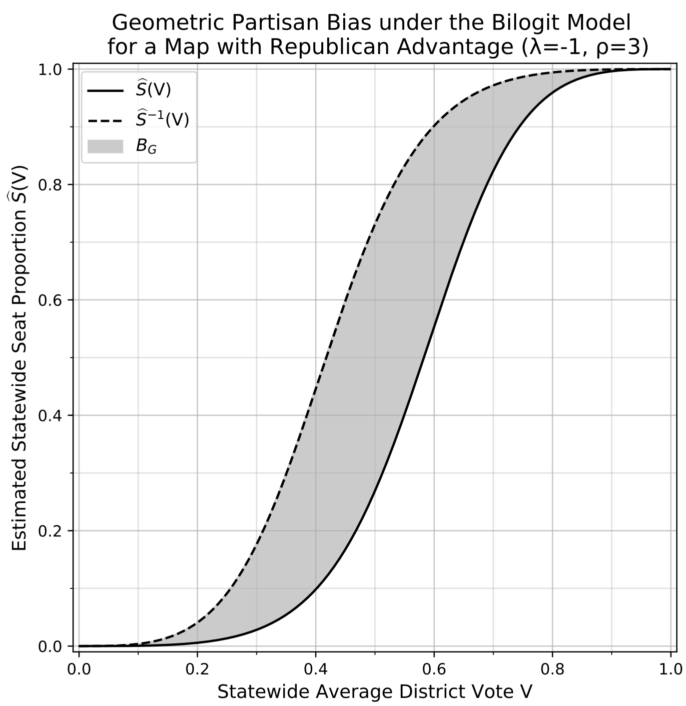


Figure 2.2: An example demonstrating the geometric measure of partisan bias B_G for a seats-votes curve estimated using the bilogit functional form.

[4]In Nagle's original development of geometric bias measure, he computes it using the seats-votes curve drawn under the variable partisan swing assumption.

Because the domain and range of both the estimated seats-votes curve and its inverse represent proportions and are thus always between 0 and 1, this geometric bias measure can be used to compare the election maps for states with different numbers of seats[5]. An additional strength of this measure is that it can reasonably include the tails of the seats-votes curves, where noncompetitive and packed districts that may be undetected by the median-mean metric may be present [41].

Note that, unlike the bias point estimate and median-mean measures, the *direction* of the bias in an electoral system cannot be determined by the sign of the geometric bias measure, which is always positive. To determine the direction of the bias, one must compare the value of $\widehat{S}(0.5)$ from the estimated seats-votes curve to the midpoint of the graph; if $\widehat{S}(0.5) < 0.5$, B_G captures a Republican advantage, and if $\widehat{S}(0.5) > 0.5$, B_G represents a Democratic advantage.

2.2.4 Measures That Do Not Capture Partisan Symmetry

The metrics discussed in this literature review are hardly the only ones that have been used to measure partisan bias in the literature surrounding the mathematical detection of partisan gerrymandering. However, when characterizing the absence of bias using the partisan symmetry standard for fairness, the discussed measures can accurately quantify deviation from this symmetry in ways that others fall short.

For example, consider the *efficiency gap* introduced by Stephanopoulos and McGhee, which is sometimes used to assess districting plans and is occasionally confused for a measure of partisan symmetry [50]. In this context, *wasted votes* are defined as the sum of (1) the number of votes ℓ cast for a losing candidate and (2) the number of votes w cast for a winning candidate above the threshold of 50% plus one vote. The efficiency gap EG then calculates the disparity in how "efficient" each party's votes are in electing candidates of their choice by finding the difference between the losing and winning parties' wasted votes and dividing by the total number of votes, which we denote to be N. In other words, if Party A is the losing party and Party B is the winning party, then

$$\mathrm{EG} = \frac{(\ell_A + w_A) - (\ell_B + w_B)}{N}. \tag{2.12}$$

[5] The seats-votes graphs constructed by Nagle show the *number* of seats on the vertical axis rather than the *proportion* of seats. In order to compare B_G for state maps with different numbers of districts using his graphs, one must divide his geometric bias measure by the number of seats. The estimated proportion of seats $\widehat{S}(V)$ is used here both for consistency and for more readily comparable measures without this additional step.

When EG > 0, the measure indicates that Party A wasted more votes than Party B and thus the electoral system is in favor of the latter. In a "fair system," one would expect this gap to be small, implying that wasted votes are equally divided between the two parties. Like median-mean, the efficiency gap is valuable because it is based solely on observed election outcomes and does not require estimates from the theoretical seats-votes curve. It also conveys an interesting aspect of an electoral system and can be helpful in assessing the likelihood of gerrymandering, as a districting plan that intentionally packs and cracks its voters often results in a large number of wasted votes for the disadvantaged party in comparison to the party advantaged by the gerrymander. However, Stephanopolous and McGhee themselves specify that the efficiency gap is a quite different metric than partisan bias, and Katz et al. demonstrate that it fails as a measure of partisan symmetry because it does not correspond to a single-valued seats-votes function. Katz et al. can be credited with identifying several such measures that do indeed quantify various aspects of an electoral system but do not reflect partisan symmetry, including *declination* and *lopsided outcomes* [33]. While worthy of further study, they do not pertain to the primary objective of this paper.

Another metric that has been used to assess partisan bias, albeit sometimes informally, is the *deviation from proportional representation*,

$$\mathrm{PRD}(V^O) = \widehat{S}(V^O) - V^O. \tag{2.13}$$

This measure follows the intuitive idea that a party that secures $P\%$ of the statewide votes should receive roughly $P\%$ of the seats. The idea is often reflected in gerrymandering claims that view a map as more suspect the further away from proportional representation its election outcomes lie. The topic of proportional representation has been of some discussion in the political arena, most recently surfacing in the court case *Rucho v. Common Cause.* The court produced the opinion that the Equal Protection Clause of the Fourteenth Amendment does not grant a Constitutional right to proportional representation [9]. Despite any controversy on the court opinion, it is worth noting that this measure is not quite as rigorous as the other bias measures in identifying partisan unfairness. Nagle posits that deviation from proportional representation is a strong diagnostic when the proportion of votes won is greater than 0.5 but the proportion of seats won is less than 0.5; however, if the proportions of votes and seats won are *both* less than 0.5, the measure is insufficient [41]. In addition, King states that proportional representation may not even be the ideal for an electoral system, as it does not allow for higher levels of competitiveness between districts [34]. While it is easy to understand, Katz et al. demonstrate that it also fails as a general measure of partisan symmetry [33]. This measure produces a coherent seats-votes curve and is a special case of partisan symmetry when $\text{PRD}(V^O) = 0$; however, nonzero values of $\text{PRD}(V^O)$ can still occur even when the partisan bias measure $\beta(V) = 0$. It is thus an insufficient measure of partisan symmetry, which is used as the standard for fairness throughout this thesis.

2.2.5 A Note on Measures of Competitiveness

Another aspect of election maps discussed throughout the literature is the idea of *competitiveness*. Scholars have not reached a consensus on whether more or less competitiveness is preferable in an attempt to achieve partisan fairness across election districts; however, it is important to recognize that competitiveness, which also characterized as responsiveness or representation such as in the parameter ρ defined in Subsection 2.1.3, is a distinct measure from partisan bias [35]. Competitiveness is certainly related to partisan bias in that districts with large numbers of packed and cracked voters, which are typically biased, will exhibit less competitiveness than they might if they were drawn fairly. Despite this, Nagle points out that maximizing competitiveness alone is not necessarily an obvious way to achieve fair districting, and it is even less so a diagnostic for assessing gerrymandering on its own [41]. For example, South Carolina requires a certain level of incumbency protection that prevents state legislators running for reelection from being intentionally written out of their own districts, which results in districts with a somewhat lower competitiveness measure [26]. On the other hand, the League of Women Voters holds a position on redistricting policy that explicitly rejects incumbency protection [36]. There is thus a lack of agreement on whether the protection of incumbents is actually harmful or not in the drawing of fair election maps. As a result, a measure that indicates lack of competitiveness is not necessarily a diagnostic for partisan bias.

Analyzing competitiveness of districting plans can, however, be useful when paired alongside another metric that assesses partisan symmetry. For example, Chikina et al. use the variance of the district vote proportions alongside the median-mean metric to assess the competitiveness of the 2011 Pennsylvania Congressional maps, with variance defined in context as

$$\textsc{Var} = \frac{\sum (v_d^O)^2 - \frac{\left(\sum v_d^O\right)^2}{L}}{L} = \frac{1}{L}\sum_{d=1}^{L} \left(v_d^O\right)^2 - \left(V^O\right)^2. \tag{2.14}$$

High values of variance represent anti-competitive districting plans and low values represent competitive plans [22]. The authors are careful to point out, however, that high-variance districtings can exist which do not favor either party, so the variance measure of competitiveness is not a rigorous enough diagnostic to be utilized on its own.

2.2.6 Bias Measures in Summary

As demonstrated in this section, there are a variety of measures that appear in the literature to assess various aspects of gerrymandered election maps. However, only some of them assess partisan bias as it relates to the definition of partisan symmetry, which is considered to be the standard definition of fairness for purposes of this thesis. A summary of the partisan bias measures that are utilized in the analysis of the South Carolina election maps (see Chapter 4) is presented in Table 2.1 for reference.

Measure	**Symbol**	**Meaning**
Median-Mean	MM	MM < 0 indicates Republican advantage
Bias Point Estimate	$\hat{\beta}(V)$	$\hat{\beta}(V) < 0$ indicates Republican advantage
Geometric Bias	B_G	$\widehat{S}(0.5) < 0.5$ and $0 < B_G \leq 1$ indicate Republican advantage

Table 2.1: A summary of the partisan bias measures considered for the analysis presented in Chapter 4, with the Democratic party serving as the reference party.

While the median-mean metric has been used extensively in the literature, the bias point estimate and geometric bias measures were developed much more recently and are thus not widely used. For this reason, the work in this thesis is presented to shed light on how the latter two measures perform as indicators of partisan bias compared to the more commonly understood metric provided by MM when evaluated on the same map. In addition, we seek to understand how the values of such measures compare between gerrymandered versus fair maps, as exemplified by the 2011 and 2018 Pennsylvania redistricting plans, respectively. Understanding the similarities and differences between these various measures of partisan fairness is crucial in order to maintain transparency about how they can be applied to evaluate election maps.

2.3 Outlier Analysis

Once one or several measures of partisan bias are chosen for analysis of an election map, the question remains of whether or not the statistic obtained is considered "extreme" enough to be considered too unfair. This question is further complicated by the observation that election maps will often generate some level of bias independent of efforts made by mapmakers to gerrymander; for some states, the political geography alone is enough to exhibit asymmetry. One example of how this occurs is the tendency of Democratic voters to be clustered in densely populated urban areas, thus weakening the power of their votes [20]. The concern that a measure of asymmetry alone is not enough to reliably assess partisan intention to gerrymander was expressed by Justice Kennedy in *LULAC v. Perry*, prompting advocates of the fight to end gerrymandering to figure out how this obstacle may be overcome when it is one of the most reliable ways to detect partisan bias as currently exists [5].

One method that has been developed by several mathematicians and adapted for specific state analyses is that of *outlier analysis*. This method fixes the political geography of the state — including the population and votes cast in each precinct — and subjects all possible maps drawn to the same criteria as the map in question, such as compactness and contiguity specifications. A large sample of redistricting plans is then generated through a selected algorithm to produce "a large set of legally viable maps" that conveys districting plans that are possible given the conditions under which the original map was drawn [33]. If the original map then exhibits more bias than the majority of maps generated, exhibiting outlier values of the bias metric imposed on each map, then it is likely that the map was drawn with some partisan intent and can be considered reasonably suspect for gerrymandering.

2.3.1 Simulation Approach

Chen and Rodden demonstrate the effectiveness of the outlier analysis method by applying it to the 2012 Florida Congressional map [20]. In their algorithm, they simulate a large sample of maps by starting with the current precinct map and allowing each precinct to be its own "district." One of these small districts is then randomly selected and aggregated with its nearest-centroid neighbor to form a new district, subject to population equality constraints, and this process is repeated until the number of districts is equal to the number of apportioned Congressional or legislative seats present.

This process is repeated multiple times to generate maps that are valid districting plans under the criteria of the state. In Chen and Rodden's analysis, they were able to generate a sample of 1,000 possible electoral maps with 24 districts using the 7,349 precincts (or clusters of precincts) in the 2012 Florida map. They then used these simulated maps to analyze how many seats would have been won under the alternative districting plans compared to the original map. The enacted plan of the Florida legislature won more Republican seats than *all* simulated plans, implying that the mapmakers likely drew the plan with partisan intent.

2.3.2 Markov Chain Approach

The approach utilized in this paper and applied to the Congressional and legislative maps of South Carolina follows that of Chikina et al. in their analysis of the 2011 Pennsylvania Congressional map in *LWV v. PA*. In their 2017 publication in *PNAS*, Chikina et al. use Markov chain theory to describe a method of detecting gerrymandering by generating a large sample of maps and testing the probability that a current districting plan came from an unbiased distribution of possible maps [22].

One may recall that a discrete-time Markov chain $\mathcal{M}$ on a state space Σ is a sequence of random variables $X_0, X_1, \ldots$ taking values in Σ such that each step of the chain to a new state depends only on the previous state of the chain in what is known as the Markov property [46]. In other words, for states $i_0, i_1, \ldots i_{k-1}, i, j \in \Sigma$,

$$P\left(X_{k+1} = j \mid X_0 = i_0,\ X_1 = i_1,\ \ldots,\ X_{k-1} = i_{k-1},\ X_k = i\right) = P\left(X_{k+1} = j \mid X_k = i\right). \qquad (2.15)$$

We call the likelihood of transitioning from state i to state j in a single step of the chain the *transition probability* from i to j, denoted $p_{i,j}$. These transition probabilities form a $|\Sigma| \times |\Sigma|$ matrix $\mathcal{P}$ called the *probability transition matrix* for the Markov chain $\mathcal{M}$.

Stated in terms of an application to detecting gerrymandering, a Markov chain $\mathcal{M}$ can be defined as a sequence of random possible districtings of an election map, starting at the initial map X_0, where the state space Σ represents all possible valid districtings (i.e., all ways of drawing district boundary lines subject to redistricting guideline constraints). The transition probabilities correspond to the probabilities of moving from one districting to another by randomly swapping the district membership of a single precinct[6] that lies along the current boundary of a district. The manner in which these transition probabilities are defined[7] ensures that $\mathcal{M}$ converges to a uniform stationary distribution, so taking a random walk along the chain will result in a sample of election maps that are all given equal weight; in other words, there is an equally likely chance of generating any particular districting under the current conditions.

A challenge when dealing with Markov chains applied to real-world situations is that, even when a stationary distribution is known to exist, it is difficult to estimate how many steps are required for the chain to converge to it. The time it takes for a run of a Markov chain to converge is referred to as the *mixing time*, and heuristically, sampling from the stationary distribution is sometimes performed by taking a random walk along the chain for some large number of steps and "hoping that sufficient mixing has occurred" [22]. Chikina et al., however, define a more rigorous approach that allows one to determine whether a given map was drawn from the stationary distribution knowing nothing beyond the reversibility of the chain, which provides an effective way of handling a Markov chain with an unknown mixing time. The Markov chain $\mathcal{M}$ described above and in Appendix A is reversible as a result of its construction, and this property combined with its unknown mixing time make it an ideal candidate for an application of this approach.

To approach this problem, a real-valued label function $\omega : \Sigma \to \mathbb{R}$ is first defined on the state space such that, for every state $i \in \Sigma$, the label function value $\omega(i)$ associates a numeric measure with that step of the chain. In context of analyzing an electoral map, this label function can be chosen to be one of the bias measures discussed in Section 2.2, resulting in a bias measure being calculated for every possible districting in the state space.

Paraphrased in context, the theorem developed and proved by Chikina et al. states that, under the null hypothesis that the current map was drawn from the stationary distribution of all possible maps (i.e., $X_0 \sim \pi$), the probability that we observe an electoral map *at least* as biased as the current map among the set of all possible districtings is at most

$$p = \sqrt{\frac{2b}{2^n + 1}}, \tag{2.16}$$

where b is the number of maps with bias measures at least as "extreme" as the observed map and

[6] Precincts are used because they are the smallest geographical unit for which state voting data is available.

[7] Both the description of the transition probabilities for $\mathcal{P}$ and the proof of the resulting properties of the Markov chain $\mathcal{M}$ are described in Appendix A.

2^n is the total number of new maps generated, corresponding to the number of steps taken in a random walk along the Markov chain. Chikina et al. call this test the $\sqrt{\varepsilon}$ test, where $\varepsilon = \frac{b}{2^n+1}$, or the fraction of maps in the sample that are as unusual or more so than the original map[8]. Being able to impose a bound of $\sqrt{2\varepsilon}$ on this probability is powerful, given the complexity of the state space and unknown mixing time of the chain.

To illustrate an application of the $\sqrt{\varepsilon}$ test, consider the analysis of the 2011 Pennsylvania Congressional map. Using the median-mean bias metric as a label function for the Markov chain, generated maps were subjected to a constraint of 2% or less deviation from equal district populations and a threshold for the worst compactness scores allowed was set. With the algorithm written to randomly choose precincts, perform swaps, and calculate the label function values at each step, the chain was run for 2^{40} steps and the probability was found to be $p = 3.3896 \cdot 10^{-5}$, which corresponds to approximately 632 out of *one trillion* maps exhibiting bias measures as extreme as those of the current map. This leads one to conclude that, for any randomly generated district drawn under the same conditions as a reasonable map, the probability of obtaining a map as biased as or more biased than the original is small, which serves as strong evidence that the map is unusual and hence suggests partisan intent. This analysis was ultimately used by Pegden in his expert witness report in *LWV v. PA* [43].

The choice of label function for the Markov chain is no small matter, and ω should be chosen with careful consideration based on the properties of the electoral map one wishes to assess. Chikina et al. list two criteria for what might be considered a "good" label function [22]:

1. The label function should likely be different for gerrymandered election maps than it is for typical maps.

2. The label function should be sensitive enough to detect small changes in the adjusting of district boundary lines.

For example, in the Pennsylvania analysis, Chikina et al. employ the use of two label functions: median-mean (Equation 2.10) and variance[9] (Equation 2.14). Median-mean is chosen because, in

[8]Chikina et al.'s naming of the $\sqrt{\varepsilon}$ test follows from the definition of an ε-outlier, described in Appendix B. Details are also provided to assist the reader in understanding the proof of the theorem in the *PNAS* paper.

[9]High values for variance represent anti-competitive districtings, while low values correspond to competitive districtings. In the Markov chain analysis, the *negative* variance of district-wide vote proportions is used, with values that are more negative corresponding to highly anti-competitive districtings. This is to correspond with the directionality we see in most outlier values of bias measures; for example, for the median-mean measure, more negative values of MM corresponding to districtings that have a Republican advantage.

addition to being an agreed-upon standard throughout gerrymandering literature, it fits the first criterion of a label function fairly well; it represents the skewness of the vote distribution and a deviation from partisan symmetry, and as a result, outlier values of MM are likely to reflect gerrymandering. The disadvantage of using median-mean is that it is a rather slow-changing function with respect to the number of steps in the chain, causing it to only moderately satisfy the second criterion. Though it will vary slightly with alterations in the district lines, the mean district vote proportion V is relatively stable for a map regardless of how the districts are drawn. Because of this, the value of MM only changes significantly when the median changes, and this only occurs if a precinct swap is made to one specific district out of the L districts (or two districts if L is even).

On the other hand, the negative variance of district-wide Democratic vote proportions is suspected by Chikina et al. to be "a good label function from the standpoint of the first characteristic ... but a great label function from the standpoint of the second characteristic" [22]. This is because it is very sensitive to small changes in districtings and, in contrast to MM, changes to *any* district will directly affect it. While this measure of competitiveness alone might not necessarily be sufficient to suggest a map is gerrymandered for the reasons stated in Subsection 2.2.5, it can be powerful when paired with a measure of partisan symmetry like median-mean, as outlier values correspond with the highly anti-competitive districts we might expect to see in a gerrymandered map.

Chikina et al. limit their analysis to the use of only two metrics to emphasize that the theorem applies regardless of which label function is chosen and that they did not need to look far to find a label function that exhibited evidence of extreme bias for the Pennsylvania map. However, they propose that it may be possible to choose label functions that are a better fit for the data and achieve higher statistical significance for gerrymandered maps. In fact, they even suggest that the geometric B_G measure developed by Nagle might be a bias measure that fits both of their criteria for rigorous label functions.

In the analysis presented in Chapter 4 of his thesis, we follow Chikina et al.'s use of median-mean as a label function for the Markov chain approach to outlier analysis, as it is a recognized measure of partisan bias with widespread historical usage. We also perform the analysis using geometric partisan bias B_G under both the uniform and variable partisan swing assumptions, as well as the bias point estimate $\beta(V)$, as alternative label functions. These measures are compared to one another on the 2011 Pennsylvania Congressional map, a known gerrymander, and the redrawn 2018 map that resulted from the proceedings of *LWV v. PA*, which is considered to be fair. The results of the Pennsylvania analyses are then used to inform the interpretation of results from similar methods applied to the 2011 Congressional and legislative maps for South Carolina. This study allows for insights to be drawn as to how outlier analysis results vary for biased versus unbiased maps and how different label functions perform under outlier analysis compared to one another.

Chapter 3

About the Data

To conduct a thorough assessment of one or more election maps for partisan fairness, information is needed about precinct-level voting data, the state population, and the boundary lines of the districting plans in question. Unfortunately, as is the case in South Carolina, these data may exist independently of one another and must be gathered from several sources. Further, it is not uncommon for there to be discrepancies between the data sets, which must therefore undergo an intense data cleaning process to be usable for analysis. The data needed for this analysis as well as necessary modifications that were made are detailed in this chapter. All shapefile manipulations described were performed in ArcMap 10.6.1, and data management was primarily handled through Python 2.7 scripts and Microsoft Excel. Associated files for this thesis are available in a public GitHub repository for this project at `https://github.com/vagnozzia408/gerrymandering_public`.

3.1 The District Maps

The first piece of data needed for any gerrymandering analysis is the set of election maps themselves. The shapefiles for the 2011 Pennsylvania Congressional map that was disputed in *LWV v. PA* were acquired via personal correspondence with Maria Chikina, and the shapefiles for the remedial plan that took effect in 2018 were obtained through the Pennsylvania Judicial System website [53]. Each map contains eighteen Congressional districts. In this thesis, these alternative maps serve as the standards for significantly biased and reasonably fair, respectively, in light of the decision of the Pennsylvania Supreme Court regarding the original map.

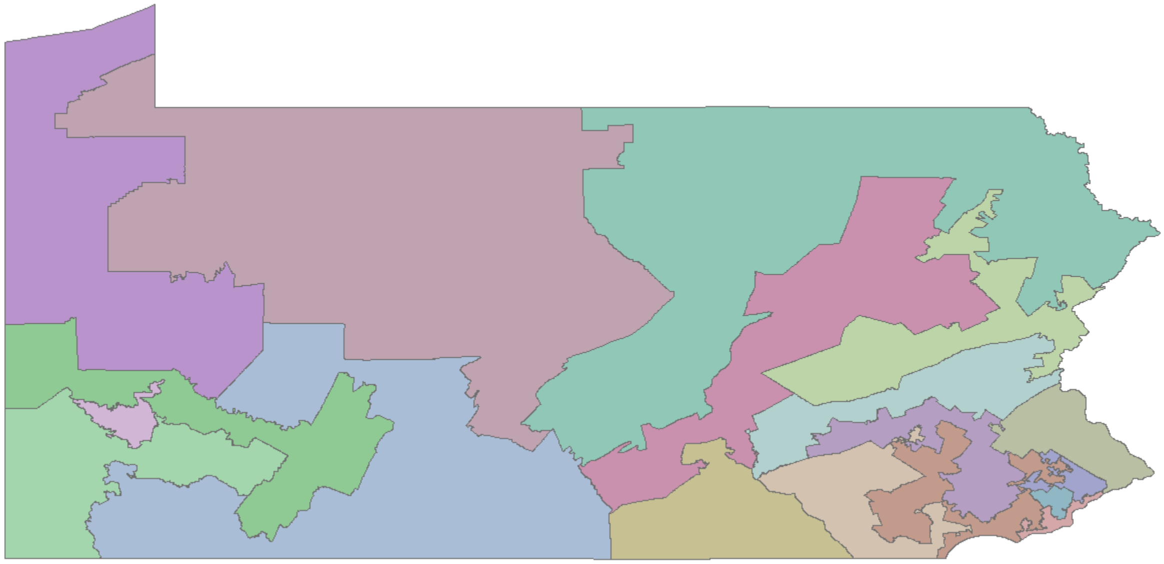


Figure 3.1: The original 2011 Pennsylvania Congressional map drawn during the redistricting cycle immediately following the 2010 Census.

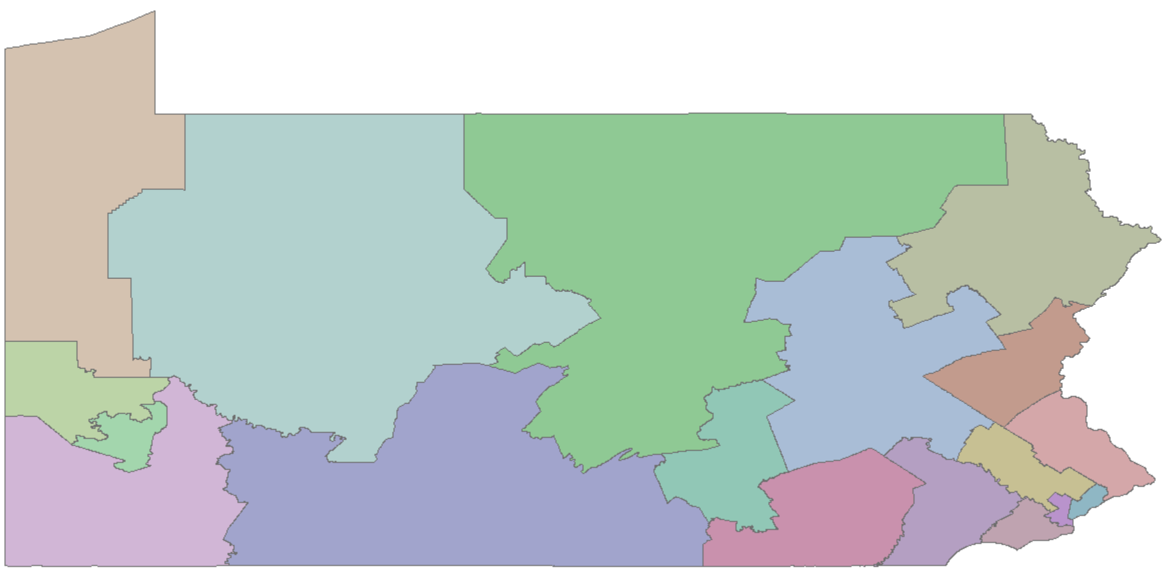


Figure 3.2: The remedial Congressional map drawn for Pennsylvania that took place in 2018 after the 2011 plan was decided by the state court to be a gerrymander.

We also analyze the 2011 South Carolina Congressional map and the state legislative districtings of the South Carolina Senate and South Carolina House of Representatives. These three maps were drawn following the 2010 Census and will remain unchanged until the conclusion of the 2020 Census. Shapefiles for each district map were obtained from the U.S. Census Bureau [54, 55, 56].

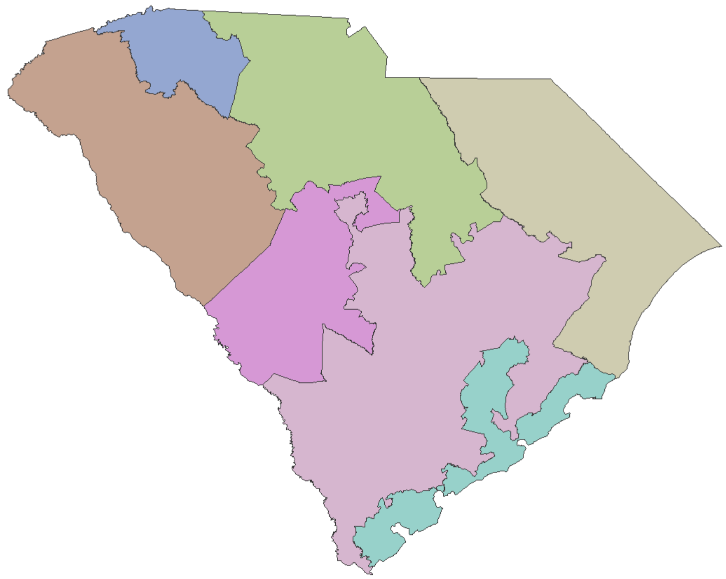
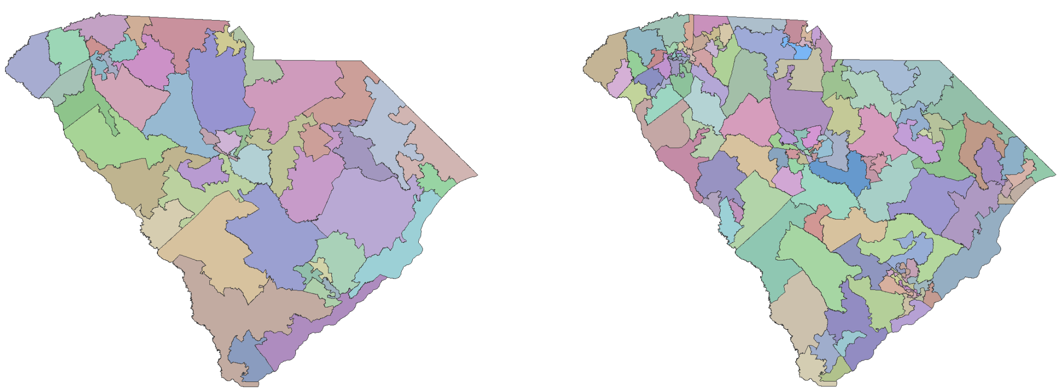


Figure 3.3: The 2011 South Carolina Congressional map, which contains seven districts.



(a) 2011 Senate Map (46 Districts)

(b) 2011 House Map (124 Districts)

Figure 3.4: The 2011 South Carolina legislative districting plans.

A notable feature of the 2011 Congressional map in South Carolina is that, in contrast with other state maps that have been analyzed through outlier analysis, it has significantly fewer Congressional districts, with only seven seats. Little appears in the literature about how partisan bias measures and outlier analysis perform for election maps with a small number of districts, and while not a primary aspect of this study, a potential experiment to test whether fewer districts affects the sensitivity of such methods in detecting partisan bias is proposed in Chapter 5.

3.2 The Voting Data

Both the practice of partisan gerrymandering and the methods for detecting it rely on an estimation of voting patterns among the constituents in a state. This presents a significant challenge, as individual voter preferences can change between elections and be strongly dependent on current events, the candidates, shifting political ideals, and any other manner of exogeneous influences on the populace (defined in Subsection 2.1.1). However, though individual voter preferences are difficult to estimate, statewide voter preferences as a whole are assumed to be relatively stable.

New data for general elections are only available every two years, which is a sufficient amount of time for statewide voting shifts to occur, so gerrymandering analysis typically uses the election results that would have taken place most recently before the contested districting was drawn. Further, because estimating a distribution for voter preferences can be a difficult task, the results of the election for a *single* race are usually chosen as a proxy for voting preferences across the state. Pegden states in his expert witness report for *LWV v. PA* that a particular election will serve as a good proxy if (1) it is a statewide race in which all voters will be choosing between the same candidates, (2) there is no incumbent, and (3) it is among the most recent election results that would have been available to mapmakers during the redistricting process for the map in question [43]. In addition, because a two-party system is assumed for this analysis and third-party candidates seldom earn a large number of votes in statewide elections, an appropriate proxy should have a relatively small percentage of votes cast for candidates not running on a Republican or Democratic ballot. Though all proxies are ultimately imperfect, Pegden argues that the use of a proxy makes it more difficult to detect gerrymandering, and thus an analysis which yields evidence of partisan intent in the redistricting process holds more weight, as it is able to detect unfairness despite an imperfect approximation of voting behavior. The 2010 Pennsylvania senate race was chosen as the proxy for analysis in *LWV v. PA*, meeting the above characteristics, and the vote counts for this race were included as precinct attributes in the precinct shapefiles provided by Chikina et al., discussed in Section 3.3.

The South Carolina Election Commission (SCEC) makes statewide voting data from general elections dating back to 1968 publicly available for download online in CSV format. Data were obtained for the 2010 gubernatorial election, which most closely fits the criteria for a reasonable proxy [49]. In this statewide race in which neither candidate was running for reelection, Republican candidate Nikki Haley defeated Democratic candidate Vincent Sheheen in a 51.37%–46.91% statewide vote. The 1.72% of votes cast for third-party candidates is relatively small, and was thus omitted from the vote totals for this analysis. Omitting these third-party votes adjusted the statewide vote proportion to 52.27% Republican and 47.73% Democratic, which had no impact on the outcome of the vote in each district or the state overall.

Voting data is broken down by precinct, the smallest geographical unit for which election results are made available. Absentee votes and other "virtual precinct" ballots, however, are aggregated by county. These votes accounted for 11.67% of the total number of votes cast in the election, which was too significant to ignore, so these votes in each county were allocated to the precincts in the county by the proportion of registered voters in each. All vote totals were rounded to the nearest integer value of votes, which did not alter the overall statewide vote proportions.

3.3 The Precinct Maps

Voting precinct shapefiles for Pennsylvania were provided by Maria Chikina and include the precinct-level vote counts from the proxy election as attributes for each of the 9,060 precincts. In South Carolina, however, voting precinct shapefiles do not contain information about election outcomes. The maps of precinct lines are maintained by a different government office than the SCEC, which manages election data, and as a result the information exists separately and must be merged.

Shapefiles of the voting precincts in South Carolina were obtained via personal correspondence[1] with Victor Frontroth at the South Carolina Revenue and Fiscal Affairs (RFA) Office [27]. In addition to the precinct names, boundary lines, and the counties to which they belong, each precinct in the shapefile has its population as an attribute. The precinct map provided is that which most closely corresponds to the November 2010 General Election.

The precincts in the shapefiles available do not have exact one-to-one correspondence with the precincts in the SCEC vote totals, so an exact join of the voting data to the geographic precincts was not possible. One discrepancy is that the naming convention for precincts between the RFA Office and the SCEC is not at all consistent. For example, discrepancies may include using the word

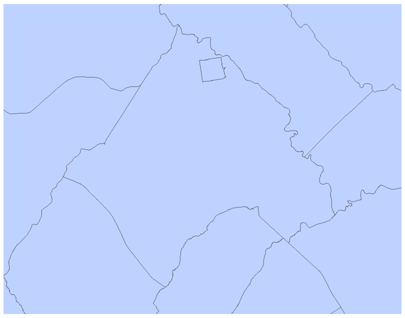
[1] While voting precinct maps for each county in South Carolina are available on the SC Revenue and Fiscal Affairs Office website at `http://rfa.sc.gov/mapping/districts?o=5`, data are in PDF format and precinct shapefiles are not made publicly available.

"and" versus an ampersand, using the abbreviation "Mt." versus spelling out "Mount," or using the abbreviation "No." for "number" versus using a hash symbol. These differences, while inconvenient, are relatively straightforward to resolve.

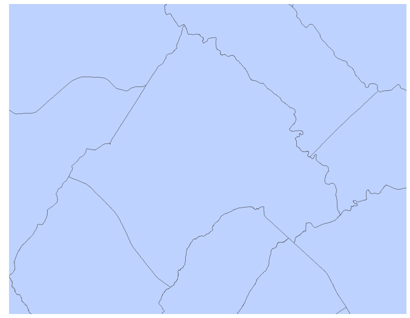
A more challenging issue arises from the fact that precinct maps are determined by individual counties and can change quite frequently. As a result, the precinct map most closely corresponding to the election was still not an exact match for the precincts reported in the SCEC data. Several precincts in the shapefile, for example, had been split following the 2010 General Election and needed to be aggregated to correspond with the precinct voting data. Others had been aggregated only following the election, and the geographic areas needed to be split into distinct regions corresponding to the earlier precincts. In some cases, precincts had even been renamed.

Fortunately, even though the shapefiles available do not capture the South Carolina precincts exactly as they were in November 2010 at the time of the General Election, there are ways to resolve these discrepancies as well. Any time a precinct is altered within a county, it is recorded in Title 7 - Elections, Chapter 7 - Polling Precincts and Voting Places, of the South Carolina Code of Laws [32]. Further, additional shapefiles were obtained from RFA for precinct maps from early 2010 and from 2014 to provide reference points before and after the election and capture some of the changes to precinct boundaries. Several iterations of data cleaning were performed to resolve the discrepancies between the voting data and shapefiles, which involved comparing the main precinct map against the others provided by RFA and using the South Carolina Code of Laws from various years to corroborate any changes made.

Once the precincts from the voting data and the shapefile matched, the SCEC data were joined to the RFA shapefile so that each geographic precinct included the respective numbers of Democratic and Republican votes as attributes. Minor topological errors in the map were resolved in ArcMap, and any precincts contained wholly inside another were merged with the surrounding precinct in which they formed a "hole." This latter adjustment was made so that swapping the district membership of an outer, surrounding precinct does not violate the contiguity principle for a districting by forcing the inner precinct to become disconnected from its other precinct neighbors in the district. Population and voter data from inner precincts were aggregated with those of their surrounding precincts.



(a) Precinct with a Hole



(b) Inner Precinct Merged with Outer

Figure 3.5: An example of a precinct “hole” that was merged to the surrounding precinct.

The resulting South Carolina precinct map consisted of a total of 2,128 voting precincts.

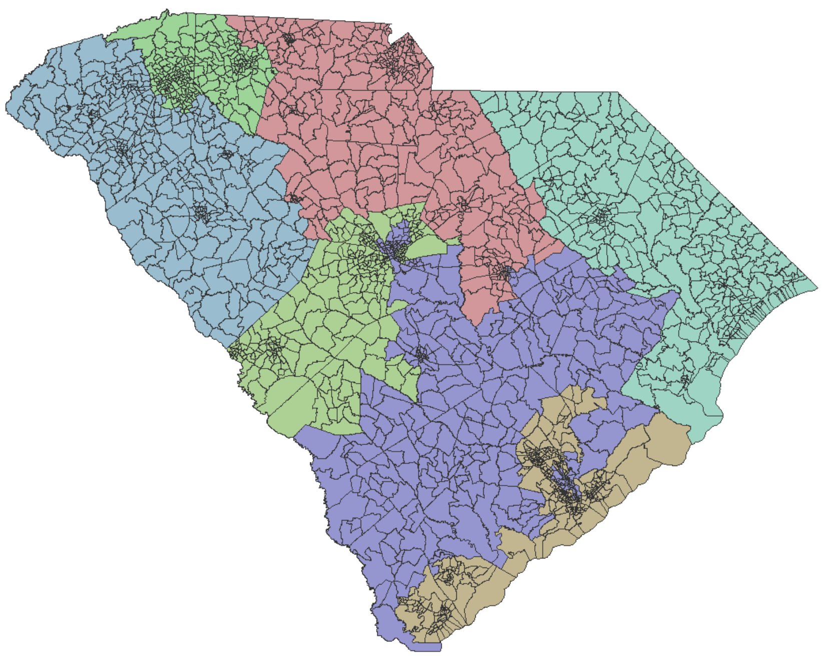


Figure 3.6: The resulting precinct map of South Carolina.

