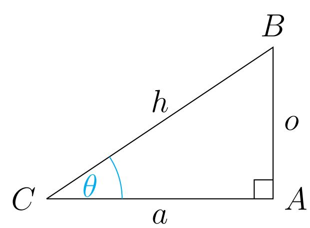
1.4: Trig and Inverse Trig Functions

Learning Objectives. Upon successful completion of Section 1.4, you will be able to...

- Answer conceptual questions involving trigonometric functions and their inverses.
- Evaluate trigonometric functions.
- Solve trigonometric equations.
- Evaluate inverse trigonometric functions.
- Graph trigonometric functions (i.e. a general sketch).

Trigonometric Functions

Right-Triangle Trigonometry. Right triangles possess ratios that depend only on the central angle of the triangle, denoted here as θ.



These ratios are **functions** of the central angle and have their own names: sine, cosine, tangent, cosecant, secant, and cotangent. Sine, cosine, and tangent are defined below.

$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h} \qquad \tan\theta = \frac{o}{a} = \frac{\sin\theta}{\cos\theta}$$

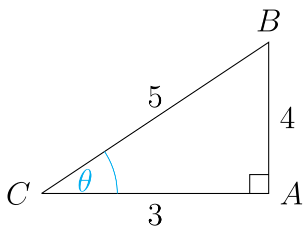
Cosecant, secant, and cotangent are known as **reciprocal** trig functions.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{h}{o} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{h}{a} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

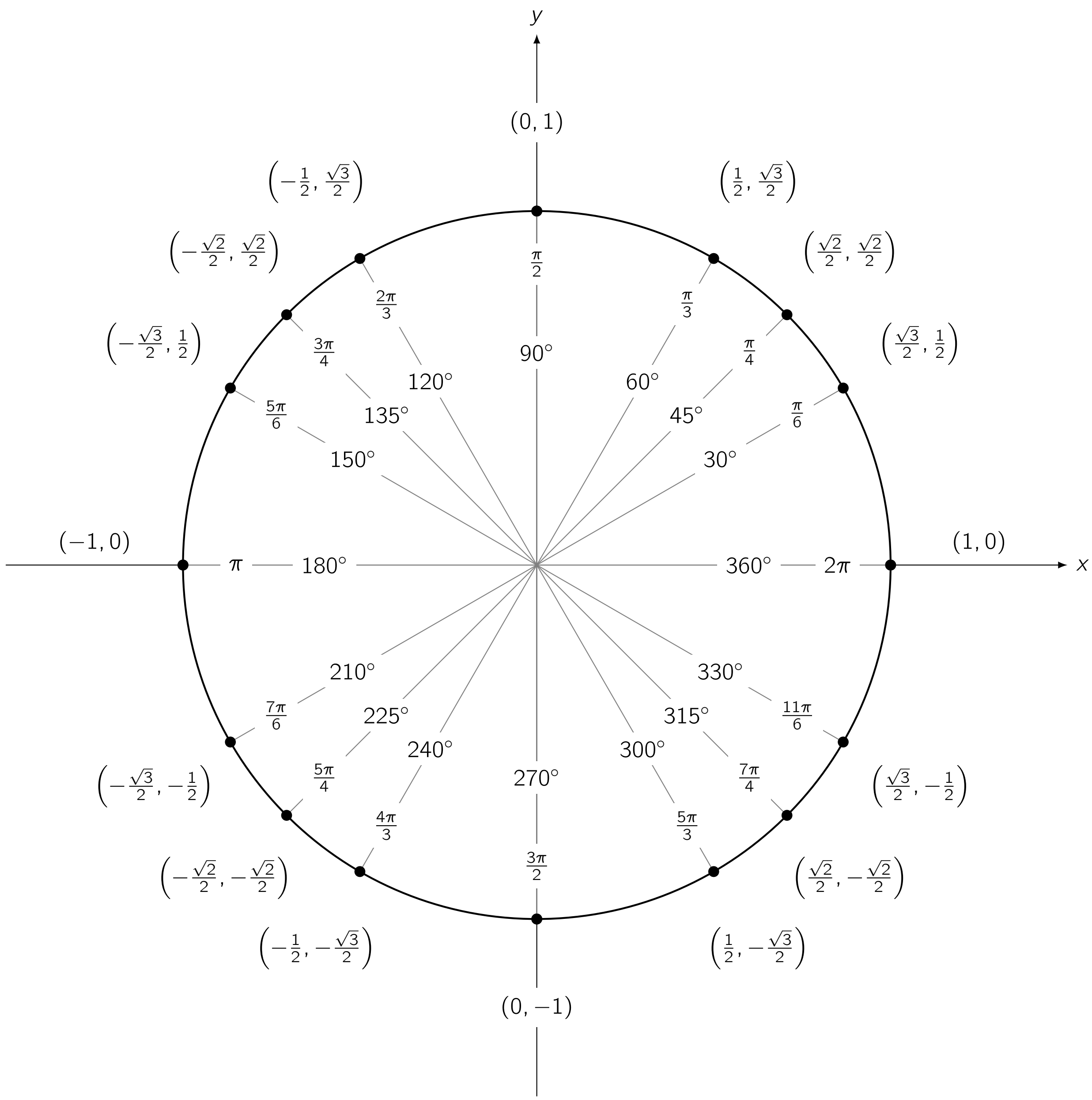
Be careful not to confuse reciprocal trig functions with *inverse* trig functions, defined later.

