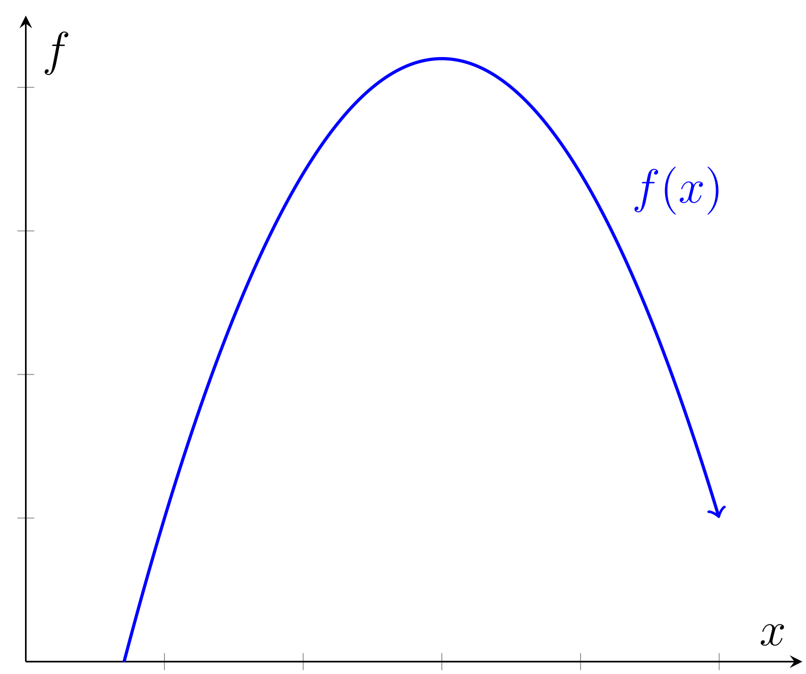
3.1: Introduction to the Derivative

Learning Objectives. Upon successful completion of Section 3.1, you will be able to...

- Answer conceptual questions involving tangent lines and derivatives.
- Solve applications involving the use of limits to calculate derivatives.
- Use limit definitions to find equations of tangent lines.
- Use limit definitions to evaluate derivatives at given points.
- Compute average and instantaneous rates of change from graphs and tables.
- Determine functions given limits of difference quotients.

The Idea Behind Derivatives

One of the major questions in calculus is: How can we calculate the slope of a tangent line? We know that we can calculate an *average* rate of change by finding the slope of a secant line. How can we calculate an *instantaneous* rate of change?



View an interactive demo here: `https://www.desmos.com/calculator/2vmz7bdvgo`

The Derivative at a Point

Definition. The line tangent to the curve $y = f(x)$ at $x = a$ has slope

$$m_{\tan} = \lim_{x\to a} m_{\sec} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists. This limit is called the **derivative** of f at the point a and is denoted by $f'(a)$, read as "f prime of a."

This value, when it exists, is sometimes called the **instantaneous rate of change** or the **slope of the curve**.

Because derivatives are, by definition, limits, there is a useful property of limits that can help us understand derivatives.

Theorem. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$. Then $L = M$. In other words, if a limit exists, it is unique.

Now consider the function $f(x) = \dfrac{\sin x}{x}$. Note that f is not continuous at $x = 0$ because $f(0)$ is undefined. However, the following function is continuous at $x = 0$...

$$f'(x) = \lim_{t\to x} \frac{\sin t}{t}$$

By the uniqueness of limits, we can say that the derivative is a **function**. In other words, for every point a, there is at most one value of $f'(a)$.

✍ **Example.** Find the equation of the line tangent to $f(x) = \sqrt{3x}$ at $x = 3$.

✍ **Example.** Find the equation of the line tangent to $f(x) = \dfrac{1}{x}$ at $x = 2$.

✍ **Example.** Find the equation of the line tangent to $f(x) = x^2 + 2$ at $x = 0$.

Alternative Definition of the Derivative. If we let h be the distance between x and a, i.e. $h = x - a$, then

$$x = a + h \text{ and } x \to a \iff h \to 0,$$

leading us to an equivalent formulation of the limit definition of a derivative at a point.

Definition. The **derivative** of a function f at $x = a$, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

Definition. If $f'(a)$ exists, we say that f is **differentiable** at $x = a$.

✍ **Example.** Find the equation of the line tangent to $f(x) = 2x^2 - 3x$ at $x = 1$.

✍ **Example.** Suppose that $y = -\frac{1}{2}x + 5$ is tangent to f at $x = 2$. Find $f'(2)$.