11.2: Properties of Power Series

Learning Objectives. Upon successful completion of Section 11.2, you will be able to...

- Answer conceptual questions involving power series.
- Find the interval and radius of convergence of power series.
- Combine power series.
- Find a power series by integrating or differentiating a known power series.
- Write a power series representation of a given function.
- Find a function represented by a given power series.

Introduction

Recall the definition of a power series from the previous section.

Definition. A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where a and c_k are real numbers and x is a variable. The c_k's are the **coefficients** of the power series and a is the **center** of the power series.

One of our goals in this section is to answer the question: **When does a given power series converge?**

- For each fixed x, the power series is an infinite sum of ____________________ that we can test for convergence or divergence.
- The power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ will converge for some values of x and diverge for other values. We want to find ___________________________ for which the series converges.
- The power series always converges at _________________________, because for $x = a$, $\sum_{k=0}^{\infty} c_k(x-a)^k = c_0$.
- The sum of the power series (instead of being a number S) is ____________________ whose domain is the set of all x-values for which the series converges.

Definition. The set of x-values for which a power series converges is called its **interval of convergence**.

Definition. The **radius of convergence** of a power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

✍ **Example.** For what values of x does $\displaystyle\sum_{n=0}^{\infty} x^n$ converge?

✍ **Example.** Find the interval of convergence and radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

✍ **Example.** Find the interval of convergence and radius of convergence for $\sum_{n=1}^{\infty} n!(2x-1)^n$.

In these three examples, we have actually observed the only three possible types of sets of x-values for which a power series is convergent. These possibilities are summarized in the theorem below.

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities.

1. The series converges only when $x = a$.

2. The series converges for all x.

3. There is a positive number R $(0 < R < \infty)$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

✍ **Example.** Find the interval of convergence and radius of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$.

Combining Power Series

Theorem. Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sums/Differences:** The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.
2. **Multiplication by a Power:** Suppose m is an integer such that $k+m \geq 0$, for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$, for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim\limits_{x \to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k \big(h(x)\big)^k$ converges to the composite function $f\big(h(x)\big)$, for all x such that $h(x)$ is in I.

✍ **Example.** Given $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$, for $|x| < 1$, find the power series and interval of convergence for the function...

(a) $\dfrac{1}{1+x^2}$

(b) $\dfrac{x}{8-x^3}$

✍ **Example.** Find the function represented by the series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{4^k}$, and find the interval of convergence of the series.

Differentiating and Integrating Power Series

Theorem. Suppose that the power series $\sum c_k(x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

1. Then f is differentiable (which implies continuous) for $|x-a| < R$, and f' is found by differentiating the power series f term by term; that is,
$$f'(x) = \sum kc_k(x-a)^{k-1},$$
for $|x-a| < R$.

2. The indefinite integral f is found by integrating the power series for f term by term; that is,
$$\int f(x)\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$
for $|x-a| < R$, where C is an arbitrary constant.

✍ **Example.** Given $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$, for $|x| < 1\ldots$

(a) Find the power series for $f'(x)$ and its interval of convergence. Also identify what function the power series represents.

✍ **Example (continued).** Given $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$, for $|x| < 1\ldots$

(b) Find the power series for $\int f(x)\,dx$ and its interval of convergence. Also identify what function the power series represents.

✍ **Example.** Find the power series representation for $g(x) = -\dfrac{1}{(1+x)^2}$ centered at 0 by differentiating or integrating the power series for $f(x) = \dfrac{1}{1+x}$.

✍ **Example.** We found a power series representation for $\dfrac{1}{1+x^2}$ on page 148:

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

(a) Use this power series to find a power series representation centered at 0 for $\arctan(x)$. Give the interval of convergence for the resulting series.

(b) Use the series in Part (a) to find a power series representation centered at 0 for $f(x) = \arctan(4x^2)$ and find the interval of convergence.