6.7: Physical Applications

Learning Objectives. Upon successful completion of Section 6.7, you will be able to...

- Find the mass of a thin bar with a given density function.
- Find the work done given constant force.
- Find the work done given a variable force function $f(x)$.
- Solve work problems involving springs and Hooke's law.
- Solve work problems involving lifting ropes/chains/cables.
- Solve work problems involving pumping water.
- Solve applications involving pressure.
- Solve applications involving hydrostatic force.

Density and Mass

Density is the concentration of mass in an object. Usually density indicates mass per volume (e.g., kg/m^3) and an object with uniform density satisfies the equation

$$\text{mass} = \text{density} \cdot \text{volume}.$$

If the density varies with the object, we need to use calculus. The problem of finding the mass of a 2- or 3-dimensional object given a density function that varies by position requires multi-variable calculus. So we will look at the problem of finding the mass of a 1-dimensional object using calculus. For 1-dimensional objects, we use *linear density* (e.g., kg/m).

Goal: Suppose a 1-dimensional object, such as a thin bar or wire, is represented by the interval $a \leq x \leq b$. Find the mass of the object given that the object's linear density $\rho(x)$ varies along its length.

✍ **Example.** A thin bar is represented by the interval $0 \leq x \leq \pi$. Find the mass of this bar if its density is given by $\rho(x) = 1 + \sin x$.

Work

Work is an important concept for determining the amount of energy needed to perform various tasks. **Work** is done when a force moves an object.

Examples of situations where we want to find the amount of work done include...

- the work needed to lift a heavy object
- the work needed to wind up a heavy chain
- the work needed to pump water up and out of a tank
- the work needed to stretch or compress a spring

Work Done by a Constant Force. If an object is moved a distance d in the direction of an applied constant force F, then the **work** done by the constant force is

$$W = F \cdot d.$$

Note: Force is mass times acceleration, $F = m \cdot a$.

Note on Units.

	SI Metric System
Displacement	meter (m)
Mass	kilogram (kg)
Force	newton (N = kg · m/s^2)
Work	joule (J = N · m)

✍ **Example.** How much work is required to move an object from $x = 0$ to $x = 10$ (measured in meters) in the presence of a constant force of 3 N acting along the x-axis?

We know that $W = F \cdot d$ for a constant force. What if the force *varies*?

Suppose an object moves along the x-axis in the positive direction from $x = a$ to $x = b$. At each point, a force $F(x)$ acts on the object. How do we find the work done?

Work Done by a Variable Force. If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** done by the variable force as the object is moved from $x = a$ to $x = b$ is

Work and Springs

For problems involving springs, we will need the following law from physics.

Hooke's Law. The force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$F(x) = kx$, where k is a positive constant called the **spring constant**.

✍ **Example.** Suppose a force of 30 N is required to stretch and hold a spring 0.2 m from its equilibrium position. How much work is required to compress the spring 0.4 m from its equilibrium position? Assume Hooke's Law is obeyed.

✍ **Example.** A spring requires 100 J of work to be stretched 0.5 m from its equilibrium position. How much work is required to stretch the spring 1.25 m from its equilibrium position? Assume Hooke's Law is obeyed.

Work and Lifting Problems

Now we will look at finding the work needed to lift an object (such as a rope, cable, or chain) when the motion is vertical and the force is due to gravity. We use that $F = mg$, where $g = 9.8$ m/s^2 is the acceleration due to gravity. We'll need calculus to find the work done when lifting ropes, chains, and cables because different parts of the rope/chain/cable will be lifted different distances.

Finding work done in lifting problems: Suppose we have a chain of length L meters with constant density ρ kg/m hanging vertically from a scaffolding platform at a construction site. How can we find the work done in lifting the chain to the platform?

✍ **Example.** A heavy rope 20 m long hangs over the edge of a building 40 m high. The rope has a linear density of 2 kg/m.

(a) How much work is done in pulling the rope to the top of the building?

(b) How much work is done in pulling half the rope to the top of the building?

(c) A 30 kg load is attached to the bottom of the rope and lifted to the top of the building. How much work is done?

Work and Pumping Liquids Out of Tanks

Finding work done in pumping problems: Suppose a fluid (such as water) with density ρ is pumped out of a tank to a height h above the bottom of the tank. How much work is required, assuming the tank is full of water?

✍ **Example.** A circular swimming pool has a diameter of 8 meters, the sides are 2 meters high, and the depth of the water is 1.5 meters. The density of water is 1000 kg/m^3.

(a) How much work is required to pump all of the water out over the side?

(b) How much work is required to pump half the water out over the side?

Force and Pressure

We want to determine the force exerted on a surface by a body of water. We will also need to know about pressure. **Pressure** is a force per unit area (e.g., N/m^2).

Idea: Say we have water putting pressure on a surface of area A m^2 that is h meters below the surface. Then the pressure on the surface is computed as

$$\text{pressure} = \frac{\text{force}}{A} = \frac{\text{volume} \cdot \text{density} \cdot g}{A} = \frac{Ah\rho g}{A} = \rho g h$$

and is called the **hydrostatic pressure** (pressure of water at rest). Note that hydrostatic pressure has the same magnitude in all directions, so the hydrostatic pressure on a vertical wall in a pool at depth h is the same as the hydrostatic pressure on a horizontal surface at depth h.

How do we find the force on a vertical wall, such as the face of a dam, assuming the water completely covers the face of the dam and the water level is at the top of the dam?

✍ **Example.** A plate shaped like an isosceles triangle with a height of 1 m is placed on a vertical wall 1 m below the surface of a pool filled with water. Compute the force on the plate.