

8.9: Improper Integrals

Learning Objectives. Upon successful completion of Section 8.9, you will be able to...

- Answer conceptual questions involving improper integrals.
- Evaluate improper integrals with an infinite limit of integration.
- Evaluate improper integrals with unbounded integrands.
- Find areas and volumes using improper integrals.
- Use the comparison test to determine whether an integral converges or diverges.

Motivating Application

The energy required to launch a rocket from the surface of Earth ($R = 6370$ km from the center of Earth) to an altitude H is given by an integral of the form

$$\int_R^{R+H} \frac{k}{x^2} dx,$$

where k is a constant that includes the mass of the rocket, the mass of Earth, and the gravitational constant. Suppose we want to launch the rocket to an arbitrarily large altitude H so that it escapes Earth's gravitational field. Then the energy required is

$$\int_R^{\infty} \frac{k}{x^2} dx.$$

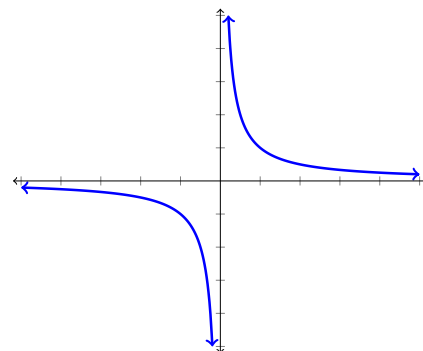
How could we evaluate this integral? Let's think back to one of our fundamental theorems.

Second Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$ and suppose that F is an antiderivative for f on the same interval. Then

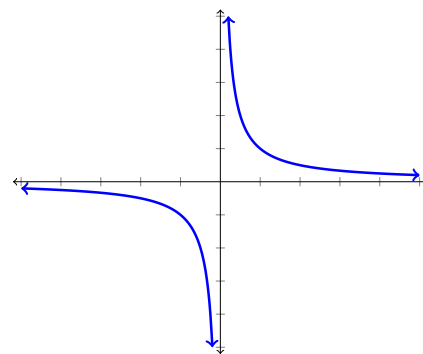
$$\int_a^b f(x) dx = \int_a^b F'(x) dx = F(b) - F(a).$$

✦ **Example.** Consider the following two integrals. Does the Second Fundamental Theorem of Calculus apply? Why or why not?

① $\int_1^{\infty} \frac{1}{x} dx$



② $\int_0^1 \frac{1}{x} dx$



Improper Integrals of Type 1

(a) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx =$$

(b) If $f(x)$ is continuous on $(-\infty, b]$, then

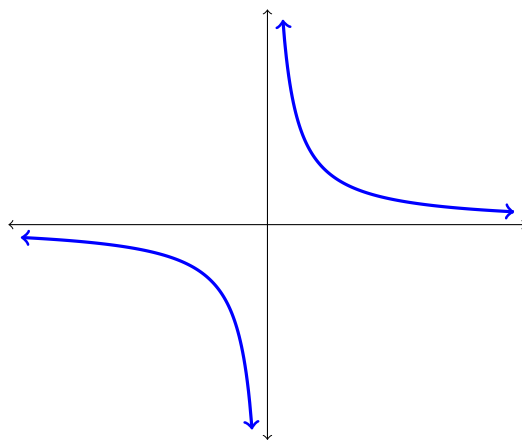
$$\int_{-\infty}^b f(x) dx =$$

We say that an integral **converges** if the limit exists. Otherwise, we say it **diverges**.

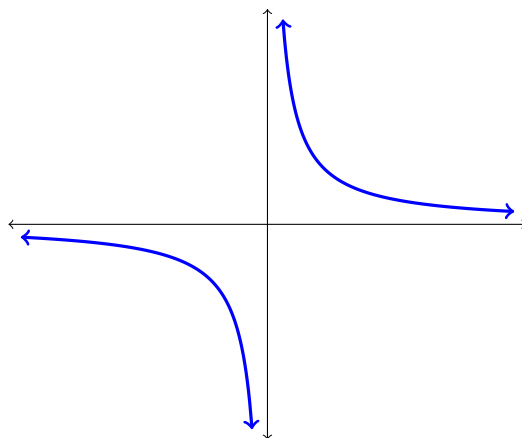
(c) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx =$$

▣ **Example.** Find the area under the curve $y = \frac{1}{x}$ for $x \geq 1$.



