10.6: Alternating Series

Learning Objectives. Upon successful completion of Section 10.6, you will be able to...

- Answer conceptual questions involving alternate series.
- Apply the alternating series test if possible. Otherwise, apply a different appropriate test.
- Use Theorem 10.18 to find an upper bound for the error in using the n^{th} partial sum to estimate the value of the series.
- Determine the number of terms needed to ensure a given error.
- Estimate the value of an alternating series.
- Determine if a series converges absolutely, converges conditionally, or diverges.

Introduction

Both the Integral Test and the Comparison Tests apply only to series with *positive terms*. An important type of series with positive and negative terms is called an **alternating** series.

✍ **Examples.** Consider the terms of each of the following series.

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\,e^{2/n}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{n}$$

Definition. An **alternate series** is an infinite sum of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1}\, b_n \text{ or } \sum_{n=1}^{\infty} (-1)^{n}\, b_n \text{ with } b_n > 0.$$

The terms of an alternating series will *alternate* between positive and negative values.

Alternating Series Test. Consider an alternating series defined as above. If both of the following are true...

1. $b_{n+1} \leq b_n$ for all $n \geq N$, and
2. $\lim_{n\to\infty} b_n = 0$,

then the series converges.

Thinking about the conditions of the Alternating Series Test. Is it possible to have a sequence that decreases but does not converge to 0? Give an example or explain why not.

Is it possible to have a sequence that converges to 0 but doesn't decrease? Give an example or explain why not.

Idea behind the proof of the Alternating Series Test. Why do the conditions of the test imply convergence?

✍ **Examples.** Determine whether the following series converge or diverge.

1. The **alternating harmonic series**: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

✍ **Examples (continued).**

2 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}e^{2/n}$

3 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$

✍ **Examples (continued).**

4 $\displaystyle\sum_{n=0}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{1+\sqrt{n}}$

Estimating Sums

Recall that the error in estimating the sum S of a convergent series by S_n is the **remainder** $R_n = S - S_n$. The Integral Estimation Theorem allows us to find the bounds on the remainder,

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

where $f(x)$ is continuous, positive, and decreasing for $x \ge 1$ and $f(n) = a_n$. This theorem will not work for alternating series because the terms are, by definition, not always positive, so we'll need a different theorem.

Alternating Series Estimation Theorem. If $S = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies (1) $b_{n+1} \le b_n$ and (2) $\displaystyle\lim_{n\to\infty} b_n = 0$, then…

✍ **Example.** Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$.

(a) If we use the sum of the first three terms to estimate the sum of the series, what is the bound on the error?

(b) How many terms of the series do we need to add in order to estimate the sum so that $\left|\text{error}\right| < 1/10^6$?

Absolute Convergence and Conditional Convergence

So far, we have only considered *whether* a series converges or diverges, but now we will also consider the *type* of convergence. The types of convergence we're interested in has to do with the relationship between the convergence/divergence of $\sum a_n$ and $\sum |a_n|$.

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Important Notes:

- If a series $\sum a_n$ is absolutely convergent, then it is convergent.
- This means that if $\sum |a_n|$ converges, then $\sum a_n$ converges.
- So if $\sum a_n$ is absolutely convergent, then both $\sum |a_n|$ and $\sum a_n$ converge.
- **<u>Caution:</u>** If $\sum a_n$ converges, $\sum |a_n|$ may converge or diverge.

Definition. A series $\sum a_n$ is called **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Steps to Determine if a Series is Absolutely Convergent, Conditionally Convergent, or Divergent.

1. Look at $\sum |a_n|$. Determine if this series converges or diverges.
   (a) If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent and we are done.
   (b) If $\sum |a_n|$ diverges, then go to Step 2.
2. Look at a_n. Determine if this series converges or diverges.
   (a) If $\sum a_n$ converges, then $\sum a_n$ is conditionally convergent.
   (b) If $\sum a_n$ diverges, then $\sum a_n$ is divergent.

✍ **Examples.** Determine if each of the following series is absolutely convergent, conditionally convergent, or divergent.

① $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

✍ **Examples (continued).** Determine if each of the following series is absolutely convergent, conditionally convergent, or divergent.

3 $\displaystyle\sum_{n=1}^{\infty}(-1)^n \arctan(n)$

4 $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(4n)}{4^n}$