3.4 Markov Chain Input Files

To generate the input file used in the $\sqrt{\varepsilon}$ test for outlier analysis, additional calculations were made to include all necessary attributes used in a run of the Markov chain. In particular, information is needed about the precincts and their relationships to one another in order to perform swaps, so an adjacency list must be generated for the precinct map under each districting plan being analyzed. The resulting attribute table for the precinct data was then formatted as a tab-delimited text file as per the specifications used in the Pennsylvania analysis by Chikina et al. A summary of relevant precinct attributes for the South Carolina maps can be found in Table 3.1[2].

Attribute	**Description**
`PSN`	Unique ID for each precinct (indexed $0, \ldots, 2127$)
`nb`	List of the indices of neighboring precincts in clockwise order
`sp`	List of corresponding shared perimeter lengths (in m) of neighboring precincts
`area`	Area of the precinct (in m^2)
`pop`	Population of the precinct
`voteA`	Number of votes cast for Democratic (reference party) candidate
`voteB`	Number of votes cast for Republican candidate
`congD`	Congressional district in which the precinct belongs (indexed $1, \ldots, 7$)
`senD`	Senate district in which the precinct belongs (indexed $1, \ldots, 46$)
`houseD`	House district in which the precinct belongs (indexed $1, \ldots, 124$)
`county`	Name of the county to which the precinct belongs

Table 3.1: Precinct attributes utilized in the Markov Chain outlier analysis.

Distinct text files containing the precinct-level data were generated for the 2011 South Carolina Congressional district map, the state Senate district map, and the state House district map, as well as for the alternative districing plans for the Pennsylvania Congressional map. These five input files capture all relevant information required for the outlier analysis detailed in Chapter 4.

[2]The neighbor calculations for `nb` and `sp` were performed using data exported from ArcMap and manipulated using an auxiliary Python script. Precincts that touch only at a point were not considered to be neighbors. The numbers designating `congD`, `senD`, and `houseD` were assigned by performing an overlap of the precinct shapefile with each of the district shapefiles and assigning precincts to the district containing the largest percentage of its area. (The district membership of a few precincts that fell between districts was manually adjusted to maintain the principle of district contiguity.) The specific length and area units for `sp` and `area` are unimportant as long as they are consistent between the two attributes.

Chapter 4

Analysis and Results

Recall that the goal of this thesis is twofold: (1) to provide a state-specific analysis of South Carolina to evaluate the likelihood of partisan gerrymandering, and (2) to increase transparency in the use and interpretation of partisan bias measures and outlier analysis as a diagnostic for unfair maps. To accomplish the second goal, the following questions are explored:

- How do bias measures and outlier analysis results compare for gerrymandered versus fair maps?
- How do bias measures and outlier analysis results compare to one another on a given map?

4.1 Methodology

The methods utilized in this analysis follow those of Chikina et al. used in *LWV v. PA*. Namely, the $\sqrt{\varepsilon}$ test described in Subsection 2.3.2 was applied to assess the fairness of the South Carolina Congressional, Senate, and House district maps drawn following the 2010 Census, which have been in place since the redistricting process concluded in 2011. The test was also applied to the 2011 Pennsylvania Congressional map to replicate the results presented in *LWV v. PA*, as well as the remedial 2018 Pennsylvania plan, which had not been analyzed prior to this study. Recall that the $\sqrt{\varepsilon}$ test is conducted on each map by using a discrete time Markov chain to generate a large sample of possible districtings that could occur for the state, given the appropriate redistricting constraints. Although it is possible that maps are repeated in the sample, the uniform stationary distribution of the Markov chain constructed ensures that each possible map is considered with equal weight [22, 43]. The sample of maps generated despite the unknown mixing time of the chain allows us to determine an upper bound on the probability of observing a map as biased as the current map with respect to some label function, under the assumption that the current map was drawn from the stationary distribution of all "typical" districtings. If this probability is extremely small, there is then evidence to suggest that the current map is highly unusual and likely to have been crafted with partisan intent.

Specifically, the null hypothesis to be tested for each map with respect to a given label function is

$$H_0 : X_0 \sim \pi,$$

where X_0 is the initial election map and π is the uniform stationary distribution of "typical" districtings. For this analysis, we use a significance level of $\alpha = 0.001$ to judge the results of the outlier analysis; achieving statistical significance in this context means that there is a less than 0.1% likelihood of observing a map as biased as the current map by chance alone. While this significance level is disputable, as court definitions of how unlikely a map's bias must be before it is considered a gerrymander may vary, it was chosen in accordance with the results of *LWV v. PA*. In the analysis of the 2011 Pennsylvania Congressional map, the $\sqrt{\varepsilon}$ test yielded p-values of at most 0.001 and was deemed acceptable evidence of partisan gerrymandering by the Pennsylvania Supreme Court. We thus consider $\alpha = 0.001$ to be a reasonable threshold for significance in this analysis.

Multiple partisan bias measures described in Section 2.2 were selected to provide a more comprehensive and rigorous analysis of the maps than has previously been available, as well as to observe how such measures that have not been previously used as label functions perform under the Markov chain outlier analysis. The maps were subjected to a series of differing constraints that restrict the possible districtings generated to those that follow state and federal redistricting guidelines as described in Section 1.1.

4.1.1 Selection of Bias Measures

Several partisan bias measures were chosen to serve as label functions of the Markov chain. All of the measures chosen correspond with the definition of partisan symmetry, which is defined as the standard for fairness in this thesis. The properties of "good" label functions — those that are likely to take on different values for gerrymandered districts and be sensitive to small changes in the districtings — were considered when choosing appropriate measures to utilize in the chain. For the same reasons stated by Chikina et al., median-mean (Equation 2.10) was used as the baseline measure to replicate the analysis performed in Pennsylvania. The performance of the median-mean metric as an indicator of potential partisan gerrymandering is relatively well-known and observable from the *LWV v. PA* analysis, and thus this bias measure was chosen to be consistent with previous literature. Using median-mean as the standard for what we expect in a label function, we should expect any other reasonable label function chosen in this context to perform similarly to median-mean as an indicator of bias.

Along with median-mean, three alternative partisan bias measures were added to provide a more comprehensive analysis of the South Carolina maps. The bias point estimate $\hat{\beta}(V)$ from Equation 2.9 is considered due to the fact that it is directly defined as the deviation from partisan symmetry, which is considered for purposes of this paper to be the standard by which partisan "fairness" is quantified. Because $\hat{\beta}(V)$ is computed from an estimation of the seats-votes curve, the uniform partisan swing assumption used by Katz et al. is also applied here when computing the bias point estimate. This measure is expected to fare well from the standpoint of the first criterion of a good label function, as extreme quantities of $\hat{\beta}(V)$ are associated with the large deviation from symmetry expected in gerrymandered maps. The bias point estimate is a rather slow-changing function, however, and will only moderately satisfy the second criterion at best. This slow-changing nature results from the fact that its value changes only when shifts in district-level vote proportions cause a seat to flip between parties.

The other two measures used are variations of the geometric bias measure B_G defined by Nagle and suggested by Chikina et al. as a potentially good measure of partisan bias that meets both criteria of a good label function [22, 41]. Because this measure relies on an estimation of the seats-votes curve, we consider the curves constructed using the outcome of a single election under both the uniform partisan swing (UPS) and variable partisan swing[1] (VPS) assumptions, described in Methods 2 and 3 of Subsection 2.1.3. These two estimated curves are then used to compute the corresponding geometric bias measures for each sample election map. To distinguish between these two values, the measures are denoted $B_{G,U}$ and $B_{G,V}$ to identify the measure of B_G as it is computed under the UPS and VPS assumptions, respectively. The primary difference between the geometric bias measures and those used previously for outlier analysis is that B_G does not have an

[1]Recall that Nagle's original usage is defined under the variable partisan swing assumption.

associated sign indicating the direction of bias; rather, it is a quantity measuring *overall* deviation from partisan symmetry. The direction of bias can be determined for the initial map by graphing the seats-votes curve, but this direction is not observed for each sample map generated along the Markov chain.

Comparing several bias measures across the same map when both applied initially and utilized in outlier analysis makes it possible to identify how sensitive such measures are to partisan bias. We would expect for various measures that all purport to measure partisan asymmetry to perform reasonably similarly for the same election map, with none giving a particular "advantage" in detecting bias over the others. Again, because median-mean is considered in the literature to be a reasonable indicator of bias, results for alternative label functions will be compared against those for MM to gain insight into their performance.

4.1.2 Constraints

When performing Markov chain outlier analysis in this context, there are several constraints that may be imposed as the chain takes a series of steps to generate a sample of possible election maps. Each of these constraints is typically chosen to correspond with an associated redistricting guideline that occurs at the federal or state level. The constraints used in this analysis are described as follows.

- *Roughly Equal Populations.* To comply with the redistricting principle of maintaining approximately equipopulous districts, bounds on the maximum deviation from the ideal district population (total population divided by number of districts) may be specified. The bounds used in this analysis for Congressional maps were tested at 1% and 2%, representing relatively small deviations that correspond to the state guidelines of having district populations as equal as possible. Senate and House maps were subjected to population deviations of 5% and 2.5%, respectively, to match the guidelines of each legislative judiciary committee [26, 45].

- *Compactness Thresholds.* Though compactness is somewhat ill-defined in the South Carolina redistricting guidelines, one can specify bounds on how "bizarre" districts may be among the sample of generated maps by using more well-defined mathematical measures. This bound is typically chosen to be a value just beyond that of the current districting, ensuring that any generated map is no worse than the current one with regard to compactness. The measures chosen for this analysis mirror those used in the Pennsylvania analysis. The first is the *perimeter* threshold, which is computed by calculating the sum of the district perimeters,

$$\sum_{d=1}^{L} P_d, \tag{4.1}$$

where P_d is the perimeter in the specified units for district d. The second compactness measure used is the *inverse Polsby-Popper* threshold (referred to as the L^1 compactness metric in Chikina et al.), which is computed for a given districting as

$$\sum_{d=1}^{L} \frac{1}{C_d}, \tag{4.2}$$

where the *Polsby-Popper measure*[2] C_d for a district d finds the ratio of the district's area A_d to the area of a circle with a circumference equal to the district's perimeter P_d and is computed as

$$C_d = \frac{4\pi A_d}{P_d^2}. \tag{4.3}$$

- *County Preservation.* One may require that any counties which are contained fully within a district in the initial map be preserved when making precinct swaps to generate new maps, meaning that precincts whose swaps break the county across districts will not be chosen. This constraint is more important for larger Congressional districts than it is for Senate and House districts, which are in general smaller and may be less likely to contain entire counties. This constraint is one way of maintaining the redistricting principle of preserving communities of interest, so maps were analyzed both with and without this constraint. It should be noted, though, that in the 2011 South Carolina maps, district boundary lines frequently cut across precincts, so this constraint may not be as realistic as some of the others.

- *Frozen Districts.* Minority-majority districts — districts in which the majority of constituents are non-white or racial or ethnic minorities — that were drawn in compliance with the Voting Rights Act can be frozen to ensure that minority demographics maintain representation in Congress. This applies only to Congressional districts, as Senate and House districts do

[2]The Polsby-Popper metric always has values between 0 and 1, with a circle having a "perfect" compactness score of 1, which is impossible to observe for every district in an election map. The sum of the *inverse* Polsby-Popper measures for the districts is used so an *upper bound* can be placed on the compactness threshold to be consistent with other metrics used.

not typically have minority-majority districts. This constraint may also be considered to correspond with the redistricting principle of preserving communities of interest, so the South Carolina Congressional map was analyzed both with and without this constraint.

4.2 Code Used to Perform Analysis

The C++ code used to perform the analysis was obtained from Pegden's faculty webpage, where it is available for public use, along with the input files used in the Pennsylvania analysis [21]. The code takes as input a tab-delimited text file containing the precinct adjacency list for a given district map as described in Section 3.4. Then, using the observed election map as the initial state of the Markov chain, successive precinct swaps are simulated to generate a sample of possible election maps. For each map, the value of the partisan bias measure specified as the label function is computed and recorded as either "more unusual" or "less unusual" than the original map in terms of its degree of bias. The p-value stated in Equation 2.16 is also computed at every step, representing the probability of observing an election map with a label function value at least as extreme as the original map. Output reports containing the bias measure of an intermediate map in the sample, the proportion ε of generated sample maps at least as biased as the original, and the associated p-value are produced every 2^{22} steps so the behavior of the chain can be observed.

4.2.1 Modifications to Code

The code as it was originally utilized in Pennsylvania includes the median-mean measure, as well as options to set the four constraints described in Subsection 4.1.2. Additional measures of deviation from partisan symmetry, however, were not found to have been used as label functions for the Markov chain prior to this analysis, so the algorithms for computing the measures for a given election map were coded into the program. The three measures for which code was added are:

- The bias point estimate from the seats-votes curve, $\hat{\beta}(V)$;
- Geometric partisan bias under the UPS assumption, $B_{G,U}$; and
- Geometric partisan bias under the VPS assumption, $B_{G,V}$.

Recall that Methods 2 and 3 in Subsection 2.1.3 describe the algorithms used to construct the estimated seats-votes curve $\widehat{S}(V)$ under the uniform and variable partisan swing assumptions, respectively. Constructing the seats-votes curve is necessary to find its inverse $\widehat{S}^{-1}(V)$ and to subsequently calculate the corresponding measure of geometric bias B_G. These algorithms, however, do not consider the case when a tie is observed in an electoral district. Exact ties are seldom observed in election outcomes, but when generating *billions* of sample maps by randomly swapping precincts between districts and calculating the hypothetical election outcomes using precinct vote counts, encountering a sample map where a tie does occur is unsurprising.

To make these algorithms suitable to calculate $B_{G,U}$ and $B_{G,V}$ for each step of the Markov chain, one additional modification to the code was needed to allow for a way of breaking ties when they occur. The code was adjusted so that, if a precinct swap results in a tie in some district d with district vote proportion $v_d = 0.5$, the equivalent of a random coin flip is performed to determine whether to allocate the seat to Democrats or Republicans. In other words, if $v_d = 0.5$ for some $d \in \{1, \ldots, L\}$ on a given step of the Markov chain, then:

- Randomly generate a zero or a one with equal probability.
  - If the random number generated is zero, adjust the district vote share v_d to $v_d - 1$ vote.
  - If the random number generated is one, adjust the district vote share v_d to $v_d + 1$ vote.
- Proceed with analysis using the adjusted vote shares.

This is, in fact, how some tied elections for legislative offices are handled in the United States, with twenty-seven states legally determining winners by drawing lots or some other random process, such as a coin toss [42]. While South Carolina is among the states that hold runoff elections in the event of a tie, the method of determining the winner of a tied election at random is common enough that it was considered a reasonable and computationally straightforward way to modify this analysis in a manner that could apply to any state.

4.2.2 Verification and Implementation

To verify that the code was running properly, the 2011 Pennsylvania Congressional map provided by Chikina et al. was run through the chain program to ensure that the resulting p-values for median-mean aligned with those reported in the *LWV v. PA* analysis. The alternative label functions discussed in Subsection 4.2.1 were also applied to the original Pennsylvania map. The $\sqrt{\varepsilon}$ test was then implemented on the 2018 Pennsylvania Congressional map and the 2011 South

Carolina maps for Congressional, state Senate, and state House districts, each under differing sets of redistricting constraints and with respect to each of the alternative label functions.

All instances of the code were run on the Clemson University Palmetto Cluster. In total, 288 instances were run to observe the outcome of the test for varying combinations of maps, label functions, and constraints. The geometric measures of partisan bias are more computationally expensive than the others, so the instances of the code that used $B_{G,U}$ and $B_{G,V}$ as label functions were run separately from and for fewer steps than those using median-mean and the bias point estimate. Similarly, both South Carolina legislative maps contain a much larger number of districts (46 for Senate and 124 for House), which causes the chain to run much more slowly, so a smaller number of steps was chosen. While the $\sqrt{\varepsilon}$ test will work for any number of steps taken along the Markov chain, with larger numbers of steps yielding more powerful results, it should be noted that even 2^{35} generated maps is still a large sample of possible districtings, representing over 34 billion maps.

Maps	**Bias Measures**	**Constraints**	**Steps**
PA 2011, PA 2018, SC Congressional	MM, $\hat{\beta}(V)$	Compactness, Frozen District, Population, County Preservation	2^{40}
PA 2011, PA 2018, SC Congressional	$B_{G,U}$, $B_{G,V}$	Compactness, Frozen District, Population, County Preservation	2^{38}
SC Senate, SC House	MM, $\hat{\beta}(V)$	Compactness, Population, County Preservation	2^{38}
SC Senate, SC House	$B_{G,V}$	Compactness, Population, County Preservation	2^{37}
SC Senate	$B_{G,U}$	Compactness, Population, County Preservation	2^{37}
SC House	$B_{G,U}$	Compactness, Population, County Preservation	2^{35}

Table 4.1: Summary of the analyses run on the Palmetto Cluster.

4.3 The Standard: Pennsylvania

Though the analysis of the 2011 Pennsylvania map in *LWV v. PA* exhibits the type of results we might expect to observe for gerrymandered maps, there is scant empirical information in the literature about what results to expect for "fair" maps. For example, we know from the 2011

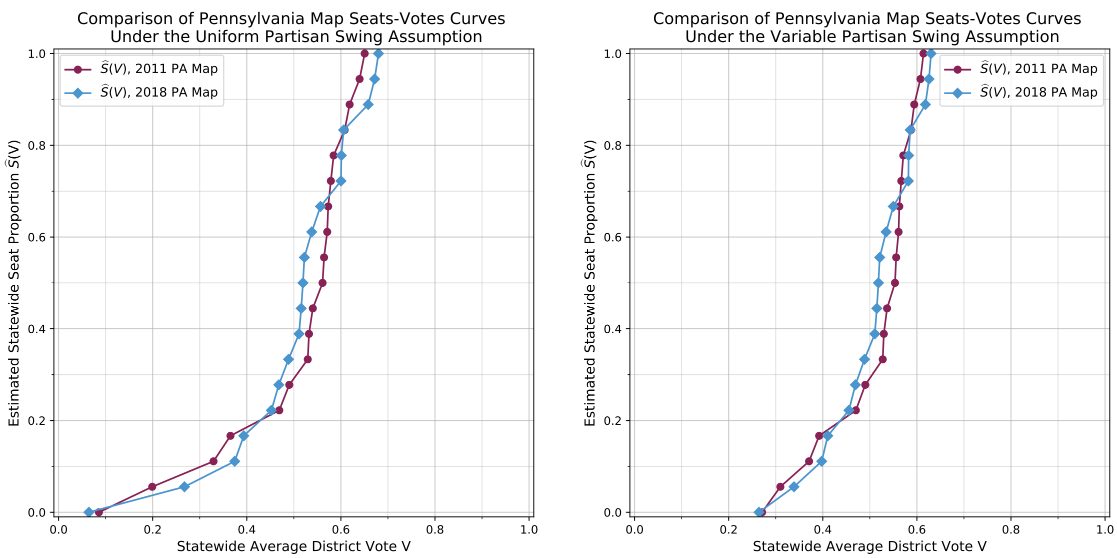
Pennsylvania map that the p-values resulting from the outlier analysis will likely be quite small for gerrymandered maps, as the p-values reported by Chikina et al. were less than 0.001 for all constraint combinations they tested [22, 43]. However, for reasonably fair maps, how many maps of a worse or equally bad bias measure would need to be observed in a sample before the current election map is considered to not be unusual? Correspondingly, would we expect to see p-values hovering around 0.05, just outside traditionally accepted bounds for significance? Or would these p-values be much larger, indicating that a large proportion of maps exhibited partisan bias as bad as our original map? As both the original map determined to be gerrymandered and the redrawn map are publicly available, Pennsylvania provides us with a reasonable basis for comparison. The $\sqrt{\varepsilon}$ test was run on the Pennsylvania precinct map using both the 2011 and 2018 Congressional districting plans, which are considered unfair and fair, respectively. The resulting range of p-values across different combinations of constraints imposed on the map allow us to observe how the outlier analysis performs for biased and unbiased maps, all other things being held equal.

As results are compared for the two Pennsylvania Congressional maps, it is noteworthy to observe how election outcomes would have fared under the 2018 redistricting plan. While the new plan drew much attention because Democrats won a significant 9 out of 18 seats in the 2018 General Election (as opposed to the 5 out of 18 won in 2012 under the old plan), this result occurred after nearly eight years of voting shifts and changes to the political climate. When subjected to the same voting proxy used in Chikina et al.'s analysis of the 2011 Congressional map, one can observe that the new plan results in Democrats winning 6 out of 18 seats, which is only one additional seat than was won under the original legislature-produced plan in the 2012 General Election. In other words, using the same voting data that would have been available to the state legislature at the time the gerrymandered plan was produced, it can be ascertained that, had the legislature produced the more fair plan, Democrats still would have only won six seats and not nine. This remark is not to discount the positive effects of redistricting on producing a more unbiased map, but rather to keep the comparative bias of the original plan and effects of the redrawn plan on election outcomes in perspective.

4.3.1 Initial Characteristics of the Pennsylvania Maps

Before the outlier analysis results are examined, we can observe several initial characteristics of the Pennsylvania election maps, starting with the estimated seats-votes curves under both the

UPS and VPS assumptions, which can be seen in Figure 4.1. The estimated seats-votes curves are strikingly similar in the $0.4 < V < 0.6$ region under both assumptions; however, deviations from similarity occur in the tails. Particularly, the tails of the curves drawn under the VPS in Subfigure 4.1b indicate more conservative estimates of the degree of partisan swing required to shift a seat from Democrat to Republican, or vice versa.



(a) Uniform Partisan Swing Assumption

(b) Variable Partisan Swing Assumption

Figure 4.1: Comparison of the estimated seats-votes curves for the 2011 and 2018 Pennsylvania Congressional maps, drawn under both the UPS and VPS assumptions.

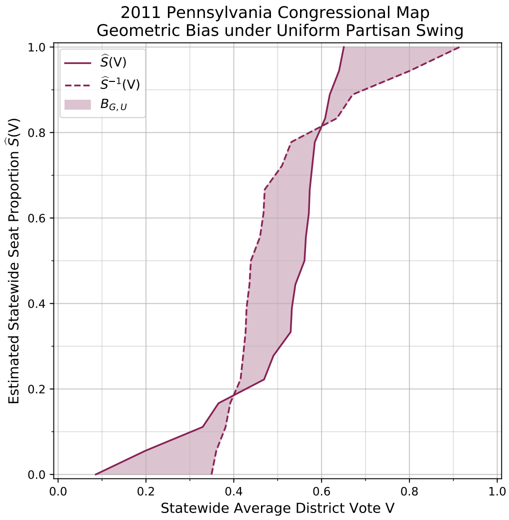
Focusing on the effect of the new redistricting plan on the estimated seats-votes curve, recall from Subsection 2.2.3 that nonzero geometric bias represents a Republican advantage when $\widehat{S}(0.5) < 0.5$, which can be seen for each of the curves in Figure 4.1. By comparing the curves for the 2011 and 2018 Pennsylvania maps on the same graph, we can observe that under both assumptions, the graph of $\widehat{S}(V)$ for the 2018 redistricting plan is shifted to the left of the 2011 plan curve and is closer to passing through the $(0.5, 0.5)$ midpoint. This shift reflects an overall improvement in partisan symmetry of the Pennsylvania election map under the redrawn plan.

The initial values of the selected bias measures for Pennsylvania under both redistricting plans can also be computed and are summarized in Table 4.2. While it is necessary to perform outlier analysis in order to assess whether these values are "bad enough" to be considered the result of gerrymandering, the initial values are still beneficial in providing a first look at the bias of a map, as nonzero values of median-mean, the bias point estimate, and geometric partisan bias can serve as useful indicators of partisan asymmetry.

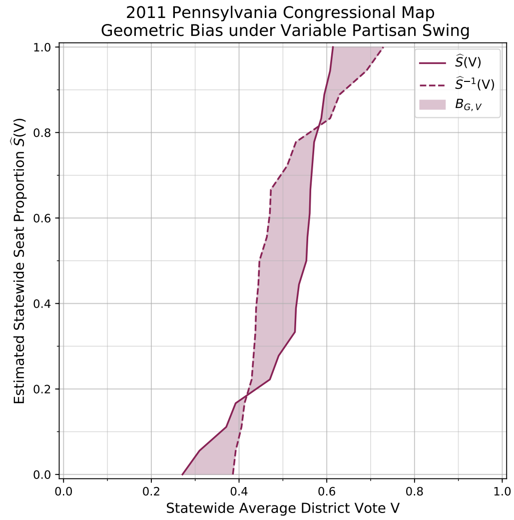
Map	MM	$\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
2011 Plan	-0.0626	-0.2222	0.0942	0.0675
2018 Plan	-0.0209	-0.1389	0.0525	0.0353

Table 4.2: Initial bias measure values for the 2011 and 2018 PA Congressional maps.

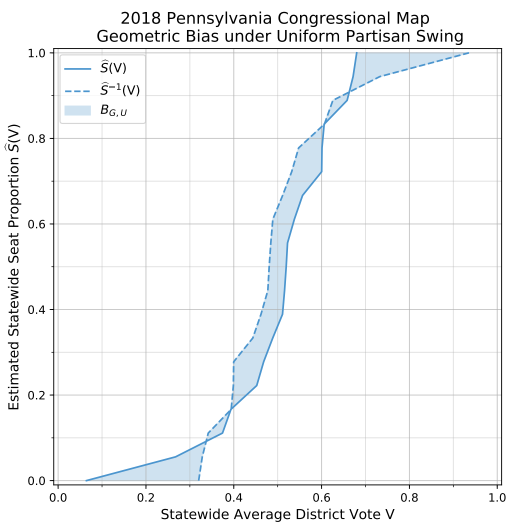
The new redistricting plan resulted in improvement in all four measures of partisan bias shown in Table 4.2, shifting each closer to zero. The values of median-mean and the bias point estimate remain negative for the 2018 plan, representing a Republican advantage, but some natural bias in the election map is not unreasonable to observe, and the degree of bias is noticeably less under the 2018 plan for both measures. Similarly, the positive values of the two geometric bias measures under the 2018 plan still indicate some asymmetry, which we know is in favor of Republicans by looking at the seats-votes curves, but the degree of asymmetry is substantially reduced. The values of $B_{G,U}$ and $B_{G,V}$ are also of particular interest because they can be compared visually. In Figure 4.2, the inverted seats-votes curves are represented by dashed lines, and the area between the original and inverted curves, which represents geometric bias, is shaded on each graph. We can see that the area between $\widehat{S}(V)$ and $\widehat{S}^{-1}(V)$ for the 2018 redistricting plan is noticeably smaller than that of the original gerrymandered plan, regardless of which partisan swing assumption is used to generate the curves.



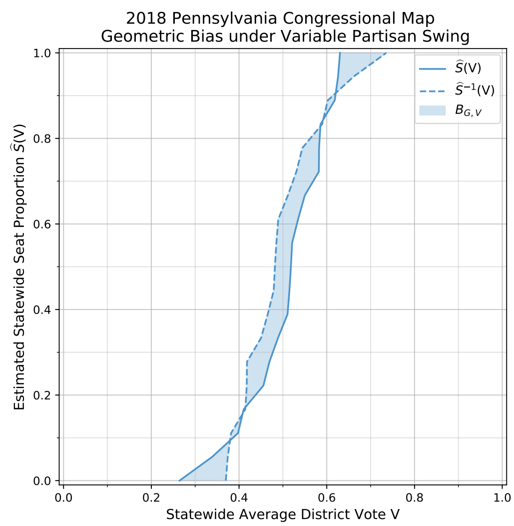
(a) $B_{G,U}$ for the Original Pennsylvania Map



(b) $B_{G,V}$ for the Original Pennsylvania Map



(c) $B_{G,U}$ for the Redrawn Pennsylvania Map



(d) $B_{G,V}$ for the Redrawn Pennsylvania Map

Figure 4.2: Comparison of the geometric bias measures for each Pennsylvania redistricting plan.

4.3.2 Outlier Analysis for Pennsylvania Maps

The initial values alone are enough for us to see that the original Pennsylvania map consistently demonstrates more partisan bias than its redrawn 2018 counterpart when subjected to a selection of partisan bias measures. But how do these maps compare under outlier analysis? Can the 2018 Pennsylvania map still be considered fair when compared to the sample of possible maps that could have been drawn instead?

The results of the outlier analysis conducted would suggest that it can. If we consider a map to be fair, then when we generate a large sample of typical districtings, we should expect to observe many maps that have bias measures similar to ours. Correspondingly, if we assume that our map is typical in the sense that it is drawn according to the uniform stationary distribution of the Markov chain, the probability of observing a map exhibiting similar bias measures should not be so small that it represents a very unlikely event. Thus, p-values resulting from the outlier analysis of a fair districting should be consistently higher than those for a decided gerrymandered districting, and this is indeed what we observe when comparing the opposing Pennsylvania redistricting plans.

Table 4.3 displays the results of the $\sqrt{\varepsilon}$ test, with the p-values for the 2011 and 2018 Pennsylvania maps side-by-side under each partisan bias measure and for each set of relevant constraints. For all but six of the 64 comparisons, the p-values for the redrawn 2018 map are larger than those of the original legislature-drawn plan, indicating that the new plan is more typical with respect to randomly drawn Congressional maps for Pennsylvania and less likely to have been intentionally crafted with partisan intent. Highlighted cells represent p-values below the 0.001 threshold, and we see that the $\sqrt{\varepsilon}$ test achieves significance for far more instances of the 2011 map than it does for the 2018 map across all partisan bias measures, indicating that the original Pennsylvania map is an outlier with respect to the sample of typical districtings. From this comparison, we can infer that, even though a few specific conditions may result in significance, fair maps such as the remedial 2018 plan will most often demonstrate overwhelming non-significance across multiple constraint combinations.

Note that, in some cases, the table reports p-values greater than one. Although this is nonsensical from a probabilistic standpoint, it is interpretable in context of the $\sqrt{\varepsilon}$ test. As defined by Equation 2.16, a p-value greater than one corresponds to more than half of the sample maps exhibiting as much bias as our original map.

Constraints				MM		$\hat{\beta}(V)$		$B_{G,U}$		$B_{G,V}$	
Compactness Threshold	Freeze District 2?	Population Threshold	Preserve Counties?	PA 2011	PA 2018	PA 2011	PA 2018	PA 2011	PA 2018	PA 2011	PA 2018
Inverse Polsby-Popper	No	0.01	No	0.00005	0.31040	0.00207	0.06457	0.00850	0.06487	0.00007	0.00819
			Yes	0.00023	0.00278	0.00051	0.00330	0.00795	0.11182	0.00003	0.00003
		0.02	No	0.00002	0.30114	0.00100	0.06473	0.00805	0.06890	0.00003	0.00616
			Yes	0.00001	0.00423	0.00156	0.00763	0.00794	0.11518	0.00001	0.00066
	Yes	0.01	No	0.00039	0.66938	0.00540	0.29299	0.13528†	0.12463	0.00004	0.01077
			Yes	0.00024	0.00613	0.00095	0.00435	0.13232†	0.11411	0.00003	0.00028
		0.02	No	0.00019	0.66945	0.00717	0.30393	0.13541†	0.12469	0.00004	0.01058
			Yes	0.00009	0.00672	0.00135	0.00442	0.12095†	0.11447	0.00003	0.00040
Perimeter	No	0.01	No	0.00011	1.24750	0.17281	0.93685	0.12351	1.00430	0.00003	1.17700
			Yes	0.00007	1.01280	0.10318	0.27867	0.12827	0.94898	0.00001	0.84303
		0.02	No	0.00008	1.20990	0.18738	0.90883	0.12331	1.12520	0.00003	1.26020
			Yes	0.00010	0.93730	0.09244	0.27541	0.12321	0.87924	0.00001	0.79592
	Yes	0.01	No	0.00004	1.22950	0.06582	0.96476	0.12588	1.01310	0.00002	1.26420
			Yes	0.00008	0.95192	0.06895	0.20674	0.12788†	0.11354	0.00005	0.01156
		0.02	No	0.00015	1.17390	0.13033	0.88923	0.12648	1.13210	0.00000	1.31490
			Yes	0.00008	0.92962	0.06058	0.20812	0.18365†	0.11337	0.00006	0.01583

Table 4.3: Resulting p-values from the $\sqrt{\varepsilon}$ test for the PA Congressional maps. Highlighted p-values indicate significance at $\alpha = 0.001$. A dagger (†) denotes cases in which the p-value for the gerrymandered plan is greater than that of the remedial plan.

Closer examination of the results shads light on how the different partisan bias measures compare to one another as label functions for a given map. Median-mean, which is considered a reliable indicator of partisan bias, achieves significance for all combinations of constraints shown in Table 4.3. We can observe that $B_{G,V}$ likewise achieves significant p-values for all configurations of traditional redistricting criteria. This indicates that $B_{G,V}$ is an alternative reliable indicator of partisan bias that performs similarly to median-mean. On the other hand, for the biased 2011 map, $\hat{\beta}(V)$ only detects significance for three of the 16 constraint combinations, while $B_{G,U}$ does not indicate significance at all. This demonstrates that $\hat{\beta}(V)$ and $B_{G,U}$ are less sensitive to detecting partisan bias under outlier analysis.

The poor performance of $\hat{\beta}(V)$ in comparison with median-mean is not necessarily unexpected. As mentioned in Subsection 4.1.1, the bias point estimate is a slow-changing function, its value altered only when a sufficient number of precinct swaps have been made to shift a seat from one party to another. From our results, we can see that it likely changes *too* slowly to be considered sensitive to small changes in district boundaries, failing the second criterion of a good label function.

On the other hand, given the similarity of the seats-votes curves drawn under the UPS and VPS assumptions, it is surprising that $B_{G,U}$ and $B_{G,V}$ show such drastically different levels of sensitivity to partisan bias on the same map. A possible reason for the reduced sensitivity of $B_{G,U}$ could be that assuming a uniform partisan swing removes an additional level of variability in voter behavior from being taken into account when graphing the seats-votes curve. Allowing the variable shifts in district-level voting outcomes through the VPS assumption may better reflect hypothetical voting patterns and, as a result, produce a better estimate of bias through the $B_{G,V}$ measure. However, further investigation is needed to fully understand this difference in performance.

In addition, it is worth reiterating that the geometric bias measure differs from median-mean in that it is not a signed measure and thus does not indicate the direction of partisan bias. This means that achieving significance using a geometric bias measure such as $B_{G,V}$ indicates that a small proportion of the sample maps generated exhibited as or more partisan bias *towards either party*. In contrast, significance under MM as the label function indicates that a small proportion of the maps exhibited as or more bias against Democratic voters, but does not consider how biased any of the sample maps are against Republican voters. One might argue that this quality makes the variable geometric bias measure $B_{G,V}$ superior to that of median-mean in the assessment of election maps, as it quantifies overall deviation from partisan symmetry and may relate more closely to the ideal of achieving districting plans that are fair for all constituents. However, in the context of bringing a suspected gerrymander to court, it may be seen as advantageous to focus on the particular party that is supposed to be disadvantaged by the districting plan in question.

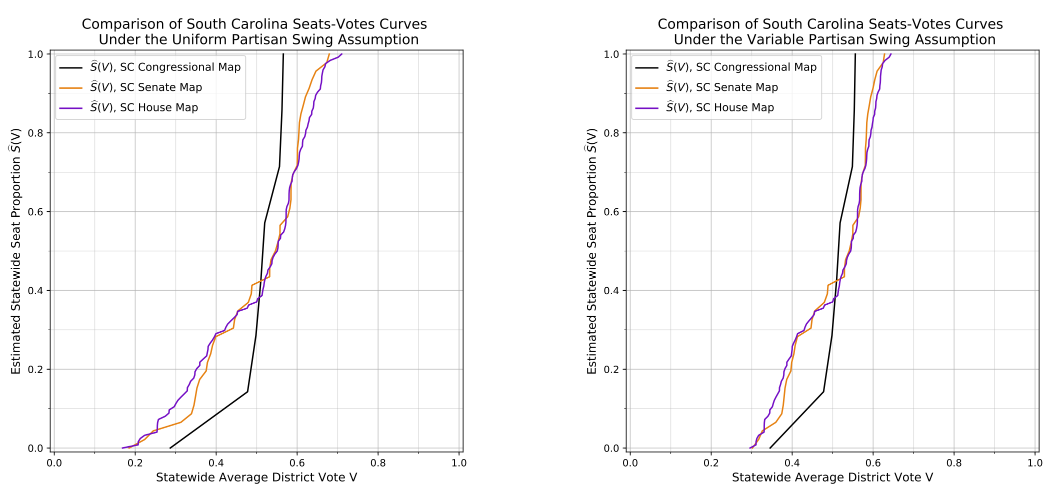
An understanding of how the various bias measures perform as label functions on fair and unfair maps informs how we proceed with the analysis of the 2011 South Carolina maps. If any of the three maps exhibits extreme partisan bias, we should expect to see the $\sqrt{\varepsilon}$ test produce p-values below the significance threshold for all or the majority of relevant constraint combinations with respect to MM and $B_{G,V}$. On the other hand, if any of the maps does not indicate extreme bias, while we may observe a few constraint combinations for which $p < 0.001$, the majority of the p-values will indicate non-significance.

4.4 The Initial South Carolina Maps

With the results of the 2011 and 2018 Pennsylvania maps in mind, we proceed with an initial look at the three South Carolina election maps analyzed in this thesis.

4.4.1 Estimated Seats-Votes Curve

The estimated seats-votes curves for the South Carolina maps are shown in Figure 4.3.



