

8.5: Partial Fractions

Learning Objectives. Upon successful completion of Section 8.5, you will be able to...

- Answer conceptual questions involving partial fractions.
- Set up or find a partial fraction decomposition.
- Evaluate integrals involving partial fractions with only simple linear factors.
- Evaluate integrals involving partial fractions with repeated linear factors.
- Evaluate integrals involving partial fractions with irreducible quadratic factors (simple or repeated).
- Evaluate integrals involving improper rational functions (where long division is needed first).
- Use partial fractions to find areas, volumes, or arc lengths.
- Evaluate integrals involving partial fractions that require a preliminary step (such as a change of variables).

Introduction

Consider the integral $\int \frac{x+5}{x^2+x-2} dx$. Will any of our previous integration techniques work?

- Antiderivative Rules
- U-Substitution
- Algebraic Manipulation
- Integration by Parts
- Trig Integral
- Trig Substitution

We can factor the denominator and write $\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$, for some $A, B \in \mathbb{R}$.

This will result in two “simpler” integrals that we can evaluate.

The Idea of Partial Fraction Decomposition

The technique of **partial fraction decomposition** allows us to rewrite a rational function as a sum of two simpler rational functions. Recall that a rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials.

When can we use partial fraction decomposition (PFD) for a rational function? To determine this, we look at the degree of the numerator (n) and degree of the denominator (k).

- PFD can be used when we have a **proper** rational function ($n < k$).
- If the rational function is **improper** ($n \geq k$), perform long division before PFD.

✎ **Example.** Consider the improper rational function $\frac{x^4 + x + 5}{x^3 + 3}$.

Steps for Partial Fraction Decomposition

Given a rational function...

- ① Determine if the function is proper or improper. If improper, do long division.
- ② If proper, factor the denominator as much as possible.
- ③ Determine the *form* of the proper rational function based on the denominator.
 - Simple linear form
 - Repeated linear form
 - Irreducible quadratic form
 - Repeated irreducible quadratic
- ④ Solve for the constants needed.

Forms in Partial Fraction Decomposition

① Simple Linear

⚡ **Example.** $f(x) = \frac{3}{x^3 - x^2 - 12x}$

② Repeated Linear

⚡ **Example.** $f(x) = \frac{x^2 - 5x + 16}{(2x + 1)(x - 2)^2}$

⚡ **Example.** $f(x) = \frac{x + 2}{x^2(x + 1)^3(2x + 3)}$

③ Irreducible Quadratic

⚡ **Example.** $f(x) = \frac{10}{(x - 1)(x^2 + 9)}$

③ Irreducible Quadratic (continued)

⚡ **Example.** $f(x) = \frac{3+x}{x^2(2x^2-x-1)}$

④ Repeated Irreducible Quadratic

⚡ **Example.** $f(x) = \frac{1}{x(x^2+4)^2}$

⚡ **Example.** Write out the form of the partial fraction decomposition of the function:

1. $f(x) = \frac{x^4}{(x+3)^3(x^2-x+1)}$

2. $f(x) = \frac{3x}{(x-1)(x^4+4x^2+4)}$

Integrating Rational Functions Using Partial Fractions

▮ Example. $\int \frac{3}{x^3 - x^2 - 12x} dx$

▮ **Example.** $\int \frac{x^2 - 5x + 16}{(2x + 1)(x - 2)^2} dx$

▣ **Example.** $\int \frac{10}{(x-1)(x^2+9)} dx$

⚡ **Example.** $\int \frac{1}{x(x^2 + 4)^2} dx$

How can **u-substitution** be used to convert an integrand into a rational function so we can apply partial fraction decomposition?

▮ **Example.** $\int \frac{e^{2x}}{e^{2x} + 3e^x + 2} dx$

▮ **Example.** $\int \frac{\sqrt{1 + \sqrt{x}}}{x} dx$