8.9: Improper Integrals

Learning Objectives. Upon successful completion of Section 8.9, you will be able to...

- Answer conceptual questions involving improper integrals.
- Evaluate improper integrals with an infinite limit of integration.
- Evaluate improper integrals with unbounded integrands.
- Find areas and volumes using improper integrals.
- Use the comparison test to determine whether an integral converges or diverges.

Motivating Application

The energy required to launch a rocket from the surface of Earth ($R = 6370$ km from the center of Earth) to an altitude H is given by an integral of the form

$$\int_R^{R+H} \frac{k}{x^2}\,dx,$$

where k is a constant that includes the mass of the rocket, the mass of Earth, and the gravitational constant. Suppose we want to launch the rocket to an arbitrarily large altitude H so that it escapes Earth's gravitational field. Then the energy required is

$$\int_R^{\infty} \frac{k}{x^2}\,dx.$$

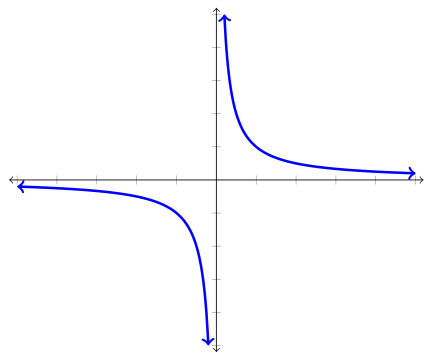
How could we evaluate this integral? Let's think back to one of our fundamental theorems.

Second Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$ and suppose that F is an antiderivative for f on the same interval. Then

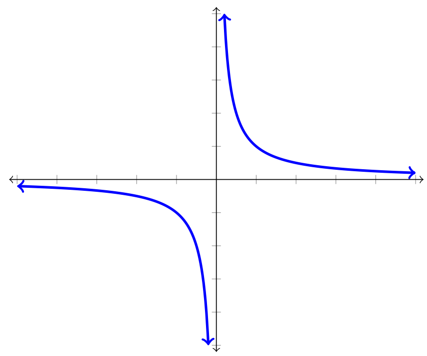
$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

✍ **Example.** Consider the following two integrals. Does the Second Fundamental Theorem of Calculus apply? Why or why not?

① $\displaystyle\int_1^\infty \frac{1}{x}\,dx$



② $\displaystyle\int_0^1 \frac{1}{x}\,dx$



Improper Integrals of Type 1

(a) If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx =$$

(b) If $f(x)$ is continuous on $(-\infty,b]$, then

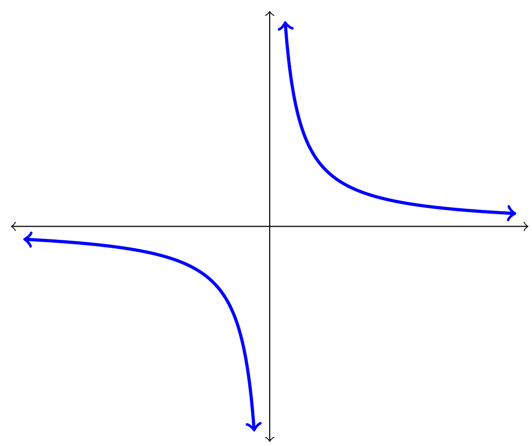
$$\int_{-\infty}^b f(x)\,dx =$$

We say that an integral **converges** if the limit exists. Otherwise, we say it **diverges**.

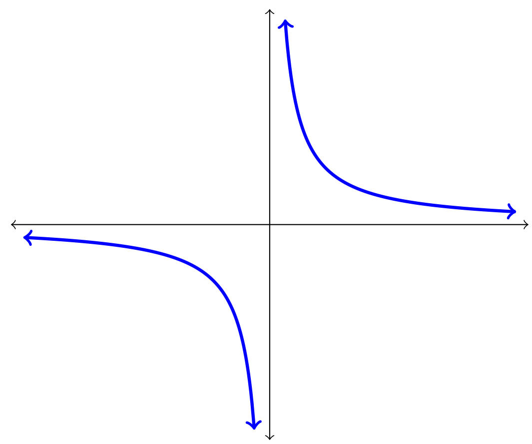
(c) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx =$$

✍ **Example.** Find the area under the curve $y = \dfrac{1}{x}$ for $x \geq 1$.



✍ **Example.** Find the volume of the solid generated when $y = \dfrac{1}{x}$ for $x \geq 1$ is rotated about the x-axis.



Fact: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ **converges** for ________________ and **diverges** for ________________.

✍ **Example.** Evaluate $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^4+1}}\,dx$.

Improper Integrals with an Unbounded Integrand (Type 2)

(a) If $f(x)$ is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx =$$

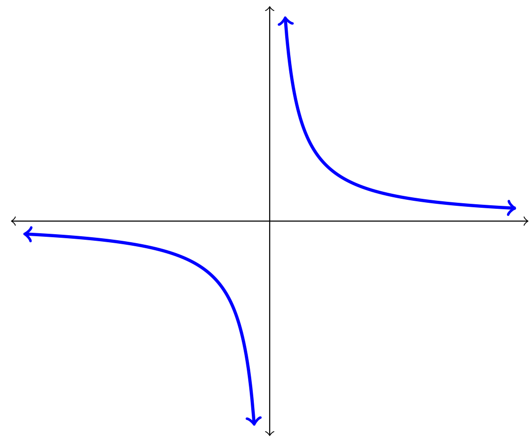
(b) If $f(x)$ is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx =$$

(c) If $f(x)$ is continuous on $[a, b]$ except at the interior point p where f is unbounded, then

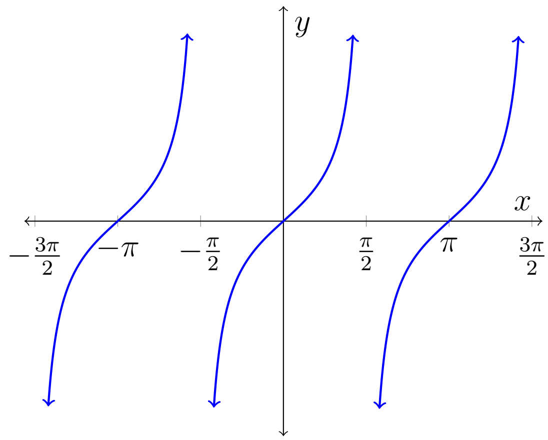
$$\int_a^b f(x)\,dx =$$

✍ **Example.** Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$.



✍ **Example.** Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$.

✍ **Example.** Evaluate $\displaystyle\int_0^{\pi/2} \tan\theta \, d\theta$.



✍ **Example.** Evaluate $\int_0^1 \ln x \, dx$.

Comparison Theorem for Improper Integrals

Comparison Theorem. Suppose f, g are continuous functions with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then

If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then

Caution: We have to be careful with what we conclude using the Comparison Theorem.

If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then

If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then

✍ **Example.** Determine whether $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges or diverges.

Recall: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ **converges** for $p > 1$ and **diverges** for $p \leq 1$.

✍ **Example.** Determine whether $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ converges or diverges.

✍ **Example.** Determine whether $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$ converges or diverges.