2.5: Limits at Infinity

Learning Objectives. Upon successful completion of Section 2.5, you will be able to...

- Evaluate limits at infinity.
- Answer conceptual questions involving end behavior and horizontal asymptotes.
- Find horizontal and vertical asymptotes of functions.
- Find slant asymptotes and sketch graphs of rational functions.
- Determine end behavior of transcendental functions and sketch their graphs.
- Solve applications involving limits used to find steady states.
- Sketch graphs of functions given information about end behavior.

Introduction to Limits at Infinity

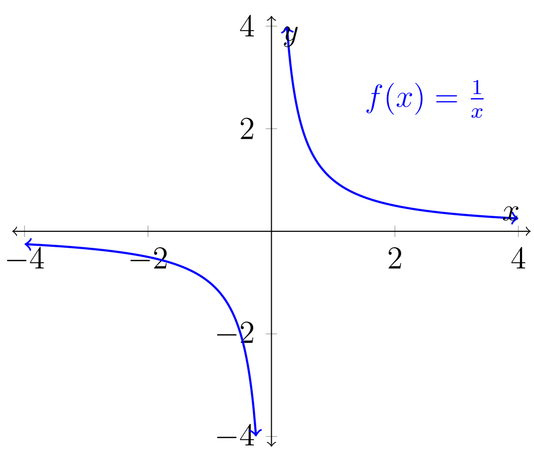
In a **limit at infinity**, the independent (x) value is allowed to become boundless in either the positive or negative direction. If we attempt direct substitution in such limits, we often see the $\frac{\infty}{\infty}$ indeterminate form. Limits at infinity example the "end behavior" of a function.

Example. Consider again the function $f(x) = \dfrac{1}{x}$.

x	10	100	1,000	10,000	100,000	1,000,000
$1/x$	0.1	0.01	0.001	0.0001	0.00001	0.000001

As $x \to \infty$, $\dfrac{1}{x} \to 0$. In other words, $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$.

We could similarly show that $\displaystyle\lim_{x\to-\infty} \frac{1}{x} = 0$.



Horizontal Asymptotes

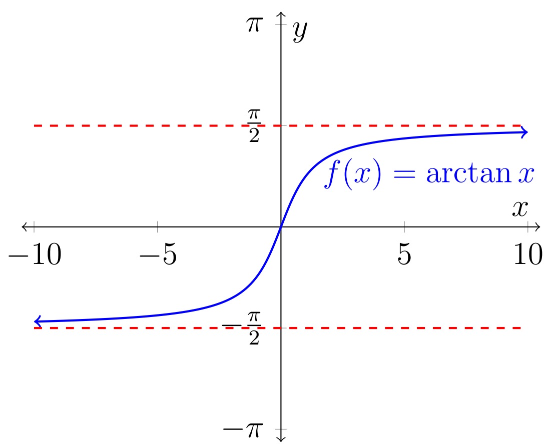
Definition We say that f is **asymptotic** to g if and only if

$$\lim_{x\to\pm\infty} |f(x) - g(x)| = 0$$

where f and g are any two functions defined as $x \to \infty$.

Definition. The line $y = L$ is called a **horizontal asymptote** (H.A.) if and only if $\lim\limits_{x\to\pm\infty} f(x) = L$ where $L \in \mathbb{R}$.

Example. The inverse tangent function has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.



✍ **Example.** If $k \in \mathbb{R}$, $k \neq 0$, evaluate the limit $\lim\limits_{x\to\infty} \dfrac{1}{k} \arctan x$.

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to-\infty} \frac{3e^{-x}-1}{2e^{-x}}$$

Polynomials.

Definition. A **polynomial** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$$

where each $a_i \in \mathbb{R}$ and each power is a nonnegative integer, i.e. $\{0, 1, 2, 3, \ldots\}$. The domain and range of every polynomial is $\mathbb{R}$.

Examples.

- $f(x) = 3x^6 + 2x^4 + x^2 - 5$ is a polynomial.
- $g(x) = 3x^{-1} + 2x^2 - 3x$ is not a polynomial (negative exponent).
- $h(x) = x^2 + x^{\frac{1}{2}}$ is not a polynomial (fractional exponent).

Polynomials have the following properties for limits at infinity.