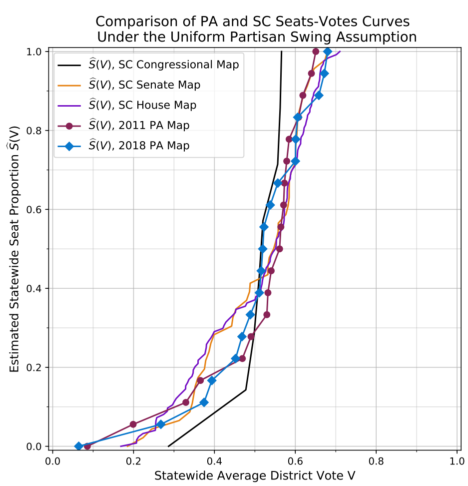
(a) Uniform Partisan Swing Assumption

(b) Variable Partisan Swing Assumption

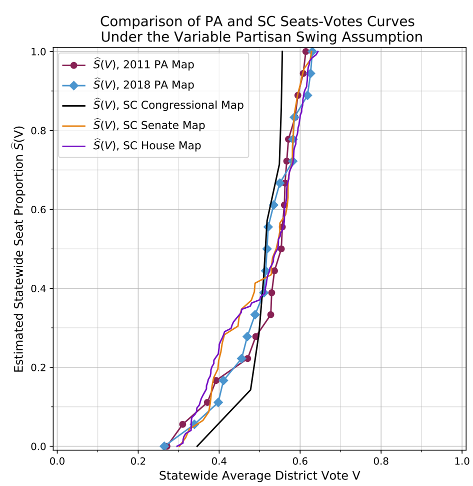
Figure 4.3: Comparison of the estimated seats-votes curves for the 2011 South Carolina election maps, drawn under both the UPS and VPS assumptions.

We can see that the graph of $\widehat{S}(V)$ for the South Carolina Congressional map, denoted with a black line, is quite close to crossing through the $(0.5, 0.5)$ midpoint, indicating that the map, at least at first glance, does not exhibit gross deviation from partisan symmetry in the region immediately around $V = 0.5$. It is noteworthy that the curve is rather steep under both assumptions, indicating that it does not take a large shift in the statewide average district vote V to switch seats from one party to the other. The seats-votes curves for the state Senate and House maps, which are nearly identical to one another, appear slightly further to the right of the graphical midpoint and display a more gradual relationship between V and the estimated seat share. All three maps, however, do indicate some degree of partisan bias advantaging Republicans.

Overlaying the curves from both South Carolina and Pennsylvania on the same graph allows us to further observe their similarities and differences. While $\widehat{S}(V)$ for the South Carolina Congressional map bears much similarity to the curve for the 2018 Pennsylvania Congressional map in roughly the $0.45 < V < 0.55$ region, the South Carolina Senate and House maps more closely resemble the 2011 Pennsylvania map in this same region. The South Carolina Congressional map also deviates more starkly in the tail regions compared with the other four maps, while the state Senate and House maps more closely align with what we observe in Pennsylvania. It is possible that this is due, in part, to the small number of Congressional seats allotted to South Carolina, which results in a less smooth curve.



(a) Uniform Partisan Swing Assumption



(b) Variable Partisan Swing Assumption

Figure 4.4: Seats-votes curves for both Pennsylvania and South Carolina overlaid for comparison.

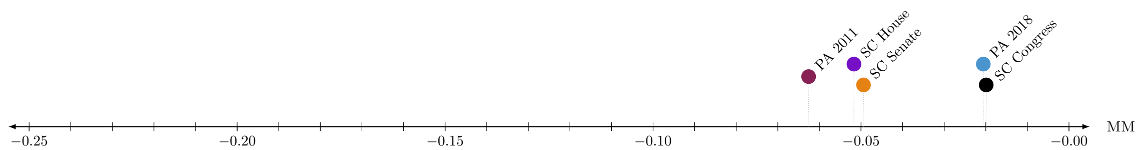
4.4.2 Initial Bias Measures

Table 4.4 summarizes the initial values for each of the four label functions imposed on the Congressional and legislative district maps in South Carolina, with measures from Pennsylvania included for comparison.

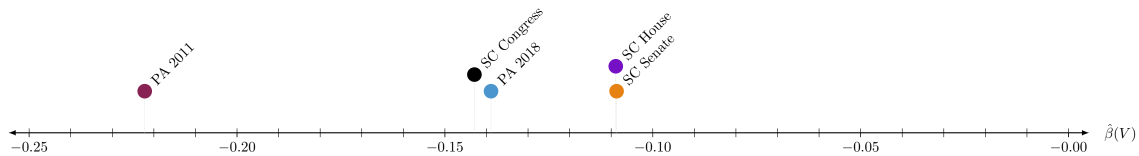
Map	MM	$\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
SC Congressional	-0.0199	-0.1429	0.0479	0.0377
SC Senate	-0.0494	-0.1087	0.0512	0.0392
SC House	-0.0517	-0.1089	0.0568	0.0430
PA 2011	*−0.0626*	*−0.2222*	*0.0942*	*0.0675*
PA 2018	*−0.0209*	*−0.1389*	*0.0525*	*0.0353*

Table 4.4: Initial bias measure values for each of the three 2011 South Carolina election maps.

We can quickly observe that the original Pennsylvania map exhibits significantly more partisan bias with respect to all four initial bias measures than any of the South Carolina maps. The South Carolina Congressional map, in fact, has initial bias measures that are strikingly similar to those of the fair remedial Pennsylvania plan, indicating even less bias from the standpoint of median-mean, the bias point estimate, and geometric bias under the UPS assumption. The South Carolina House and Senate maps, on the other hand, are a bit more variable, indicating less bias than all the Congressional maps with respect to $\hat{\beta}(V)$, bias that more closely resembles the gerrymandered Pennsylvania map with respect to MM, and similar degrees of bias for both $B_{G,U}$ and $B_{G,V}$. The relative scales of the initial bias measure values for each map are displayed in Figures 4.5 and 4.6.

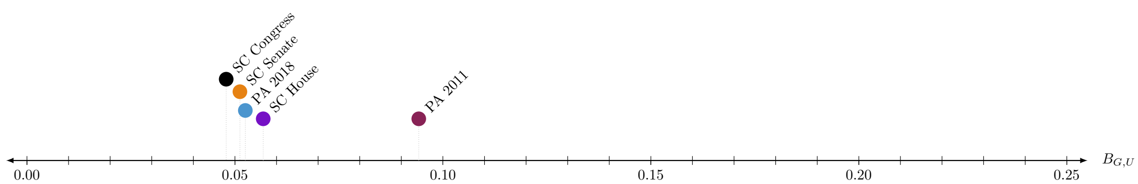


(a) Median-Mean

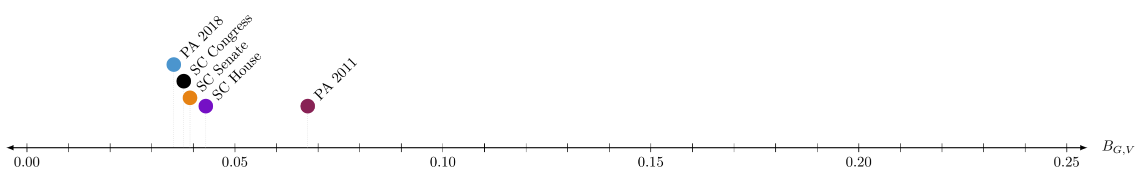


(b) Bias Point Estimate

Figure 4.5: Comparison of initial MM and $\hat{\beta}(V)$ for South Carolina and Pennsylvania.



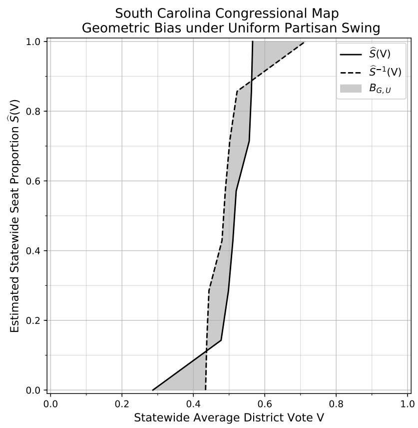
(a) Geometric Bias under UPS



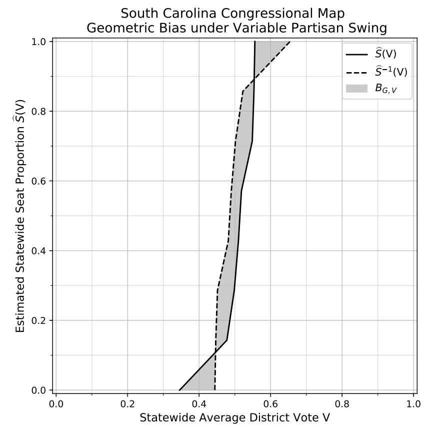
(b) Geometric Bias under VPS

Figure 4.6: Comparison of initial $B_{G,U}$ and $B_{G,V}$ for South Carolina and Pennsylvania.

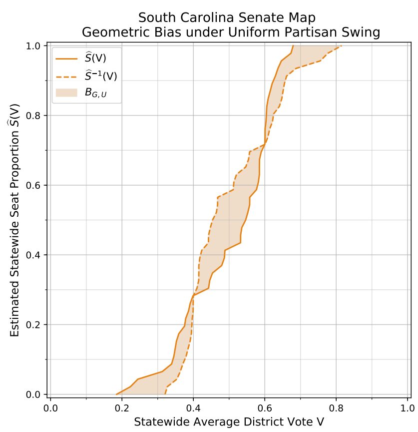
The amount of partisan bias can again be emphasized visually by looking at the corresponding graphs for $B_{G,U}$ and $B_{G,V}$ in Figure 4.7 and noting how the shaded areas compare to those in Figure 4.2. The shaded area of the South Carolina Congressional map bears much resemblance in its width and shape to that of the 2018 Pennsylvania plan. The state Senate and House maps appear to be somewhere between the geometric bias measure graphs of the two Pennsylvania plans, but the initial values of $B_{G,U}$ and $B_{G,V}$ reveal that the accumulated area between the seats-votes curve and its inverse is relatively closer to that of the fair Pennsylvania plan.



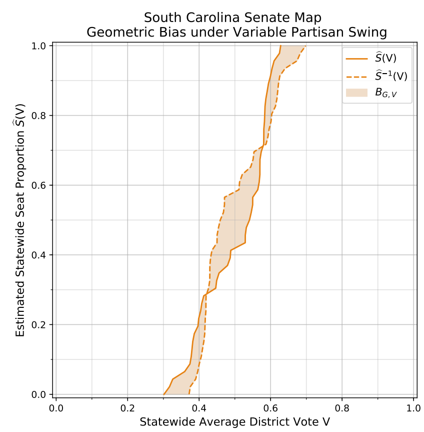
(a) $B_{G,U}$ for the SC Congressional Map



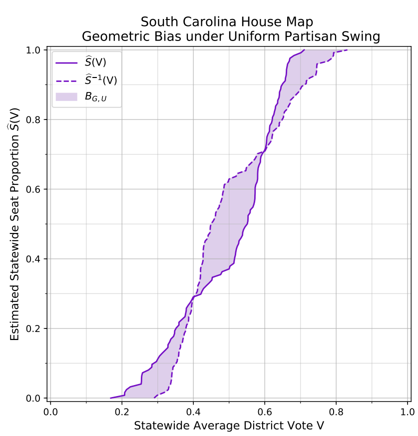
(b) $B_{G,V}$ for the SC Congressional Map



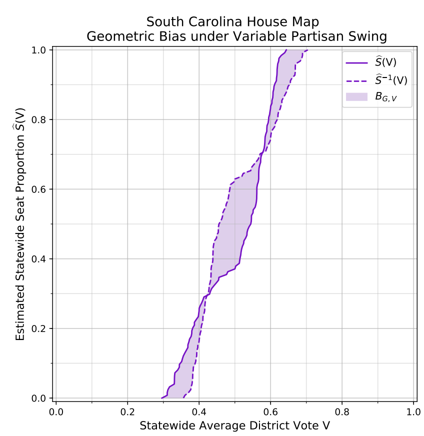
(c) $B_{G,U}$ for the SC Senate Map



(d) $B_{G,V}$ for the SC Senate Map



(e) $B_{G,U}$ for the SC Senate Map



(f) $B_{G,V}$ for the SC Senate Map

Figure 4.7: Comparison of the geometric bias measures for each South Carolina election map.

Despite the similarity of the initial bias measures in South Carolina to those of 2018 Pennsylvania map, we again turn to outlier analysis to test the likelihood of gerrymandering. This is necessary because state election maps carry several distinctive characteristics. For instance, when considering the SC Congressional map, the shape of the state itself, the distribution of population and voting preferences across precincts, and the fewer Congressional seats are all features in contrast with those of Pennsylvania. These factors likely help determine the sample of all maps that can be drawn in the state, so it is possible that, while MM $= -0.0199$ appears conservative compared to Pennsylvania, the majority of Congressional maps you *could* draw for South Carolina exhibit even less bias. Every electoral map will exhibit some inherent partisan bias, so it is most beneficial to compare possible plans within a given state as opposed to comparing one state's map to another.

4.5 Outlier Analysis Results for South Carolina

All constraint combinations under which the $\sqrt{\varepsilon}$ test was performed were selected from those described in Subsection 4.1.2 in accordance with the South Carolina Senate and House Judiciary Committee Guidelines on redistricting [26, 45]. Both sets of guidelines require that districts be compact in shape, so for each of the three South Carolina maps, both the perimeter threshold and the inverse Polsby-Popper threshold were considered. Guidelines also require that communities of interest be taken into account, so runs of the code were conducted with and without the requirement to preserve counties that are fully contained inside existing districts, and the SC Congressional map was tested with and without freezing Congressional District 6, the minority-majority district drawn in compliance with the Voting Rights Act. Lastly, both sets of guidelines impose a maximum allowable percent deviation from population equality of districts, and the bounds on population equality were chosen with respect to those specified in the documents. Results for reasonable alternative combinations of constraints are included in Appendix C, but the most relevant constraints are presented in Tables 4.5–4.7.

One thing that we observe almost immediately is that the bias point estimate and geometric bias under the UPS assumption do not achieve significance for any of the tested constraint combinations, corroborating our evidence from the Pennsylvania maps that $\hat{\beta}(V)$ and $B_{G,U}$ do not perform as reliably as indicators of partisan bias as do MM and $B_{G,V}$. However, with the exception of one case for the South Carolina Senate map, using median-mean as a label function does not achieve

Constraints				p-values			
Compactness Threshold	Freeze District 6?	Population Threshold	Preserve Counties?	MM	$\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	No	0.01	No	0.45312	0.11908	0.03790	0.01086
			Yes	0.08114	0.60432	0.04132	0.00011
		0.02	No	0.47728	0.10221	0.03773	0.01139
			Yes	0.04133	0.46509	0.04401	0.00000
	Yes	0.01	No	0.78627	1.32270	0.08286	0.05560
			Yes	1.41420	1.41420	0.00332	0.00316
		0.02	No	0.79339	1.32550	0.08596	0.06245
			Yes	1.41350	1.41420	0.08432	0.07029
Perimeter	No	0.01	No	0.62437	0.06283	0.03720	0.00761
			Yes	0.06836	0.72337	0.04004	0.00010
		0.02	No	0.53587	0.06621	0.03455	0.00666
			Yes	0.02984	0.48391	0.04485	0.00000
	Yes	0.01	No	0.82927	1.27810	0.08393	0.04951
			Yes	0.01034	1.41420	0.01195	0.01149
		0.02	No	0.82010	1.28420	0.07454	0.04079
			Yes	1.41420	1.41420	1.39930	1.39850

Table 4.5: Outlier analysis results for the South Carolina Congressional map. Highlighted p-values indicate significance at $\alpha = 0.001$.

significance for the vast majority of constraints imposed on the three maps. Given that median-mean is an agreed-upon standard for detecting partisan bias, the overall lack of significant results indicates that there is little evidence of gerrymandering in South Carolina. The geometric bias measure under the VPS assumption does indicate bias for four of the constraint combinations imposed on the South Carolina Congressional map, all of which have the county preservation requirement in common and do not require the minority-majority district to be unchanged. However, given that these instances only comprise a fraction of the possible constraint combinations, these four cases do not indicate sufficient evidence to conclude beyond a reasonable doubt that the South Carolina Congressional map is a gerrymander.

Constraints			p-values			
Compactness Threshold	Population Threshold	Preserve Counties?	MM	$\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	0.05	No	0.00612	0.08274	0.11096	0.00471
		Yes	0.00620	0.08815	0.09939	0.00506
Perimeter	0.05	No	0.00749	0.08081	0.10423	0.00459
		Yes	0.00095	0.05906	0.08331	0.00174

Table 4.6: Outlier analysis results for the South Carolina Senate map. Highlighted p-values indicate significance at $\alpha = 0.001$.

Constraints			p-values			
Compactness Threshold	Population Threshold	Preserve Counties?	MM	$\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	0.025	No	0.03802	1.30550	0.56883	0.46723
		Yes	0.02701	1.39880	0.90689	1.13480
Perimeter	0.025	No	0.03822	1.22670	0.50180	0.35868
		Yes	0.00833	1.39500	0.65659	0.43324

Table 4.7: Outlier analysis results for the South Carolina House map. Highlighted p-values indicate significance at $\alpha = 0.001$.

4.6 Summary of Analysis

The results of this analysis provide us with several insights into the utilization of partisan bias measures and outlier analysis as a diagnostic for partisan gerrymandering. We observe that the $\sqrt{\varepsilon}$ test on non-gerrymandered maps yields higher p-values than those resulting from significantly biased maps, indicating that a relatively large proportion of sample maps generated have bias measure values at least as extreme as the original map in question. These p-values typically fall outside reasonable significance thresholds used in hypothesis testing.

We also observe that certain partisan bias measures are more sensitive to detecting partisan bias than others. Median-mean and geometric bias under the VPS assumption perform consistently as indicators of partisan bias, while the bias point estimate and geometric bias under the UPS assumption are not sensitive enough to changes in districtings to perform well as label functions.

Using these insights to inform the analysis of the South Carolina Congressional, state Senate, and state House maps, we conclude that there is insufficient evidence of partisan gerrymandering in South Carolina.

Chapter 5

Discussion

The research presented in this thesis bears relevance both to the current context in which it takes place and to the future work to be done in the area of election map analysis. This chapter presents a contextualization of the results, limitations of this analysis, implications for the use of such methods, and directions for future work.

5.1 Contextualizing the Results

Based on the initial bias measure values for the South Carolina maps, we can see that there is some natural bias exhibited by the election maps as indicated by negative values of MM and $\hat{\beta}(V)$ and positive values of $B_{G,U}$ and $B_{G,V}$. South Carolina is a historically "red" state, where Democratic voters tend to cluster in urban centers and Republican voters tend to be spread out in rural areas, which contributes to the natural bias in favor of Republicans that we observe. However, the fact that the $\sqrt{\varepsilon}$ test should suggest that South Carolina is not gerrymandered despite nonzero initial partisan bias measure values may not necessarily be surprising. Recall from Section 1.3 that the federal preclearance requirement for legislature-drawn election maps was in place when the 2011 maps were drawn, even though it has since been rendered unenforceable. This preclearance requirement was initially a provision to keep states in compliant with the Voting Rights Act by incorporating an extra measure to prevent racially discriminatory districting plans. The Democratic voting tendencies of racial minority citizens is well-documented in research, showing that in the last forty years, both African-American and Hispanic voters have tended to support Democratic candidates [31, 38]. This correlation may shed light on the impact of the federal preclearance requirement on the partisan bias of election maps; in other words, maps that fairly represent minority voters are likely to also be fair with respect to partisan symmetry.

5.2 Limitations of This Project

One key limitation of this analysis — and of gerrymandering detection methods in general — is that both the partisan bias measures discussed and methods of outlier analysis currently available assume a two-party system. This seems to work reasonably well in some cases; for example, only a modest 1.72% of the votes cast in the 2010 gubernatorial race chosen as a proxy for the South Carolina analysis were for third-party candidates, making an assumed two-party system appropriate. However, this assumption is limiting and perhaps oversimplifies voting behavior.

An additional limitation is the time required to perform outlier analysis in this manner. The C++ code used to simulate the Markov chain and perform the $\sqrt{\varepsilon}$ test can take a relatively long time to run and requires uninterrupted computing time. The runtime varies not only with the number of steps taken along the chain, but also with the computational complexity of the bias measure(s) chosen, the constraints imposed on the map, and the number of precincts and election districts in the map. For instance, the code producing outlier analysis results for the South Carolina Congressional map ran from 2–400 hours, while that for the state Senate map ran from 150–950 hours (see Appendix D for a complete summary of runtimes). Some of the code for the state House map under certain constraints even exceeded the maximum runtime of 1440 hours, or sixty days, allotted on the Palmetto Cluster and terminated shortly after completing 2^{35} steps along the chain. The amount of time and resources required to run the analysis presents a challenge for assessing proposed election maps proactively before they are established in an electoral system.

These methods provide a useful way to identify extremely biased maps but cannot be utilized to generate samples from the stationary distribution of possible maps [22]. It is also worth mentioning that there are specific instances, though presumably rare, under which the $\sqrt{\varepsilon}$ test could yield results that do not fully reflect how unusual a districting is with respect to the sample of possible districtings. One distinction of Chikina et al.'s approach is that it tests a map's status as a *local* outlier as opposed to a global one. While this is powerful, as it does not require knowledge about the mixing time of the Markov chain, problems could arise for chains with disconnected state spaces. A disconnected state space may occur if, for example, the bounds on population inequality are restrictive enough to prevent precinct swaps that are required to access other reasonable districtings. Such chains will never mix, and though the $\sqrt{\varepsilon}$ test still applies, it can only report on the outlier status of the map with respect to the other possible maps generated from the portion of the state space it traverses. Cho and Rubinstein-Salzedo demonstrate two examples of this phenomenon, one in which a map is identified as a local outlier but is not a global outlier with respect to all possible maps [24].

Despite these limitations, there is still strong rationale behind using the $\sqrt{\varepsilon}$ test. The application of a global outlier test would assume that mapmakers construct gerrymandered maps by identifying a map which is a global optimum with respect to its measure of bias. However, Chikina et al. point out that solving such a problem is highly inefficient; if formulated as a planar-graph partitioning problem, the identification of a map that is a global optimum with respect to partisan bias is NP-hard [23]. Further, the state space of all feasible maps against which the gerrymandered map is an outlier is extremely complicated, and it follows that the enumeration of this state space that would be required to identify the gerrymandered map as a global outlier is #P-hard (read "number-P hard") [28]. In practicality, rather than attempting to solve this global optimization problem, it is more reasonable that mapmakers will search for local optimum by starting with a map and making iterative changes to increase the degree of partisan bias in their party's favor until it can no longer be significantly improved. Local outlier tests such as the $\sqrt{\varepsilon}$ test are rigorous alternatives to global outlier techniques that still get at the heart of gerrymandering claims and more closely mirror a plausible way in which a gerrymander might be constructed. The test by Chikina et al. identifies cases in which the partisan bias of a map quickly falls apart as the map is modified along the Markov chain, indicating the precise and careful crafting of district boundaries that is exhibited by gerrymandered maps.

Lastly, the usefulness of this approach for assessing election maps from a legal standpoint is still up for debate. Making an informed decision about how to use the results of this approach requires some degree of mathematical understanding, which could create a barrier to its use as a legal standard by which to evaluate the likelihood of gerrymandering. In the end, it is up to state courts to decide whether significant results from the $\sqrt{\varepsilon}$ test are enough to constitute sufficient evidence of a gerrymander.

5.3 Implications

Outlier analysis is one of the most rigorous, feasible, and mathematically sound methods of analyzing a map for the likelihood of partisan gerrymandering. As such, it carries implications for practice. One such implication is the necessity that researchers utilizing such an approach take time to understand what the results can and cannot say about the likelihood of partisan intent in the design of election maps. A second and perhaps more crucial implication is the ethical use of the diagnostic. The fact that there are different acceptable partisan bias measures that all indicate asymmetry in the seats-votes curve, but perform differently with respect to outlier analysis, could lead someone trying to demonstrate the fairness of a map to use a less-sensitive label function such as $\hat{\beta}(V)$ to achieve non-significant results and argue that it passes the $\sqrt{\varepsilon}$ test and is thus unbiased, when in reality the map might demonstrate overwhelming bias with respect to another label function. Chikina et al. acknowledge the gravity of this concern, emphasizing that "care must be taken to choose a 'canonical' label function ω, so that there is no concern that ω was carefully crafted" in response to the original map [22]. Given that median-mean has historical usage as an indicator of partisan bias, is similarly recognized as such by modern scholarship, and has been successfully utilized as a label function for this test in a court of law through *LWV v. PA*, any measure of partisan asymmetry used as a label function should demonstrate results that are similar to those produced by median-mean, such as $B_{G,V}$.

5.4 Future Work

There is much left to be done as a continuation of this project. One area for future work is to further investigate why certain bias measures perform differently from one another as label functions. Of particular interest is the differences between the geometric bias measure B_G under both the uniform and variable partisan swing assumptions. As hypothesized in Chapter 4, it is suspected that the differing degrees of partisan voting shifts across districts that are accounted for by the VPS assumption provides an extra dimension for estimating voter behavior that is lost by the UPS assumption, making it a more accurate indicator of partisan bias.

In future work, the algorithm for computing geometric bias in the C++ code will be adjusted to correspond to the directionality we see in other measures of partisan asymmetry with respect to partisanship. Specifically, Equation 2.11 will be altered to be

$$B_G = \begin{cases} \displaystyle\int_0^1 \left|\widehat{S}(v) - \widehat{S}^{-1}(v)\right| dv & \text{if } \widehat{S}(0.5) > 0.5, \text{ and} \\ \\ \displaystyle -\int_0^1 \left|\widehat{S}(v) - \widehat{S}^{-1}(v)\right| dv & \text{if } \widehat{S}(0.5) < 0.5, \end{cases} \tag{5.1}$$

where negative values of B_G indicate a Republican advantage. Doing so would be advantageous from the standpoint of achieving additional transparency and clarity in the use of partisan bias measures as label functions of the Markov chain.

Also of interest is how the $\sqrt{\varepsilon}$ test of outlier analysis compares with the simulation approach developed by Chen and Rodden with respect to the label functions used in this analysis [20]. In practice, approaches to quantify the effects of gerrymandering rarely stand on their own; for example, both Chen and Rodden's approach and that of Chikina et al. were used to argue for the redrawing of the Pennsylvania Congressional map [6]. It would be valuable to see how these methods compare to one another and whether the results are consistent for South Carolina maps. Also, because the simulation algorithm for generating sample maps is distinct from the Markov chain approach, this method may generate additional sample maps that are not part of the state space visited by the chain, which could help mitigate the problem posed by Cho and Rubinstein-Salzedo in which the state space is disconnected and viable maps are not considered in the $\sqrt{\varepsilon}$ test. Incorporating both approaches to outlier analysis would produce an even more comprehensive and rigorous analysis.

Given the significant runtime required for this analysis, it is natural to question whether the $\sqrt{\varepsilon}$ test can be performed more efficiently. One direction to explore is the option of running instances of the Markov chain in parallel to generate the same number of sample maps in a fraction of the time. The C++ code provided by Pegden has an option for doing so, but it is not immediately clear how the theory presented in the *PNAS* paper by Chikina et al. corresponds to this method; more precisely, it is not directly stated how running multiple chains in parallel from the same initial state X_0 alters the upper bound on the probability that X_0 is an outlier. Care would need to be taken to understand how simulating multiple trajectories from the initial state affects the analysis results. Another direction would be to analyze whether the number of steps required for the chain to achieve a stable significance level can be reduced. This question arises from the behavior of the chain that was observed in this analysis. For the results in Chapter 4, the p-value at a given point of the chain was plotted against the number of steps taken at that point. In many cases, the p-values become stable within 2^{35} steps, after which the significance level does not change drastically regardless of how many additional sample maps are generated (e.g., Subfigures 5.1a and 5.1c). In a few instances, however, the p-values bounce around for the first several billion steps, then steadily increase (Subfigure 5.1b) or decrease (Subfigure 5.1d). It would be valuable to further analyze these results to investigate whether there is a set of conditions under which running the chain for a shorter period of time is sufficient to achieve a reliable level of significance. Doing so could shave orders of magnitude off the runtime and increase the practicality of the approach.

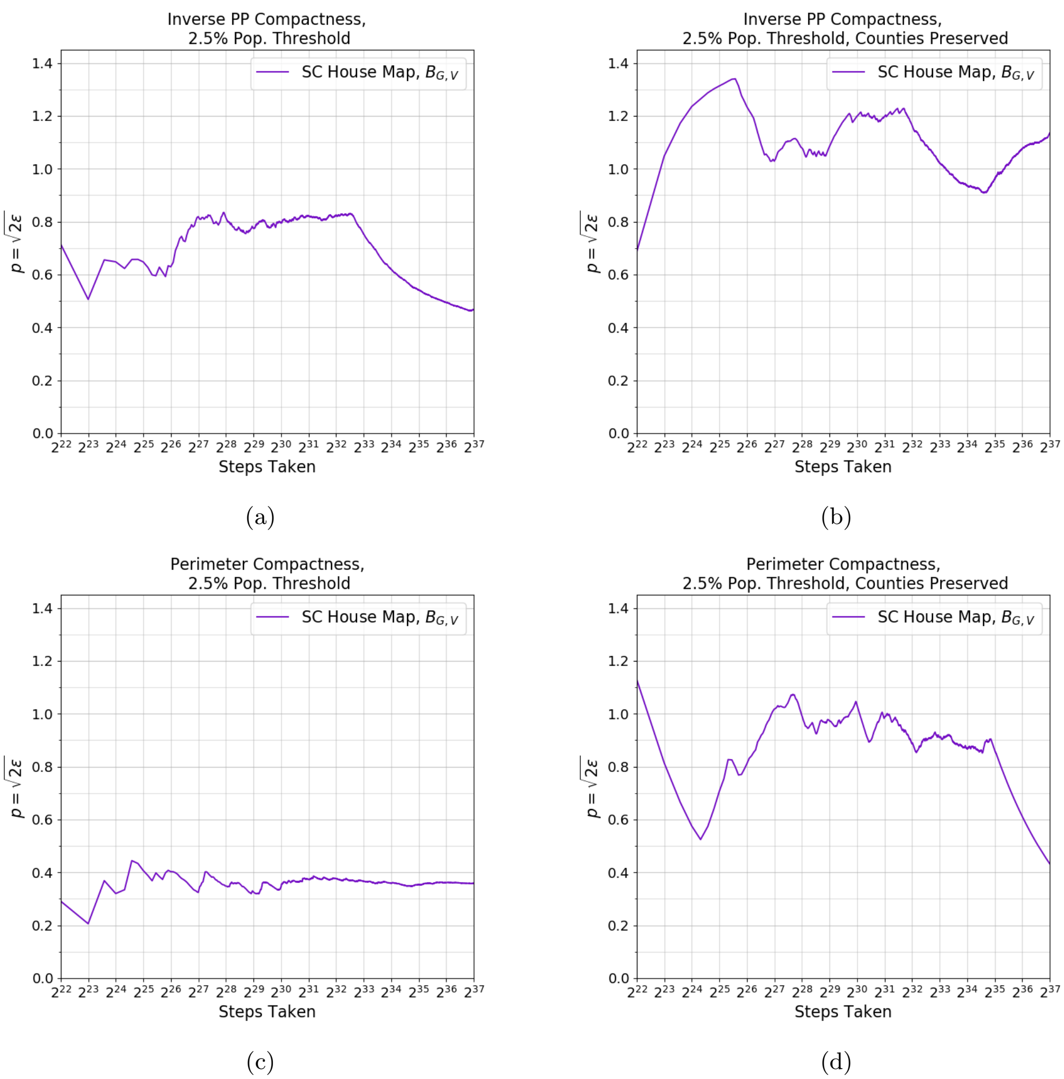


Figure 5.1: Semi-log plots (base 2) of p-values vs. steps taken along the Markov chain for $B_{G,V}$ on the SC House map.

As noted in this thesis, a characteristic that sets the South Carolina Congressional map apart from that of Pennsylvania, but is by no means unique, is its small number of Congressional districts. Future work should be dedicated to further exploring the question of how having few districts in an election map may affect the $\sqrt{\varepsilon}$ test. It takes a greater number of precinct shifts to actually flip a Congressional seat from one party to another for a state with a few districts versus many, and it is suspected that this could affect the sensitivity of the analysis with respect to certain label functions. The following experiment is proposed to test the hypothesis that partisan bias measures of election maps become less sensitive to gerrymandering in states with fewer election districts.

1. Overlay South Carolina with a grid of squares. For each square:
   - Assign 0 if it does not overlap with SC.
   - Assign 1 if it does overlap with SC.
2. Eliminate squares with a value of zero. We will be left with a grid version of South Carolina. (Ideally, the number of squares will approximately equal the number of precincts in SC.)
3. Calculate the percent overlap of each precinct with each grid square.
4. Using the observed values from the underlying South Carolina map, assign a population, Democratic votes, and Republican votes to each grid square proportionally by area.
5. For each grid square, randomly generate an election outcome.
   - Let D_i be a random variable representing the number of Democratic votes cast in square i.
   - $D_i \sim Binom(n_i, p_i)$, where n_i is the number of voters in square i and p_i is the probability that a single voter in square i votes for a Democratic candidate.
   - Estimate p_i using $\hat{p_i}$, the observed proportion of Democratic votes in square i.
   - $n_i \geq 30$ for all voting precincts; thus, by the Central Limit Theorem, $D_i \sim N_i\left(n_i p_i, n_i p_i\left(1 - p_i\right)\right)$.
   - Generate a random draw D_i^O from the distribution of D_i.
   - The simulated proportion of Democratic votes cast in square i is now $D_i^O n_i$.

6. For each randomly generated election, calculate the seats won and the value of each bias measure under *several different districting plans* imposed on the grid map. These districting plans may be generated using:
   - Chikina et al.'s Markov chain method [22],
   - Chen and Rodden's simulation method [20], or
   - Some alternative method.
7. Scale both the population of South Carolina and the number of grid squares by enough to allow for one additional Congressional seat. Repeat the experiment.

This process would be repeated a finite number of times until the map has been tested with a reasonably large number of seats in the electorate. For example, iterations could be performed until the map population had been scaled up to roughly correspond to fifty-three Congressional districts; this would be a reasonable place to terminate the experiment, as it is the largest number of Congressional seats apportioned to a given state in the 2010 redistricting cycle, held by California.

The last, but perhaps most important, area of future work mentioned in this thesis is the application of these results and insights to the upcoming redistricting cycle. The 2020 Census and subsequent redistricting process are just around the corner, and new election maps will be proposed that will govern each state's electoral system for the next ten years. Hopefully, the insights gained by this study will be utilized to assist in the process of assessing proposed districting plans, ultimately helping to establish electoral systems that fairly represent their constituents and uphold the principles of a just democracy.

Appendices

Appendix A Markov Chain Used for Outlier Analysis

A formal description of the Markov chain used for outlier analysis is presented in this appendix, building on the description of the transition probabilities in Chikina et al. and drawing on principles of Markov chain theory [22, 46, 51].

Let $\mathcal{M} = X_0, X_1, \ldots$ be a Markov chain on state space $\Sigma = \{i_0, i_1, \ldots, i_{k-1}, i, j\}$, where Σ represents the set of all possible districtings. Also let $N_{\max} > 0$ be the theoretical maximum number of possible transitions from any state in Σ. The transition probabilities from state i to state j, for any $i, j \in \Sigma$, are defined as follows:

1. Let S_i be the set of all valid transitions from state i, which is the set of all pairs (p, D), where p is a precinct in district D_p and $D \neq D_p$ is a district adjacent to precinct p. Let $N_{S_i} = |S_i| > 0$ be the size of this set.
2. Remain in state i with probability $1 - \frac{N_{S_i}}{N_{\max}}$. Continue with probability $\frac{N_{S_i}}{N_{\max}}$.
3. Choose one pair $(p_0, D_0) \in S_i$ uniformly at random; that is, with probability $\frac{1}{N_{S_i}}$.
4. Transition to state j by changing the district membership of p_0 from D_{p_0} to D_0 if the resulting districting falls within the constraints imposed on the election maps. If not, remain in the current state.

Note that the probability of transitioning from state i to state j for any $i \neq j$ is

$$\begin{aligned}
p_{i,j} &= P(\text{continue} \cap \text{swap}) \\
&= P(\text{swap} \mid \text{continue}) \cdot P(\text{continue}) \\
&= \frac{1}{N_{S_i}} \cdot \frac{N_{S_i}}{N_{\max}} \\
&= \frac{1}{N_{\max}},
\end{aligned}$$

which is independent of the starting state i. Thus, the probability transition matrix $\mathcal{P}$ is stated element-wise as

$$p_{i,j} = \begin{cases} 1 - \frac{N_{S_i}}{N_{\max}} & \text{for } i = j \text{, corresponds to remaining in state } i, \\ \frac{1}{N_{\max}} & \text{for } i \neq j \text{, } j \text{ corresponds to valid transition, and} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

The construction of the transition matrix yields several properties of the chain.

Proposition 1. $\mathcal{M}$ converges to a stationary distribution π.

Proof. $N_{\max}$ is chosen such that $N_{\max} > N_{S_i}$ for all i, making the diagonal entries of $\mathcal{P}$ to be $p_{i,i} = 1 - \frac{N_{S_i}}{N_{\max}} > 0$. Thus some power of the transition matrix $\mathcal{P}$ has all positive entries; that is, for some $n \geq 1$, we have that $p^n_{i,j} > 0$ for all i, j. By definition, the Markov chain $\mathcal{M}$ is regular, implying that it converges to a stationary distribution π. □

Proposition 2. The stationary distribution π is unique.

Proof. Because $\mathcal{M}$ is regular, all states in Σ communicate and thus $\mathcal{M}$ is irreducible, which implies that the stationary distribution π is unique. □

Proposition 3. The stationary distribution π is uniform.

Proof. Note the following properties of $\mathcal{M}$:

- The state space Σ of the Markov chain is finite. (Even though the number of possible districtings is enormous, there is a finite number of possible ways to draw the district boundary lines when doing so using precinct swaps.)
- Because the diagonal elements of $\mathcal{P}$ are strictly positive ($p_{i,i} > 0$), $\mathcal{M}$ is aperiodic.
- $\mathcal{P}$ is symmetric by construction.

In addition, note that the sum of row i of $\mathcal{P}$ is

$$\begin{aligned}\sum_j p_{i,j} &= p_{i,i} + \sum_{i \neq j} p_{i,j} \\ &= 1 - \frac{N_{S_i}}{N_{\max}} + N_{S_i}\left(\frac{1}{N_{\max}}\right) \\ &= 1,\end{aligned}$$

and by the symmetry of $\mathcal{P}$, the columns also sum to one. Thus, $\mathcal{M}$ is doubly stochastic.

Because $\mathcal{M}$ is finite, irreducible, aperiodic, and doubly stochastic, the stationary distribution π is uniform with $\pi_i = \frac{1}{|\Sigma|}$ for all i. $\square$

Proposition 4. $\mathcal{M}$ is reversible.

Proof. The symmetry of $\mathcal{P}$ and uniform stationary distribution π imply that

$$\pi_i p_{i,j} = \pi_j p_{j,i} \text{ for all } i, j,$$

and thus $\mathcal{M}$ is reversible and the $\sqrt{\varepsilon}$ test applies. $\square$

In application, it is nearly impossible to observe the theory above with actual election maps because the state space Σ of all possible districtings is enormous. For illustration purposes, a simplified example of the Markov chain described in this appendix is presented below.

