5.2: The Definite Integral

Learning Objectives. Upon successful completion of Section 5.2, you will be able to...

- Answer conceptual questions involving definite integrals.
- Sketch the graph of a curve and approximate the net area using Riemann sums.
- Use geometry to find area and net area of a described region.
- Write definite integrals from the limits of sums.
- Sketch the graph of an integrand and use geometry to evaluate the definite integral.
- Evaluate definite integrals using properties of definite integrals.
- Use the limit definition of the definite integral to evaluate definite integrals.

Introduction to the Definite Integral

The limit we have been working with is so fundamental to calculus that it has a name.

Definition. The unique value defined by the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

on the interval $[a, b]$ is called the **definite integral** of f from a to b and is denoted by

The process of finding this limit, if it exists, is called **integration**. The function f is called the **integrand**. The number a is the **lower limit** of integration and the number b is the **upper limit**. The differential dx indicates the **variable of integration**.

A function f is said to be **integrable** on $[a, b]$ if $\int_a^b f(x)\,dx$ exists.

Continuity Implies Integrability.

$$f \text{ is continuous on } [a,b] \implies f \text{ is integrable on } [a,b].$$

In the last section, we approximated the area under $f(x) = x^2 + 1$ on the interval $[0, 2]$. Using a uniform partition and right-hand endpoints, we showed that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^{*2} + 1\right) \Delta x = \cdots = \frac{14}{3},$$

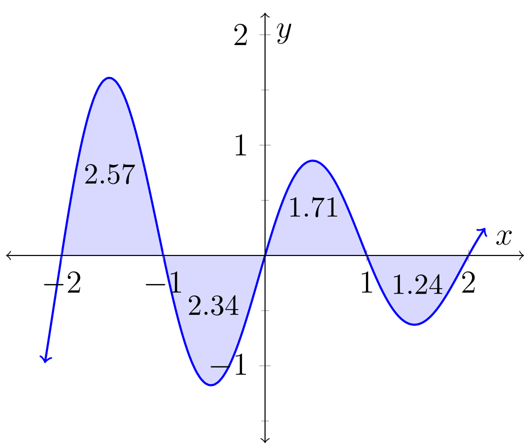
where $x_i^* = \frac{2i}{n}$ and $\Delta x = \frac{2}{n}$. We will show that, using the definite integral, we can write...

Interpreting the Definite Integral. In general, the definite integral measures the *net area* under the curve.

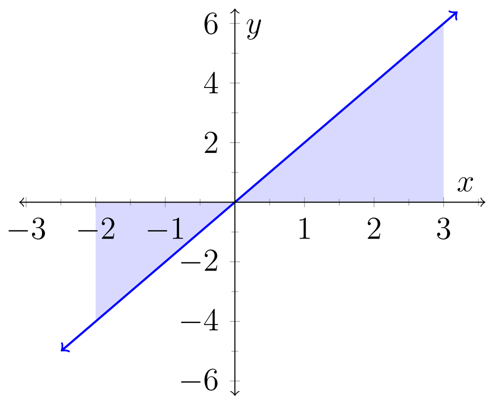
$$\int_a^b f(x)\, dx = \text{ area above } x\text{-axis} - \text{area below } x\text{-axis}$$

If f is nonnegative on $[a, b]$, we may interpret the value as the area under the curve.

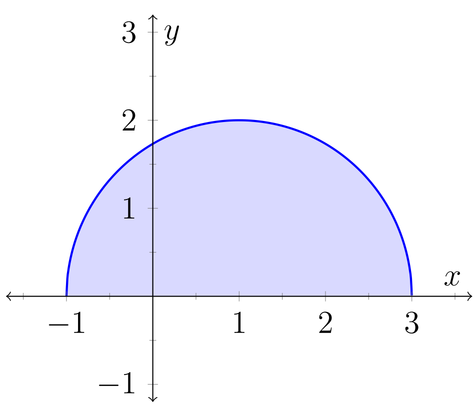
✍ **Example.** The areas of the shaded regions for the graph of f are given below. Express these areas using definite integrals.



✍ **Example.** Use the graph below to evaluate $\displaystyle\int_{-2}^{3} 2x\,dx$.



✍ **Example.** Use the graph below to evaluate $\displaystyle\int_{-1}^{3} \sqrt{4-(x-1)^2}\,dx$.



Properties of the Definite Integral.

- $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ for $k \in \mathbb{R}$
- $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $c \in (a,b)$

✍ **Example.** Evaluate $\int_0^1 \big(f(x) - 2g(x)\big)\,dx$ given that

$$\int_0^1 f(x)\,dx = e \text{ and } \int_1^0 g(x)\,dx = \pi.$$

✍ **Example.** Use only the fact that $\int_0^4 3x(4-x)\,dx = 32$ and the properties of integrals to evaluate the following.

1. $\int_4^0 3x(4-x)\,dx$

2. $\int_0^4 x(x-4)\,dx$

3. $\int_0^4 6x(4-x)\,dx$

4. $\int_0^8 3x(4-x)\,dx$

✍ **Example.** Use geometry and properties of integrals to evaluate

$$\int_0^1 \left(2x + \sqrt{1 - x^2} + 1\right)\, dx.$$

Note: $\sqrt{1-x^2}$ represents an upper semicircle with $r = 1$ centered at the origin.

✍ **Example.** Evaluate the following definite integrals given the following.

$$\int_1^9 f(x)\, dx = -1 \qquad \int_7^9 f(x)\, dx = 5 \qquad \int_7^9 h(x)\, dx = 4$$

1. $\displaystyle\int_1^9 -2f(x)\, dx$

2. $\displaystyle\int_7^9 \big(2f(x) - 3h(x)\big)\, dx$

3. $\displaystyle\int_9^1 f(x)\, dx$

4. $\displaystyle\int_9^7 \big(h(x) - f(x)\big)\, dx$