4.6: Linear Approximation and Differentials

Learning Objectives. Upon successful completion of Section 4.6, you will be able to...

- Answer conceptual questions involving linear approximation and differentials.
- Find a linear approximation function.
- Write a linear approximation function, and estimate the value of a function and evaluate the error.
- Choose a value ot minimize error and use it to write a linear approximation of a quantity.
- Graph a function and its linear approximation to identify overestimates and underestimates.
- Solve applications involving linear approximations.
- Given a function, write a differential expression expressing the change in the dependent variable as a function of a change in the independent variable.

Linear Approximation

Differentiable functions are said to be **locally linear**. In other words, if we "zoom in" really closely, the function may appear to be linear. Recall the definition of a linear function.

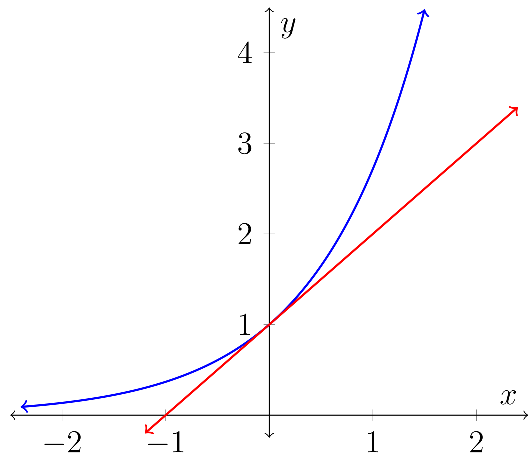
Definition. A **linear function** has the form $f(x) = mx + b$, where m is the slope and $(0, b)$ is the y-intercept. The domain and range is $\mathbb{R}$.

Motivation. Suppose f is differentiable at $x = c$ and let L represent the tangent line centered at $x = c$. If x is "near" c, then

$$\left|f(x) - L(x)\right| \approx 0 \iff f(x) \approx L(x).$$

In other words, the linear function defined by the tangent line at $x = c$ may serve as a "good" approximation to the function f so long as we don't stray too far from the center.

Linear approximation is especially useful in situations when f involves non-algebraic operations that are difficult (or impossible) to evaluate by hand.



$f(x) \approx L(x)$ for x "near" the center of the tangent line

Definition. The **linearization**, L, of f at $x = c$ is defined by

$$y - f(c) = f'(c)(x - c) \implies \boxed{L(x) = f(c) + f'(c)(x - c)}$$

so long as f is continuous and differentiable at $x = c$.

This is sometimes referred to as the *degree-1 Taylor polynomial*.

For purposes of approximation, the linearization takes the place of the function. For example, to estimate $f(4.2)$, we can find $L(4.2)$ instead.

✍ **Example.** Write the linearization of $f(x) = e^x$ at $x = 0$. Use this to find a linear approximation for $\sqrt[5]{e}$.

✍ **Example (continued).** Suppose we'd now like to approximate e^2 using the previous linearization. Explain why the approximation will be less accurate.

✍ **Example.** Find a linear approximation for $f(2.1)$ if $f(x) = 12 - x^2$.

✍ **Example.** Find a linear approximation for $\sqrt[3]{25}$.

✍ **Example.** Find a linear approximation for $\sqrt{500}$.

Error and Concavity. The *concavity* of a function can help explain the **accuracy** of a linear approximation.

- If f'' is "small" at the center of the linearization, the curvature is only slight, so the tangent line "hugs" the graph relatively tightly. This leads to a *smaller* error in the approximation.
- If f'' is "large" at the center of the linearization, the curvature is very pronounced, so the tangent line does not "hug" the graph for very long, and the error in the approximation will be *larger*.

Hence, a linear approximation works best when f'' is relatively small in magnitude.

Once we have an approximation, we are able to say whether the approximation is an *underestimate* or *overestimate* of the true function value.

Theorem. Suppose f is twice-differentiable at $x = c$.

- If $f''(c) < 0$, then $L(c)$ is an overestimate of $f(c)$.
- If $f''(c) > 0$, then $L(c)$ is an underestimate of $f(c)$.

✍ **Example.** Find a linear approximation for $\ln(2)$ and determine if the approximation is an underestimate or overestimate.

Differentials

While linear approximation allows us to estimate the y-value itself, **differentials** allow us to estimate the *change* in the y-value, Δy, for a small change in the x-value, Δx.

Using a linear approximation, we could say that $\Delta y \approx \Delta L$.

If $x = c$ changes to $x = c + \Delta x$, then __________________.

We introduce the **infinitesimals** dy and dx to replace ΔL and Δx.

Definition. The difference $f(x + dx) - f(x)$ is called the **increment** or **change** in f and is denoted by Δy.

Definition. If f is differentiable, the product $f'(x)dx$ is called the **differential** and is denoted by dy, i.e. $dy = f'(x)dx$.

✍ **Example.** Let $y = e^x$. Suppose that x increases from $x = 0$ to $x = \frac{1}{4}$. By about how much will the y-value change?

✍ **Example.** Let $y = \arctan(x)$. Suppose that x decreases from $x = 1$ to $x = \frac{9}{10}$. By about how much will the y-value change?

✍ **Example.** Let $y = \sin(x)$. Suppose that x increases from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{3} + 0.05$. By about how much will the y-value change?

✍ **Example.** Use a differential to approximate $\sqrt{146}$.

✍ **Example.** Suppose the radius of a circle decreases from $r = 5$ cm to $r = 4.25$ cm. By about how much will the area of the circle change?

Accuracy. Much like in linearization, the accuracy of an estimate using differentials relies on "staying close" to the center value of $x = c$. dx should be "small" in order for the approximation to be useful.

For example, consider the following differentials for $y = e^x$ centered at $c = 0$. The values of dy are fairly accurate estimates of Δy when dx is small, but as dx grows larger, the differential is less accurate.

dx	dy	Δy
0.01	0.01	0.0101
0.10	0.10	0.1052
0.50	0.50	0.6487
1.00	1.00	1.7183
2.00	2.00	6.3891

In fact, $\lim\limits_{dx \to \pm\infty} \left|\Delta y - dy\right| = \infty$.