Consider a square region with two districts — a purple district P and a blue district B — which are each allocated two equally sized precincts, denoted $p_1, \ldots, p_4$. The original districting of the region can be represented as shown.

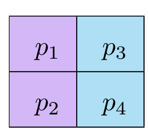


Figure A.1: An example of a region with two election districts and four precincts.

Each of the cells in Figure A.2 on the following page represents a possible districting in the state space Σ, taking into account the restriction that precincts touching at a point are not considered contiguous. For each state i, the set of possible transitions S_i and the size of that set N_{S_i} are included.

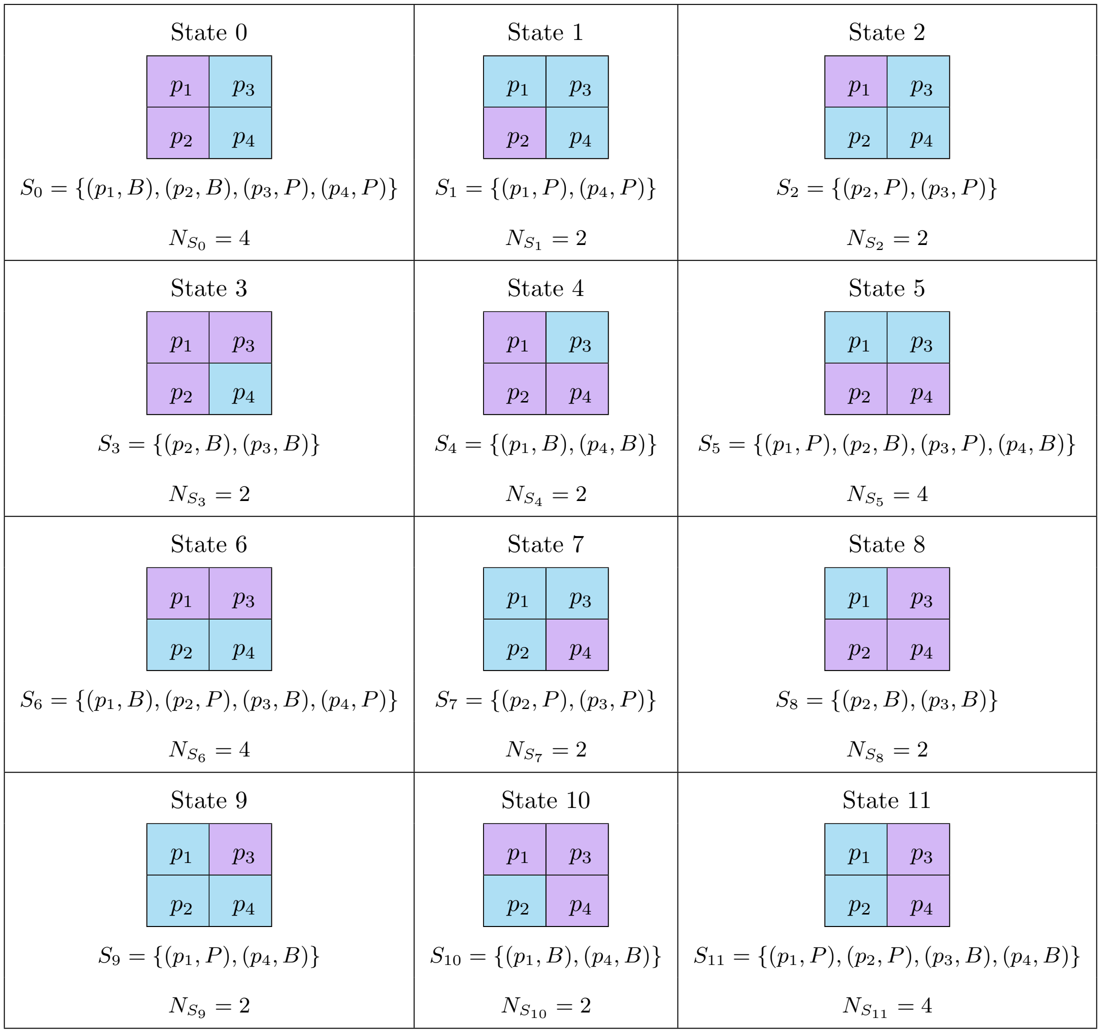


Figure A.2: All possible states, or valid districtings, for the example election map in Figure A.1.

The graph of allowable transitions between the above states (districtings) is shown in Figure A.3, with loops that indicate remaining in a state omitted for clarity.

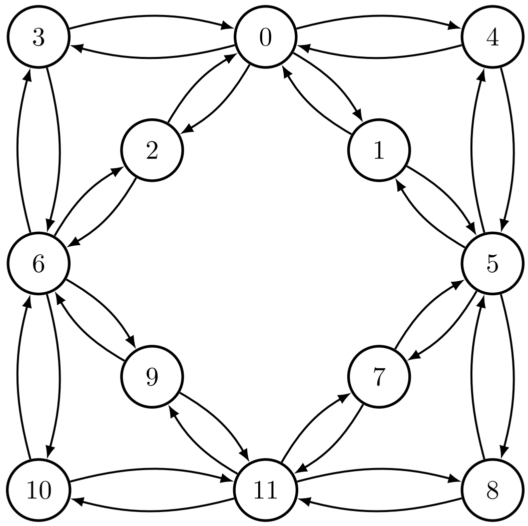


Figure A.3: A graph showing possible transitions between the states enumerated in Figure A.2.

We choose $N_{\max}$ to be the possible number of total configurations of precincts between the two districts, which is (number of precincts) $\times$ (number of districts) $= 4 \times 2 = 8$. This ensures that $N_{\max} > N_{S_i}$ for all i, as we know that this number includes invalid districtings that violate the contiguity principle. Then, using the construction of the chain defined by Chikina et al. and described above, the resulting probability transition matrix $\mathcal{P}$ can be written as follows.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} &
\begin{bmatrix}
1/2 & 1/8 & 1/8 & 1/8 & 1/8 & & & & & & & \\
1/8 & 3/4 & & & & 1/8 & & & & & & \\
1/8 & & 3/4 & & & & 1/8 & & & & & \\
1/8 & & & 3/4 & & & 1/8 & & & & & \\
1/8 & & & & 3/4 & 1/8 & & & & & & \\
 & 1/8 & & & 1/8 & 1/2 & & 1/8 & 1/8 & & & \\
 & & 1/8 & 1/8 & & & 1/2 & & & 1/8 & 1/8 & \\
 & & & & & 1/8 & & 3/4 & & & & 1/8 \\
 & & & & & 1/8 & & & 3/4 & & & 1/8 \\
 & & & & & & 1/8 & & & 3/4 & & 1/8 \\
 & & & & & & 1/8 & & & & 3/4 & 1/8 \\
 & & & & & & & 1/8 & 1/8 & 1/8 & 1/8 & 1/2
\end{bmatrix}
\end{array}
$$

With this relatively small example, we can observe the convergence of the chain to its stationary distribution. To do so, the probability transition matrix $\mathcal{P}$ was entered into MatLab R2018b.

```
P = [.500 .125 .125 .125 .125 0.00 0.00 0.00 0.00 0.00 0.00 0.00;
        .125 .750 0.00 0.00 0.00 .125 0.00 0.00 0.00 0.00 0.00 0.00;
        .125 0.00 .750 0.00 0.00 0.00 .125 0.00 0.00 0.00 0.00 0.00;
        .125 0.00 0.00 .750 0.00 0.00 .125 0.00 0.00 0.00 0.00 0.00;
        .125 0.00 0.00 0.00 .750 .125 0.00 0.00 0.00 0.00 0.00 0.00;
        0.00 .125 0.00 0.00 .125 .500 0.00 .125 .125 0.00 0.00 0.00;
        0.00 0.00 .125 .125 0.00 0.00 .500 0.00 0.00 .125 .125 0.00;
        0.00 0.00 0.00 0.00 0.00 .125 0.00 .750 0.00 0.00 0.00 .125;
        0.00 0.00 0.00 0.00 0.00 .125 0.00 0.00 .750 0.00 0.00 .125;
        0.00 0.00 0.00 0.00 0.00 0.00 .125 0.00 0.00 .750 0.00 .125;
        0.00 0.00 0.00 0.00 0.00 0.00 .125 0.00 0.00 0.00 .750 .125;
        0.00 0.00 0.00 0.00 0.00 0.00 0.00 .125 .125 .125 .125 .500]
```

Successive powers of $\mathcal{P}$ were then computed to observe the chain's behavior. After four steps, one can observe that all entries of the transition matrix are positive, showing that $\mathcal{P}$ is indeed a regular Markov chain.

.1973	.1563	.1563	.1563	.1563	.0684	.0684	.0098	.0098	.0098	.0098	.0020
.1563	.4033	.0439	.0439	.0869	.1563	.0098	.0439	.0439	.0010	.0010	.0098
.1563	.0439	.4033	.0869	.0439	.0098	.1563	.0010	.0010	.0439	.0439	.0098
.1563	.0439	.0869	.4033	.0439	.0098	.1563	.0010	.0010	.0439	.0439	.0098
.1563	.0869	.0439	.0439	.4033	.1563	.0098	.0439	.0439	.0010	.0010	.0098
.0684	.1563	.0098	.0098	.1563	.1973	.0020	.1563	.1563	.0098	.0098	.0684
.0684	.0098	.1563	.1563	.0098	.0020	.1973	.0098	.0098	.1563	.1563	.0684
.0098	.0439	.0010	.0010	.0439	.1563	.0098	.4033	.0869	.0439	.0439	.1563
.0098	.0439	.0010	.0010	.0439	.1563	.0098	.0869	.4033	.0439	.0439	.1563
.0098	.0010	.0439	.0439	.0010	.0098	.1563	.0439	.0439	.4033	.0869	.1563
.0098	.0010	.0439	.0439	.0010	.0098	.1563	.0439	.0439	.0869	.4033	.1563
.0020	.0098	.0098	.0098	.0098	.0684	.0684	.1563	.1563	.1563	.1563	.1973

The mixing time for this chain is 93 steps, at which one can observe that all entries of $\mathcal{P}^{93}$ are equal, at least to the precision of four decimal places.

.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833
.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833	.0833

We can thus see that the Markov chain converges to the uniform stationary distribution

$$\pi = \left[\tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\ \tfrac{1}{12}\right],$$

indicating that the long-run proportion of time the Markov chain will spend in a given state is the same for any districting, and thus all possible maps in the state space are given equal weight.

Appendix B Explanation of the Chikina et al. Proof

This appendix is a supplement to the proof of the theorem in Chikina et al. used to define the $\sqrt{\varepsilon}$ test [22]. It is merely an effort to assist the reader in understanding the proof by providing additional detail. We first begin with two key definitions.

Definition 1. A real number α_0 is an ε**-outlier** among $\alpha_0, \alpha_1, \ldots, \alpha_k$ if there are, at most, $\varepsilon(k+1)$ indices i for which $\alpha_i \leq \alpha_0$.

Definition 2. σ_j is ℓ**-small** among $\sigma_0, \sigma_1, \ldots, \sigma_k$ if there are, at most, ℓ indices $i \neq j$ among $0, \ldots, k$ such that the label of σ_i is, at most, the label of σ_j.

Following from these definitions, note that for a Markov chain $X_0, X_1, X_2, \ldots$ on the state space $i_0, i_1, \ldots, i_k \in \Sigma$ with real-valued label function $\omega : \Sigma \to \mathbb{R}$, the following are equivalent:

(a) The label of X_0 is an ε-outlier among the labels for $X_0, X_1, \ldots, X_k$.

(b) X_0 is $\lfloor \varepsilon(k+1) - 1 \rfloor$-small among $X_0, X_1, \ldots, X_k$.

We now proceed with the statement of the theorem proposed in the text by Chikina et al.

Theorem 1. Let $\mathcal{M} = X_0, X_1, \ldots$ be a reversible Markov chain with a stationary distribution π and state space Σ, and suppose the states of $\mathcal{M}$ have real-valued labels defined by $\omega : \Sigma \to \mathbb{R}$. If $X_0 \sim \pi$, then for any fixed k, the probability that the label of X_0 is an ε-outlier among the list of labels observed in the trajectory $X_0, X_1, \ldots, X_k$ is, at most, $\sqrt{2\varepsilon}$. [22]

Proof. Let π denote any stationary distribution for $\mathcal{M}$ and suppose that the initial state X_0 is distributed $X_0 \sim \pi$, so that $X_s \sim \pi$ for all s. For $0 \leq j \leq k$ and state $i \in \Sigma$, we define the following probabilities:

$$\rho^k_{j,\ell} := P\left(X_j \text{ is } \ell\text{-small among } X_0, \ldots, X_k\right) \tag{B.1}$$

and

$$\rho^k_{j,\ell}(i) := P\left(X_j \text{ is } \ell\text{-small among } X_0, \ldots, X_k \,|\, X_j = i\right). \tag{B.2}$$

Observe that, because $X_s \sim \pi$ for all s, we also have that

$$\rho^k_{j,\ell}(i) = P\left(X_{s+j} \text{ is } \ell\text{-small among } X_s, \dots, X_{s+k} \,|\, X_{s+j} = i\right). \tag{B.3}$$

Note the following observations.

Observation 1: $\rho^k_{j,\ell}(i) = \rho^k_{k-j,\ell}(i)$.

Proof. Because $\mathcal{M} = X_0, X_1, \dots$ is stationary (with all X_s's having the same distribution) and reversible,

$$P\left((X_0, \dots, X_k) = (i_0, \dots, i_k)\right) = P\left((X_0, \dots, X_k) = (i_k, \dots, i_0)\right)$$

for any fixed sequence of states $(i_0, \dots, i_k)$.

Thus, for $0 \leq j \leq k$, any sequence $(i_0, \dots, i_k)$ for which $i_j = i$ and i_j is ℓ-small corresponds to an equiprobable sequence $(i_k, \dots, i_0)$ for which $i_{k-j} = i$ and i_{k-j} is ℓ-small. That is,

$$P\left(X_j \text{ is } \ell\text{-small among } X_0, \dots, X_k \,|\, X_j = i\right) = P\left(X_{k-j} \text{ is } \ell\text{-small among } X_0, \dots, X_k \,|\, X_{k-j} = i\right).$$

$\therefore\ \rho^k_{j,\ell}(i) = \rho^k_{k-j,\ell}(i)$. $\square$

Observation 2: $\rho^k_{j,2\ell}(i) = \rho^j_{j,\ell}(i) \cdot \rho^{k-j}_{0,\ell}(i)$.

Proof. Consider the events that X_j is ℓ-small among $X_0, \ldots, X_j$ and among $X_j, \ldots, X_k$. Note that the two events are conditionally independent when conditioning on $X_j = i$.

The probability of the first event, by definition from Equation B.1, is $\rho^j_{j,l}(i)$.

If we consider $\rho^{k-j}_{0,\ell}(i)$ and apply the definition from Equation B.2, we have that

$$\begin{aligned}\rho^{k-j}_{0,\ell}(i) &= P\left(X_s \text{ is } \ell\text{-small among } X_s, \ldots, X_{s+k-j} | X_s = i\right) \\ &\overset{(\text{let } s=j)}{=} P\left(X_j \text{ is } \ell\text{-small among } X_j, \ldots, X_k | X_j = i\right),\end{aligned}$$

which is the probability of the second event.

Finally, the probability of both events occurring is the probability that X_j is 2ℓ-small among $X_0, \ldots, X_k$; that is,

$$P\left(X_j \text{ is } 2\ell\text{-small among } X_0, \ldots, X_k\right) = \rho^k_{j,2\ell}.$$

$\therefore\ \rho^k_{j,2\ell}(i) = \rho^j_{j,\ell}(i) \cdot \rho^{k-j}_{0,\ell}(i)$ by definition of conditional independence. ☐

It follows that

$$\rho^k_{j,2\ell}(i) = \rho^j_{j,\ell}(i) \cdot \rho^{k-j}_{0,\ell}(i) = \rho^j_{0,\ell}(i) \cdot \rho^{k-j}_{0,\ell}(i) \geq \left(\rho^k_{0,\ell}(i)\right)^2, \tag{B.4}$$

where the first equality follows from Observation 2, the second equality from Observation 1, and the inequality from the fact that $\rho^k_{j,\ell}(i)$ is monotone non-increasing in k for a fixed j, ℓ, and i.

Observe that $\rho^k_{j,\ell} = P\left(X_j \text{ is } \ell\text{-small among } X_0, \ldots, X_k\right) = E\left[\rho^k_{j,\ell}(X_j)\right]$, where expectation is taken over the random choice of $X_j \sim \pi$. Thus, taking the expectation of both sides in Equation B.4, we find that

$$\rho^k_{j,2\ell} = E\left[\rho^k_{j,2\ell}(i)\right] \geq E\left[\left(\rho^k_{0,\ell}(i)\right)^2\right] \geq \left(E\left[\rho^k_{0,\ell}(i)\right]\right)^2 = \left(\rho^k_{0,\ell}\right)^2, \tag{B.5}$$

where the second inequality follows from Cauchy-Schwarz.

Summing the left- and right-hand sides of Equation B.5 over j, we obtain

$$\sum_{j=0}^{k} \rho_{j,2\ell}^{k} \geq \sum_{j=0}^{k} \left(\rho_{0,\ell}^{k}\right)^{2} = (k+1)\left(\rho_{0,\ell}^{k}\right)^{2}. \tag{B.6}$$

For $0 \leq j \leq k$, let ξ_j be the indicator variable defined to be 1 whenever X_j is 2ℓ-small among $X_0, \ldots, X_k$ and 0 otherwise. Then there are at most $2\ell + 1$ 2ℓ-small terms; that is,

$$\sum_{j=0}^{k} \xi_j \leq 2\ell + 1. \tag{B.7}$$

Recall that we can express a probability as the expectation of an indicator function. Thus, we have

$$\sum_{j=0}^{k} \rho_{j,2\ell}^{k} = \sum_{j=0}^{k} E(\xi_j) = E\left[\sum_{j=0}^{k} \xi_j\right] \leq E\left[2\ell + 1\right] = 2\ell + 1, \tag{B.8}$$

where the second equality follows from the linearity of expectation, and the inequality follows from Equation B.7. Combining Equations B.6 and B.8, we have

$$2\ell + 1 \geq (k+1)\left(\rho_{0,\ell}^{k}\right)^{2}, \tag{B.9}$$

and solving for the probability, we obtain

$$\rho_{0,\ell}^{k} \leq \sqrt{\frac{2\ell+1}{k+1}}. \tag{B.10}$$

Using the equivalent Definitions 1 and 2, we have that $\ell = \lfloor \varepsilon(k+1) - 1 \rfloor \leq \varepsilon(k+1) - 1$, and thus

$$2\ell + 1 \leq 2\varepsilon(k+1) - 1 \leq 2\varepsilon(k+1). \tag{B.11}$$

Substituting into Equation B.10, we thus have

$$\rho_{0,\ell}^{k} \leq \sqrt{\frac{2\ell+1}{k+1}} \leq \sqrt{\frac{2\varepsilon(k+1)}{k+1}} = \sqrt{2\varepsilon}. \tag{B.12}$$

$\therefore$ The label of X_0 is an ε-outlier among the labels for $X_0, X_1, \ldots, X_k$ with probability at most $\sqrt{2\varepsilon}$.

$\square$

Appendix C Outlier Analysis Results

The tables included in this appendix summarize the results of the outlier analysis for the five maps considered in this thesis. For each map, separate tables are included for each bias measure, and each row of a given table represents the results obtained from one run of the chain under the specified constraints. The initial value of the given bias measure for the map under analysis is included in the header of each table. The tables contain the following pieces of information.

- *Compactness Threshold.* Specifies whether a threshold on the maximum allowable compactness score was imposed and, if so, whether the Inverse Polsby-Popper (Equation 4.1) or Perimeter (Equation 4.2) threshold was used. For maps that use one of these thresholds, the initial compactness score of the map was calculated and the threshold set to be between 2% and 4% higher than this initial score. This threshold value is denoted beneath the constraint in the tables. Note that some threshold values (such as the Perimeter thresholds for the South Carolina maps) are much larger than others. This does not necessarily indicate that these maps show extreme violations of compactness compared to the others; rather, this is a result of the different geographic units used for each map. Maps for which no bounds on compactness were set have this constraint labeled as None.

- *Freeze District.* Specifies whether a minority-majority district was frozen to prevent swapping precincts to or from that district. This applies only to Congressional plans that have a designated minority-majority district drawn in compliance with the Voting Rights Act.

- *Population Threshold.* Specifies the maximum allowable proportion difference between any two districts for a given districting. Precincts that violate this condition if swapped between districts cannot be selected.

- *Preserve Counties.* Specifies whether counties fully contained in districts were left unchanged. If so, precincts residing in such counties cannot be chosen for district membership swaps.

- *Mean.* The Markov chain code prints output every 2^{22} steps, providing information about a subsample of maps and allowing one to observe the behavior of the chain. Mean designates the average bias measure value of the generated subsample. The subsample size n is specified in each table; if n is denoted with an asterisk (*), it indicates that some of the rows contain descriptive statistics for up to seven fewer sample maps due to an issue with the output file.

- *Std. Dev.* Reports the standard deviation of the bias measure values from the subsample.
- *More Biased Maps.* The number of maps observed in the full sample of maps generated that contain a bias measure value more extreme than that of the original map.
- *Total Steps.* Total number of steps taken by the chain, equal to the number of sample maps generated.
- *ε.* Proportion of maps more biased than the original map. Also denotes the value of ε for which the original districting is an ε-outlier (see Definition 1).
- *p-value.* Probability of observing a map as biased as the original map with respect to a given bias measure under the assumption that the original map is among the population of "typical" districtings. Where a p-value is shown as being greater than one, recall that the p-value is an upper bound on significance by Theorem 1 and is computed according to Equation 2.16 as $p = \sqrt{2\varepsilon}$. Thus, this result indicates that $\varepsilon > 0.5$, or more than half the generated maps exhibited bias as "extreme" as the current map. Values which fall below the designated significance level of $\alpha = 0.001$ are highlighted in each table.

Additional constraint combinations are included in this appendix that were not presented in the main results section of Chapter 4. The results presented in the main body of the thesis are those most relevant to the redistricting guidelines per state. For example, both Pennsylvania and South Carolina guidelines require that election districts be relatively compact, so it is most reasonable to consider cases where compactness is constrained in some manner. Similarly, guidelines for South Carolina legislative districts specify explicit bounds on the percent population difference between districts, so considering alternative population thresholds is not as insightful as considering those that are in place for the redistricting process. However, in order to be fully transparent in my analysis, all of the results are included here.

Note that some of the alternative constraint combinations presented achieve significance, whereas those most closely corresponding to the appropriate redistricting guidelines did not. This does not, however, alter the conclusion drawn that there is little evidence of partisan gerrymandering in South Carolina. For this conclusion to change, we would need to observe significance across all or most constraints with respect to reliable label functions (namely, median-mean and variable geometric bias as shown in Chapter 4), as we do with the original Pennsylvania gerrymander.

Constraints				Results for 2011 Pennsylvania Congressional Map (MM = −0.062641)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	−0.010753	0.006645	1,178	1,099,511,627,780	$1.0714 \cdot 10^{-9}$	$4.6290 \cdot 10^{-5}$
			Yes	−0.022661	0.005486	30,256	1,099,511,627,785	$2.7518 \cdot 10^{-8}$	$2.3460 \cdot 10^{-4}$
		0.02	No	−0.009799	0.006656	184	1,099,511,627,778	$1.6735 \cdot 10^{-10}$	$1.8295 \cdot 10^{-5}$
			Yes	−0.020540	0.006004	35	1,099,511,627,778	$3.1832 \cdot 10^{-11}$	$7.9790 \cdot 10^{-6}$
	Yes	0.01	No	−0.011643	0.006818	85,771	1,099,511,627,776	$7.8008 \cdot 10^{-8}$	$3.9499 \cdot 10^{-4}$
			Yes	−0.021909	0.005408	30,574	1,099,511,627,778	$2.7807 \cdot 10^{-8}$	$2.3583 \cdot 10^{-4}$
		0.02	No	−0.011806	0.006894	20,696	1,099,511,627,777	$1.8823 \cdot 10^{-8}$	$1.9403 \cdot 10^{-4}$
			Yes	−0.019244	0.005314	4,917	1,099,511,627,784	$4.4720 \cdot 10^{-9}$	$9.4573 \cdot 10^{-5}$
Perimeter (125)	No	0.01	No	−0.026474	0.007850	7,113	1,099,511,627,776	$6.4692 \cdot 10^{-9}$	$1.1375 \cdot 10^{-4}$
			Yes	−0.037286	0.005455	2,563	1,099,511,627,785	$2.3310 \cdot 10^{-9}$	$6.8279 \cdot 10^{-5}$
		0.02	No	−0.027507	0.007481	3,094	1,099,511,627,777	$2.8140 \cdot 10^{-9}$	$7.5020 \cdot 10^{-5}$
			Yes	−0.037553	0.004510	5,147	1,099,511,627,778	$4.6812 \cdot 10^{-9}$	$9.6759 \cdot 10^{-5}$
	Yes	0.01	No	−0.025329	0.007399	923	1,099,511,627,781	$8.3946 \cdot 10^{-10}$	$4.0975 \cdot 10^{-5}$
			Yes	−0.031154	0.005834	3,424	1,099,511,627,788	$3.1141 \cdot 10^{-9}$	$7.8919 \cdot 10^{-5}$
		0.02	No	−0.027328	0.007398	12,700	1,099,511,627,776	$1.1551 \cdot 10^{-8}$	$1.5199 \cdot 10^{-4}$
			Yes	−0.034267	0.004464	3,786	1,099,511,627,799	$3.4433 \cdot 10^{-9}$	$8.2986 \cdot 10^{-5}$
None	No	0.01	No	−0.024193	0.010118	64,856,749	1,099,511,627,779	$5.8987 \cdot 10^{-5}$	$1.0862 \cdot 10^{-2}$
			Yes	−0.032384	0.008652	18,785,792	1,099,511,627,776	$1.7086 \cdot 10^{-5}$	$5.8456 \cdot 10^{-3}$
		0.02	No	−0.024304	0.010126	65,919,843	1,099,511,627,780	$5.9954 \cdot 10^{-5}$	$1.0950 \cdot 10^{-2}$
			Yes	−0.020617	0.007916	1,424	1,099,511,627,777	$1.2951 \cdot 10^{-9}$	$5.0894 \cdot 10^{-5}$
	Yes	0.01	No	−0.023276	0.009677	34,637,487	1,099,511,627,777	$3.1503 \cdot 10^{-5}$	$7.9376 \cdot 10^{-3}$
			Yes	−0.030281	0.008921	2,844,294	1,099,511,627,778	$2.5869 \cdot 10^{-6}$	$2.2746 \cdot 10^{-3}$
		0.02	No	−0.024808	0.009593	33,613,893	1,099,511,627,777	$3.0572 \cdot 10^{-5}$	$7.8194 \cdot 10^{-3}$
			Yes	−0.029265	0.009164	2,312,871	1,099,511,627,780	$2.1035 \cdot 10^{-6}$	$2.0511 \cdot 10^{-3}$

Table C.1: Results for median-mean on the 2011 Pennsylvania Congressional map.

Constraints				**Results for 2011 Pennsylvania Congressional Map** $\left(\hat{\beta}(V) = -0.222222\right)$					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	-0.050555	0.028815	2,357,386	1,099,511,627,780	$2.1440 \cdot 10^{-6}$	$2.0708 \cdot 10^{-3}$
			Yes	-0.082152	0.016324	142,187	1,099,511,627,785	$1.2932 \cdot 10^{-7}$	$5.0856 \cdot 10^{-4}$
		0.02	No	-0.046622	0.028534	546,988	1,099,511,627,778	$4.9748 \cdot 10^{-7}$	$9.9748 \cdot 10^{-4}$
			Yes	-0.067890	0.024085	1,330,742	1,099,511,627,778	$1.2103 \cdot 10^{-6}$	$1.5558 \cdot 10^{-3}$
	Yes	0.01	No	-0.064557	0.033745	16,047,408	1,099,511,627,776	$1.4595 \cdot 10^{-5}$	$5.4028 \cdot 10^{-3}$
			Yes	-0.079477	0.022059	492,347	1,099,511,627,778	$4.4779 \cdot 10^{-7}$	$9.4635 \cdot 10^{-4}$
		0.02	No	-0.066168	0.034422	28,224,921	1,099,511,627,777	$2.5670 \cdot 10^{-5}$	$7.1653 \cdot 10^{-3}$
			Yes	-0.075510	0.022596	998,237	1,099,511,627,784	$9.0789 \cdot 10^{-7}$	$1.3475 \cdot 10^{-3}$
Perimeter (125)	No	0.01	No	-0.131809	0.035877	16,417,691,532	1,099,511,627,776	$1.4932 \cdot 10^{-2}$	$1.7281 \cdot 10^{-1}$
			Yes	-0.162808	0.016155	5,852,728,860	1,099,511,627,785	$5.3230 \cdot 10^{-3}$	$1.0318 \cdot 10^{-1}$
		0.02	No	-0.137191	0.035134	19,303,122,154	1,099,511,627,777	$1.7556 \cdot 10^{-2}$	$1.8738 \cdot 10^{-1}$
			Yes	-0.165181	0.013499	4,697,878,928	1,099,511,627,778	$4.2727 \cdot 10^{-3}$	$9.2441 \cdot 10^{-2}$
	Yes	0.01	No	-0.121356	0.030022	2,381,993,384	1,099,511,627,781	$2.1664 \cdot 10^{-3}$	$6.5824 \cdot 10^{-2}$
			Yes	-0.142810	0.024013	2,613,684,916	1,099,511,627,788	$2.3771 \cdot 10^{-3}$	$6.8951 \cdot 10^{-2}$
		0.02	No	-0.132905	0.033273	9,338,160,768	1,099,511,627,776	$8.4930 \cdot 10^{-3}$	$1.3033 \cdot 10^{-1}$
			Yes	-0.153585	0.017776	2,017,511,233	1,099,511,627,799	$1.8349 \cdot 10^{-3}$	$6.0579 \cdot 10^{-2}$
None	No	0.01	No	-0.108277	0.041623	8,200,737,487	1,099,511,627,779	$7.4585 \cdot 10^{-3}$	$1.2214 \cdot 10^{-1}$
			Yes	-0.125426	0.027172	1,125,332,126	1,099,511,627,776	$1.0235 \cdot 10^{-3}$	$4.5243 \cdot 10^{-2}$
		0.02	No	-0.108859	0.041895	8,495,220,874	1,099,511,627,780	$7.7264 \cdot 10^{-3}$	$1.2431 \cdot 10^{-1}$
			Yes	-0.098252	0.032885	260,215,841	1,099,511,627,777	$2.3666 \cdot 10^{-4}$	$2.1756 \cdot 10^{-2}$
	Yes	0.01	No	-0.106383	0.039148	5,224,267,756	1,099,511,627,777	$4.7514 \cdot 10^{-3}$	$9.7483 \cdot 10^{-2}$
			Yes	-0.117258	0.027017	302,986,744	1,099,511,627,778	$2.7556 \cdot 10^{-4}$	$2.3476 \cdot 10^{-2}$
		0.02	No	-0.115962	0.040535	11,411,020,882	1,099,511,627,777	$1.0378 \cdot 10^{-2}$	$1.4407 \cdot 10^{-1}$
			Yes	-0.117946	0.028002	551,763,460	1,099,511,627,780	$5.0183 \cdot 10^{-4}$	$3.1680 \cdot 10^{-2}$

Table C.2: Results for the bias point estimate on the 2011 Pennsylvania Congressional map.

Constraints				**Results for 2011 Pennsylvania Congressional Map** ($B_{G,U} = 0.094239$)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	0.014850	0.004952	9,935,538	274,877,906,944	$3.6145 \cdot 10^{-5}$	$8.5024 \cdot 10^{-3}$
			Yes	0.026151	0.004501	8,696,292	274,877,906,947	$3.1637 \cdot 10^{-5}$	$7.9545 \cdot 10^{-3}$
		0.02	No	0.013698	0.004605	8,900,567	274,877,906,944	$3.2380 \cdot 10^{-5}$	$8.0474 \cdot 10^{-3}$
			Yes	0.026355	0.004563	8,657,757	274,877,906,951	$3.1497 \cdot 10^{-5}$	$7.9368 \cdot 10^{-3}$
	Yes	0.01	No	0.040663	0.008505	2,515,393,543	274,877,906,947	$9.1509 \cdot 10^{-3}$	$1.3528 \cdot 10^{-1}$
			Yes	0.044942	0.008496	2,406,321,020	274,877,906,949	$8.7541 \cdot 10^{-3}$	$1.3232 \cdot 10^{-1}$
		0.02	No	0.040956	0.008875	2,520,172,043	274,877,906,944	$9.1683 \cdot 10^{-3}$	$1.3541 \cdot 10^{-1}$
			Yes	0.045737	0.007603	2,010,531,361	274,877,906,945	$7.3143 \cdot 10^{-3}$	$1.2095 \cdot 10^{-1}$
Perimeter (125)	No	0.01	No	0.054479	0.009530	2,096,627,572	274,877,906,948	$7.6275 \cdot 10^{-3}$	$1.2351 \cdot 10^{-1}$
			Yes	0.066825	0.008991	2,261,378,600	274,877,906,948	$8.2268 \cdot 10^{-3}$	$1.2827 \cdot 10^{-1}$
		0.02	No	0.053882	0.012365	2,089,936,881	274,877,906,944	$7.6031 \cdot 10^{-3}$	$1.2331 \cdot 10^{-1}$
			Yes	0.067622	0.009319	2,086,379,354	274,877,906,951	$7.5902 \cdot 10^{-3}$	$1.2321 \cdot 10^{-1}$
	Yes	0.01	No	0.056195	0.009043	2,177,958,035	274,877,906,946	$7.9234 \cdot 10^{-3}$	$1.2588 \cdot 10^{-1}$
			Yes	0.060296	0.008499	2,247,427,989	274,877,906,953	$8.1761 \cdot 10^{-3}$	$1.2788 \cdot 10^{-1}$
		0.02	No	0.056684	0.009470	2,198,782,056	274,877,906,945	$7.9991 \cdot 10^{-3}$	$1.2648 \cdot 10^{-1}$
			Yes	0.060416	0.011529	4,635,670,796	274,877,906,945	$1.6864 \cdot 10^{-2}$	$1.8365 \cdot 10^{-1}$
None	No	0.01	No	0.049738	0.012076	1,649,719,156	274,877,906,946	$6.0016 \cdot 10^{-3}$	$1.0956 \cdot 10^{-1}$
			Yes	0.056945	0.010229	1,468,035,062	274,877,906,951	$5.3407 \cdot 10^{-3}$	$1.0335 \cdot 10^{-1}$
		0.02	No	0.050667	0.012129	1,718,190,812	274,877,906,944	$6.2507 \cdot 10^{-3}$	$1.1181 \cdot 10^{-1}$
			Yes	0.056222	0.011537	2,519,756,482	274,877,906,946	$9.1668 \cdot 10^{-3}$	$1.3540 \cdot 10^{-1}$
	Yes	0.01	No	0.052586	0.009655	2,298,396,244	274,877,906,944	$8.3615 \cdot 10^{-3}$	$1.2932 \cdot 10^{-1}$
			Yes	0.055119	0.007994	1,675,356,760	274,877,906,945	$6.0949 \cdot 10^{-3}$	$1.1041 \cdot 10^{-1}$
		0.02	No	0.052877	0.009795	2,284,621,942	274,877,906,944	$8.3114 \cdot 10^{-3}$	$1.2893 \cdot 10^{-1}$
			Yes	0.055613	0.008122	1,693,627,761	274,877,906,952	$6.1614 \cdot 10^{-3}$	$1.1101 \cdot 10^{-1}$

Table C.3: Results for uniform geometric bias on the 2011 Pennsylvania Congressional map.

Constraints				Results for 2011 Pennsylvania Congressional Map ($B_{G,V} = 0.067460$)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 65537^*$)	Std. Dev. ($n = 65537^*$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	0.012261	0.003001	747	274,877,906,944	$2.7176 \cdot 10^{-9}$	$7.3723 \cdot 10^{-5}$
			Yes	0.021338	0.002424	111	274,877,906,947	$4.0382 \cdot 10^{-10}$	$2.8419 \cdot 10^{-5}$
		0.02	No	0.011362	0.002626	94	274,877,906,944	$3.4197 \cdot 10^{-10}$	$2.6152 \cdot 10^{-5}$
			Yes	0.021389	0.002597	26	274,877,906,951	$9.4587 \cdot 10^{-11}$	$1.3754 \cdot 10^{-5}$
	Yes	0.01	No	0.025191	0.002007	233	274,877,906,947	$8.4765 \cdot 10^{-10}$	$4.1174 \cdot 10^{-5}$
			Yes	0.028607	0.002195	126	274,877,906,949	$4.5839 \cdot 10^{-10}$	$3.0278 \cdot 10^{-5}$
		0.02	No	0.025366	0.002013	234	274,877,906,944	$8.5129 \cdot 10^{-10}$	$4.1262 \cdot 10^{-5}$
			Yes	0.029502	0.001874	126	274,877,906,945	$4.5839 \cdot 10^{-10}$	$3.0278 \cdot 10^{-5}$
Perimeter (125)	No	0.01	No	0.038489	0.004864	118	274,877,906,948	$4.2928 \cdot 10^{-10}$	$2.9301 \cdot 10^{-5}$
			Yes	0.046974	0.004724	26	274,877,906,948	$9.4587 \cdot 10^{-11}$	$1.3754 \cdot 10^{-5}$
		0.02	No	0.038017	0.004655	91	274,877,906,944	$3.3106 \cdot 10^{-10}$	$2.5732 \cdot 10^{-5}$
			Yes	0.047373	0.004852	26	274,877,906,951	$9.4587 \cdot 10^{-11}$	$1.3754 \cdot 10^{-5}$
	Yes	0.01	No	0.038279	0.004249	71	274,877,906,946	$2.5830 \cdot 10^{-10}$	$2.2729 \cdot 10^{-5}$
			Yes	0.041941	0.003636	389	274,877,906,953	$1.4152 \cdot 10^{-9}$	$5.3201 \cdot 10^{-5}$
		0.02	No	0.038744	0.004324	1	274,877,906,945	$3.6380 \cdot 10^{-12}$	$2.6974 \cdot 10^{-6}$
			Yes	0.041004	0.003205	427	274,877,906,945	$1.5534 \cdot 10^{-9}$	$5.5739 \cdot 10^{-5}$
None	No	0.01	No	0.035568	0.007668	92	274,877,906,946	$3.3469 \cdot 10^{-10}$	$2.5873 \cdot 10^{-5}$
			Yes	0.040255	0.006116	26	274,877,906,951	$9.4587 \cdot 10^{-11}$	$1.3754 \cdot 10^{-5}$
		0.02	No	0.036107	0.007558	10,411	274,877,906,944	$3.7875 \cdot 10^{-8}$	$2.7523 \cdot 10^{-4}$
			Yes	0.038894	0.006728	26	274,877,906,946	$9.4587 \cdot 10^{-11}$	$1.3754 \cdot 10^{-5}$
	Yes	0.01	No	0.035393	0.004635	199	274,877,906,944	$7.2396 \cdot 10^{-10}$	$3.8051 \cdot 10^{-5}$
			Yes	0.037465	0.003723	130	274,877,906,945	$4.7294 \cdot 10^{-10}$	$3.0755 \cdot 10^{-5}$
		0.02	No	0.035648	0.004799	1	274,877,906,944	$3.6380 \cdot 10^{-12}$	$2.6974 \cdot 10^{-6}$
			Yes	0.037947	0.003892	130	274,877,906,952	$4.7294 \cdot 10^{-10}$	$3.0755 \cdot 10^{-5}$

Table C.4: Results for variable geometric bias on the 2011 Pennsylvania Congressional map.

