3.4: The Product and Quotient Rules

Learning Objectives. Upon successful completion of Section 3.4, you will be able to...

- Answer conceptual questions involving the product and quotient rules.
- Find derivatives using two different methods.
- Find derivatives of products and quotients of functions involving exponentials.
- Find derivatives of products and quotients of algebraic expressions.
- Find derivatives using the extended power rule.
- Find slopes and equations of tangent lines of functions involving products and quotients.
- Solve applications involving the product rule and quotient rule.
- Find higher order derivatives of products and quotients.
- Find derivatives of products and quotient rules using given values or graphs.

The Product Rule

To differentiate a product of two functions, we use a technique called the **product rule**.

The Product Rule. For two differentiable functions f and g, the derivative of their product fg is

$$(fg)' = f'g + fg'.$$

Alternatively, the product rule can be expressed as follows.

For example, consider the function $y = x^2e^x$.

$$\frac{d}{dx}\left(x^2e^x\right) = 2xe^x + x^2e^x$$

Proof of the Product Rule. Suppose f and g are both differentiable and let $h(x) = f(x)g(x)$.

$$\begin{aligned}
h'(x) &= \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)g(x+h) + f(x)g(x+h)}^{=0} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)\big(f(x+h) - f(x)\big) + f(x)\big(g(x+h) - g(x)\big)}{h} \\
&= \lim_{h\to 0} g(x+h) \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{=f'(x)} + f(x) \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{=g'(x)} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

□

✍ **Example.** Find the derivative of $f(x) = x^4 e^x$.

✍ **Example.** Find the derivative of $f(x) = 2e^x\,(x^3 - 3)$.

✍ **Example.** Determine a formula for the n^{th} derivative of $f(x) = xe^x$.

✍ **Example.** Find the derivative of $g(x) = 6x^2\left(3 + 2x^{-3}\right)$.

✍ **Example.** Find the derivative of $h(x) = \dfrac{70 + 2x^2}{x^3}$.

The Quotient Rule

The Quotient Rule. For two differentiable functions f and g, where $g(x) \neq 0$, the derivative of their quotient $\frac{f}{g}$ is

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Alternatively, the quotient rule can be expressed as follows.

Note that subtraction is not commutative, i.e. $x - y \neq y - x$, so the order of subtraction in the numerator is critical.

✍ **Example.** Evaluate the expression $\dfrac{d}{dx}\left(\dfrac{e^x}{x+1}\right)$.

✍ **Example.** Find the derivative of $y = \dfrac{x+2}{3x^2 - x}$.

✍ **Example.** Find the derivative of $g(x) = \dfrac{2 + xe^x}{x^2 - 2}$.

✍ **Example.** Find the equation of the line tangent to $y = \dfrac{2x}{x+1}$ at $x = 2$.

✍ **Example.** if $h(x) = f(x)g(x)$, find $h'(2)$ given the following information.

$$f(2) = -3 \qquad f'(2) = -2 \qquad g(2) = 4 \qquad g'(2) = 7$$

✍ **Example.** Suppose f is a differentiable function. Find an expression for the derivative of each of the following functions.

① $y = x^2 f(x)$

② $y = \dfrac{\sqrt{x} + xf(x)}{e^x}$

✍ **Example.** Find the derivative for each of the following functions.

① $g(x) = \dfrac{2x^5 + x - 1}{e^x + 9}$

② $f(x) = \sqrt{x}\,(3x^7 - 2)$