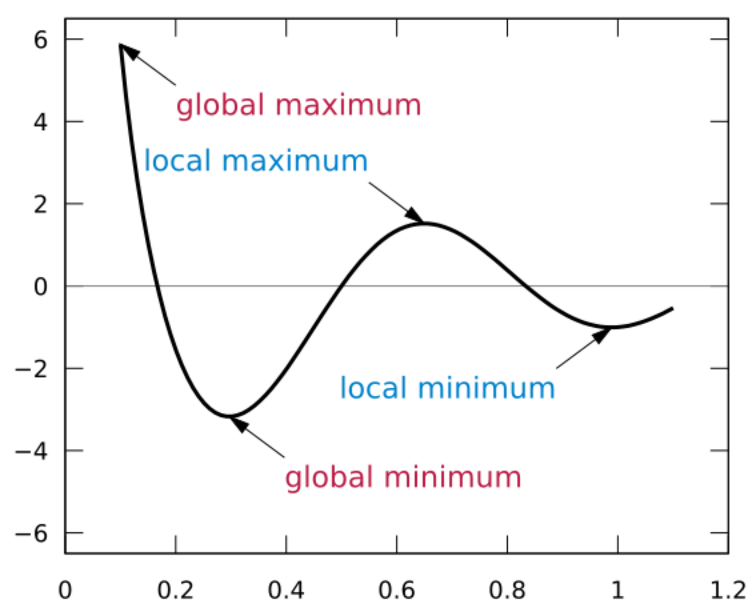
4.1: Maxima and Minima

Learning Objectives. Upon successful completion of Section 4.1, you will be able to...

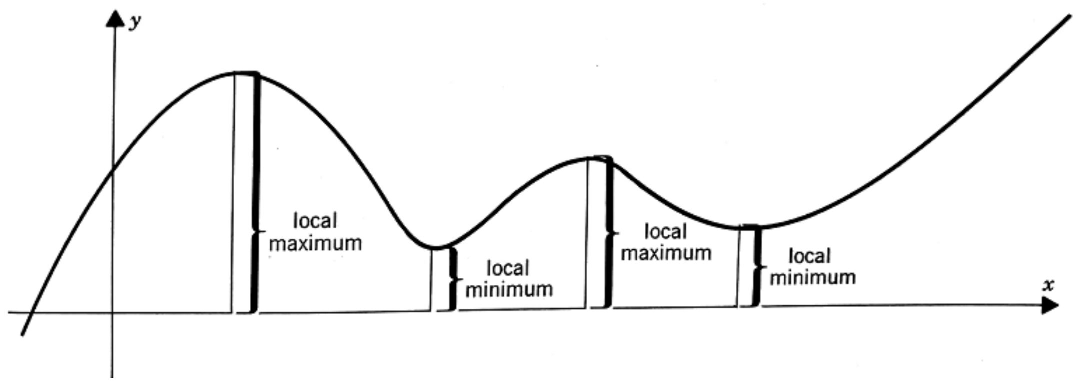
- Answer conceptual questions involving maxima and minima.
- Use a graph to identify absolute and/or local extrema.
- Sketch the graph of a function on an interval satisfying given properties.
- Locate critical points of functions.
- Determine the existence, location, and value of absolute extrema on a given interval of a function.
- Solve applications involving maxima and minima.

Motivation

An important practical application of calculus is **optimization**, which is concerned with *how large* or *how small* a certain quantity of interest can be. To set the foundation for solving optimization problems, this section will introduce the concepts of the **maximum** and **minimum** values of a function.



Definition. The maximum and minimum values of f are collectively referred to as the **extreme values** or the **extrema** of f.



Definition. The number $f(c)$ is called a **local maximum** if it is the largest y-value in a "window" around $x = c$.

The number $f(c)$ is called a **local minimum** if it is the smallest y-value in a "window" around $x = c$.