Constraints				Results for 2018 Pennsylvania Congressional Map (MM = −0.020890)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	−0.010186	0.006717	52,967,254,374	1,099,511,627,777	$4.8173 \cdot 10^{-2}$	$3.1040 \cdot 10^{-1}$
			Yes	−0.009919	0.002282	4,237,150	1,099,511,627,792	$3.8537 \cdot 10^{-6}$	$2.7762 \cdot 10^{-3}$
		0.02	No	−0.010479	0.006314	49,856,217,351	1,099,511,627,780	$4.5344 \cdot 10^{-2}$	$3.0114 \cdot 10^{-1}$
			Yes	−0.009480	0.002284	9,858,133	1,099,511,627,795	$8.9659 \cdot 10^{-6}$	$4.2346 \cdot 10^{-3}$
	Yes	0.01	No	−0.015529	0.007383	246,330,104,691	1,099,511,627,776	$2.2404 \cdot 10^{-1}$	$6.6938 \cdot 10^{-1}$
			Yes	−0.008085	0.004540	20,666,430	1,099,511,627,776	$1.8796 \cdot 10^{-5}$	$6.1312 \cdot 10^{-3}$
		0.02	No	−0.015123	0.007819	246,382,899,657	1,099,511,627,776	$2.2408 \cdot 10^{-1}$	$6.6945 \cdot 10^{-1}$
			Yes	−0.007549	0.004648	24,858,930	1,099,511,627,792	$2.2609 \cdot 10^{-5}$	$6.7244 \cdot 10^{-3}$
Perimeter (125)	No	0.01	No	−0.026606	0.007215	855,616,172,608	1,099,511,627,786	$7.7818 \cdot 10^{-1}$	1.2475
			Yes	−0.021446	0.004618	563,901,344,853	1,099,511,627,777	$5.1287 \cdot 10^{-1}$	1.0128
		0.02	No	−0.025710	0.007528	804,754,958,677	1,099,511,627,777	$7.3192 \cdot 10^{-1}$	1.2099
			Yes	−0.020504	0.004692	482,979,471,740	1,099,511,627,783	$4.3927 \cdot 10^{-1}$	$9.3730 \cdot 10^{-1}$
	Yes	0.01	No	−0.025990	0.007145	831,022,950,237	1,099,511,627,782	$7.5581 \cdot 10^{-1}$	1.2295
			Yes	−0.019873	0.005917	498,162,176,507	1,099,511,627,780	$4.5308 \cdot 10^{-1}$	$9.5192 \cdot 10^{-1}$
		0.02	No	−0.024810	0.007524	757,561,256,622	1,099,511,627,776	$6.8900 \cdot 10^{-1}$	1.1739
			Yes	−0.019528	0.006078	475,098,566,851	1,099,511,627,778	$4.3210 \cdot 10^{-1}$	$9.2962 \cdot 10^{-1}$
None	No	0.01	No	−0.023227	0.009987	646,994,513,737	1,099,511,627,779	$5.8844 \cdot 10^{-1}$	1.0848
			Yes	−0.019184	0.004949	353,901,192,411	1,099,511,627,778	$3.2187 \cdot 10^{-1}$	$8.0234 \cdot 10^{-1}$
		0.02	No	−0.023330	0.009971	649,990,164,971	1,099,511,627,776	$5.9116 \cdot 10^{-1}$	1.0873
			Yes	−0.019411	0.004876	366,784,283,247	1,099,511,627,776	$3.3359 \cdot 10^{-1}$	$8.1681 \cdot 10^{-1}$
	Yes	0.01	No	−0.024086	0.009383	684,826,865,663	1,099,511,627,776	$6.2285 \cdot 10^{-1}$	1.1161
			Yes	-0.017052	0.006243	252,882,650,404	1,099,511,627,776	$2.3000 \cdot 10^{-1}$	$6.7823 \cdot 10^{-1}$
		0.02	No	−0.024208	0.009339	691,441,909,943	1,099,511,627,776	$6.2886 \cdot 10^{-1}$	1.1215
			Yes	−0.017587	0.006077	274,982,904,607	1,099,511,627,779	$2.5010 \cdot 10^{-1}$	$7.0724 \cdot 10^{-1}$

Table C.5: Results for median-mean on the 2018 Pennsylvania Congressional map.

Constraints				Results for 2018 Pennsylvania Congressional Map $\left(\hat{\beta}(V) = -0.138889\right)$					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	-0.051486	0.029453	2,292,141,143	1,099,511,627,777	$2.0847 \cdot 10^{-3}$	$6.4571 \cdot 10^{-2}$
			Yes	-0.007071	0.022508	5,985,899	1,099,511,627,792	$5.4441 \cdot 10^{-6}$	$3.2997 \cdot 10^{-3}$
		0.02	No	-0.053097	0.028211	2,303,390,815	1,099,511,627,780	$2.0949 \cdot 10^{-3}$	$6.4729 \cdot 10^{-2}$
			Yes	-0.003718	0.021712	31,973,281	1,099,511,627,795	$2.9080 \cdot 10^{-5}$	$7.6262 \cdot 10^{-3}$
	Yes	0.01	No	-0.071806	0.033911	47,194,164,820	1,099,511,627,776	$4.2923 \cdot 10^{-2}$	$2.9299 \cdot 10^{-1}$
			Yes	-0.034954	0.023407	10,405,579	1,099,511,627,776	$9.4638 \cdot 10^{-6}$	$4.3506 \cdot 10^{-3}$
		0.02	No	-0.070800	0.035637	50,782,368,893	1,099,511,627,776	$4.6186 \cdot 10^{-2}$	$3.0393 \cdot 10^{-1}$
			Yes	-0.032735	0.024182	10,732,751	1,099,511,627,792	$9.7614 \cdot 10^{-6}$	$4.4185 \cdot 10^{-3}$
Perimeter (125)	No	0.01	No	-0.140446	0.032024	482,516,271,814	1,099,511,627,786	$4.3885 \cdot 10^{-1}$	$9.3685 \cdot 10^{-1}$
			Yes	-0.096015	0.023944	42,691,801,277	1,099,511,627,777	$3.8828 \cdot 10^{-2}$	$2.7867 \cdot 10^{-1}$
		0.02	No	-0.136770	0.033632	454,087,845,326	1,099,511,627,777	$4.1299 \cdot 10^{-1}$	$9.0883 \cdot 10^{-1}$
			Yes	-0.093477	0.024953	41,699,023,566	1,099,511,627,783	$3.7925 \cdot 10^{-2}$	$2.7541 \cdot 10^{-1}$
	Yes	0.01	No	-0.141995	0.031477	511,696,714,973	1,099,511,627,782	$4.6539 \cdot 10^{-1}$	$9.6476 \cdot 10^{-1}$
			Yes	-0.082871	0.029131	23,497,605,989	1,099,511,627,780	$2.1371 \cdot 10^{-2}$	$2.0674 \cdot 10^{-1}$
		0.02	No	-0.135526	0.032737	434,713,050,957	1,099,511,627,776	$3.9537 \cdot 10^{-1}$	$8.8923 \cdot 10^{-1}$
			Yes	-0.080895	0.029738	23,812,657,193	1,099,511,627,778	$2.1657 \cdot 10^{-2}$	$2.0812 \cdot 10^{-1}$
None	No	0.01	No	-0.107445	0.042127	258,490,881,786	1,099,511,627,779	$2.3510 \cdot 10^{-1}$	$6.8571 \cdot 10^{-1}$
			Yes	-0.079843	0.028863	37,485,070,175	1,099,511,627,778	$3.4092 \cdot 10^{-2}$	$2.6112 \cdot 10^{-1}$
		0.02	No	-0.108019	0.041558	261,809,480,282	1,099,511,627,776	$2.3811 \cdot 10^{-1}$	$6.9009 \cdot 10^{-1}$
			Yes	-0.081029	0.029168	41,111,414,969	1,099,511,627,776	$3.7391 \cdot 10^{-2}$	$2.7346 \cdot 10^{-1}$
	Yes	0.01	No	-0.118074	0.041629	363,956,190,414	1,099,511,627,776	$3.3102 \cdot 10^{-1}$	$8.1365 \cdot 10^{-1}$
			Yes	-0.060814	0.029859	6,487,287,117	1,099,511,627,776	$5.9002 \cdot 10^{-3}$	$1.0863 \cdot 10^{-1}$
		0.02	No	-0.118489	0.041673	367,038,960,002	1,099,511,627,776	$3.3382 \cdot 10^{-1}$	$8.1709 \cdot 10^{-1}$
			Yes	-0.064198	0.029804	10,134,670,811	1,099,511,627,779	$9.2174 \cdot 10^{-3}$	$1.3578 \cdot 10^{-1}$

Table C.6: Results for the bias point estimate on the 2018 Pennsylvania Congressional map.

Constraints				**Results for 2018 Pennsylvania Congressional Map** ($B_{G,U} = 0.052525$)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	0.015307	0.029216	578,293,049	274,877,906,944	$2.1038 \cdot 10^{-3}$	$6.4866 \cdot 10^{-2}$
			Yes	0.013908	0.004530	1,718,476,520	274,877,906,944	$6.2518 \cdot 10^{-3}$	$1.1182 \cdot 10^{-1}$
		0.02	No	0.014860	0.008156	652,406,673	274,877,906,944	$2.3734 \cdot 10^{-3}$	$6.8898 \cdot 10^{-2}$
			Yes	0.014540	0.005098	1,823,287,965	274,877,906,944	$6.6331 \cdot 10^{-3}$	$1.1518 \cdot 10^{-1}$
	Yes	0.01	No	0.019902	0.005963	2,134,665,993	274,877,906,944	$7.7659 \cdot 10^{-3}$	$1.2463 \cdot 10^{-1}$
			Yes	0.016502	0.005353	1,789,731,356	274,877,906,957	$6.5110 \cdot 10^{-3}$	$1.1411 \cdot 10^{-1}$
		0.02	No	0.020359	0.006252	2,136,869,807	274,877,906,946	$7.7739 \cdot 10^{-3}$	$1.2469 \cdot 10^{-1}$
			Yes	0.016399	0.005113	1,800,903,548	274,877,906,954	$6.5516 \cdot 10^{-3}$	$1.1447 \cdot 10^{-1}$
Perimeter (125)	No	0.01	No	0.053434	0.009804	138,631,329,650	274,877,906,945	$5.0434 \cdot 10^{-1}$	1.0043
			Yes	0.050072	0.011066	123,773,391,193	274,877,906,944	$4.5028 \cdot 10^{-1}$	$9.4898 \cdot 10^{-1}$
		0.02	No	0.055705	0.009549	174,006,607,879	274,877,906,944	$6.3303 \cdot 10^{-1}$	1.1252
			Yes	0.048402	0.028266	106,247,762,045	274,877,906,948	$3.8653 \cdot 10^{-1}$	$8.7924 \cdot 10^{-1}$
	Yes	0.01	No	0.053595	0.007931	141,049,873,754	274,877,906,949	$5.1314 \cdot 10^{-1}$	1.0131
			Yes	0.037124	0.009544	1,771,703,806	274,877,906,955	$6.4454 \cdot 10^{-3}$	$1.1354 \cdot 10^{-1}$
		0.02	No	0.055209	0.007913	176,141,537,680	274,877,906,946	$6.4080 \cdot 10^{-1}$	1.1321
			Yes	0.035462	0.008874	1,766,432,283	274,877,906,947	$6.4262 \cdot 10^{-3}$	$1.1337 \cdot 10^{-1}$
None	No	0.01	No	0.049820	0.012290	110,736,493,418	274,877,906,944	$4.0286 \cdot 10^{-1}$	$8.9762 \cdot 10^{-1}$
			Yes	0.050231	0.011825	120,831,018,509	274,877,906,945	$4.3958 \cdot 10^{-1}$	$9.3764 \cdot 10^{-1}$
		0.02	No	0.050036	0.012192	112,541,449,522	274,877,906,946	$4.0942 \cdot 10^{-1}$	$9.0490 \cdot 10^{-1}$
			Yes	0.050783	0.011532	126,402,243,947	274,877,906,946	$4.5985 \cdot 10^{-1}$	$9.5901 \cdot 10^{-1}$
	Yes	0.01	No	0.052841	0.010797	142,938,271,326	274,877,906,944	$5.2001 \cdot 10^{-1}$	1.0198
			Yes	0.038339	0.009368	2,721,929,147	274,877,906,957	$9.9023 \cdot 10^{-3}$	$1.4073 \cdot 10^{-1}$
		0.02	No	0.053312	0.011098	146,801,602,043	274,877,906,944	$5.3406 \cdot 10^{-1}$	1.0335
			Yes	0.039883	0.009249	2,238,901,187	274,877,906,946	$8.1451 \cdot 10^{-3}$	$1.2763 \cdot 10^{-1}$

Table C.7: Results for uniform geometric bias on the 2018 Pennsylvania Congressional map.

Constraints				**Results for 2018 Pennsylvania Congressional Map** ($B_{G,V} = 0.035278$)					
Compactness Threshold	Freeze Dist. 2?	Population Threshold	Preserve Counties?	Mean ($n = 65537^*$)	Std. Dev. ($n = 65537^*$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (160)	No	0.01	No	0.012615	0.003125	9,212,610	274,877,906,944	$3.3515 \cdot 10^{-5}$	$8.1872 \cdot 10^{-3}$
			Yes	0.010947	0.001442	116	274,877,906,944	$4.2201 \cdot 10^{-10}$	$2.9052 \cdot 10^{-5}$
		0.02	No	0.012361	0.002981	5,213,600	274,877,906,944	$1.8967 \cdot 10^{-5}$	$6.1591 \cdot 10^{-3}$
			Yes	0.011352	0.002001	60,625	274,877,906,944	$2.2055 \cdot 10^{-7}$	$6.6416 \cdot 10^{-4}$
	Yes	0.01	No	0.014993	0.003720	15,951,006	274,877,906,944	$5.8029 \cdot 10^{-5}$	$1.0773 \cdot 10^{-2}$
			Yes	0.011870	0.002283	11,035	274,877,906,957	$4.0145 \cdot 10^{-8}$	$2.8336 \cdot 10^{-4}$
		0.02	No	0.015388	0.004148	15,390,816	274,877,906,946	$5.5991 \cdot 10^{-5}$	$1.0582 \cdot 10^{-2}$
			Yes	0.011787	0.002308	21,781	274,877,906,954	$7.9239 \cdot 10^{-8}$	$3.9809 \cdot 10^{-4}$
Perimeter (125)	No	0.01	No	0.037702	0.004884	190,411,706,710	274,877,906,945	$6.9271 \cdot 10^{-1}$	1.1770
			Yes	0.032429	0.005609	97,678,034,705	274,877,906,944	$3.5535 \cdot 10^{-1}$	$8.4303 \cdot 10^{-1}$
		0.02	No	0.039063	0.005052	218,257,945,981	274,877,906,944	$7.9402 \cdot 10^{-1}$	1.2602
			Yes	0.031573	0.005915	87,065,256,391	274,877,906,948	$3.1674 \cdot 10^{-1}$	$7.9592 \cdot 10^{-1}$
	Yes	0.01	No	0.038547	0.004150	219,660,268,074	274,877,906,949	$7.9912 \cdot 10^{-1}$	1.2642
			Yes	0.025835	0.005817	18,368,714	274,877,906,955	$6.6825 \cdot 10^{-5}$	$1.1561 \cdot 10^{-2}$
		0.02	No	0.039518	0.004305	237,617,554,076	274,877,906,946	$8.6445 \cdot 10^{-1}$	1.3149
			Yes	0.024902	0.005513	34,451,923	274,877,906,947	$1.2534 \cdot 10^{-4}$	$1.5833 \cdot 10^{-2}$
None	No	0.01	No	0.035499	0.007726	145,114,561,032	274,877,906,944	$5.2792 \cdot 10^{-1}$	1.0275
			Yes	0.032618	0.006386	105,626,747,368	274,877,906,945	$3.8427 \cdot 10^{-1}$	$8.7666 \cdot 10^{-1}$
		0.02	No	0.035627	0.007600	147,856,173,116	274,877,906,946	$5.3790 \cdot 10^{-1}$	1.0372
			Yes	0.032985	0.006246	110,832,763,218	274,877,906,946	$4.0321 \cdot 10^{-1}$	$8.9801 \cdot 10^{-1}$
	Yes	0.01	No	0.038008	0.007043	183,521,246,637	274,877,906,944	$6.6765 \cdot 10^{-1}$	1.1555
			Yes	0.026508	0.005909	6,854,107,460	274,877,906,957	$2.4935 \cdot 10^{-2}$	$2.2332 \cdot 10^{-1}$
		0.02	No	0.038316	0.007100	186,584,714,594	274,877,906,944	$6.7879 \cdot 10^{-1}$	1.1652
			Yes	0.027329	0.005733	4,537,254,986	274,877,906,946	$1.6506 \cdot 10^{-2}$	$1.8169 \cdot 10^{-1}$

Table C.8: Results for variable geometric bias on the 2018 Pennsylvania Congressional map.

Constraints				Results for 2011 South Carolina Congressional Map (MM = −0.019895)					
Compactness Threshold	Freeze Dist. 6?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (47)	No	0.01	No	−0.004624	0.011929	112,874,694,676	1,099,511,627,776	$1.0266 \cdot 10^{-1}$	$4.5312 \cdot 10^{-1}$
			Yes	−0.005589	0.004556	3,619,158,136	1,099,511,627,779	$3.2916 \cdot 10^{-3}$	$8.1137 \cdot 10^{-2}$
		0.02	No	−0.005797	0.011821	125,234,856,890	1,099,511,627,777	$1.1390 \cdot 10^{-1}$	$4.7728 \cdot 10^{-1}$
			Yes	−0.003364	0.005212	939,002,839	1,099,511,627,779	$8.5402 \cdot 10^{-4}$	$4.1328 \cdot 10^{-2}$
	Yes	0.01	No	−0.017644	0.004585	339,866,208,378	1,099,511,627,792	$3.0911 \cdot 10^{-1}$	$7.8627 \cdot 10^{-1}$
			Yes	−0.020261	0.000190	1,099,506,052,118	1,099,511,627,803	$9.9999 \cdot 10^{-1}$	1.4142
		0.02	No	−0.017644	0.004722	346,055,792,952	1,099,511,627,808	$3.1474 \cdot 10^{-1}$	$7.9339 \cdot 10^{-1}$
			Yes	−0.028061	0.000597	1,098,403,610,397	1,099,511,627,868	$9.9899 \cdot 10^{-1}$	1.4135
Perimeter (6646800)	No	0.01	No	−0.010339	0.011058	214,318,260,194	1,099,511,627,780	$1.9492 \cdot 10^{-1}$	$6.2437 \cdot 10^{-1}$
			Yes	−0.006019	0.004490	2,568,913,571	1,099,511,627,776	$2.3364 \cdot 10^{-3}$	$6.8358 \cdot 10^{-2}$
		0.02	No	−0.008541	0.010756	157,867,195,774	1,099,511,627,781	$1.4358 \cdot 10^{-1}$	$5.3587 \cdot 10^{-1}$
			Yes	−0.002959	0.005424	489,663,278	1,099,511,627,832	$4.4535 \cdot 10^{-4}$	$2.9844 \cdot 10^{-2}$
	Yes	0.01	No	−0.018175	0.003934	378,063,310,870	1,099,511,627,779	$3.4385 \cdot 10^{-1}$	$8.2927 \cdot 10^{-1}$
			Yes	−0.018483	0.000019	58,796,879	1,099,511,627,949	$5.3475 \cdot 10^{-5}$	$1.0342 \cdot 10^{-2}$
		0.02	No	−0.018100	0.003978	369,748,358,295	1,099,511,627,826	$3.3628 \cdot 10^{-1}$	$8.2010 \cdot 10^{-1}$
			Yes	−0.027883	0.000104	1,099,438,764,341	1,099,511,627,787	$9.9993 \cdot 10^{-1}$	1.4142
None	No	0.01	No	−0.002004	0.014921	135,059,770,113	1,099,511,627,778	$1.2284 \cdot 10^{-1}$	$4.9565 \cdot 10^{-1}$
			Yes	0.003913	0.002656	1,118	1,099,511,627,780	$1.0168 \cdot 10^{-9}$	$4.5096 \cdot 10^{-5}$
		0.02	No	−0.001974	0.015086	137,468,204,225	1,099,511,627,777	$1.2503 \cdot 10^{-1}$	$5.0005 \cdot 10^{-1}$
			Yes	0.005272	0.005351	45,547	1,099,511,627,783	$4.1425 \cdot 10^{-8}$	$2.8784 \cdot 10^{-4}$
	Yes	0.01	No	−0.031792	0.011712	916,260,341,429	1,099,511,627,781	$8.3333 \cdot 10^{-1}$	1.2910
			Yes	−0.028360	0.000305	1,099,188,244,755	1,099,511,627,826	$9.9971 \cdot 10^{-1}$	1.4140
		0.02	No	−0.031894	0.011631	919,975,325,826	1,099,511,627,781	8.3671E-01	1.2936E+00
			Yes	−0.027367	0.002175	1,061,343,186,365	1,099,511,627,782	$9.6529 \cdot 10^{-1}$	1.3895

Table C.9: Results for median-mean on the 2011 South Carolina Congressional map.

Constraints				**Results for 2011 South Carolina Congressional Map** $\left(\hat{\beta}(V) = -0.142857\right)$					
Compactness Threshold	Freeze Dist. 6?	Population Threshold	Preserve Counties?	Mean ($n = 262145$)	Std. Dev. ($n = 262145$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (47)	No	0.01	No	-0.020802	0.058487	7,796,018,771	1,099,511,627,776	$7.0904 \cdot 10^{-3}$	$1.1908 \cdot 10^{-1}$
			Yes	-0.078602	0.036288	200,772,511,931	1,099,511,627,779	$1.8260 \cdot 10^{-1}$	$6.0432 \cdot 10^{-1}$
		0.02	No	-0.021421	0.058504	5,743,063,295	1,099,511,627,777	$5.2233 \cdot 10^{-3}$	$1.0221 \cdot 10^{-1}$
			Yes	-0.071025	0.033773	118,916,600,790	1,099,511,627,779	$1.0815 \cdot 10^{-1}$	$4.6509 \cdot 10^{-1}$
	Yes	0.01	No	-0.144014	0.036724	961,888,223,467	1,099,511,627,792	$8.7483 \cdot 10^{-1}$	1.3227
			Yes	-0.142860	0.000000	1,099,511,627,803	1,099,511,627,803	1.0000	1.4142
		0.02	No	-0.144973	0.037146	965,886,921,644	1,099,511,627,808	$8.7847 \cdot 10^{-1}$	1.3255
			Yes	-0.214094	0.003736	1,099,511,627,868	1,099,511,627,868	1.0000	1.4142
Perimeter (6646800)	No	0.01	No	-0.039797	0.057405	2,170,017,838	1,099,511,627,780	$1.9736 \cdot 10^{-3}$	$6.2827 \cdot 10^{-2}$
			Yes	-0.086305	0.037286	287,666,674,657	1,099,511,627,776	$2.6163 \cdot 10^{-1}$	$7.2337 \cdot 10^{-1}$
		0.02	No	-0.028887	0.057170	2,409,782,350	1,099,511,627,781	$2.1917 \cdot 10^{-3}$	$6.6207 \cdot 10^{-2}$
			Yes	-0.073686	0.032150	128,734,494,508	1,099,511,627,832	$1.1708 \cdot 10^{-1}$	$4.8391 \cdot 10^{-1}$
	Yes	0.01	No	-0.137394	0.038051	898,017,460,318	1,099,511,627,779	$8.1674 \cdot 10^{-1}$	1.2781
			Yes	-0.142860	0.000140	1,099,511,627,949	1,099,511,627,949	1.0000	1.4142
		0.02	No	-0.138394	0.038051	906,616,451,593	1,099,511,627,826	$8.2456 \cdot 10^{-1}$	1.2842
			Yes	-0.214279	0.000871	1,099,511,627,787	1,099,511,627,787	1.0000	1.4142
None	No	0.01	No	-0.007590	0.059593	9,200,272,177	1,099,511,627,778	$8.3676 \cdot 10^{-3}$	$1.2936 \cdot 10^{-1}$
			Yes	-0.068768	0.013693	140,086,449	1,099,511,627,780	$1.2741 \cdot 10^{-4}$	$1.5963 \cdot 10^{-2}$
		0.02	No	-0.007928	0.059758	8,744,357,753	1,099,511,627,777	$7.9529 \cdot 10^{-3}$	$1.2612 \cdot 10^{-1}$
			Yes	-0.063449	0.022980	179,787,608	1,099,511,627,783	$1.6352 \cdot 10^{-4}$	$1.8084 \cdot 10^{-2}$
	Yes	0.01	No	-0.161979	0.044525	561,360,440,111	1,099,511,627,781	$5.1055 \cdot 10^{-1}$	1.0105
			Yes	-0.214269	0.001224	1,099,511,627,826	1,099,511,627,826	1.0000	1.4142
		0.02	No	-0.164304	0.044520	583,351,903,190	1,099,511,627,781	$5.3056 \cdot 10^{-1}$	1.0301
			Yes	-0.211755	0.013217	1,099,511,627,782	1,099,511,627,782	1.0000	1.4142

Table C.10: Results for the bias point estimate on the 2011 South Carolina Congressional map.

Constraints				Results for 2011 South Carolina Congressional Map ($B_{G,U} = 0.047855$)					
Compactness Threshold	Freeze Dist. 6?	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (47)	No	0.01	No	0.014067	0.006244	197,450,221	274,877,906,949	$7.1832 \cdot 10^{-4}$	$3.7903 \cdot 10^{-2}$
			Yes	0.014494	0.003799	234,663,904	274,877,906,954	$8.5370 \cdot 10^{-4}$	$4.1321 \cdot 10^{-2}$
		0.02	No	0.014178	0.007469	195,643,246	274,877,906,944	$7.1175 \cdot 10^{-4}$	$3.7729 \cdot 10^{-2}$
			Yes	0.012428	0.003685	266,211,238	274,877,906,948	$9.6847 \cdot 10^{-4}$	$4.4011 \cdot 10^{-2}$
	Yes	0.01	No	0.036391	0.005241	943,655,225	274,877,906,946	$3.4330 \cdot 10^{-3}$	$8.2861 \cdot 10^{-2}$
			Yes	0.042379	0.000029	1,512,471	274,877,906,975	$5.5023 \cdot 10^{-6}$	$3.3173 \cdot 10^{-3}$
		0.02	No	0.036422	0.005298	1,015,558,306	274,877,906,951	$3.6946 \cdot 10^{-3}$	$8.5960 \cdot 10^{-2}$
			Yes	0.045185	0.001273	977,154,800	274,877,906,954	$3.5549 \cdot 10^{-3}$	$8.4319 \cdot 10^{-2}$
Perimeter (6646800)	No	0.01	No	0.014990	0.006659	190,238,670	274,877,906,945	$6.9208 \cdot 10^{-4}$	$3.7204 \cdot 10^{-2}$
			Yes	0.014193	0.003886	220,302,254	274,877,906,950	$8.0145 \cdot 10^{-4}$	$4.0036 \cdot 10^{-2}$
		0.02	No	0.013799	0.006215	164,051,784	274,877,906,945	$5.9682 \cdot 10^{-4}$	$3.4549 \cdot 10^{-2}$
			Yes	0.012529	0.003608	276,460,772	274,877,906,951	$1.0058 \cdot 10^{-3}$	$4.4850 \cdot 10^{-2}$
	Yes	0.01	No	0.036754	0.006031	968,231,730	274,877,906,956	$3.5224 \cdot 10^{-3}$	$8.3933 \cdot 10^{-2}$
			Yes	0.043193	0.000067	19,624,525	274,877,907,107	$7.1394 \cdot 10^{-5}$	$1.1949 \cdot 10^{-2}$
		0.02	No	0.036552	0.005527	763,615,342	274,877,906,952	$2.7780 \cdot 10^{-3}$	$7.4539 \cdot 10^{-2}$
			Yes	0.048845	0.001182	269,111,579,033	274,877,907,026	$9.7902 \cdot 10^{-1}$	1.3993
None	No	0.01	No	0.013445	0.006101	217,001,177	274,877,906,944	$7.8945 \cdot 10^{-4}$	$3.9735 \cdot 10^{-2}$
			Yes	0.013440	0.003183	341,896,067	274,877,906,944	$1.2438 \cdot 10^{-3}$	$4.9876 \cdot 10^{-2}$
		0.02	No	0.013482	0.006104	216,226,672	274,877,906,950	$7.8663 \cdot 10^{-4}$	$3.9664 \cdot 10^{-2}$
			Yes	0.014591	0.003273	448,121,175	274,877,906,945	$1.6303 \cdot 10^{-3}$	$5.7101 \cdot 10^{-2}$
	Yes	0.01	No	0.042446	0.005941	23,811,435,854	274,877,906,949	$8.6625 \cdot 10^{-2}$	$4.1623 \cdot 10^{-1}$
			Yes	0.046351	0.000244	836,105,254	274,877,906,951	$3.0417 \cdot 10^{-3}$	$7.7997 \cdot 10^{-2}$
		0.02	No	0.042479	0.005777	25,159,689,969	274,877,906,945	$9.1530 \cdot 10^{-2}$	$4.2786 \cdot 10^{-1}$
			Yes	0.046182	0.005042	76,943,979,602	274,877,906,967	$2.7992 \cdot 10^{-1}$	$7.4823 \cdot 10^{-1}$

Table C.11: Results for uniform geometric bias on the 2011 South Carolina Congressional map.

Constraints				**Results for 2011 South Carolina Congressional Map** ($B_{G,V} = 0.037697$)					
Compactness Threshold	Freeze Dist. 6?	Population Threshold	Preserve Counties?	Mean ($n = 65537^*$)	Std. Dev. ($n = 65537^*$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (47)	No	0.01	No	0.013360	0.004936	16,216,922	274,877,906,949	$5.8997 \cdot 10^{-5}$	$1.0862 \cdot 10^{-2}$
			Yes	0.011831	0.002646	1,542	274,877,906,954	$5.6098 \cdot 10^{-9}$	$1.0592 \cdot 10^{-4}$
		0.02	No	0.013475	0.004964	17,833,046	274,877,906,944	$6.4876 \cdot 10^{-5}$	$1.1391 \cdot 10^{-2}$
			Yes	0.010448	0.002417	1	274,877,906,948	$3.6380 \cdot 10^{-12}$	$2.6974 \cdot 10^{-6}$
	Yes	0.01	No	0.028526	0.003054	424,896,069	274,877,906,946	$1.5458 \cdot 10^{-3}$	$5.5601 \cdot 10^{-2}$
			Yes	0.033353	0.000025	1,368,942	274,877,906,975	$4.9802 \cdot 10^{-6}$	$3.1560 \cdot 10^{-3}$
		0.02	No	0.028534	0.003070	536,044,076	274,877,906,951	$1.9501 \cdot 10^{-3}$	$6.2452 \cdot 10^{-2}$
			Yes	0.035240	0.000879	679,115,406	274,877,906,954	$2.4706 \cdot 10^{-3}$	$7.0294 \cdot 10^{-2}$
Perimeter (6646800)	No	0.01	No	0.014146	0.005224	7,966,515	274,877,906,945	$2.8982 \cdot 10^{-5}$	$7.6134 \cdot 10^{-3}$
			Yes	0.011542	0.002728	1,382	274,877,906,950	$5.0277 \cdot 10^{-9}$	$1.0028 \cdot 10^{-4}$
		0.02	No	0.013202	0.004854	6,104,455	274,877,906,945	$2.2208 \cdot 10^{-5}$	$6.6645 \cdot 10^{-3}$
			Yes	0.010527	0.002377	1	274,877,906,951	$3.6380 \cdot 10^{-12}$	$2.6974 \cdot 10^{-6}$
	Yes	0.01	No	0.028810	0.003042	336,924,004	274,877,906,956	$1.2257 \cdot 10^{-3}$	$4.9512 \cdot 10^{-2}$
			Yes	0.034053	0.000051	18,154,572	274,877,907,107	$6.6046 \cdot 10^{-5}$	$1.1493 \cdot 10^{-2}$
		0.02	No	0.028682	0.002932	228,612,730	274,877,906,952	$8.3169 \cdot 10^{-4}$	$4.0785 \cdot 10^{-2}$
			Yes	0.038025	0.000600	268,785,517,944	274,877,907,026	$9.7784 \cdot 10^{-1}$	1.3985
None	No	0.01	No	0.012598	0.004636	6,706,853	274,877,906,944	$2.4399 \cdot 10^{-5}$	$6.9856 \cdot 10^{-3}$
			Yes	0.011830	0.002154	1,519	274,877,906,944	$5.5261 \cdot 10^{-9}$	$1.0513 \cdot 10^{-4}$
		0.02	No	0.012642	0.004628	6,244,621	274,877,906,950	$2.2718 \cdot 10^{-5}$	$6.7406 \cdot 10^{-3}$
			Yes	0.012941	0.002294	1	274,877,906,945	$3.6380 \cdot 10^{-12}$	$2.6974 \cdot 10^{-6}$
	Yes	0.01	No	0.032070	0.003547	13,127,221,652	274,877,906,949	$4.7757 \cdot 10^{-2}$	$3.0905 \cdot 10^{-1}$
			Yes	0.036137	0.000190	646,393,947	274,877,906,951	$2.3516 \cdot 10^{-3}$	$6.8579 \cdot 10^{-2}$
		0.02	No	0.032183	0.003585	14,899,858,744	274,877,906,945	$5.4205 \cdot 10^{-2}$	$3.2926 \cdot 10^{-1}$
			Yes	0.035900	0.001813	52,290,404,320	274,877,906,967	$1.9023 \cdot 10^{-1}$	$6.1682 \cdot 10^{-1}$

Table C.12: Results for variable geometric bias on the 2011 South Carolina Congressional map.

Constraints			**Results for 2011 South Carolina Senate Map** (MM = -0.049402)					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (235)	0.025	No	-0.018175	0.009236	34,133,033	274,877,906,944	$1.2418 \cdot 10^{-4}$	$1.5759 \cdot 10^{-2}$
		Yes	-0.021208	0.009685	199,478,542	274,877,906,944	$7.2570 \cdot 10^{-4}$	$3.8097 \cdot 10^{-2}$
	0.05	No	-0.013565	0.009131	5,155,421	274,877,906,945	$1.8755 \cdot 10^{-5}$	$6.1246 \cdot 10^{-3}$
		Yes	-0.013549	0.008720	5,280,342	274,877,906,948	$1.9210 \cdot 10^{-5}$	$6.1984 \cdot 10^{-3}$
Perimeter (14304300)	0.025	No	-0.019121	0.009125	41,427,080	274,877,906,945	$1.5071 \cdot 10^{-4}$	$1.7361 \cdot 10^{-2}$
		Yes	-0.023671	0.009120	306,702,744	274,877,906,945	$1.1158 \cdot 10^{-3}$	$4.7239 \cdot 10^{-2}$
	0.05	No	-0.014421	0.009040	7,718,423	274,877,906,949	$2.8079 \cdot 10^{-5}$	$7.4939 \cdot 10^{-3}$
		Yes	-0.009379	0.007610	123,793	274,877,906,951	$4.5036 \cdot 10^{-7}$	$9.4906 \cdot 10^{-4}$
None	0.025	No	-0.018580	0.008987	29,037,567	274,877,906,949	$1.0564 \cdot 10^{-4}$	$1.4535 \cdot 10^{-2}$
		Yes	-0.016481	0.008708	13,865,040	274,877,906,945	$5.0441 \cdot 10^{-5}$	$1.0044 \cdot 10^{-2}$
	0.05	No	-0.014944	0.008689	3,185,469	274,877,906,944	$1.1589 \cdot 10^{-5}$	$4.8143 \cdot 10^{-3}$
		Yes	-0.015231	0.008422	6,199,148	274,877,906,945	$2.2552 \cdot 10^{-5}$	$6.7160 \cdot 10^{-3}$

Table C.13: Results for median-mean on the 2011 South Carolina Senate map.

Constraints			Results for 2011 South Carolina Senate Map $\left(\hat{\beta}(V) = -0.108696\right)$					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (235)	0.025	No	-0.049333	0.020621	1,335,703,937	274,877,906,944	$4.8593 \cdot 10^{-3}$	$9.8583 \cdot 10^{-2}$
		Yes	-0.060612	0.020322	4,518,459,280	274,877,906,944	$1.6438 \cdot 10^{-2}$	$1.8132 \cdot 10^{-1}$
	0.05	No	-0.043003	0.022548	940,812,946	274,877,906,945	$3.4227 \cdot 10^{-3}$	$8.2736 \cdot 10^{-2}$
		Yes	-0.045071	0.021847	1,067,970,281	274,877,906,948	$3.8853 \cdot 10^{-3}$	$8.8150 \cdot 10^{-2}$
Perimeter (14304300)	0.025	No	-0.051029	0.020372	1,337,239,000	274,877,906,945	$4.8648 \cdot 10^{-3}$	$9.8639 \cdot 10^{-2}$
		Yes	-0.065016	0.019819	6,692,114,624	274,877,906,945	$2.4346 \cdot 10^{-2}$	$2.2066 \cdot 10^{-1}$
	0.05	No	-0.043866	0.022206	897,419,139	274,877,906,949	$3.2648 \cdot 10^{-3}$	$8.0806 \cdot 10^{-2}$
		Yes	-0.039035	0.021996	479,367,434	274,877,906,951	$1.7439 \cdot 10^{-3}$	$5.9058 \cdot 10^{-2}$
None	0.025	No	-0.053910	0.021897	3,086,404,084	274,877,906,949	$1.1228 \cdot 10^{-2}$	$1.4986 \cdot 10^{-1}$
		Yes	-0.049433	0.021058	1,203,197,953	274,877,906,945	$4.3772 \cdot 10^{-3}$	$9.3565 \cdot 10^{-2}$
	0.05	No	-0.048037	0.023123	2,112,518,652	274,877,906,944	$7.6853 \cdot 10^{-3}$	$1.2398 \cdot 10^{-1}$
		Yes	-0.049685	0.022444	2,229,741,933	274,877,906,945	$8.1118 \cdot 10^{-3}$	$1.2737 \cdot 10^{-1}$

Table C.14: Results for the bias point estimate on the 2011 South Carolina Senate map.

