3.2: The Derivative as a Function

Learning Objectives. Upon successful completion of Section 3.2, you will be able to...

- Answer conceptual questions involving the derivative as a function.
- Obtain the graphs of derivative functions from graphs of functions.
- Find points where functions are continuous and differentiable.
- Find derivatives of functions using limits.
- Solve applications involving derivatives as functions.
- Use graphs of functions to analyze slopes of tangent lines.
- Obtain graphs of functions from graphs of their derivative function.
- Find equations of normal lines.
- Find vertical tangent lines from graphs.

Defining the Derivative Function

In Section 3.1, we learned how to find the **derivative at a point** $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we wanted the derivative at a new point, this would require an entirely new limit. Fortunately, there is a way of calculating derivatives without having to recompute a new limit each time. Because we know that limits, and hence derivatives, are *functions*, we will work towards finding the **derivative function**. Once found, we can calculate $f'(a)$ for whatever value of a we would like.

Definition. The **derivative** of f is the function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. The process of finding this limit, i.e. finding the derivative, is called **differentiation.**

Remark. The domain of f' is contained within the domain of f.

Derivative Notation. The derivative (function) of $y = f(x)$ may be denoted in several different ways as follows.

$$f'(x) = y' = y'(x) = \frac{dy}{dx} = \frac{df}{dx}$$

The symbol $\frac{d}{dx}$ is called the **differential operator** and it instructs us to take the derivative.

$$\frac{d}{dx}(3x-5) \iff \text{“take the derivative of } 3x-5$$

The evaluation of a derivative at a point $x = a$ may be denoted as follows.

$$f'(a) = y'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a}$$

✍ **Example.** Find the derivative of $y = 3x - 5$.

✍ **Example.** Find the derivative of $f(x) = x^2 + 3x$.

✍ **Example.** Find the slope of the tangent line at three different x values for the function f in the previous example.

✍ **Example.** Evaluate the following expression.

$$\frac{d}{dx}\left(\sqrt{x}\right)$$

✍ **Example.** For $y = \sqrt{x}$, at which x-value will the slope of the tangent line be one?

✍ **Example.** Find $\dfrac{dy}{dx}$ for $y = \dfrac{1}{x+1}$.

Normal Lines

Recall that the derivative at a point represents the slope of a tangent line at that point. We have already used derivatives to find the equation of a tangent line. We can also use derivatives to identify the slope (and thus equation) of a *normal line*.

Definition. Two lines with slopes m_1 and m_2, respectively, are **perpendicular** if the slopes multiply to negative one.

$$m_1 m_2 = -1 \iff m_1 = -\frac{1}{m_2}$$

Hence, m_1 and m_2 are **negative reciprocals.**

Definition. A **normal line** to the graph of $y = f(x)$ at a point $x = c$ is perpendicular to the tangent line at that point.

$$m_T m_N = -1 \iff m_N = -\frac{1}{m_T} = -\frac{1}{f'(c)}$$

✍ **Example.** The derivative of $f(x) = x^2 + 3x$ is $f'(x) = 2x + 3$. Find the equation of the normal line to $f(x)$ at $x = 3$.

Derivative Examples

✍ **Example.** Find the derivative of $f(x) = \sqrt{2 - 2x}$.

✍ **Example.** Find the derivative of $f(x) = \dfrac{x}{x+1}$.

Differentiability

In Section 2.6, we learned about *continuity*. This property of functions has a relationship with the differentiability of a function.

Theorem. If f is differentiable at $x = c$, then it is continuous at $x = c$.

Note that continuity does <u>not</u> imply differentiability. Consider the absolute value function, $f(x) = |x|$. For the derivative to exist using its limit definition, the associated left- and right-hand limits must be equal.

$$\lim_{x \to 0^+} \frac{|x+0| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{|x+0| - |0|}{x - 0} = \lim_{x \to 0^-} -\frac{x}{x} = -1$$

Hence, $f'(0)$ does not exist although f is continuous at $x = 0$.

When Differentiability Fails. A function will fail to be differentiable in the following situations.

- Any **discontinuity** causes the derivative to not exist.
- The existence of a **corner** or **cusp** will cause the derivative to not exist. (See the absolute value function example above.)
- The existence of a **vertical tangent** will cause the derivative to not exist.
- The derivative will not exist at an **endpoint** because a two-sided limit is not possible.

✍ **Example.** Circle the locations within $[-2, 2]$ where f is not differentiable.