Remark. Note that, as with other functions, trigonometric functions must always have an argument (i.e. a function input). Writing "sin," "cos," or "tan" with no argument does not convey any mathematical meaning.

✍ **Example.** Write the six trig ratios for the following triangle.



The Unit Circle. The **unit circle** is the circle with radius 1 that is centered at the origin. The coordinates of a unit circle are given by $(\cos(\theta), \sin(\theta))$ for each θ.



The Pythagorean Theorem. The Pythagorean theorem is useful to remember when working with trig functions.

Pythagorean Theorem. For a right triangle with legs a and b and hypotenuse c, we always have that

$$a^2 + b^2 = c^2.$$

✍ **Example.** Find $\cos\theta$ and $\tan\theta$ given that $\sin\theta = \dfrac{3}{5}$ and $\theta \in \left[\frac{\pi}{2}, \pi\right]$.

Trigonometric Identities. Several trigonometric identities can often be useful to manipulate trigonometric functions.

- **Pythagorean Identities.**

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

- **Double-Angle Identities.**

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \end{aligned}$$

- **Half-Angle Identities.**

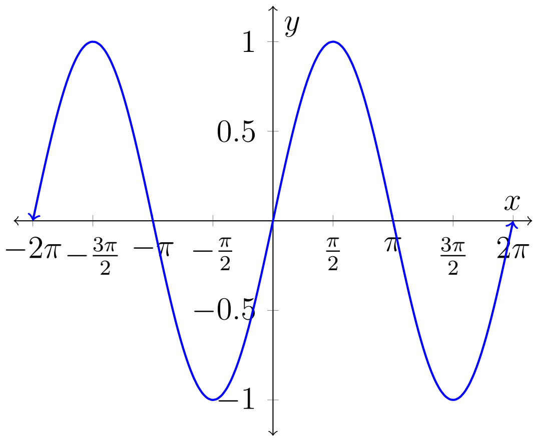
$$\begin{aligned} \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned}$$

Facts about the Trig Functions.

Definition. A function is called **periodic** if there exists $p \in \mathbb{R}$, $p \neq 0$, such that $f(x+p) = f(x)$ for all x in the domain of f.

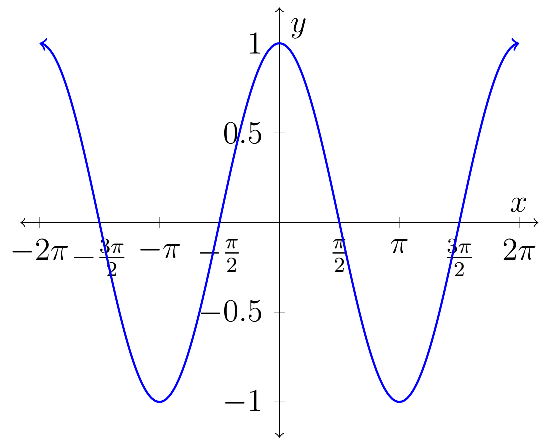
By definition, a periodic function is never one-to-one.

$\boxed{y = \sin x}$



domain: $\mathbb{R}$
range: $[-1,-1]$
period: 2π

$\boxed{y = \cos x}$

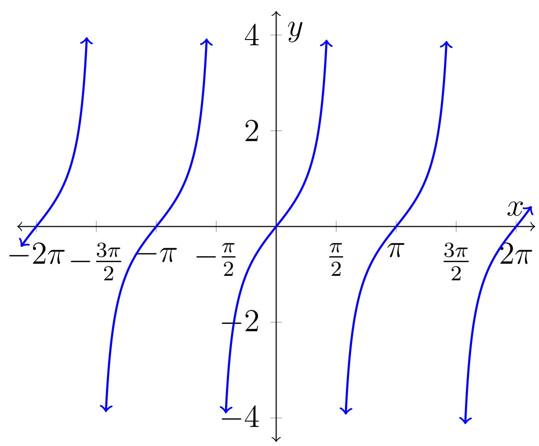


domain: $\mathbb{R}$
range: $[-1,-1]$
period: 2π

$\boxed{y = \tan x}$

domain: $\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi\right\}$
range: $\mathbb{R}$
period: π

Note the vertical asymptotes where $\cos x = 0$, i.e. at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$



function	**domain**	**range**	**period**
$\sin\theta$	$\mathbb{R}$	$[-1,1]$	2π
$\cos\theta$	$\mathbb{R}$	$[-1,1]$	2π
$\tan\theta$	$\mathbb{R}\setminus\{\frac{\pi}{2}+n\pi\}$	$\mathbb{R}$	π

function	**domain**	**range**	**period**
$\csc\theta$	$\mathbb{R}\setminus\{\pi+n\pi\}$	$(-\infty,-1]\cup[1,\infty]$	2π
$\sec\theta$	$\mathbb{R}\setminus\{\frac{\pi}{2}+n\pi\}$	$(-\infty,-1]\cup[1,\infty]$	2π
$\cot\theta$	$\mathbb{R}\setminus\{\pi+n\pi\}$	$\mathbb{R}$	π