Constraints			**Results for 2011 South Carolina Senate Map** ($B_{G,U} = 0.051240$)					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 32769$)	Std. Dev. ($n = 32769$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (235)	0.025	No	0.026618	0.007527	1,556,774,314	137,438,953,472	$1.1327 \cdot 10^{-2}$	$1.5051 \cdot 10^{-1}$
		Yes	0.030446	0.007666	3,106,943,411	137,438,953,477	$2.2606 \cdot 10^{-2}$	$2.1263 \cdot 10^{-1}$
	0.05	No	0.023761	0.007298	846,012,462	137,438,953,475	$6.1556 \cdot 10^{-3}$	$1.1096 \cdot 10^{-1}$
		Yes	0.022473	0.009145	678,888,387	137,438,953,484	$4.9396 \cdot 10^{-3}$	$9.9394 \cdot 10^{-2}$
Perimeter (14304300)	0.025	No	0.026592	0.007592	1,502,399,347	137,438,953,476	$1.0931 \cdot 10^{-2}$	$1.4786 \cdot 10^{-1}$
		Yes	0.030286	0.007613	2,961,001,808	137,438,953,476	$2.1544 \cdot 10^{-2}$	$2.0758 \cdot 10^{-1}$
	0.05	No	0.023118	0.007371	746,506,370	137,438,953,474	$5.4315 \cdot 10^{-3}$	$1.0423 \cdot 10^{-1}$
		Yes	0.021305	0.008370	476,966,310	137,438,953,475	$3.4704 \cdot 10^{-3}$	$8.3311 \cdot 10^{-2}$
None	0.025	No	0.026689	0.007734	1,658,826,816	137,438,953,472	$1.2070 \cdot 10^{-2}$	$1.5537 \cdot 10^{-1}$
		Yes	0.026169	0.007687	1,506,281,967	137,438,953,475	$1.0960 \cdot 10^{-2}$	$1.4805 \cdot 10^{-1}$
	0.05	No	0.023808	0.007675	863,189,795	137,438,953,472	$6.2805 \cdot 10^{-3}$	$1.1208 \cdot 10^{-1}$
		Yes	0.024073	0.007788	958,524,548	137,438,953,473	$6.9742 \cdot 10^{-3}$	$1.1810 \cdot 10^{-1}$

Table C.15: Results for uniform geometric bias on the 2011 South Carolina Senate map.

Constraints			**Results for 2011 South Carolina Senate Map** ($B_{G,V} = 0.039203$)					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 32769^*$)	Std. Dev. ($n = 32769^*$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (235)	0.025	No	0.019840	0.004551	6,329,248	137,438,953,472	$4.6051 \cdot 10^{-5}$	$9.5970 \cdot 10^{-3}$
		Yes	0.023095	0.004749	37,902,656	137,438,953,477	$2.7578 \cdot 10^{-4}$	$2.3485 \cdot 10^{-2}$
	0.05	No	0.017437	0.004538	1,527,297	137,438,953,475	$1.1113 \cdot 10^{-5}$	$4.7144 \cdot 10^{-3}$
		Yes	0.016849	0.004737	1,757,578	137,438,953,484	$1.2788 \cdot 10^{-5}$	$5.0573 \cdot 10^{-3}$
Perimeter (14304300)	0.025	No	0.020138	0.004646	7,298,820	137,438,953,476	$5.3106 \cdot 10^{-5}$	$1.0306 \cdot 10^{-2}$
		Yes	0.023444	0.004725	42,291,932	137,438,953,476	$3.0771 \cdot 10^{-4}$	$2.4808 \cdot 10^{-2}$
	0.05	No	0.017341	0.004650	1,450,551	137,438,953,474	$1.0554 \cdot 10^{-5}$	$4.5944 \cdot 10^{-3}$
		Yes	0.015874	0.004450	207,228	137,438,953,475	$1.5078 \cdot 10^{-6}$	$1.7365 \cdot 10^{-3}$
None	0.025	No	0.020153	0.004854	12,848,485	137,438,953,472	$9.3485 \cdot 10^{-5}$	$1.3674 \cdot 10^{-2}$
		Yes	0.019522	0.004674	2,046,503	137,438,953,475	$1.4890 \cdot 10^{-5}$	$5.4572 \cdot 10^{-3}$
	0.05	No	0.017976	0.004834	3,573,081	137,438,953,472	$2.5998 \cdot 10^{-5}$	$7.2108 \cdot 10^{-3}$
		Yes	0.018407	0.005034	7,001,800	137,438,953,473	$5.0945 \cdot 10^{-5}$	$1.0094 \cdot 10^{-2}$

Table C.16: Results for variable geometric bias on the 2011 South Carolina Senate map.

Constraints			**Results for 2011 South Carolina House Map** (MM $= -0.051680$)					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (530)	0.025	No	-0.045681	0.003158	198,618,606	274,877,906,946	$7.2257 \cdot 10^{-4}$	$3.8015 \cdot 10^{-2}$
		Yes	-0.046506	0.002526	100,271,835	274,877,906,944	$3.6479 \cdot 10^{-4}$	$2.7011 \cdot 10^{-2}$
	0.05	No	-0.025932	0.006445	1,223,513	274,877,906,945	$4.4511 \cdot 10^{-6}$	$2.9837 \cdot 10^{-3}$
		Yes	-0.027355	0.006349	2,656,279	274,877,906,945	$9.6635 \cdot 10^{-6}$	$4.3962 \cdot 10^{-3}$
Perimeter (20595500)	0.025	No	-0.045204	0.003172	200,782,922	274,877,906,945	$7.3044 \cdot 10^{-4}$	$3.8222 \cdot 10^{-2}$
		Yes	-0.045630	0.001999	9,547,089	274,877,906,945	$3.4732 \cdot 10^{-5}$	$8.3345 \cdot 10^{-3}$
	0.05	No	-0.025029	0.006621	837,630	274,877,906,944	$3.0473 \cdot 10^{-6}$	$2.4687 \cdot 10^{-3}$
		Yes	-0.024050	0.006738	203,352	274,877,906,944	$7.3979 \cdot 10^{-7}$	$1.2164 \cdot 10^{-3}$
None	0.025	No	-0.042430	0.003120	212,402,059	274,877,906,945	$7.7271 \cdot 10^{-4}$	$3.9312 \cdot 10^{-2}$
		Yes	-0.046185	0.002589	410,782,254	274,877,906,944	$1.4944 \cdot 10^{-3}$	$5.4670 \cdot 10^{-2}$
	0.05	No	-0.023894	0.006532	474,586	274,877,906,944	$1.7265 \cdot 10^{-6}$	$1.8582 \cdot 10^{-3}$
		Yes	-0.027305	0.006699	102,036	274,877,906,944	$3.7120 \cdot 10^{-7}$	$8.6163 \cdot 10^{-4}$

Table C.17: Results for median-mean on the 2011 South Carolina House map.

Constraints			**Results for 2011 South Carolina House Map** $\left(\hat{\beta}(V) = -0.108871\right)$					
Compactness Threshold	Population Threshold	Preserve Counties?	Mean ($n = 65537$)	Std. Dev. ($n = 65537$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (530)	0.025	No	-0.110594	0.003729	234,239,663,962	274,877,906,946	$8.5216 \cdot 10^{-1}$	1.3055
		Yes	-0.113022	0.002914	268,921,439,312	274,877,906,944	$9.7833 \cdot 10^{-1}$	1.3988
	0.05	No	-0.063216	0.011717	20,381,767	274,877,906,945	$7.4148 \cdot 10^{-5}$	$1.2178 \cdot 10^{-2}$
		Yes	-0.071193	0.010810	121,636,175	274,877,906,945	$4.4251 \cdot 10^{-4}$	$2.9749 \cdot 10^{-2}$
Perimeter (20595500)	0.025	No	-0.109406	0.003946	206,810,105,099	274,877,906,945	$7.5237 \cdot 10^{-1}$	1.2267
		Yes	-0.111640	0.002954	267,443,220,613	274,877,906,945	$9.7295 \cdot 10^{-1}$	1.3950
	0.05	No	-0.061058	0.011997	12,341,866	274,877,906,944	$4.4899 \cdot 10^{-5}$	$9.4762 \cdot 10^{-3}$
		Yes	-0.068176	0.011708	108,914,955	274,877,906,944	$3.9623 \cdot 10^{-4}$	$2.8151 \cdot 10^{-2}$
None	0.025	No	-0.103380	0.004228	54,569,991,312	274,877,906,945	$1.9852 \cdot 10^{-1}$	$6.3012 \cdot 10^{-1}$
		Yes	-0.107541	0.003553	164,645,852,935	274,877,906,944	$5.9898 \cdot 10^{-1}$	1.0945
	0.05	No	-0.061121	0.012343	22,565,661	274,877,906,944	$8.2093 \cdot 10^{-5}$	$1.2814 \cdot 10^{-2}$
		Yes	-0.072029	0.011406	264,749,031	274,877,906,944	$9.6315 \cdot 10^{-4}$	$4.3890 \cdot 10^{-2}$

Table C.18: Results for the bias point estimate on the 2011 South Carolina House map.

Constraints			**Results for 2011 South Carolina House Map** ($B_{G,U} = 0.056833$)						
Compactness Threshold	Population Threshold	Preserve Counties?	n	Mean	Std. Dev.	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (530)	0.025	No	10364	0.056450	0.002570	7,032,068,673	43,465,572,353	$1.6178 \cdot 10^{-1}$	$5.6883 \cdot 10^{-1}$
		Yes	11543	0.057263	0.002697	19,907,535,686	48,410,656,770	$4.1122 \cdot 10^{-1}$	$9.0689 \cdot 10^{-1}$
	0.05	No	9997	0.041561	0.006603	1,219,518,811	41,926,262,784	$2.9087 \cdot 10^{-2}$	$2.4119 \cdot 10^{-1}$
		Yes	11070	0.043062	0.006681	1,913,087,992	46,426,750,977	$4.1207 \cdot 10^{-2}$	$2.8708 \cdot 10^{-1}$
Perimeter (20595500)	0.025	No	10411	0.056162	0.002930	5,497,142,319	43,662,704,641	$1.2590 \cdot 10^{-1}$	$5.0180 \cdot 10^{-1}$
		Yes	11470	0.056668	0.002122	10,369,182,640	48,104,472,576	$2.1556 \cdot 10^{-1}$	$6.5659 \cdot 10^{-1}$
	0.05	No	10057	0.040732	0.006511	984,298,586	42,177,921,025	$2.3337 \cdot 10^{-2}$	$2.1604 \cdot 10^{-1}$
		Yes	11084	0.041660	0.006713	1,430,293,038	46,485,471,232	$3.0769 \cdot 10^{-2}$	$2.4807 \cdot 10^{-1}$
None	0.025	No	10062	0.054717	0.003300	4,680,674,634	42,198,892,546	$1.1092 \cdot 10^{-1}$	$4.7100 \cdot 10^{-1}$
		Yes	11155	0.055408	0.002877	4,366,805,083	46,783,266,816	$9.3341 \cdot 10^{-2}$	$4.3207 \cdot 10^{-1}$
	0.05	No	8834	0.040460	0.006756	868,841,093	37,048,287,233	$2.3452 \cdot 10^{-2}$	$2.1657 \cdot 10^{-1}$
		Yes	10109	0.041921	0.006737	1,361,098,391	42,396,024,832	$3.2104 \cdot 10^{-2}$	$2.5339 \cdot 10^{-1}$

Table C.19: Results for uniform geometric bias on the 2011 South Carolina House map.

Constraints			Results for 2011 South Carolina House Map ($B_{G,V} = 0.042987$)					
Compactness Threshold	Population Constraint	Preserve Counties?	Mean ($n = 32769^*$)	Std. Dev. ($n = 32769^*$)	More Biased Maps	Total Steps	ε-Outlier at $\varepsilon =$	Significant at $p =$
Inverse Polsby-Popper (530)	0.025	No	0.042264	0.000593	15,001,972,802	137,438,953,477	$1.0915 \cdot 10^{-1}$	$4.6723 \cdot 10^{-1}$
		Yes	0.043094	0.000472	88,487,498,995	137,438,953,472	$6.4383 \cdot 10^{-1}$	1.1348
	0.05	No	0.028261	0.002647	2	137,438,953,473	$1.4552 \cdot 10^{-11}$	$5.3948 \cdot 10^{-6}$
		Yes	0.029565	0.002590	4,043	137,438,953,472	$2.9417 \cdot 10^{-8}$	$2.4256 \cdot 10^{-4}$
Perimeter (20595500)	0.025	No	0.041999	0.000671	8,841,038,490	137,438,953,476	$6.4327 \cdot 10^{-2}$	$3.5868 \cdot 10^{-1}$
		Yes	0.042567	0.000281	12,898,140,871	137,438,953,473	$9.3846 \cdot 10^{-2}$	$4.3324 \cdot 10^{-1}$
	0.05	No	0.027435	0.002718	2	137,438,953,476	$1.4552 \cdot 10^{-11}$	$5.3948 \cdot 10^{-6}$
		Yes	0.028872	0.002699	5,485	137,438,953,472	$3.9909 \cdot 10^{-8}$	$2.8252 \cdot 10^{-4}$
None	0.025	No	0.040770	0.000762	146,131,085	137,438,953,472	$1.0632 \cdot 10^{-3}$	$4.6114 \cdot 10^{-2}$
		Yes	0.041517	0.000633	674,218,228	137,438,953,472	$4.9056 \cdot 10^{-3}$	$9.9051 \cdot 10^{-2}$
	0.05	No	0.027291	0.002893	401	137,438,953,472	$2.9177 \cdot 10^{-9}$	$7.6389 \cdot 10^{-5}$
		Yes	0.029491	0.002704	37,390	137,438,953,472	$2.7205 \cdot 10^{-7}$	$7.3763 \cdot 10^{-4}$

Table C.20: Results for variable geometric bias on the 2011 South Carolina House map.

Appendix D Additional Information

D.1 Markov Chain Runtimes

As mentioned in Chapter 5, one major limitation of the Markov chain outlier analysis approach to analyze election maps is the runtime required for the algorithm. Even when run on dedicated nodes on a high-performance computing resource like the Clemson University Palmetto Cluster, the C++ code used to generate and compare election maps can take a long time to run to completion. This runtime varies by not only the number of steps, but also the constraint combinations chosen, the complexity of the label function, the number of polling precincts in the state, and the number of election districts for the map in question. A complete summary of runtimes for different maps, label functions[1], and constraints is included in Tables D.1–D.3.

[1]Where the tables indicate the median-mean measure and bias point estimate as the chosen label functions, note that two additional measures that were not the primary focus of this analysis were run alongside these as label functions. Variance (Equation 2.14) was included to compare results against the Pennsylvania analysis by Chikina et al. [22]. Efficiency gap (Equation 2.12), which is not a measure of partisan bias, was included to investigate a question unrelated to this thesis. Both measures are similar to median-mean in their computational simplicity.

Constraints				PA 2011			PA 2018		
Compactness Threshold	Freeze District 2?	Population Threshold	Preserve Counties?	2^{40} Steps	2^{38} Steps		2^{40} Steps	2^{38} Steps	
				MM and $\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$	MM and $\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	No	0.02	No	349:37:40	289:51:52	188:36:42	342:41:43	285:26:10	184:29:49
			Yes	242:53:34	186:40:33	116:20:51	216:47:13	204:15:18	139:20:35
		0.01	No	339:54:14	290:53:05	186:05:36	335:26:28	281:57:51	184:21:40
			Yes	240:53:17	186:50:52	115:31:15	216:43:01	211:01:59	141:03:18
	Yes	0.02	No	300:28:13	231:48:46	148:45:12	329:57:57	271:43:00	182:46:20
			Yes	208:32:32	156:09:55	98:12:56	190:34:23	175:23:54	121:44:35
		0.01	No	302:16:11	230:41:43	147:54:04	322:26:56	273:31:17	185:12:05
			Yes	205:22:22	154:52:43	97:14:42	189:29:46	173:48:47	120:25:06
Perimeter	No	0.02	No	305:55:30	246:27:35	150:32:44	311:10:27	242:12:50	151:05:53
			Yes	239:09:32	192:11:51	116:38:38	214:57:55	179:11:12	111:16:44
		0.01	No	307:38:26	245:47:59	151:02:47	312:55:44	245:42:08	151:31:49
			Yes	243:38:01	197:14:28	119:05:49	221:59:46	180:07:57	112:17:16
	Yes	0.02	No	262:48:23	211:32:10	129:00:38	279:40:31	216:47:14	135:31:48
			Yes	193:49:55	159:56:20	97:26:05	185:23:42	155:16:16	96:35:04
		0.01	No	263:36:54	209:49:12	129:06:41	282:30:27	215:48:51	133:28:04
			Yes	200:43:15	159:48:17	97:17:35	187:33:35	154:43:39	96:29:48
None	No	0.02	No	588:04:45	453:20:01	274:08:40	591:34:38	455:47:15	275:04:42
			Yes	401:04:07	308:54:29	187:45:09	288:21:24	227:05:18	139:50:55
		0.01	No	593:29:21	454:52:58	278:33:12	587:12:52	452:10:59	279:55:30
			Yes	399:43:44	311:20:15	187:57:49	292:49:04	228:12:48	141:00:24
	Yes	0.02	No	515:10:52	400:35:05	246:12:52	545:56:49	408:32:34	248:53:46
			Yes	344:14:04	260:48:56	159:06:08	249:16:39	192:29:02	119:15:09
		0.01	No	531:23:05	402:09:20	247:07:30	535:09:41	410:05:28	250:05:02
			Yes	342:31:31	264:28:36	161:29:17	245:27:35	193:03:19	122:03:01

Table D.1: Runtimes for Pennsylvania Congressional maps.

Constraints				SC Congressional		
Compactness Threshold	Freeze District 6?	Population Threshold	Preserve Counties?	2^{40} Steps	2^{38} Steps	
				MM and $\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	No	0.02	No	178:51:40	74:39:53	61:57:14
			Yes	78:01:10	36:02:36	30:13:53
		0.01	No	181:39:52	75:31:17	61:09:45
			Yes	77:38:42	34:23:57	29:32:23
	Yes	0.02	No	51:32:22	20:18:12	17:04:46
			Yes	9:56:48	4:00:54	3:21:06
		0.01	No	52:18:34	20:11:51	17:09:07
			Yes	11:44:45	4:41:11	3:56:51
Perimeter	No	0.02	No	167:33:09	70:59:15	56:56:27
			Yes	74:02:07	33:47:59	28:48:58
		0.01	No	172:47:09	70:16:32	56:49:09
			Yes	75:43:19	33:07:06	28:24:40
	Yes	0.02	No	49:11:40	21:22:42	16:27:09
			Yes	8:35:56	3:23:36	2:52:21
		0.01	No	49:45:28	19:39:09	16:27:02
			Yes	9:45:34	3:52:41	3:15:25
None	No	0.02	No	383:01:25	163:04:41	130:42:22
			Yes	120:41:03	53:11:13	45:14:32
		0.01	No	391:35:07	164:11:49	132:01:08
			Yes	113:22:25	49:29:25	42:06:35
	Yes	0.02	No	227:10:50	91:55:50	75:32:52
			Yes	15:39:31	6:21:04	5:15:56
		0.01	No	230:40:32	93:04:22	77:22:54
			Yes	20:40:05	8:09:37	6:44:41

Table D.2: Runtimes for South Carolina Congressional map.

Constraints			SC Senate			SC House		
Compactness Threshold	Population Threshold	Preserve Counties?	2^{38} Steps	2^{37} Steps		2^{38} Steps	2^{37} Steps	
			MM and $\hat{\beta}(V)$	$B_{G,U}$	$B_{G,V}$	MM and $\hat{\beta}(V)$	$B^*_{G,U}$	$B_{G,V}$
Inverse Polsby-Popper	0.05	No	206:36:31	678:32:09	287:52:56	567:31:29	1410:02:52	979:23:04
		Yes	171:11:09	561:52:21	243:01:42	508:18:58	1410:01:43	878:55:27
	0.025	No	194:42:16	636:06:58	268:13:03	538:45:22	1410:00:22	969:20:41
		Yes	160:24:22	528:48:37	217:05:55	494:19:43	1409:59:23	882:57:10
Perimeter	0.05	No	191:13:14	691:05:39	287:30:51	551:07:30	1406:59:42	990:38:20
		Yes	160:35:21	580:44:11	245:32:15	498:37:37	1406:56:32	896:37:29
	0.025	No	190:23:09	654:10:19	268:35:08	543:38:56	1406:53:50	974:40:03
		Yes	159:05:21	550:54:56	220:03:32	490:00:29	1406:51:07	887:40:32
None	0.05	No	252:05:38	940:22:53	380:50:18	615:39:44	1410:40:04	1124:33:33
		Yes	200:51:26	735:40:02	300:08:50	543:07:05	1410:37:52	987:38:27
	0.025	No	249:14:50	914:11:16	365:12:00	549:14:20	1410:35:39	1008:58:24
		Yes	199:31:08	715:20:09	291:29:50	499:22:46	1410:33:52	913:38:53

Table D.3: Runtimes for South Carolina legislative maps.

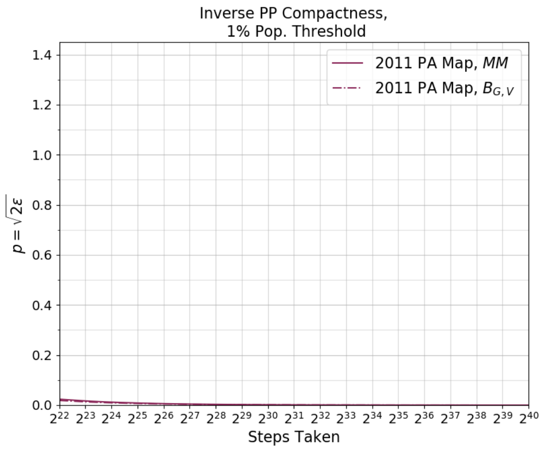
* The SC House computations for $B_{G,U}$ terminated prior to completion. The Markov chain for this column ran for approximately 2^{35} steps.

D.2 p-Value Graphs

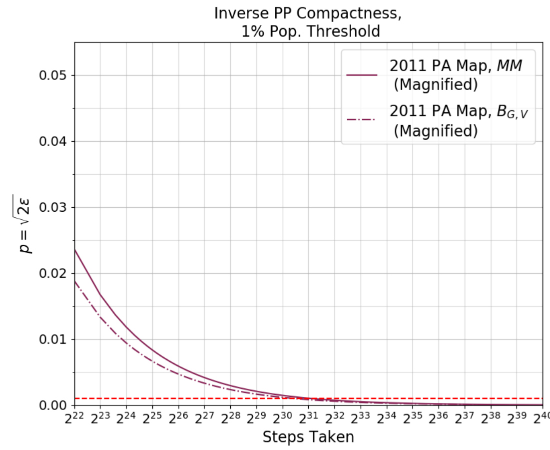
The long-term behavior of the Markov chain is also mentioned in Chapter 5, with an example for the South Carolina state House map included in Figure 5.1. For some districtings, such as the 2011 Pennsylvania Congressional plan, the p-value decreases steadily, reaching significance relatively quickly with respect to the total number of steps taken along the chain and remaining stable, regardless of which constraints are imposed on the map. Interpreted in context, this means that the more maps that are generated by the chain, the fewer exhibit bias as extreme as the original map in question, indicating that the original map is an outlier. For other maps, however, such as the example in Figure 5.1, this long-term behavior is far less evident. It remains an open question as to whether certain observable characteristics of the chain allow one to predict the convergence of the p-value. If so, one could reduce the runtime of the chain by one or several orders of magnitude without concern over loss of information.

The following figures display the p-value graphed against the number of steps on a semi-log scale, where the horizontal axis ticks indicate successive powers of two. I include the chain behavior for all five maps under both median-mean (Equation 2.10) and the variable geometric bias measure (Equation 2.11 under the VPS assumption), as these are the two label functions identified in Chapter 4 as reliable indicators of partisan bias for the $\sqrt{\varepsilon}$ test. The constraints imposed on the map are indicated above each graph. Note the values on the vertical axis of the graphs. The figures on the left include the range of all possible p-values that could result from the analysis, allowing graphs to be compared on the same scale. The right-hand figures show a magnified version of the same graph to allow the p-value behavior to be observed more closely. On the magnified graphs, the selected significance level of $\alpha = 0.001$ is indicated by a dashed red line.

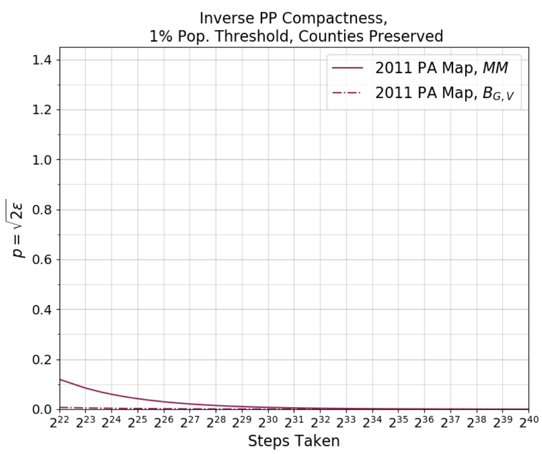
Figure D.1: p-value graphs for the 2011 PA Congressional map.



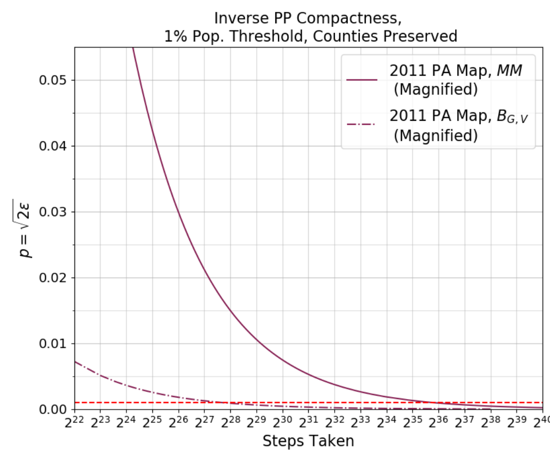
(i)



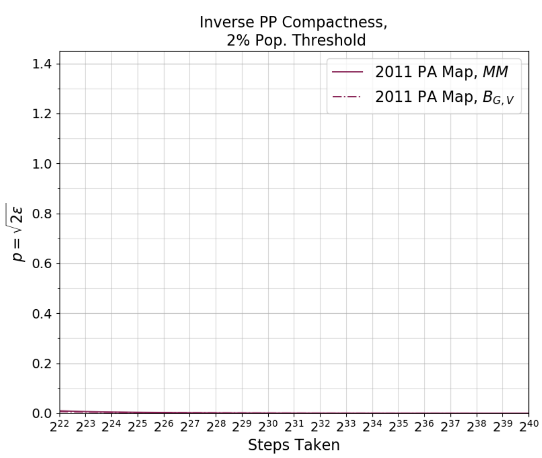
(ii)



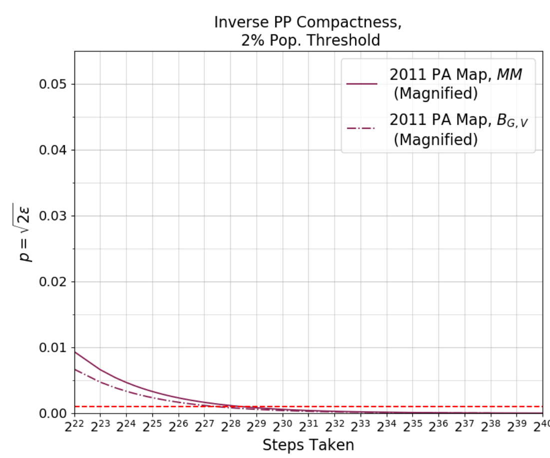
(iii)



(iv)



(v)



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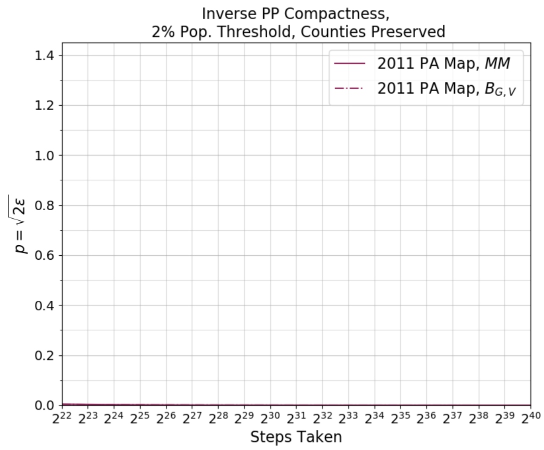

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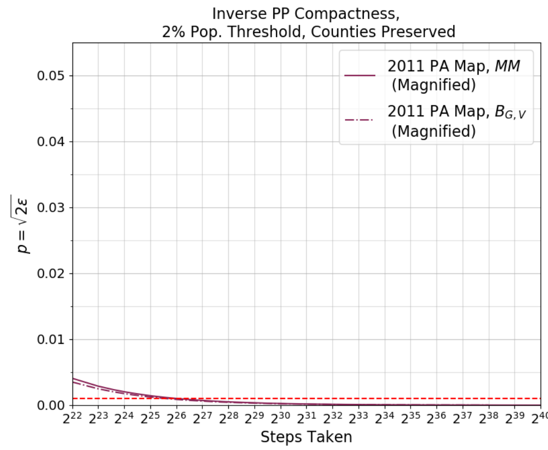

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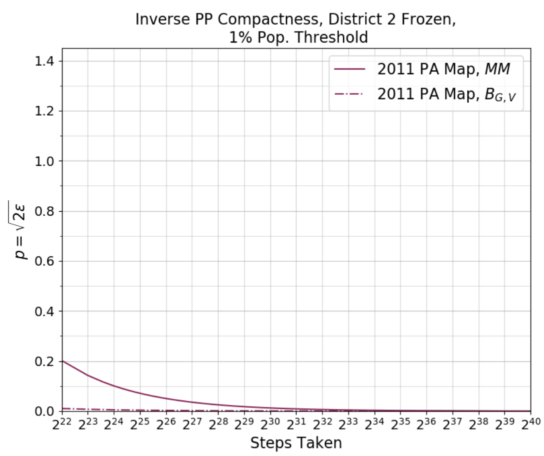
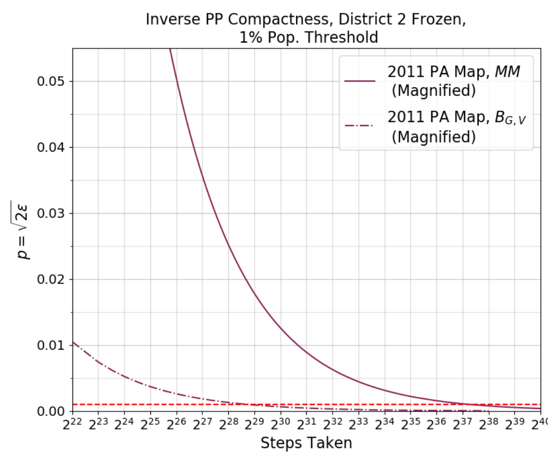
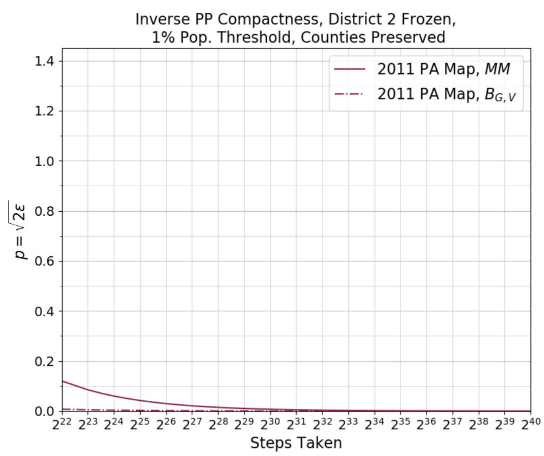


(ix)



(x)



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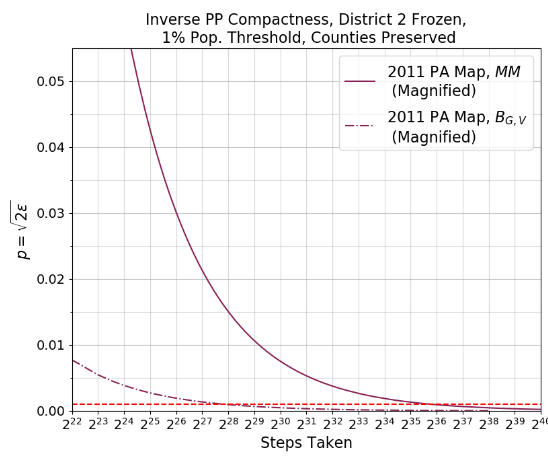
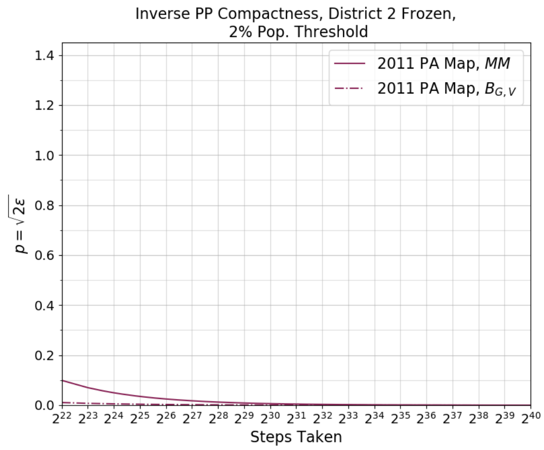


(xii)



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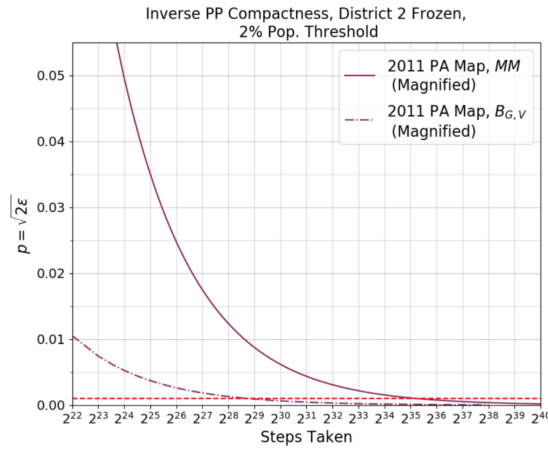

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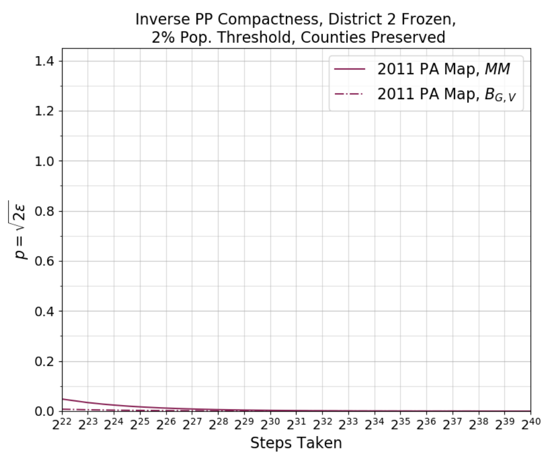

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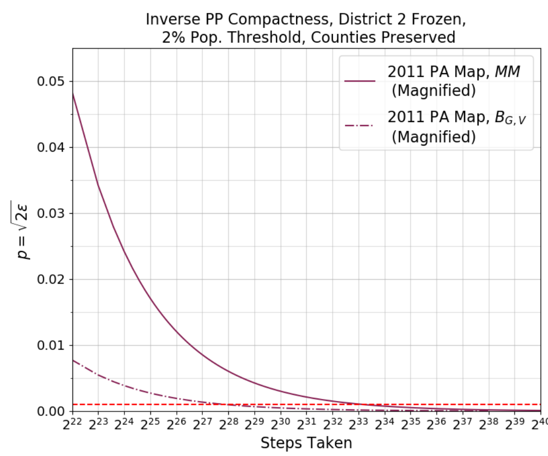

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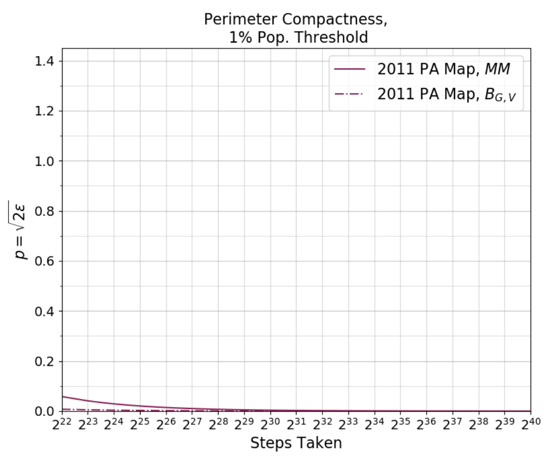

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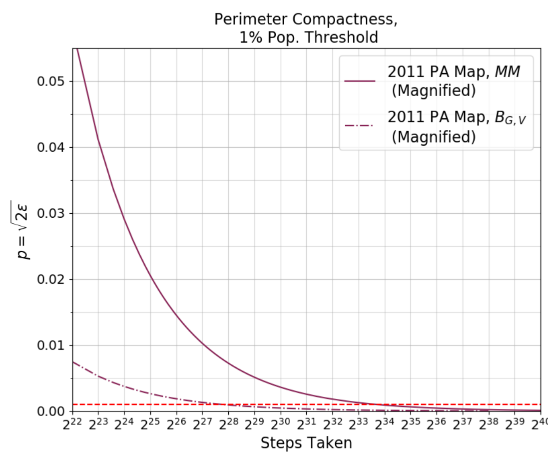

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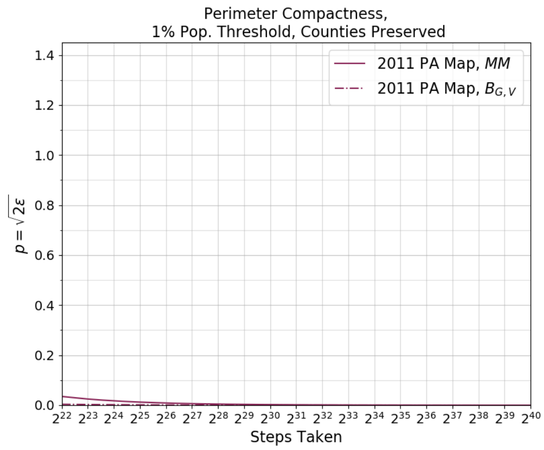

