1.3: Inverse, Exponential, and Logarithmic Functions

Learning Objectives. Upon successful completion of Section 1.3, you will be able to...

- Answer conceptual questions involving inverse, exponential, and logarithmic functions.
- Determine the largest possible intervals on which a given function has an inverse.
- Find and graph inverse functions.
- Use the Change of Base Rule to evaluate logarithms and rewrite exponential expressions.
- Evaluate logarithmic expressions.
- Solve exponential or logarithmic equations.

Exponential Functions

Motivation. Suppose a colony of rabbits doubles in size each month. Let p_0 represent the initial population and let f represent a function describing the population after some number of months. We can model this as follows.

Such a process is modeled by an **exponential function**.

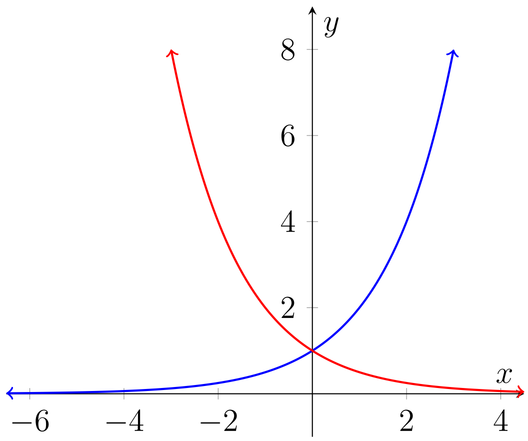
Definition. An **exponential function** is a function of the form where $a, b \in \mathbb{R}$ and

$b > 0$. Note that $f(0) = a$. The value of b is referred to as the base.

Remark. Why must we have $b > 0$? Suppose $x = \frac{1}{2}$ and $b = -2$. Then... So, $b > 0$

ensures that we avoid complex (non-real) numbers.

Below are the graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$.



If $b > 1$, then f is increasing. If $0 < b < 1$, then f is decreasing.

✍ **Example.** Is the function $h(x) = 3 \cdot \left(\frac{1}{5}\right)^x$ increasing or decreasing?

Facts about Exponential Functions

- The domain of an exponential function is $\mathbb{R}$, i.e. $(-\infty, \infty)$.
- The output of an exponential function is always positive and never zero. Hence, we express the range as $(0, \infty)$.
  - This means that the equation $b^x = 0$ has no solution.
- Exponential functions have a horizontal asymptote.
- Exponential functions are one-to-one, i.e. outputs are never repeated.
  - In other words, if $b^x = b^y$, then it must be that $x = y$.

✍ **Example.** Solve the following exponential equation.

$$2^{3x} = 2^{4x-1}$$

Fractional exponents represent roots. For instance, $x^{\frac{1}{2}} = \sqrt{x}$. The following relationship can be helpful when working with exponential functions.

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

✍ **Example.** Rewrite the following expressions in exponent form.

1. $\sqrt[4]{(2x-3)^3}$

2. $\sqrt[8]{12^{\frac{3}{8}}}$

3. $\sqrt[2x]{(1-3x)^4}$

The Natural Exponential. Of particular interest in calculus are exponential functions with the base e, i.e.

where $e \approx 2.71828$. This constant e is an irrational number, meaning that it cannot be expressed as a ratio of integers. We will see the significance of the natural exponential later.

Laws of Exponents. You should be able to apply some commonly used exponent laws.

$$\boxed{\begin{array}{cc} b^x b^y = b^{x+y} & \dfrac{b^x}{b^y} = b^{x-y} \\[2ex] (b^x)^y = b^{xy} & (ab)^x = a^x b^x \\[2ex] b^{-x} = \dfrac{1}{b^x} & b^x = \dfrac{1}{b^{-x}} \end{array}}$$

✍ **Example.** Simplify the expression $\sqrt{\sqrt{3}}\sqrt{\sqrt{12}}$.

✍ **Example.** Simplify the expression $\left(\dfrac{x^{-2}}{x^8}\right)^{-2}$.

When working with exponents, be careful to avoid a common error.

$$(x-y)^2 \neq x^2 - y^2$$

Inverse Functions

Very loosely, an inverse function has the effect of "reversing" the action of another function.

