12.3: Calculus in Polar Coordinates

Learning Objectives. Upon successful completion of Section 12.3, you will be able to...

- Answer conceptual questions involving calculus in polar coordinates.
- Find the slope of the line tangent to a polar curve at a given point.
- Find the points at which a polar curve has horizontal or vertical tangent lines.
- Find intersection points for two polar curves.
- Find the area of a region bounded by polar curves.
- Find the lengths of polar curves.

Tangents to Polar Curves

Given a polar curve $r = f(\theta)$, how do we find $\dfrac{dy}{dx}$?

✍ **Example.** Let's consider the polar curve $r = 1+\cos\theta$. Find the slope of the line tangent to the curve at $\theta = \dfrac{\pi}{2}$.

Tangents to Polar Curves. The slope $\frac{dy}{dx}$ of the line tangent to a polar curve $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Horizontal tangents occur where _________________, provided that _________________.

Vertical tangents occur where _________________, provided that _________________.

✍ **Example.** Let's again consider the polar curve $r = 1 + \cos\theta$. Find the points (r, θ) where the graph has horizontal or vertical tangents.

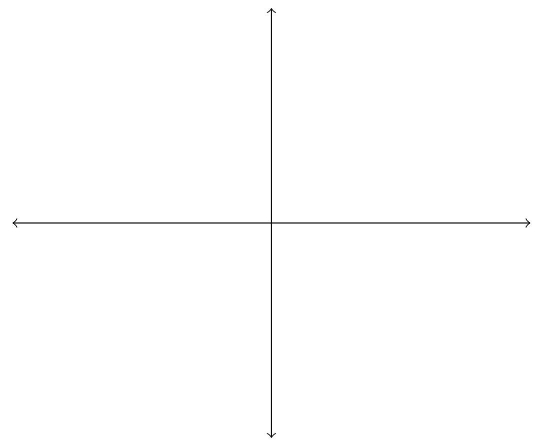
Area of a Polar Region

Note: The area of the sector of a circle is $A = \frac{1}{2}r^2\theta$.

Let R be the region bounded by $r = f(\theta)$ between $\theta = a$ and $\theta = b$ where f is positive and continuous and $0 < b - a \leq 2\pi$. How can we find the area of this region?

✍ **Example.** Consider the rose curve $r = 4\sin(3\theta)$ traced out on the interval $[0, \pi]$.

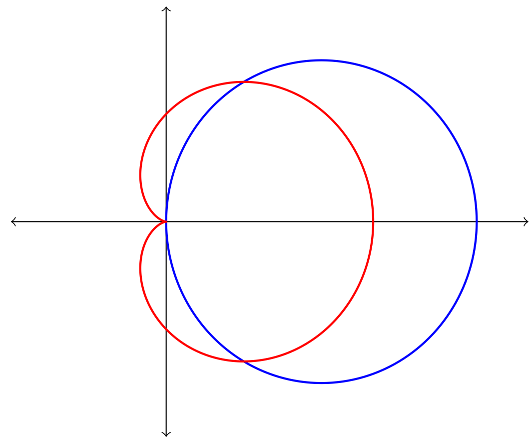
(a) Sketch the polar curve.



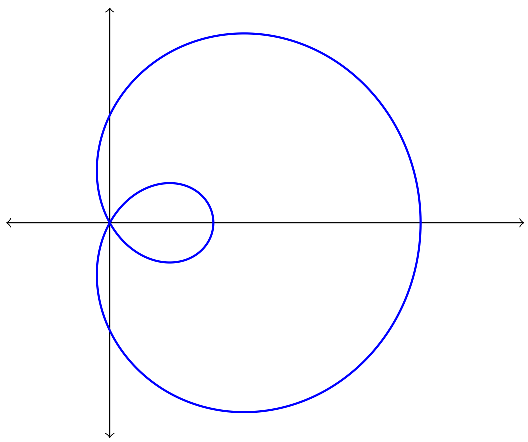
(b) Set up a few different integrals that represent the area of one petal of the rose curve.

(c) Evaluate one of the representations above to determine the area enclosed by one petal.

✍ **Example.** Consider the region that lies inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$. Set up a few different integrals that could represent the area of this region.



✍ **Example.** Set up the integral(s) representing the area inside the larger loop and outside the smaller loop of $r = \frac{1}{2} + \cos\theta$.



Polar Arc Length

Polar Arc Length. The length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

✍ **Example.** Find the length of the polar curve $r = 2\cos\theta$, $0 \le \theta \le \pi$.