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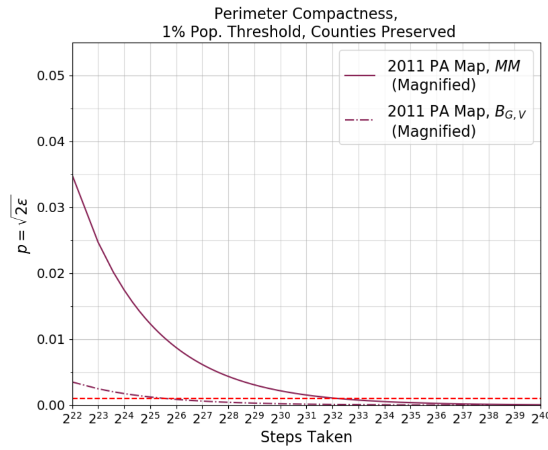
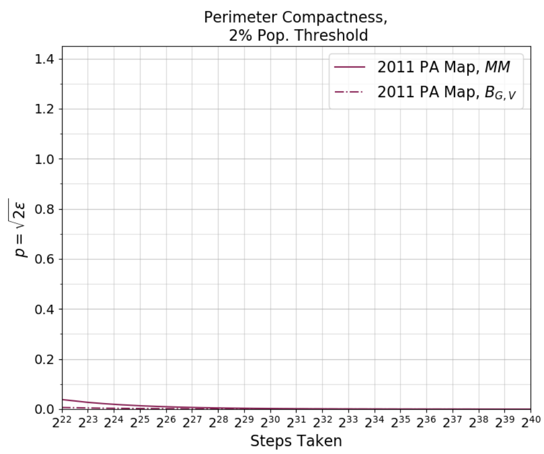
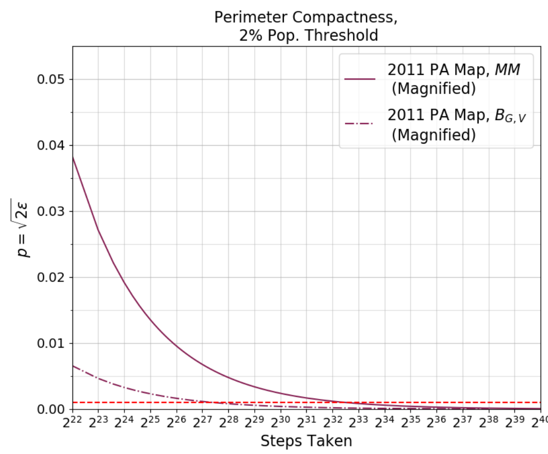


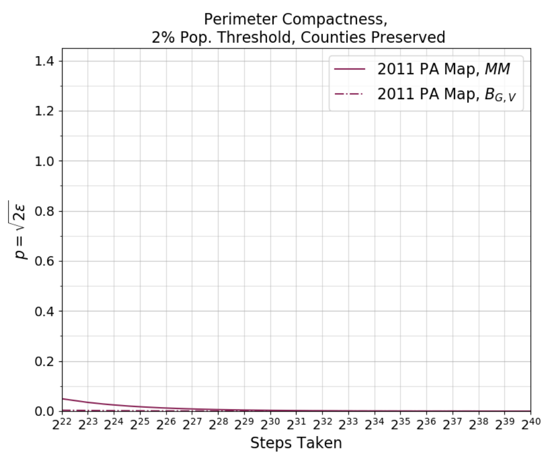
(xx)



(xxi)



(xxii)



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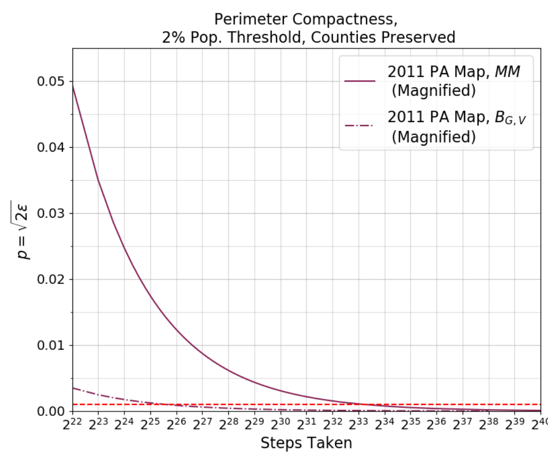

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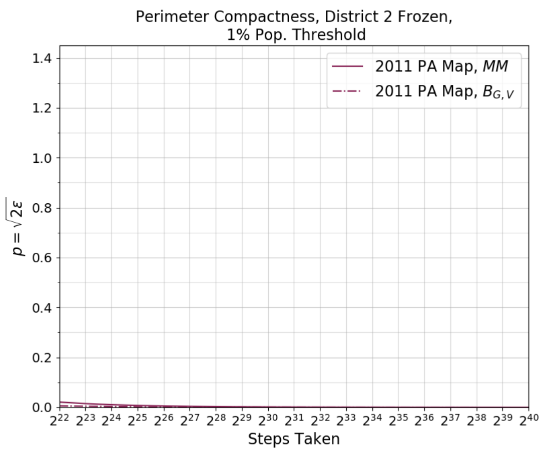
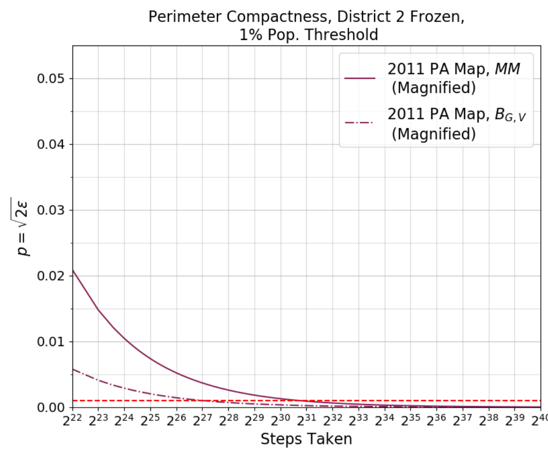


(xxv)



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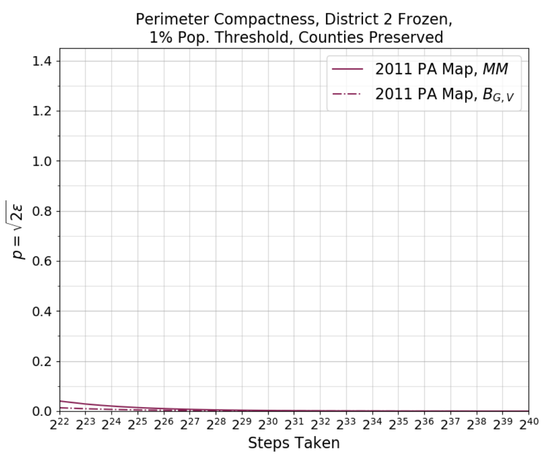

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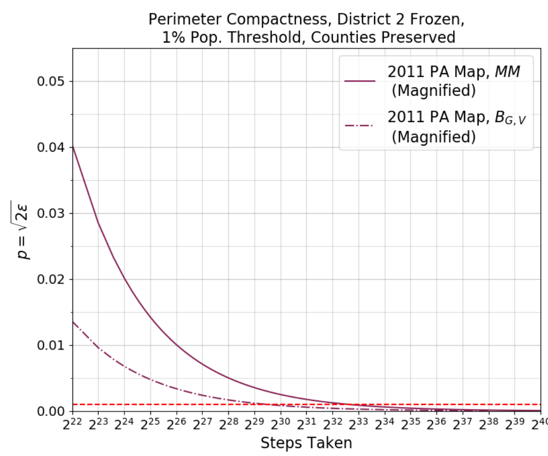

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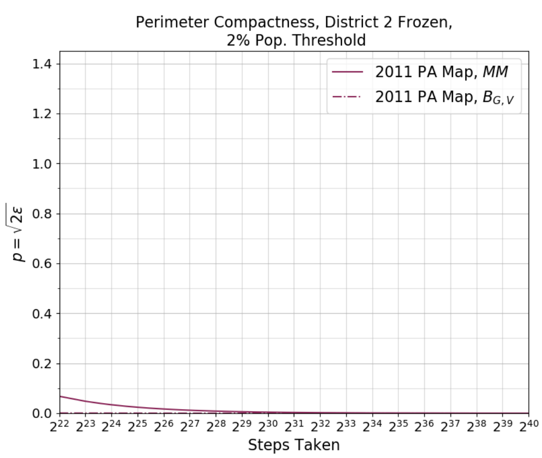

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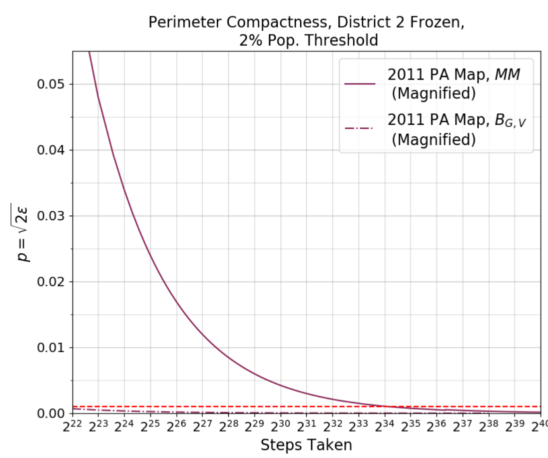
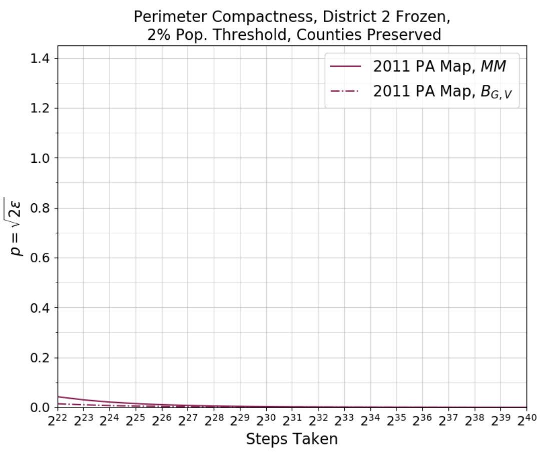
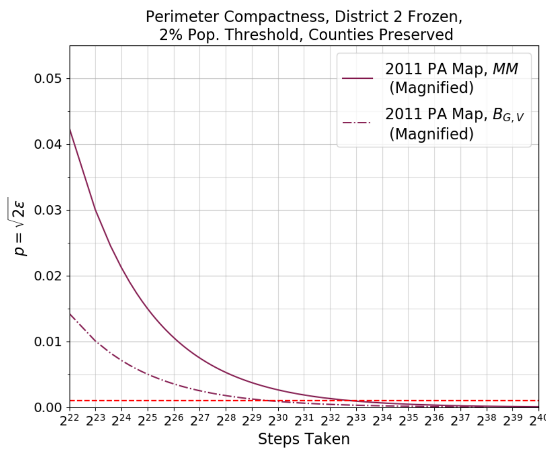


(xxx)

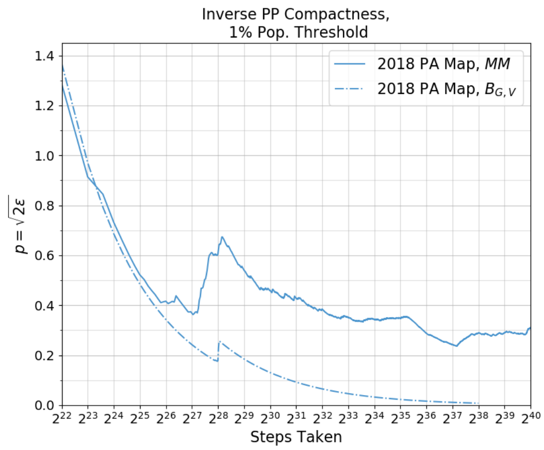

Perimeter Compactness, District 2 Frozen,
2% Pop. Threshold, Counties Preserved
2011 PA Map, MM
2011 PA Map, $B_{G,V}$
$p = \sqrt{2\varepsilon}$
Steps Taken

(xxxi)

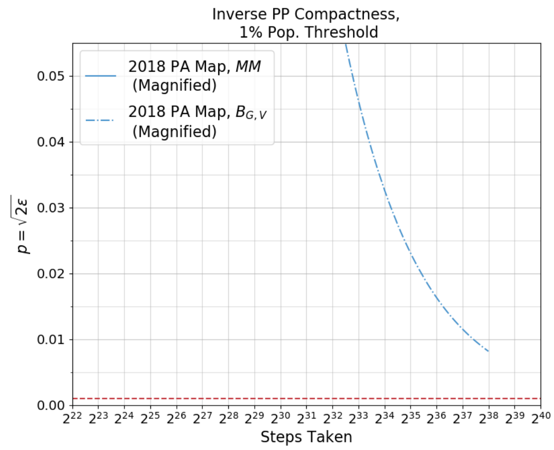

Perimeter Compactness, District 2 Frozen,
2% Pop. Threshold, Counties Preserved
2011 PA Map, MM (Magnified)
2011 PA Map, $B_{G,V}$ (Magnified)
$p = \sqrt{2\varepsilon}$
Steps Taken

(xxxii)

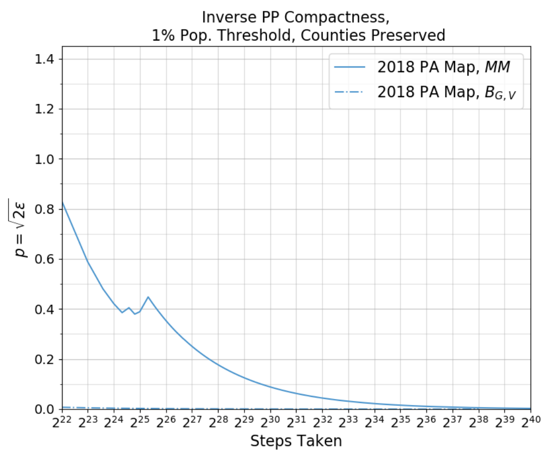
Figure D.2: p-value graphs for the 2018 PA Congressional map.



(i)



(ii)



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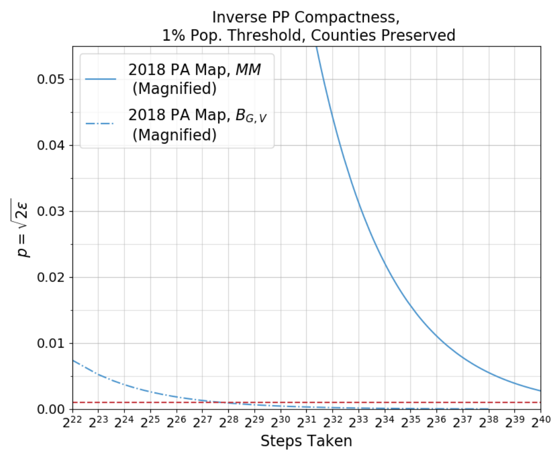
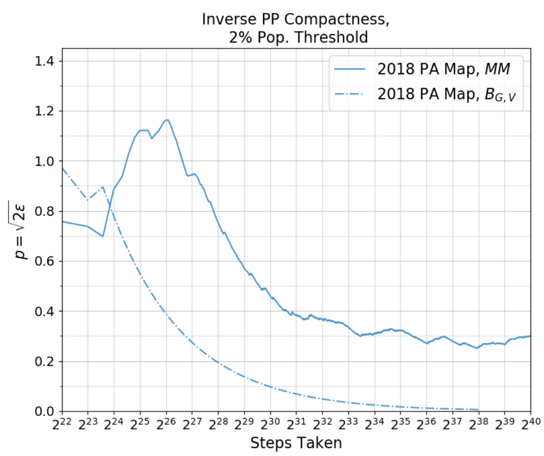
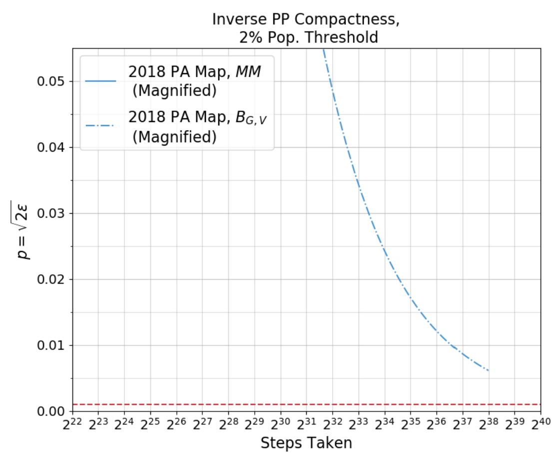


(iv)



(v)



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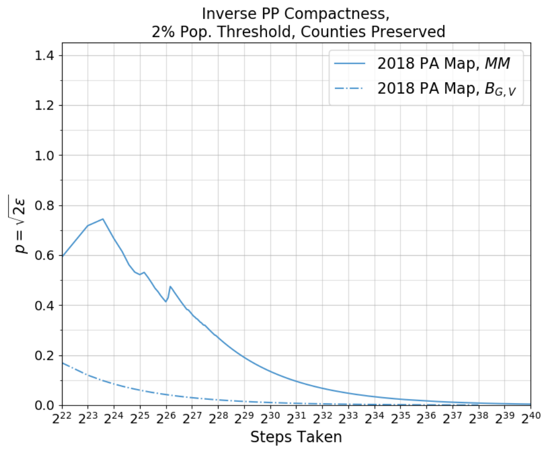

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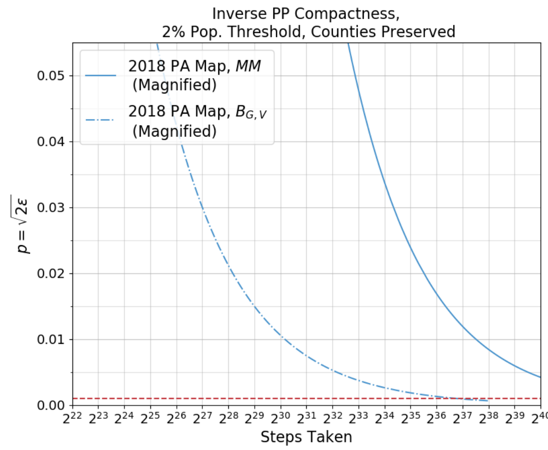

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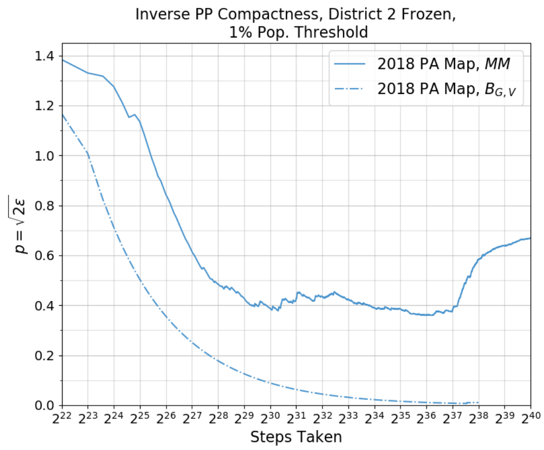

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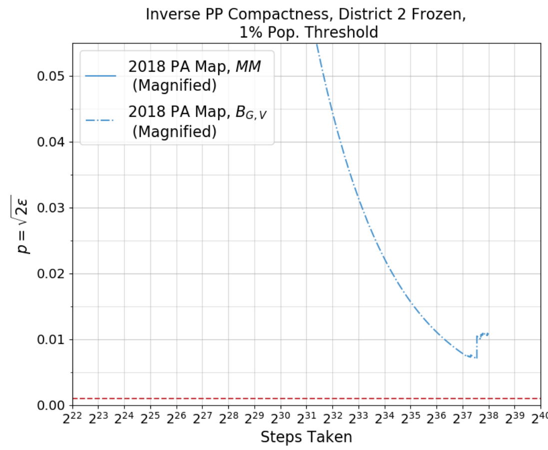

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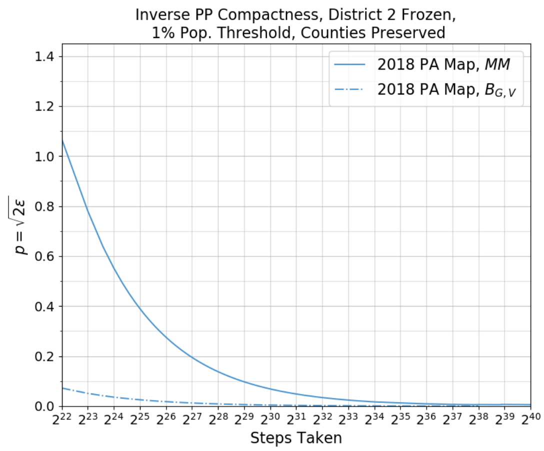

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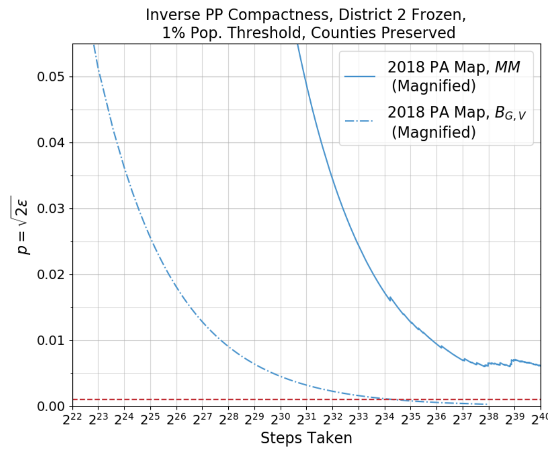

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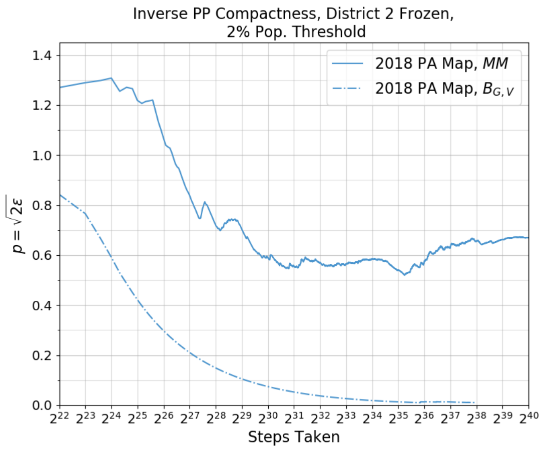

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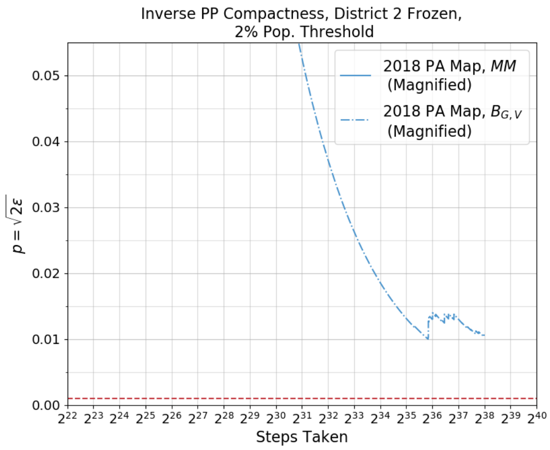

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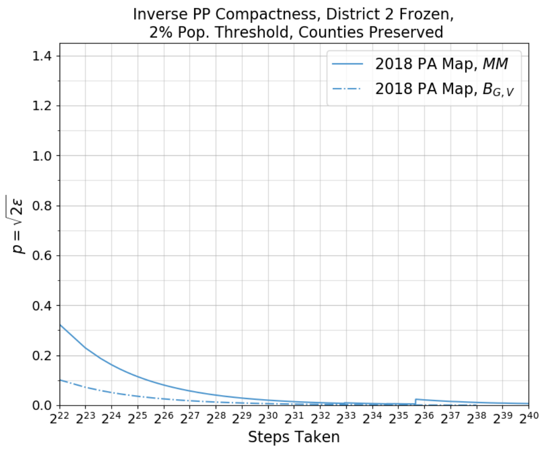

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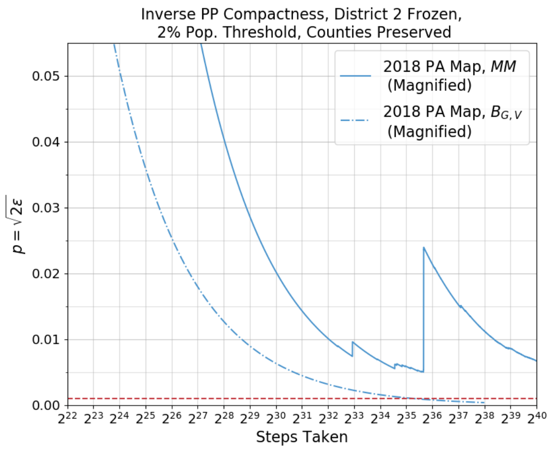

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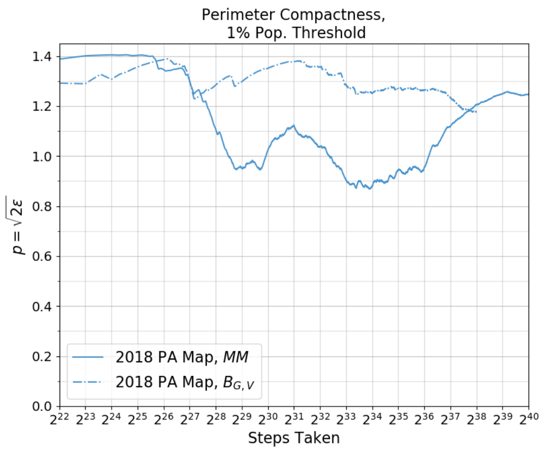

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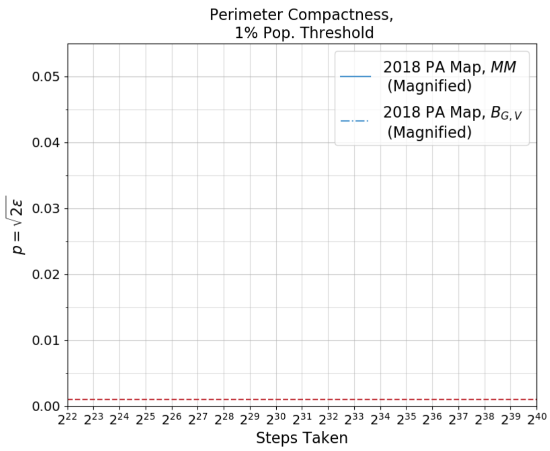

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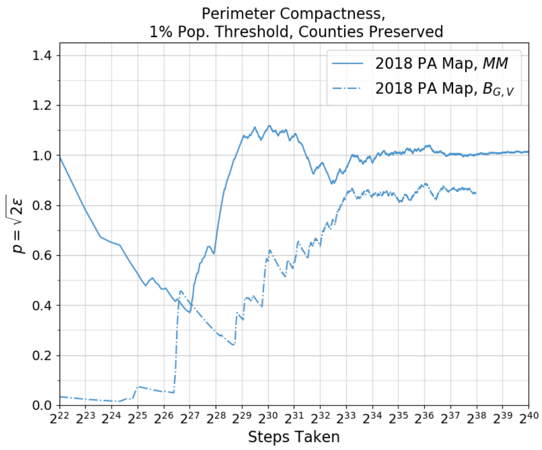

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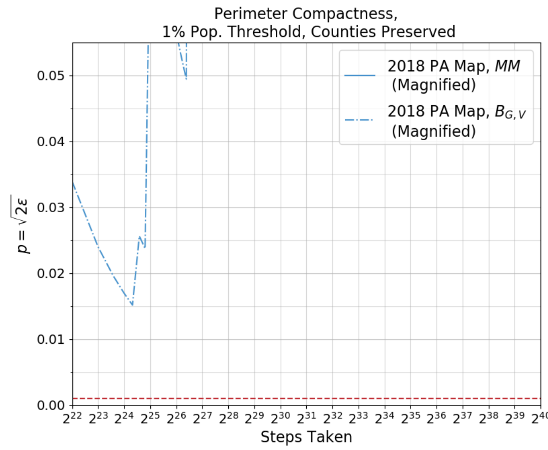

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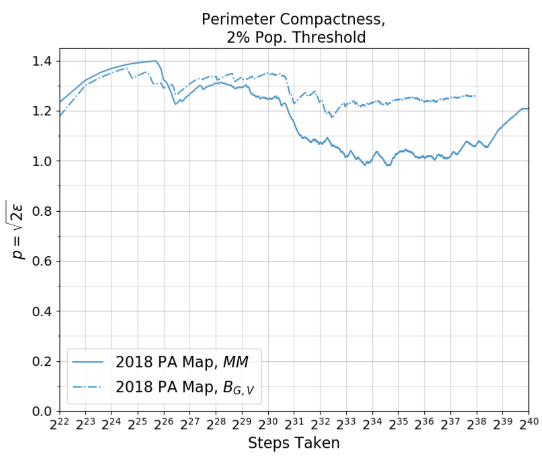

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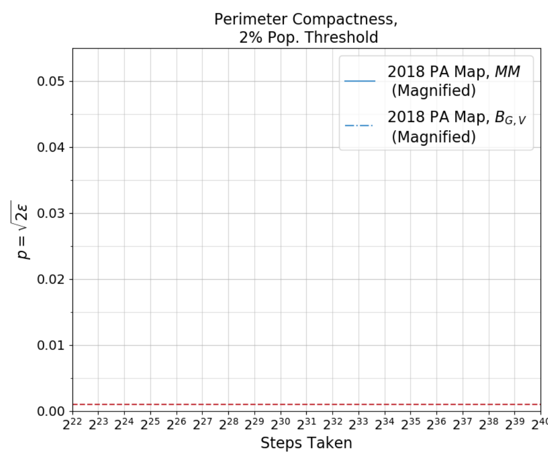

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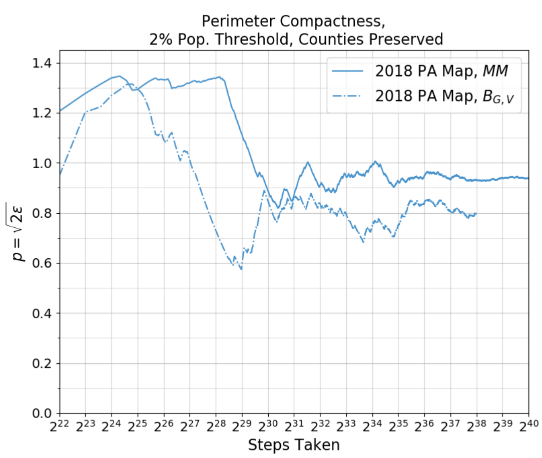

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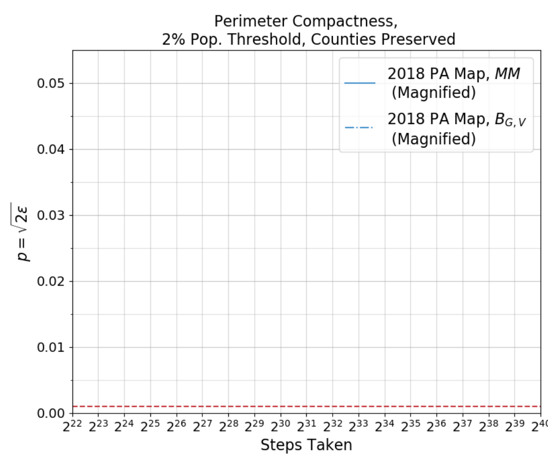

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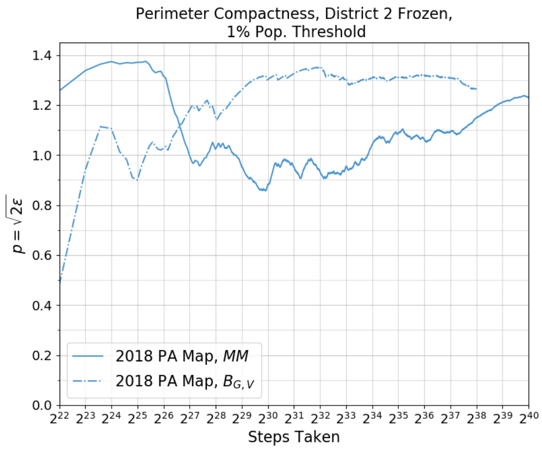

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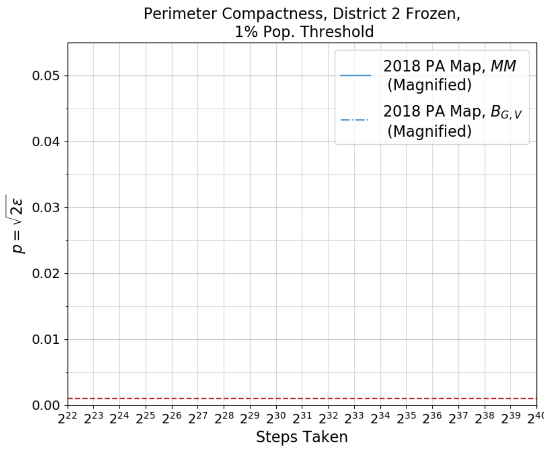

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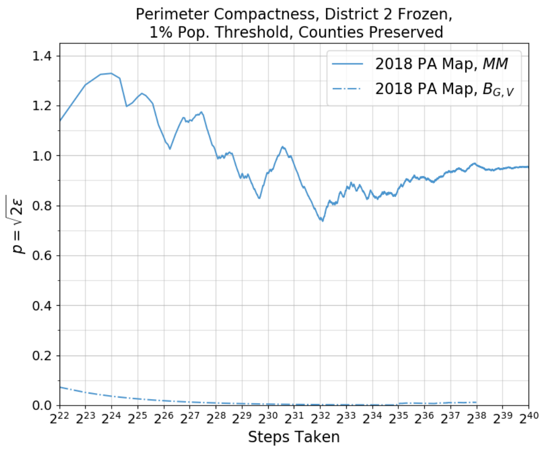

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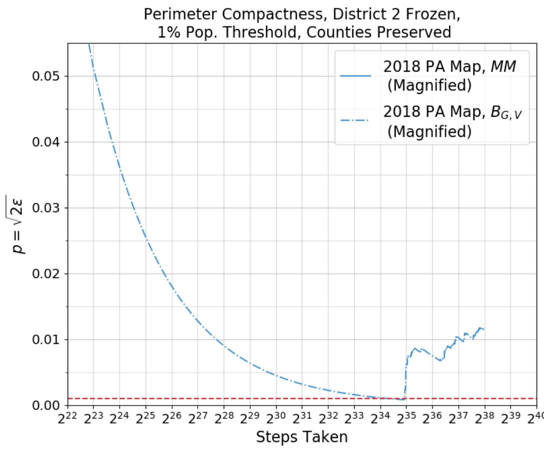

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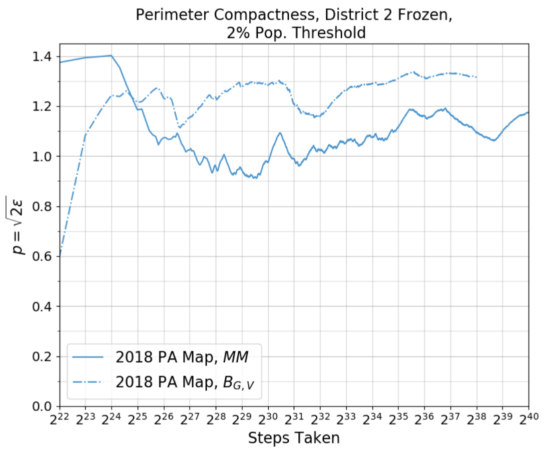

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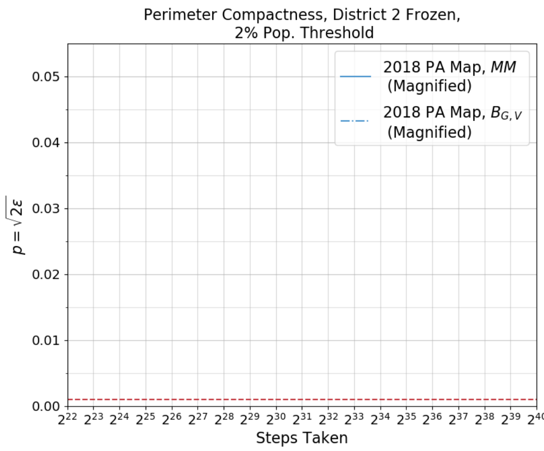

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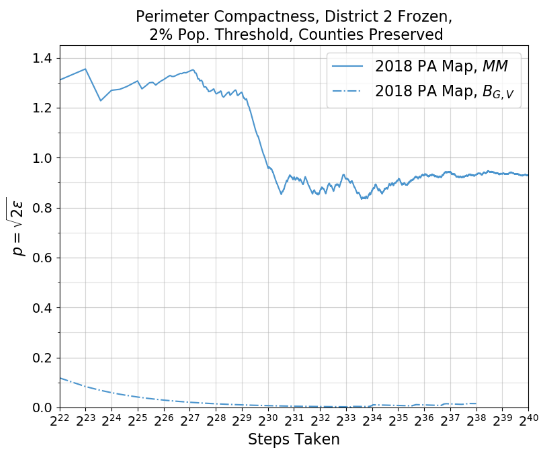
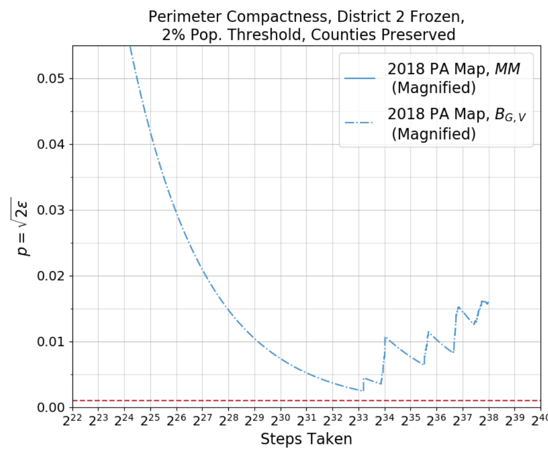
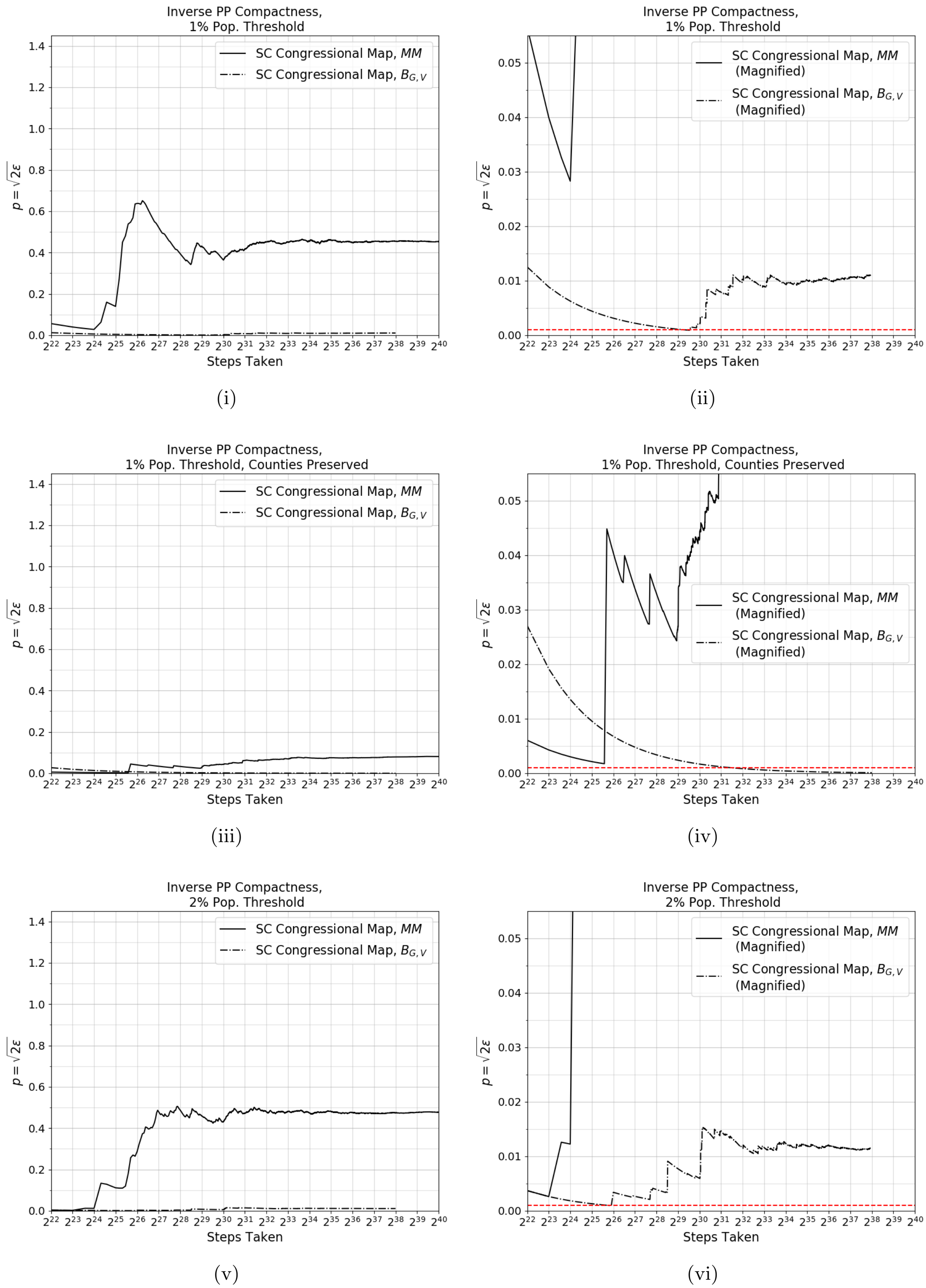

Perimeter Compactness, District 2 Frozen,
2% Pop. Threshold, Counties Preserved
2018 PA Map, MM
2018 PA Map, $B_{G,V}$
$p = \sqrt{2\epsilon}$
Steps Taken

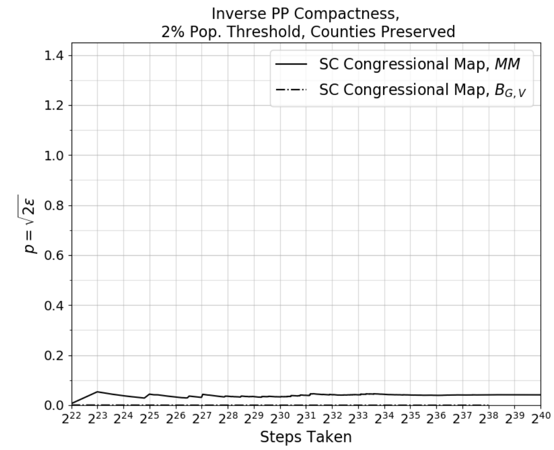
(xxxi)


Perimeter Compactness, District 2 Frozen,
2% Pop. Threshold, Counties Preserved
2018 PA Map, MM (Magnified)
2018 PA Map, $B_{G,V}$ (Magnified)
$p = \sqrt{2\epsilon}$
Steps Taken

(xxxii)

Figure D.3: p-value graphs for the 2011 SC Congressional map.