✎ **Example.** Find the volume of the solid generated when $y = \frac{1}{x}$ for $x \geq 1$ is rotated about the x -axis.



Fact: $\int_1^{\infty} \frac{1}{x^p} dx$ converges for _____ and diverges for _____.

▮ **Example.** Evaluate $\int_0^\infty \frac{x}{\sqrt{x^4+1}} dx$.

Improper Integrals with an Unbounded Integrand (Type 2)

- (a) If $f(x)$ is continuous on $(a, b]$ with $\lim_{x \rightarrow a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x) dx =$$

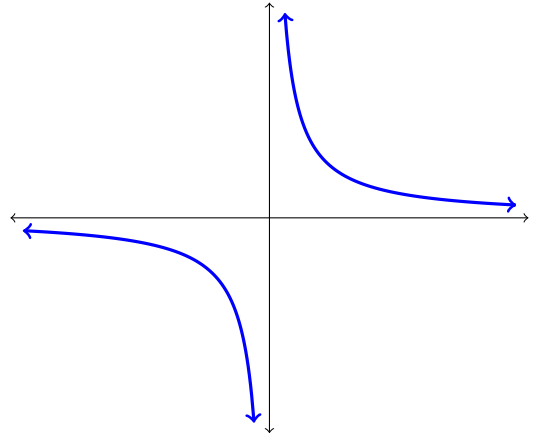
- (b) If $f(x)$ is continuous on $[a, b)$ with $\lim_{x \rightarrow b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x) dx =$$

- (c) If $f(x)$ is continuous on $[a, b]$ except at the interior point p where f is unbounded, then

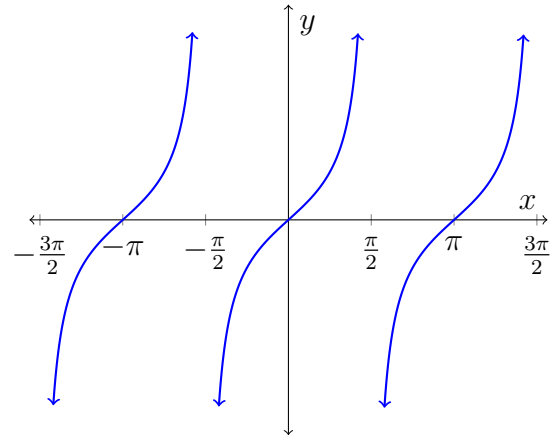
$$\int_a^b f(x) dx =$$

▮ **Example.** Evaluate $\int_{-1}^1 \frac{1}{x} dx$.



▮ **Example.** Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}} dx$.

▮ **Example.** Evaluate $\int_0^{\pi/2} \tan \theta \, d\theta$.



▮ **Example.** Evaluate $\int_0^1 \ln x \, dx$.

Comparison Theorem for Improper Integrals

Comparison Theorem. Suppose f, g are continuous functions with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

If $\int_a^\infty f(x) dx$ converges, then

If $\int_a^\infty g(x) dx$ diverges, then

Caution: We have to be careful with what we conclude using the Comparison Theorem.

If $\int_a^\infty f(x) dx$ diverges, then

If $\int_a^\infty g(x) dx$ converges, then

▮ **Example.** Determine whether $\int_1^\infty e^{-x^2} dx$ converges or diverges.

Recall: $\int_1^\infty \frac{1}{x^p} dx$ **converges** for $p \geq 1$ and **diverges** for $p \leq 1$.

▮ **Example.** Determine whether $\int_1^\infty \frac{2 + e^{-x}}{x} dx$ converges or diverges.

▮ **Example.** Determine whether $\int_1^\infty \frac{x}{x^3 + 1} dx$ converges or diverges.