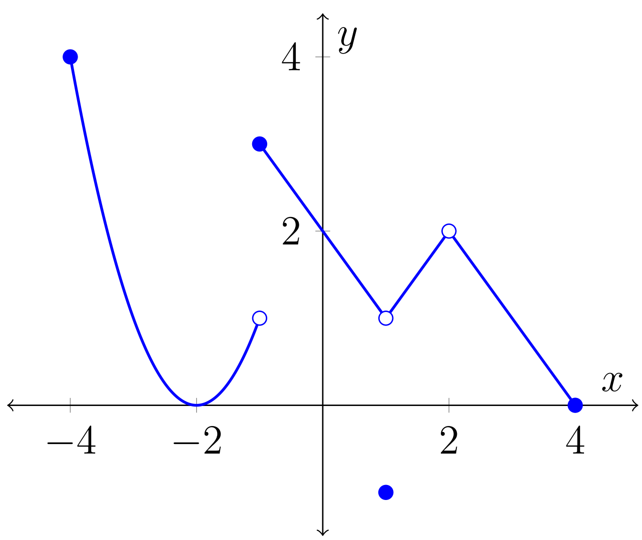
By definition, local extrema may not occur at an endpoint. Note that some definitions use the word "relative" instead of "local."

Definition. The number $f(c)$ is called a **global maximum** if it is greater than or equal to all other y-values.

The number $f(c)$ is called a **global minimum** if it is less than or equal to all other y-values.

It is possible for global extrema to occur at an endpoint. Some definitions use the word "absolute" instead of "global."

✍ **Example.** Classify each of the extrema in the graph below as a local or global maximum or minimum.

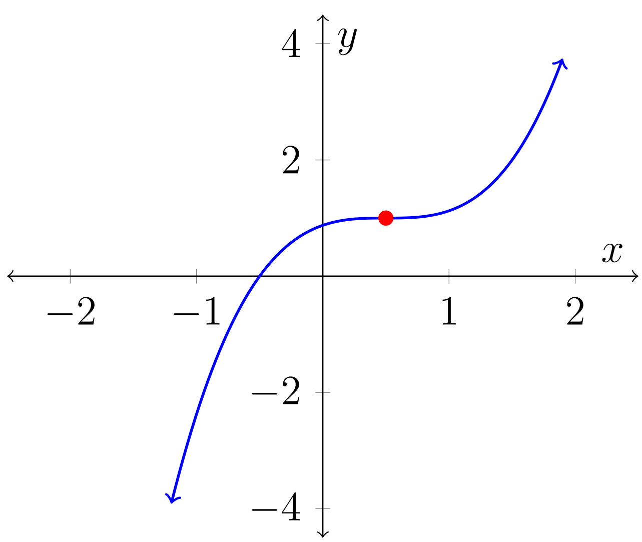


Locating Extrema

How can we locate extrema using tools we know without seeing the graph of the function?

Theorem. Suppose f attains a maximal or minimal values at $x = c$. Then it must be that either $f'(c) = 0$ or $f'(c)$ fails to exist.

Note that the converse is **not** true: The fact that $f'(c) = 0$ or $f'(c)$ fails to exist does not automatically imply the existence of a maximum or minimum. For example, consider the graph of $y = \left(x - \frac{1}{2}\right)^3 + 1$.



Critical Points. Critical points are values that give *candidates* for extrema. They allow us to determine (1) if we have a maximum or minimum and (2) if so, what type of maximum or minimum (i.e. global or local).

Definition. An *interior point* $c \in \text{Dom}\,[f]$ is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ fails to exist.

First, we will focus on finding global extrema within a closed interval.

Extreme Value Theorem (EVT). Suppose f is continuous on the closed interval $[a, b]$. Then there exists a minimum and maximum y-value on the interval.

Applying the EVT. If f is continuous on $[a, b]$, then...

1. Calculate $f(a)$ and $f(b)$.
2. Find the critical points of f on $[a, b]$, say $x = c_1$ and $x = c_2$.
3. Evaluate f at the critical points, for example $f(c_1)$ and $f(c_2)$.
4. The **global max** is the *largest* of these values and the **global min** is the *smallest* of these values.

✍ **Example.** Determine the global extrema of $g(x) = \dfrac{4x}{x^2 + 1}$ on $[0, 2]$.

✍ **Example.** Determine the global extrema of $f(x) = \ln(x^2 + 2)$ on $[-1, 2]$.