- $\lim\limits_{x\to\pm\infty} x^n = \infty$ when n is even.
- $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$ when n is odd.
- $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = \lim\limits_{x\to\pm\infty} x^{-n} = 0$.
- If p is any polynomial, then $\lim\limits_{x\to\pm\infty} p(x) = \pm\infty$.

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to-\infty}\left(3x^{-7}+7x^{3}\right)$$

Rational Functions.

Definition. A **rational function** is a function of the form

$$Q=\frac{f(x)}{g(x)},$$

where f and g are polynomials. The domain of rational functions is $\mathbb{R}$ excluding the x-values that make $g(x)=0$.

Remark. If a rational function has a horizontal asymptote, it only has one.

To find the horizontal asymptote(s) of a rational function, we evaluate the limits at infinity by dividing each term of the function by the highest degree term of the denominator.

✍ **Example.** Find the horizontal asymptote(s) of $f(x)=\dfrac{2x^3+7}{3x^3-x^2+x+7}$.

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to\infty} \frac{2x^4+5}{3x^3-2x^2+7}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to-\infty} \frac{2x^4-3x}{3x^5+2x^2}$$

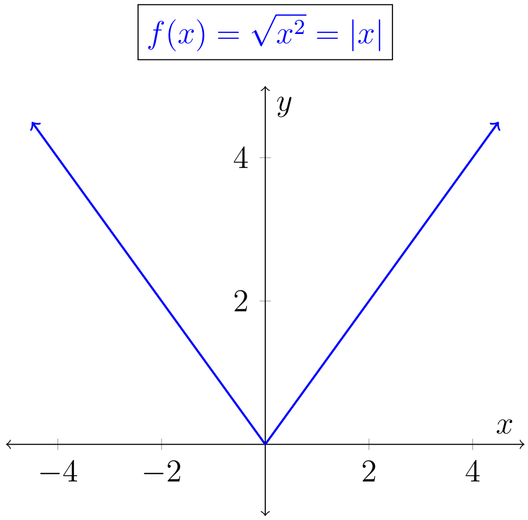
Limits at Infinity with Absolute Values. Consider the function $\sqrt{x^2}$ and note the following.

$$\text{If } x=-1, \text{ then } \sqrt{(-1)^2}=\sqrt{1}=1.$$

$$\text{If } x=1, \text{ then } \sqrt{1^2}=\sqrt{1}=1.$$

Here we see that $\sqrt{x^2}$ behaves like $|x|$, the absolute value function, which can be defined as a piecewise function.

$$\sqrt{x^2}=|x|=\begin{cases} +x & x\geqslant 0 \\ -x & x<0 \end{cases}$$



✍ **Example.** Evaluate the following limit.

$$\lim_{x\to-\infty} \frac{2x+5}{\sqrt{2x^2+5x}}$$

Oblique Asymptotes

Definition. An **oblique asymptote** of a rational function Q is a polynomial $\mathcal{O}$ of degree greater than or equal to one such that

$$\lim_{x\to\pm\infty} |Q(x) - \mathcal{O}(x)| = 0,$$

i.e. the distance between Q and $\mathcal{O}$ vanishes as $x \to \pm\infty$.

Theorem. An oblique asymptote exists only when the degree of the numerator of Q is greater than the degree of the denominator of Q.

Remark. For a rational function, an oblique asymptote and a horizontal asymptote may not coexist.

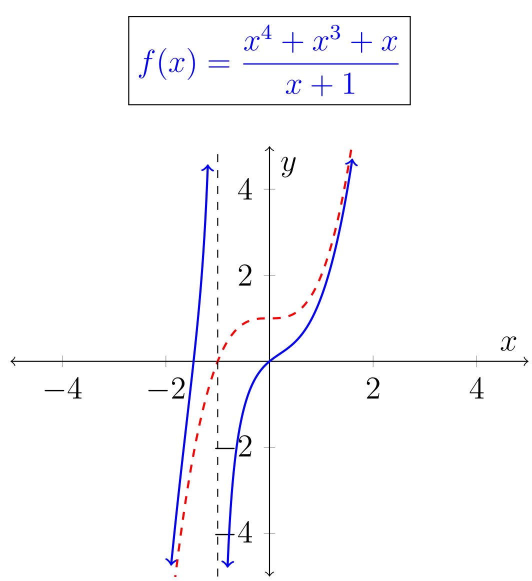
How to Find Oblique Asymptotes. We can perform *polynomial long division* on a rational function Q in order to obtain $Q(x) = \mathcal{O}(x) + R(x)$, where $\mathcal{O}$ is the quotient and R is the remainder, which may not necessarily be zero. We can then show that

$$\lim_{x \to \pm\infty} |R(x)| = 0 \iff \lim_{x \to \pm\infty} |Q(x) - \mathcal{O}(x)| = 0,$$

