4.3: What Derivatives Tell Us

Learning Objectives. Upon successful completion of Section 4.3, you will be able to...

- Answer conceptual questions involving derivatives.
- Find the intervals on which a function is increasing or decreasing.
- Use the first derivative test to locate critical points and local and absolute extrema.
- Sketch the graph of a function given properties of the function.
- Determine the concavity on intervals and find inflection points.
- Determine if critical points correspond to local maxima/minima using the second derivative test.
- Compare the graphs of a function with the graphs of its first and second derivatives.

First Derivatives

First derivatives tell us where a function is increasing or decreasing.

- If the derivative is *positive* on an interval, then the function is *increasing*.
- If the derivative is *negative* on an interval, then the function is *decreasing*.

Derivatives also allow us to classify critical points.

First Derivative Test. Suppose that $x = c$ is a critical point of a function f. If f'...

- changes from $+$ to $-$ around $x = c$, then $f(c)$ is a local maximum.
- changes from $-$ to $+$ around $x = c$, then $f(c)$ is a local minimum.
- does not change sign, then $f(c)$ is neither.

✍ **Example.** For the function $f(x) = x^3 + x^2 - x$, find the intervals on which f increases and decreases and classify all critical points.

Theorem. Suppose that f is a continuous function with one and only one critical point. If that critical point is a local extremum, then it is also a global extremum.

✍ **Example.** For $f(x) = \dfrac{e^x}{e^{2x}+1}$, find the intervals on which f increases and decreases and classify all critical points.

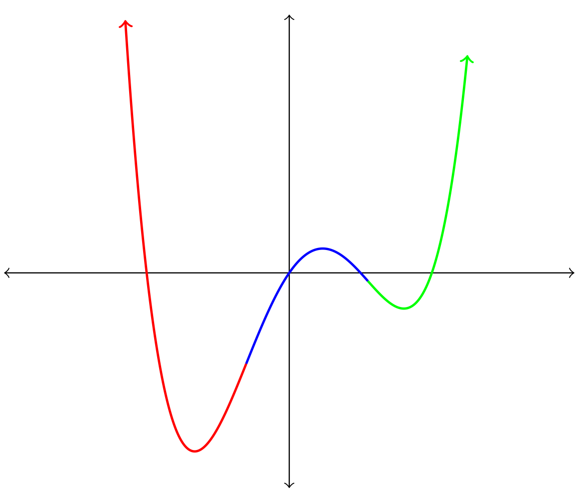
Second Derivatives

Second derivatives reveal information about the **curvature** or **concavity** of the graph of a function.

Theorem. Suppose that f is twice differentiable on an open interval.

- If $f''(x) > 0$ for all x in the interval, then f is concave up.
- If $f''(x) < 0$ for all x in the interval, then f is concave down.

Definition. If f is continuous at $x = c$ and f changes concavity at that point (i.e. f'' changes sign), then the point $(c, f(c))$ is called an **inflection point**.



To locate inflection points, we will...

1. Identify the value(s) where f'' is zero or fails to exist.
2. Identify whether f'' changes in sign around those value(s).

✍ **Example.** Consider the function $f(x) = x - 2\arctan(x)$.

1. Find the intervals where f is increasing/decreasing and classify all critical points.

2. Find the intervals of concavity and any inflection points.

If a function f is twice differentiable, then the following test can also be applied to classify a critical point.

Second Derivative Test. Suppose that $f'(c) = 0$ and that $f''(c)$ exists.

1. If $f''(c) > 0$ (i.e. f is concave up), then $f(c)$ is a local minimum.
2. If $f''(c) < 0$ (i.e. f is concave down), then $f(c)$ is a local maximum.

Note that this test does not apply if $f'(c)$ is undefined, because $f''(c)$ will also be undefined.

✍ **Example.** Determine the critical points of $f(x) = x^3 - x$ and use the second derivative test to classify the critical points.

Function f	**First Derivative** f'	**Second Derivative** f''
Increasing	Positive	—
Decreasing	Negative	—
Local Maximum	Zero or DNE, + to −	Negative
Local Minimum	Zero or DNE, − to +	Positive
Concave Up	Increasing	Positive
Concave Down	Decreasing	Negative
Inflection Point	Local Max/Min	Zero or DNE, changes sign