5.5: The Substitution Method

Learning Objectives. Upon successful completion of Section 5.5, you will be able to...

- Answer conceptual questions involving the Substitution Rule.
- Evaluate indefinite integrals using substitution when the substitution is given.
- Evaluate indefinite integrals using substitution when the substitution is not given.
- Evaluate definite integrals using substitution.
- Find areas of regions using integration that requires substitution.

U-Substitution

When we differentiate a composite function, we do so using the **chain rule**.

$$f\big(g(x)\big) \implies f'\big(g(x)\big) \cdot g'(x)$$

In integration, we often need to apply the chain rule in reverse.

$$f'\big(g(x)\big) \cdot g'(x) \implies f\big(g(x)\big) + C$$

In general, applying the chain rule in reverse may not be straightforward. In such cases, the process is usually facilitated by making what we call a "u-substitution."

Strategy for Indefinite Integrals. Suppose we have an indefinite integral of the form

$$\int f\big(g(x)\big)g'(x)\,dx.$$

(1) Set $u = g(x)$ so that $du = g'(x)\,dx$.

(2) The integral may now be expressed as $\displaystyle\int f(u)\,du$.

(3) If F is an antiderivative for f, then $\displaystyle\int f(u)\,du = F(u) + C$.

(4) We can now substitute $u = g(x)$ into F to obtain the final result.

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

✍ **Example.** Apply the u-substitution method to evaluate $\int x \sin(2x^2)\, dx$.

✍ **Example.** Apply the u-substitution method to evaluate the indefinite integrals.

1. $\int x \left(x^2+1\right)^4 \, dx$

2. $\int \frac{x}{\sqrt{4-9x^2}}\, dx$

✍ **Example (continued).** Apply the u-substitution method to evaluate the indefinite integrals.

3. $\displaystyle\int \tan x \, dx$

4. $\displaystyle\int \sqrt{2x+1} \, dx$

5. $\displaystyle\int \frac{1}{1+16x^2} \, dx$

✍ **Example.** Evaluate the integral $\displaystyle\int \sec^3(x)\tan(x)\,dx$.

Strategy for Definite Integrals. For definite integrals, the process of u-substitution is nearly the same, but we must modify the limits of integration to correspond with the change of variable.

In other words, if $u = g(x)$ and $du = g'(x)\,dx$, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

✍ **Example.** Evaluate the definite integral $\displaystyle\int_2^e \frac{1}{x(\ln x)^2}\,dx$.

✍ **Example.** Evaluate the integral $\displaystyle\int_{\ln\sqrt{3}}^{0} \frac{e^x}{1+e^{2x}}\,dx$.
Hint: $e^{2x} = (e^x)^2$.

✍ **Example.** Evaluate the integral $\displaystyle\int_{1}^{2} \frac{10x^2}{x^3+2}\,dx$.

✍ **Example.** Evaluate the integral $\displaystyle\int_{2}^{4} x\sqrt{4-x}\,dx$.

✍ **Example.** Evaluate the integral $\int_2^5 \frac{x+1}{x^2+2x}\,dx$.

✍ **Example.** Evaluate the integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x}{\sqrt{1-\cos^2 x}}\,dx$ in two ways.

Hint: For the second way, use the Pythagorean trig identity $\sin^2 x + \cos^2 x = 1$.

✍ **Example.** Evaluate the integral $\int_0^{\ln 2} \frac{e^{2x}}{1+e^x}\,dx$.

Looking Ahead. We often view differentiation and integration as "inverses" of one another, but in reality, integration is often a much more involved process than differentiation. Integration requires a collection of tools and "tricks," and the substitution method is only one of those. If you take MATH 1080, you'll learn more of these integration techniques!