10.1: An Overview of Sequences and Series

Learning Objectives. Upon successful completion of Section 10.1, you will be able to...

- Answer conceptual questions involving sequences, series, and recurrence relations.
- Find terms of sequences.
- Write recurrence relations and/or explicit formulas for the terms of sequences.
- Determine limits of sequences if the sequences converge.
- Solve applications involving sequences.
- Determine limits of sequences of partial sums if the sequences converge.

Motivation

Chapter 10 is about *infinite sequences and series.* One of the important applications of infinite series is Taylor series, which allow us to represent functions as sums of infinite series. Taylor series can allow us to...

- work with exponential, trigonometric, and logarithmic functions using just addition, subtraction, multiplication, and division
- solve differential equations
- do approximations in physics and chemistry
- represent π, e, and repeating decimals as sums of infinite series

We'll first need to understand the difference between a **sequence** and a **series**.

A **sequence** is an infinite ordered list of numbers.

A **series** is an infinite sum of numbers.

What does it mean for a sequence or series to **converge**?

- For sequences, we will use the same tools as for limits of functions at infinity.
- For series, we will need some new techniques!

Sequences

Definition. A **sequence** is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \dots, a_n, \dots\},$$

where each number in the sequence is called a **term**.

- A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \dots$, where a_1 is given.
- A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \dots$.

✍ **Examples.** Consider the following examples of sequences.

$\{1, 4, 7, 10, 13, \dots\}$

$\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

$a_n = (-1)^n$, $n = 1, 2, 3, \dots$

$a_{n+1} = 3a_n - 12$; $a_1 = 10$

Limit of a Sequence. If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases — that is, if a_n can be made arbitrarily close to L by taking a sufficiently large n — then we say that $\lim_{n\to\infty} a_n = L$ exists and the sequence **converges** to L.

If the terms of the sequence do not approach a single number as n increases, the sequence has no limit and **diverges**.

✍ **Example.** For each of the following, write the first four terms of the sequence. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.

(a) $a_n = 1 - 10^{-n}$; $n = 1, 2, 3, \ldots$

(b) $a_n = 3 + \cos(\pi n)$, $n = 1, 2, 3, \ldots$

✍ **Example.** Jack took a 200-mg dose of a painkiller at midnight. every hour, 5% of the drug is washed out of his bloodstream. Let d_n be the amount of the drug in Jack's blood n hours after the drug was taken, where $d_0 = 200$ mg. Write out the first 5 terms, and find an explicit formula for the n^{th} term, find a recurrence relation that generates the sequence, and estimate the limit of the sequence.

Infinite Series

Definition. An **infinite series** is a sum of infinitely many terms and can be represented as follows.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

✍ **Examples.** The following are examples of infinite series.

$1 + 2 + 3 + 4 + 5 + \dots$

$0.\overline{3}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

A *sequence* $\{a_n\} = \{a_1, a_2, a_3, \dots\}$ converges if $\lim_{n\to\infty} a_n = L$, where $L \in \mathbb{R}$, and diverges otherwise.

What does it mean for a *series* to converge?

Series Convergence/Divergence. Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, we say that the series **converges** if its sequence of partial sums $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ converges, meaning

We say that the series **diverges** otherwise.

✍ **Examples.** Use the definition of convergence/divergence of a series to determine if the series converges or diverges. If it converges, find its sum.

$$\sum_{i=1}^{\infty} i$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i}$$