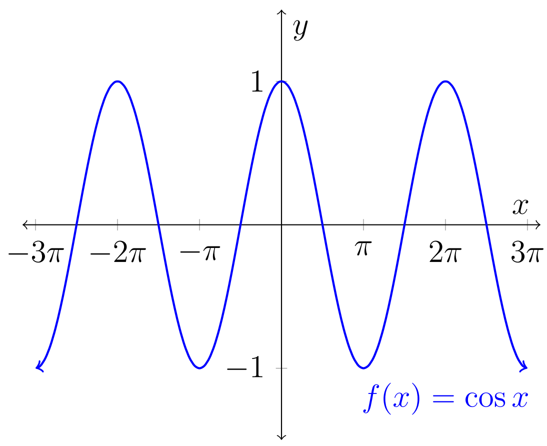
which implies that $\mathcal{O}$, the **quotient** of the polynomial long division, is the oblique asymptote.

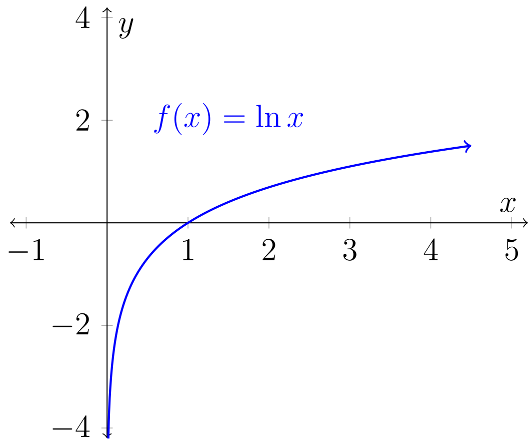
> **Remark.** If the degree n of the numerator and degree k of the denominator differ by one, i.e. $n - k = 1$, then the oblique asymptote is a line and may be referred to as a **slant asymptote**.

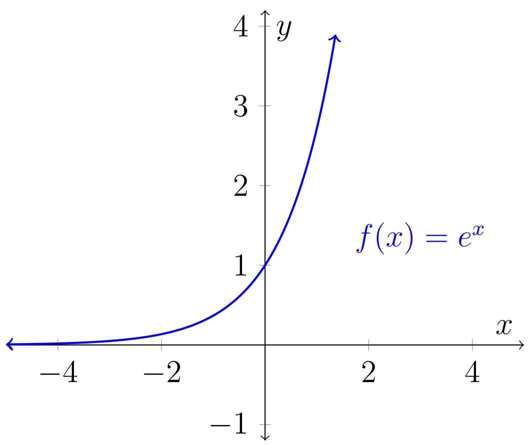
✍ **Example.** Find all asymptotes of $f(x) = \dfrac{x^4 + x^3 + x}{x + 1}$.



End Behavior of Other Notable Functions







Summary

Rational Functions at Infinity. For a rational function $f(x)$, let n be the degree of the numerator and k be the degree of the denominator.

- If $n < k$ (i.e. if the function is "bottom-heavy"), then $f(x)$ will approach zero.
- If $n > k$ (i.e. if the function is "top-heavy"), then $f(x)$ will approach $\pm\infty$.
- If $n = k$, then $f(x)$ will approach the ratio of the leading coefficients of the numerator and denominator.

Asymptotes of a Function.

- A function has a **vertical axis** (V.A.) at $x = c$ if the denominator only is zero and any limit of $f(x)$ as $x \to c$ is $\pm\infty$.
- A function has a **horizontal asymptote** (H.A.) at $y = L$ if $\lim\limits_{x\to\pm\infty} f(x) = L$.
- A function has an **oblique asymptote** (O.A.) at the quotient $\mathcal{O}(x)$ found by performing polynomial long division.