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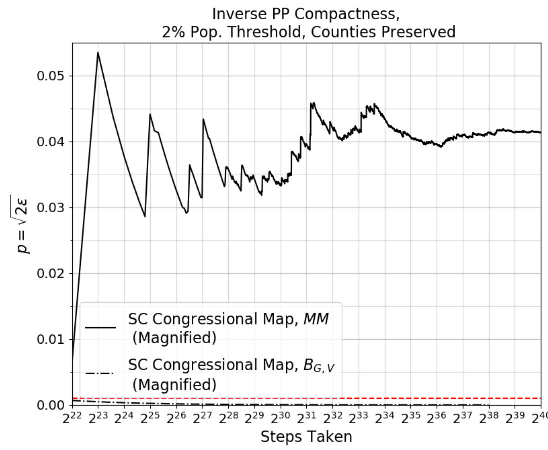

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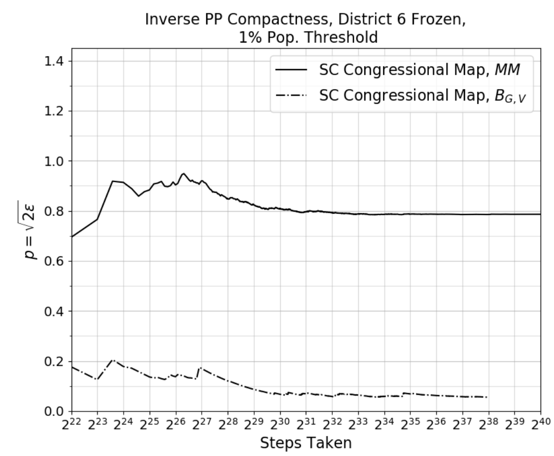

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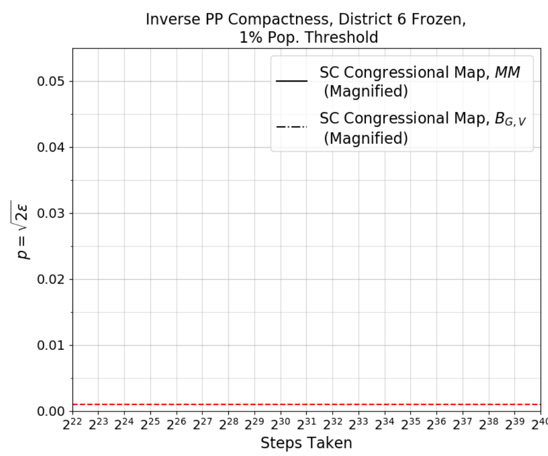

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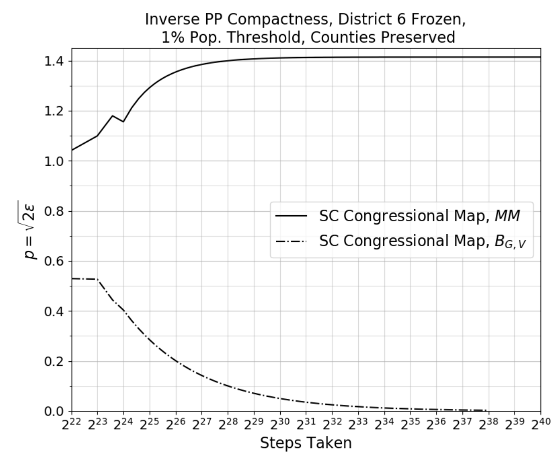

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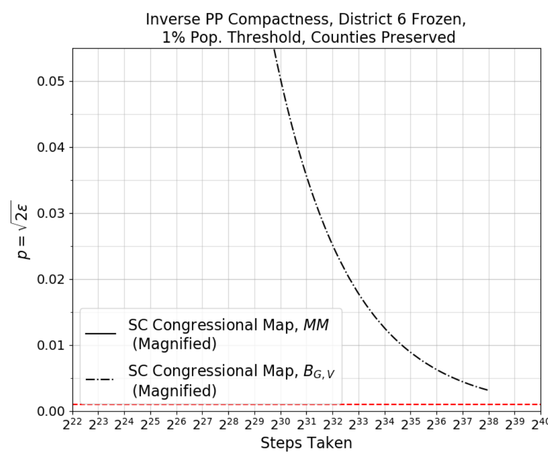

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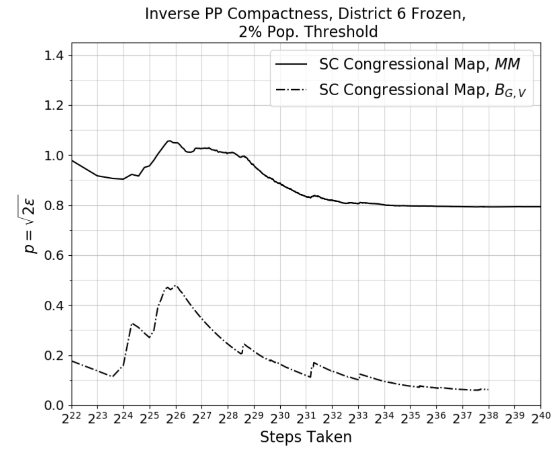

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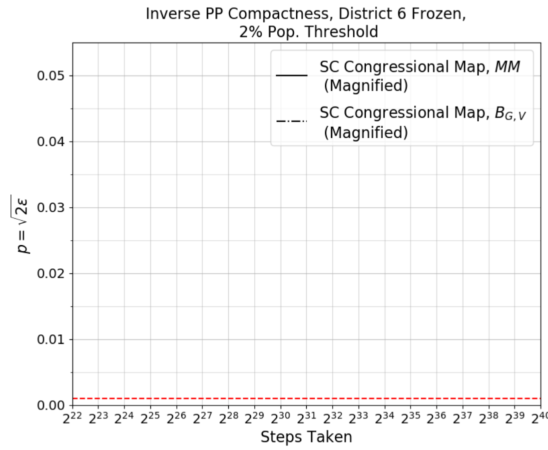

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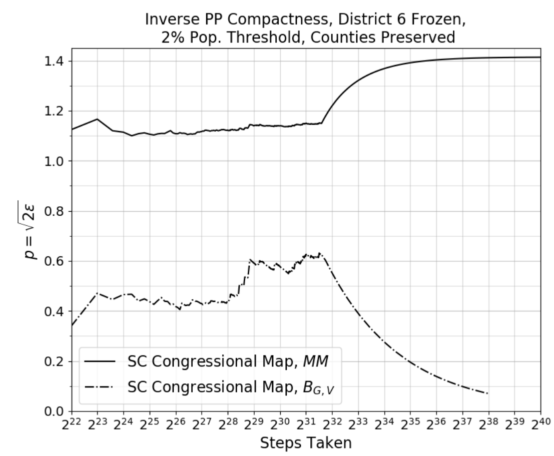

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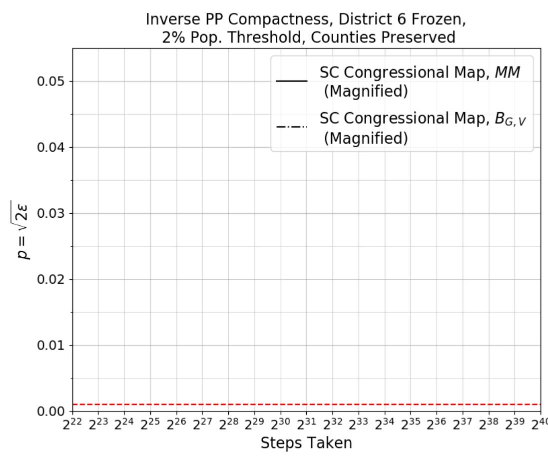

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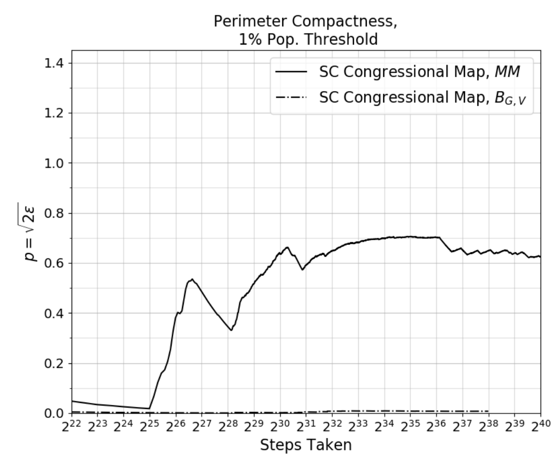

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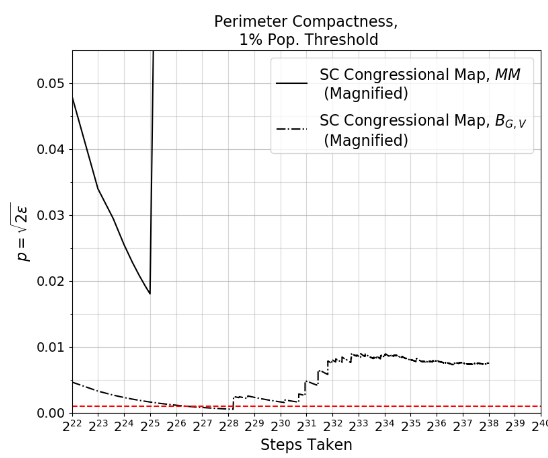

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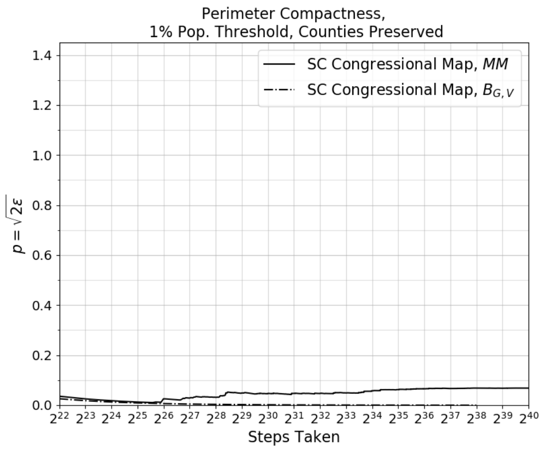

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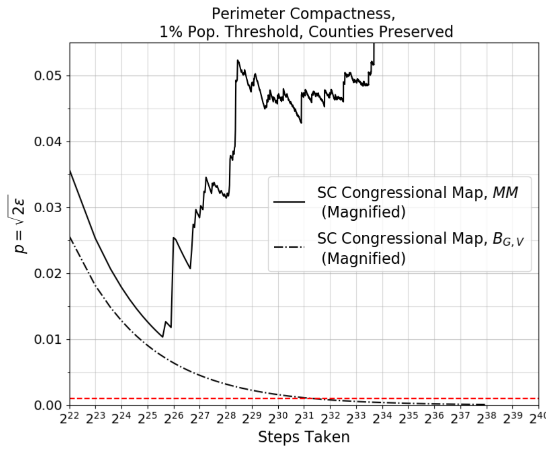

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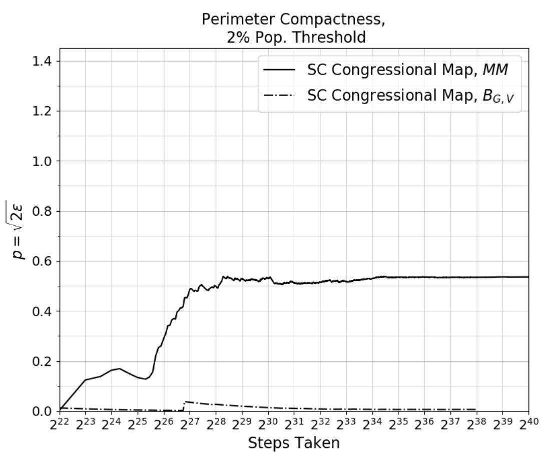

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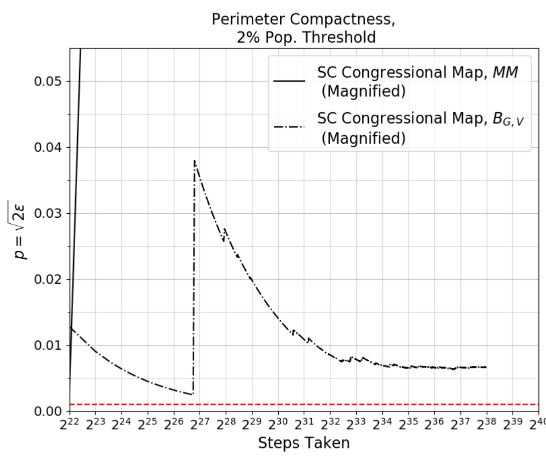

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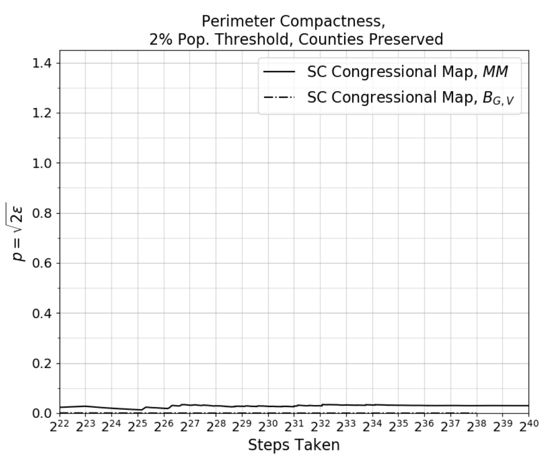

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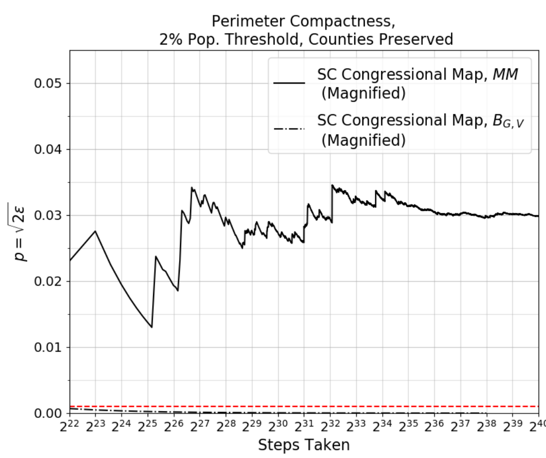

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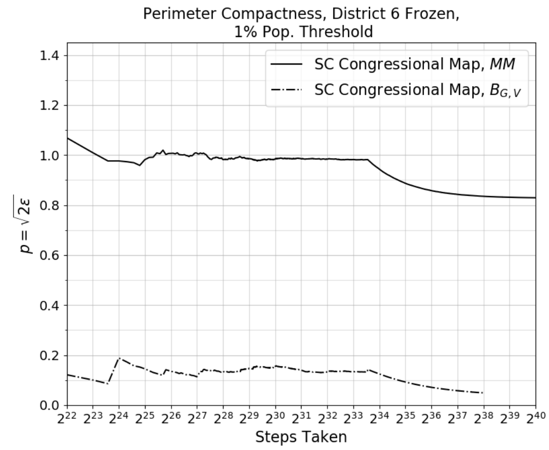

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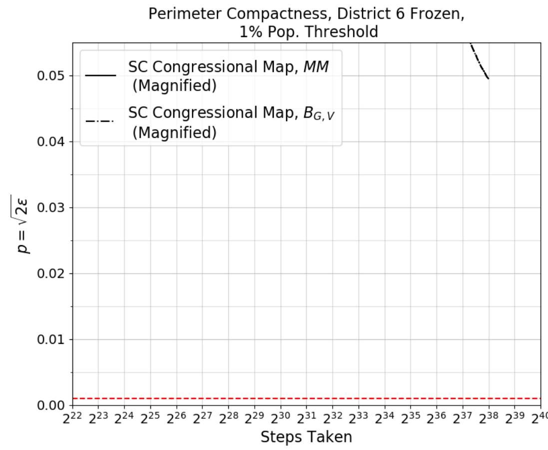

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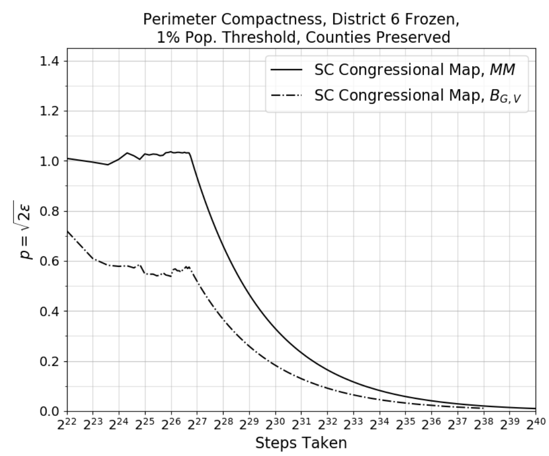

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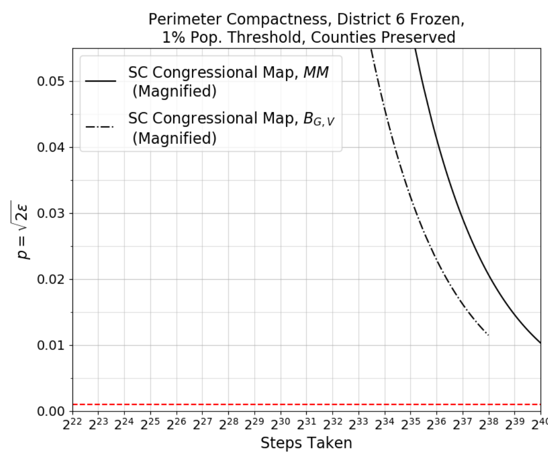

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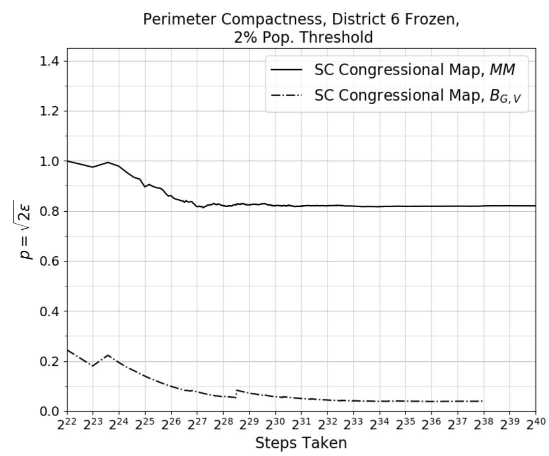

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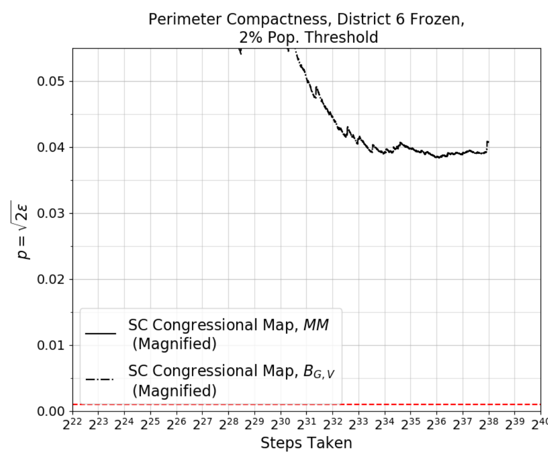

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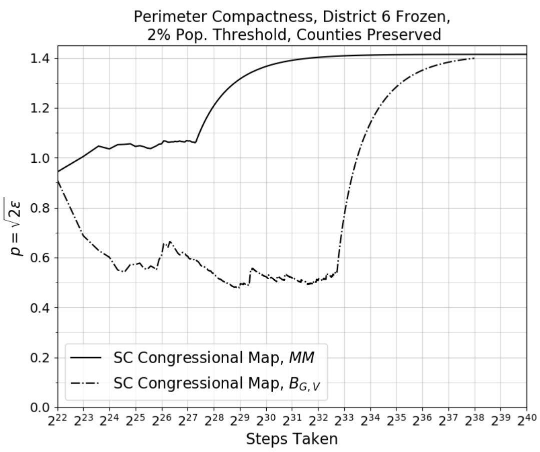
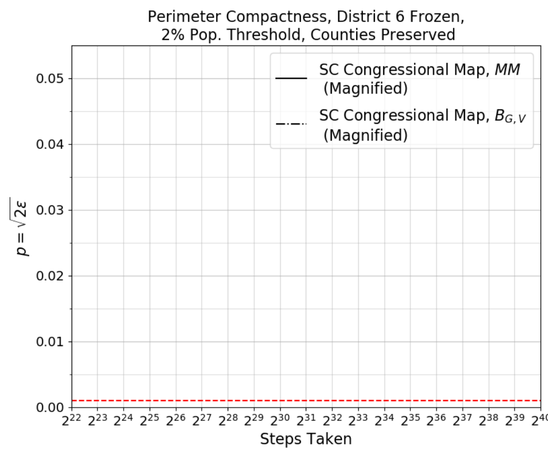


Perimeter Compactness, District 6 Frozen,
2% Pop. Threshold, Counties Preserved
$p = \sqrt{2\varepsilon}$
Steps Taken
SC Congressional Map, MM
SC Congressional Map, $B_{G,V}$

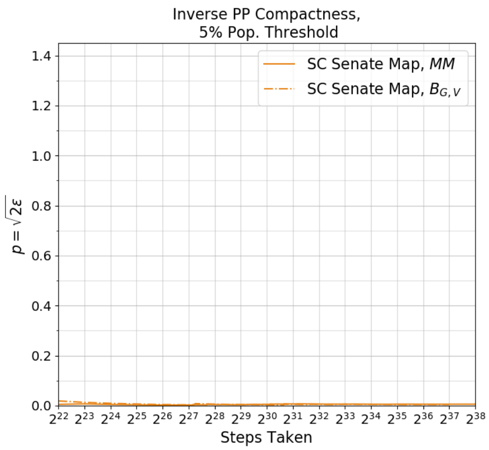
(xxxi)



Perimeter Compactness, District 6 Frozen,
2% Pop. Threshold, Counties Preserved
$p = \sqrt{2\varepsilon}$
Steps Taken
SC Congressional Map, MM (Magnified)
SC Congressional Map, $B_{G,V}$ (Magnified)

(xxxii)

Figure D.4: p-value graphs for the 2011 SC Senate map.



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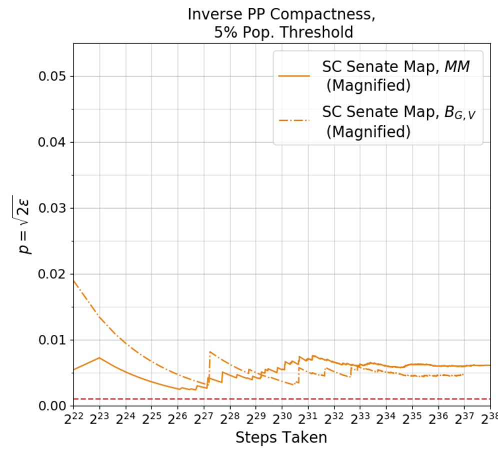

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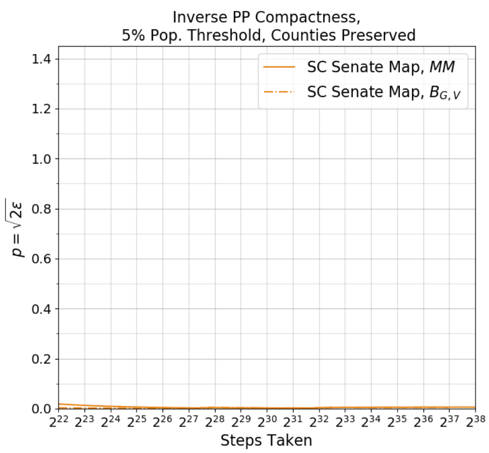

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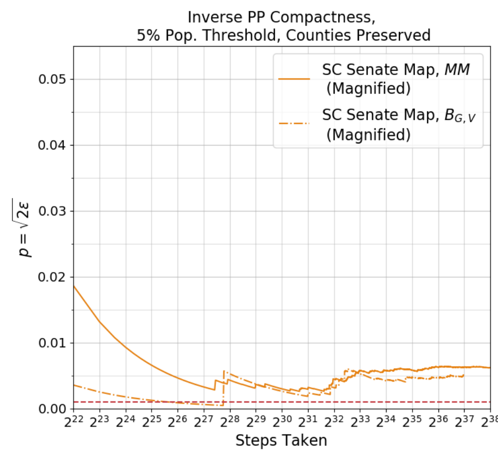

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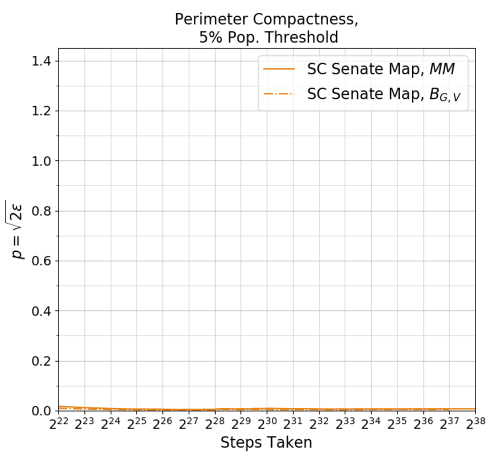

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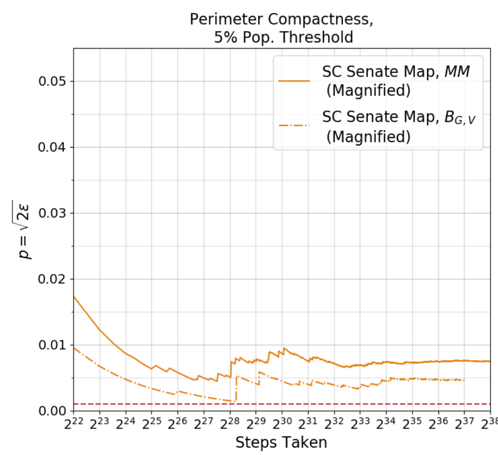

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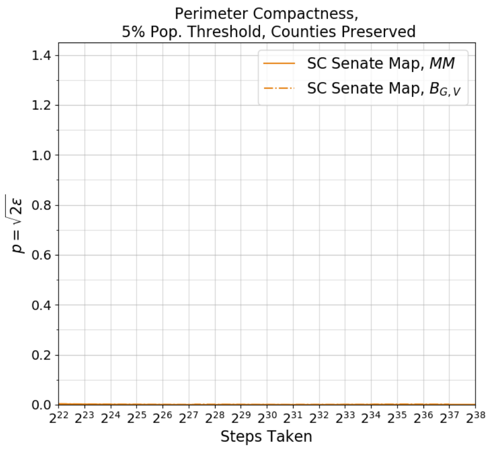
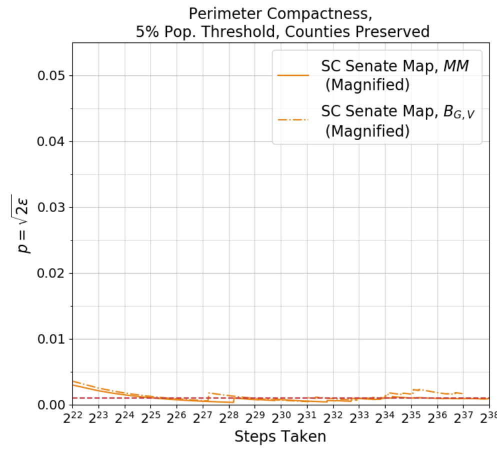


Perimeter Compactness,
5% Pop. Threshold, Counties Preserved
SC Senate Map, MM
SC Senate Map, $B_{G,V}$
$p = \sqrt{2\epsilon}$
Steps Taken

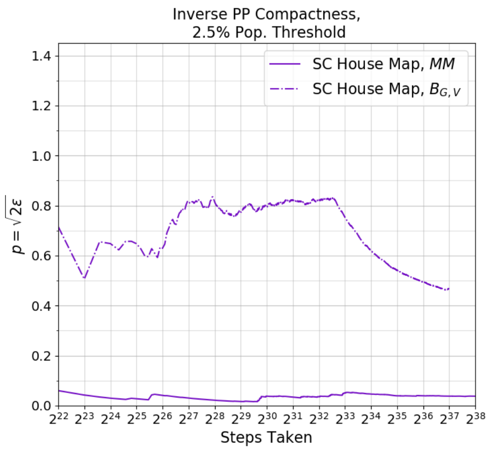
(vii)



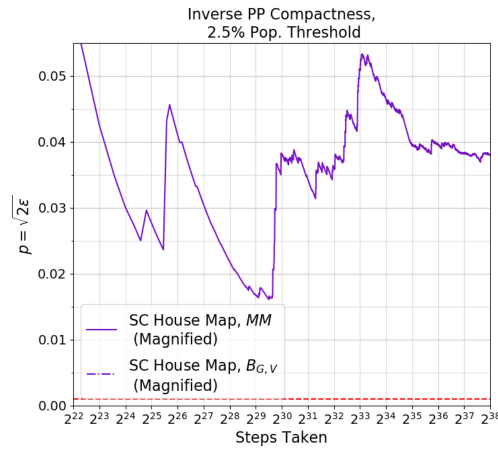
Perimeter Compactness,
5% Pop. Threshold, Counties Preserved
SC Senate Map, MM (Magnified)
SC Senate Map, $B_{G,V}$ (Magnified)
$p = \sqrt{2\epsilon}$
Steps Taken

(viii)

Figure D.5: p-value graphs for the 2011 SC House map.



(i)



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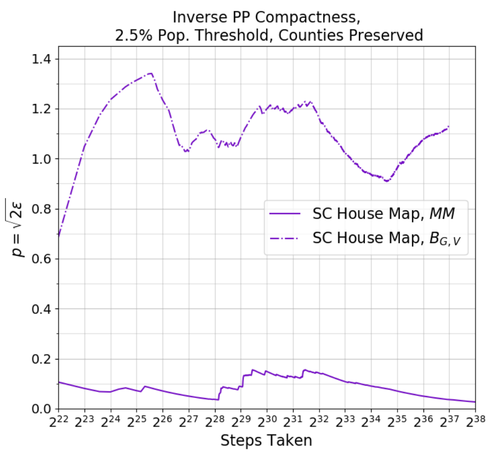

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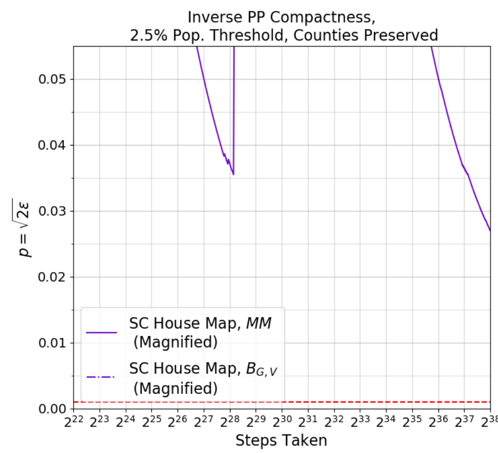

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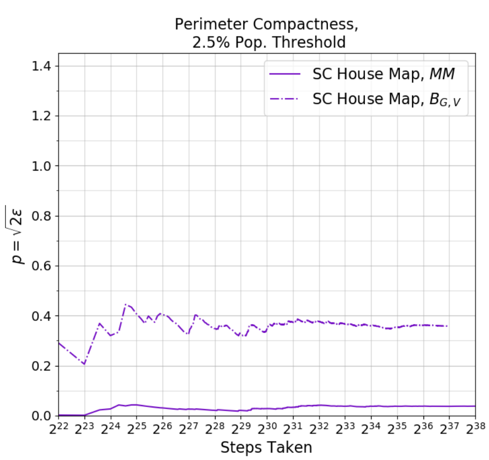

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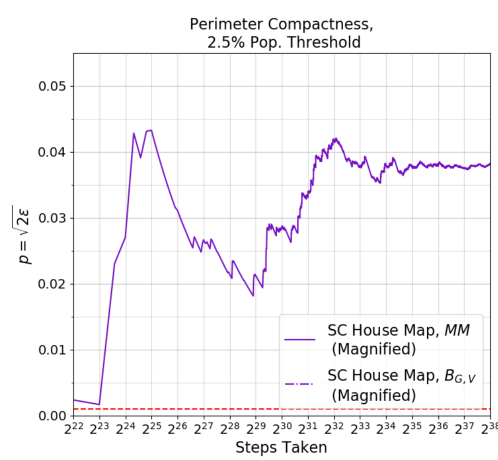

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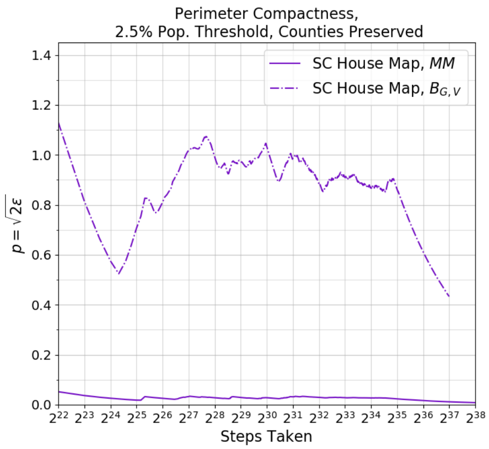

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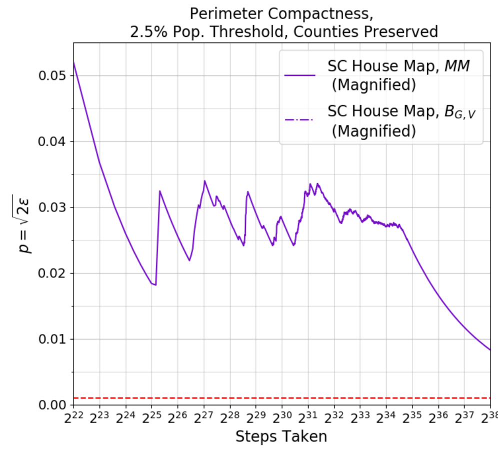

(viii)

D.3 Code and Associated Data Files

For those wishing to perform this analysis on election maps from other states or use this example for instructional purposes, all code utilized in this thesis and the associated data files for Pennsylvania and South Carolina are available at `https://github.com/vagnozzia408/gerrymandering_public`. If utilizing the code and data for published work, please include appropriate citations as necessary.

Bibliography

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