11.4: Working with Taylor Series

Learning Objectives. Upon successful completion of Section 11.4, you will be able to...

- Answer conceptual questions involving Taylor series.
- Evaluate limits using Taylor series.
- Differentiate Taylor series.
- Find power series solutions to differential equations.
- Approximate definite integrals using Taylor series.
- Approximate real numbers using Taylor series.
- Evaluate infinite series.
- Identify functions represented by power series.

Working with Taylor Series

So far, we've manipulated series by *differentiation*, *integration*, *multiplication* by a value, and making a *substitution* for x. In this section, we'll see a few more ways we can work with Taylor series.

✍ **Example.** We can use series to evaluate limits. Use series to evaluate $\lim_{x\to\infty} \dfrac{x - \ln(1+x)}{x^2}$.

Note: $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$, for $-1 < x \leq 1$.

Common Maclaurin Series (Taylor Series Centered at $a = 0$)

We will often need to manipulate series using known power series. Some common Maclaurin series are included below for reference.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \leq 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \leq x < 1$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \leq 1$$

✍ **Example.** Use a known power series to find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$.

✍ **Example.** Use a known power series to find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$.

✍ **Example.** Identify the function represented by the power series $\sum_{k=0}^{\infty} 2^k x^{2k+1}$.

✍ **Example.** We can also use Taylor series to approximate integrals. Use a Taylor series to approximate the integral $\int_0^{0.35} \arctan x \, dx$. Retain as many terms needed to ensure that the error is less than $1/10^4$.

✍ **Example.** Lastly, we can use power series to solve *differential equations*. Find a power series for the solution of the differential equation $y'(t) - 3y = 10$ with initial condition $y(0) = 2$. Then identify the function represented by the power series.