Definition. An **inverse function** of a function f is a function f^{-1} such that

if the inverse exists. Expressed differently, if $y = f(x)$, then $x = f^{-1}(y)$.

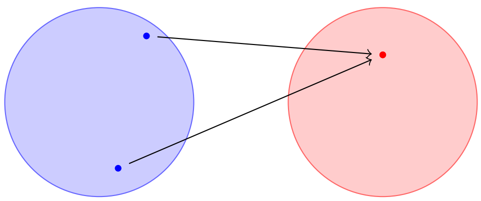
A function f is called **invertible** if f^{-1} exists.

Example. Suppose that f is invertible and $f(2) = 3$. Then $f^{-1}(3) = 2$.

Theorem. A function f has a unique inverse f^{-1} if and only if f is one-to-one and onto.

In calculus, we work with functions in such a way that all functions have the property of being **onto**, so we can generally ignore this condition in this course.

One-to-one functions pass the horizontal line test. If a function is not one-to-one, we cannot describe an inverse function.



Remark. The inverse notation is **not** an exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Finding an Inverse. Given $y = f(x)$, swap x and y, then solve for y.

✍ **Example.** Find the inverse of the function $f(x) = 2x - 4$, if it exists.

Logarithmic Functions

Exponential functions are one-to-one, so they possess unique inverses. Such inverses are called logarithms.

Definition. A **logarithmic function** is a function of the form where $a, b \in \mathbb{R}$, $b > 0$.

Remark. The following are equivalent expressions.

For instance, $2^3 = 8 \iff \log_2 8 = 3$. This also shows how to solve exponential and logarithmic equations.

✍ **Example.** Evaluate the expression $\log_{10}\left(\dfrac{1}{1000}\right)$.

Working with Logarithms. The following manipulations are often used when evaluating logarithmic expressions.

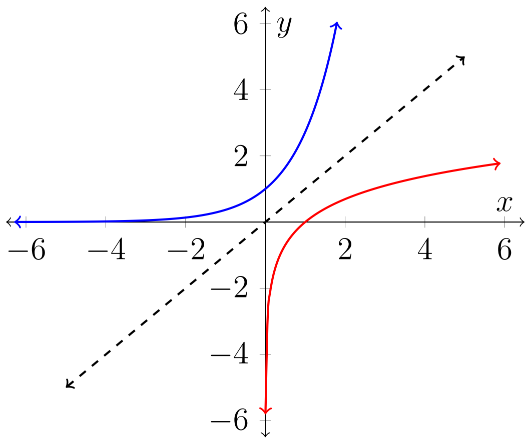
$$\log_b b = 1 \qquad b^{\log_b x} = x$$

$$\log_b 1 = 0 \qquad \log_b b^x = x$$

✍ **Example.** Solve the following equation using an inverse.

$$2^{x^2+1} = 8$$

Finding Inverses Graphically. If a function is one-to-one, then its inverse may be sketched by reflecting the graph of the function about the line $y = x$.



The Natural Logarithm. When the base of a logarithm is e, then

$$\log_e x = \ln x.$$

This is called the **natural logarithm.** Note the following properties of natural logarithms.

$$\boxed{\begin{aligned} \ln e &= \log_e e = 1 \\ \ln 1 &= \log_e 1 = 0 \end{aligned}}$$

Remark. In the case that no base is written, $\log x = \log_{10} x$.

Facts about Logarithmic Functions

- The domain of a logarithmic function is $(0, \infty)$.
- The range of a logarithmic function is $(-\infty, \infty)$.
- Logarithmic functions have a vertical asymptote.
- Notice that these properties are opposite of exponential functions.

Laws of Logarithms. You should be able to apply some commonly used logarithm laws.

1. $\log_b(xy) =$

2. $\log_b\left(x^r\right) =$

3. $\log_b\left(\dfrac{x}{y}\right) =$

✍ **Example.** Solve the following equation using properties of logarithms.

$$\log(x+21) + \frac{1}{2}\log x^2 = 2$$

✍ **Example.** Solve the following equation using properties of logarithms.

$$\ln 10 - \ln(7-x) = \ln x$$