✍ **Example.** Solve the following equation defined on the interval $[0,2\pi]$.

$$\sin^2\theta + 2\sin\theta + 2 = 1$$

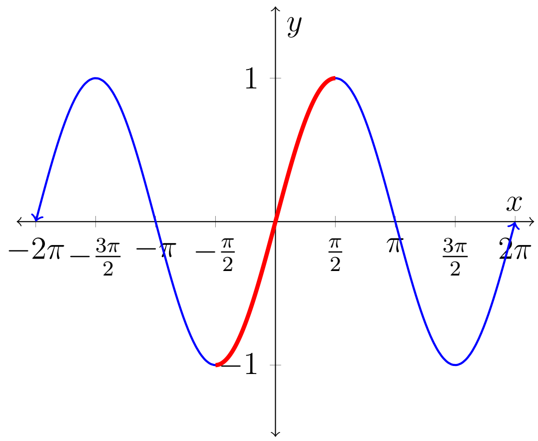
✍ **Example.** Solve the following equation defined on the interval $[0,2\pi]$.

$$\tan^2 x - \tan x = 0$$

Inverse Trigonometric Functions

The trig functions act on *angles* and return *ratios*. If we wanted a function that instead acts on a *ratio* and returns an *angle*, we would need an **inverse trig function**. Note that trig functions are periodic, so they are not one-to-one and thus do not have inverses. However, we can define an inverse trig function if we *restrict the domain* of a trig function.

Consider the sine function $y = \sin x$. We can see that it is not one-to-one on its domain $\mathbb{R}$.



However, it is one-to-one on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (highlighted in red) and spans the values of $[-1, 1]$. We can then define the inverse sine function $f(x) = \arcsin x$ with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Remark. Notationally, inverse trig functions may be expressed as

$$\sin^{-1} x = \arcsin x.$$

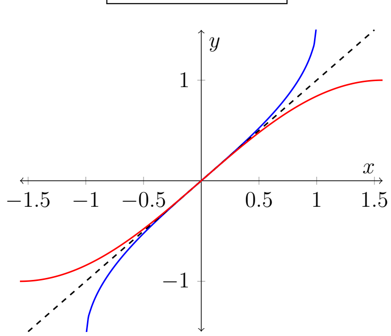
Note that the -1 in the inverse trig function notation is **not** a negative exponent.

$$\sin^{-1} x \neq \frac{1}{\sin x} = \csc x$$

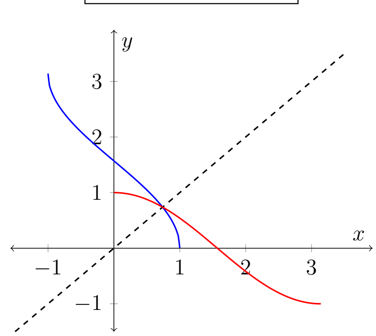
Remark. The following are equivalent expressions.

$$y = \arcsin x \iff \sin y = x$$

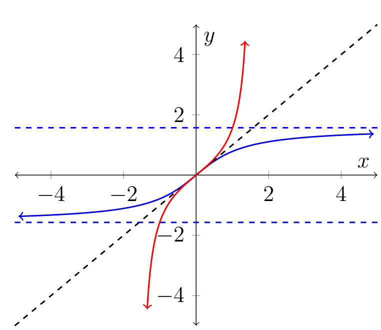
Inverse Sine



Inverse Cosine



Inverse Tangent



Inverse trig functions are achieved by forcing the original functions to be one-to-one through **domain restrictions**. The restricted domain of the original trig function becomes the range of the inverse trig function.

function	**domain**	**range**
$\arcsin\theta$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\arccos\theta$	$[-1,1]$	$[0,\pi]$
$\arctan\theta$	$\mathbb{R}$	$(-\frac{\pi}{2},\frac{\pi}{2})$

function	**domain**	**range**
$\text{arccsc}\ \theta$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$
$\text{arcsec}\ \theta$	$(-\infty,-1]\cup[1,\infty)$	$[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2}]$
$\text{arccot}\ \theta$	$\mathbb{R}$	$(0,\pi)$

✍ **Example.** Evaluate the following expressions.

1. $\arccos\left(\dfrac{1}{2}\right)$

2. $\arccos\left(-\dfrac{1}{\sqrt{2}}\right)$

3. $\cos\left(\arccos(-1)\right)$

4. $\arccos\left(\cos\left(\dfrac{7\pi}{6}\right)\right)$

✍ **Example.** Use a right triangle to simplify the expression. Assume $x > 0$.

$$\sin\left(\arccos\left(\frac{x}{2}\right)\right)$$