5.3: The Fundamental Theorems of Calculus

Learning Objectives. Upon successful completion of Section 5.3, you will be able to...

- Answer conceptual questions involving the Fundamental Theorem of Calculus.
- Given a graph and areas of designated regions, evaluate area functions.
- Find and verify area functions.
- Evaluate definite integrals using the Fundamental Theorem of Calculus.
- Find areas bounded by functions.
- Evaluate derivatives of definite integrals.
- Evaluate area functions.
- Maximize values of definite integrals and solve integral equations.

The First Fundamental Theorem of Calculus

Recall from Section 4.9 that a function F is called an **antiderivative** of f on an interval $[a, b]$ if

$$F'(x) = f(x) \text{ for all } x \in [a, b].$$

First Fundamental Theorem of Calculus. If f is continuous on $[a, b]$, then the function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$, differentiable on $[a, b]$, and has derivative

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x) \text{ for all } x \in [a, b].$$

In other words, $F'(x) = f(x) \implies F$ is an antiderivative of f.

The First Fundamental Theorem of Calculus (FTC1) establishes that *differentiation* and *integration* are, in a sense, inverse operations, much like multiplication and division. For example...

✍ **Example.** Find the local extrema for $F(x) = \int_0^x \cos(t)\,dt$ on the interval $[0, 2\pi]$.

✍ **Example.** Find the intervals of increase/decrease and the intervals of concavity for

$$F(x) = \int_1^x \left(t^2 - 7t + 10\right)\,dt.$$

✍ **Example.** Find the equation of the line tangent to

$$F(x) = \int_2^x \left(3t^2 - t\right)\,dt \text{ at } x = 2.$$

Chain Rule for Integral Functions. Let f be a continuous function defined on $[a, b]$ and let u be a continuous function defined on $[a, b]$ so that $a \leq u(x) \leq b$ for all $x \in [a, b]$. Furthermore, let F be the function defined as follows.

$$F(x) = \int_a^{u(x)} f(t)\, dt$$

Then $F'(x) = f\big(u(x)\big) \cdot u'(x)$ for all $x \in [a, b]$.

✍ **Example.** Find the derivative of $F(x) = \displaystyle\int_0^{x^3} \left(2z - z^2\right)\, dz$.

✍ **Example.** Find the derivative of each of the following functions.

① $F(x) = \displaystyle\int_1^{\ln x} e^t\, dt$

② $F(x) = \displaystyle\int_{\tan x}^{0} \frac{1}{1+k^2}\, dk$

Where does the First Fundamental Theorem lead us? We will begin to see a pattern in the following example.

✍ **Example.** Use the First Fundamental Theorem of Calculus to evaluate $\displaystyle\int_2^5 \left(e^t + 2t\right)\, dt$.

The Second Fundamental Theorem of Calculus

We now introduce the Second Fundamental Theorem of Calculus (FTC2), sometimes referred to as the *Fundamental Theorem of Integral Calculus*.

Second Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$ and suppose that F is an antiderivative for f on the same interval. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Recall the definition of the definite integral as a limit of a Riemann sum.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, dx.$$

The Second Fundamental Theorem of Calculus now allows us to evaluate integrals without dealing with this clunky limit!

Notation for Evaluating Definite Integrals. Because we are subtracting two antiderivatives, the "$+C$" may be ignored, as $C - C = 0$.

✍ **Example.** We previously found that the area under $f(x) = x^2+1$ on $[0, 2]$ is $\dfrac{14}{3}$ through the limit of a Riemann sum. Using FTC2, now find this area directly.

✍ **Example.** Evaluate the following definite integrals.

1. $\displaystyle\int_{-1}^{3} (2+x)\,dx$

2. $\displaystyle\int_{-2}^{-1} \frac{2}{3x}\,dx$

3. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sec^2(x)\,dx$

4. $\displaystyle\int_{0}^{1000} dx$

✍ **Example (continued).** Evaluate the following definite integrals.

(5) $\displaystyle\int_4^9 \frac{\sqrt{u^3}-1}{\sqrt{u^3}}\,du$

(6) $\displaystyle\int_0^{\frac{\pi}{2}} (2-3\sin\theta)\,d\theta$

(7) $\displaystyle\int_1^{\sqrt{3}} \frac{4}{1+x^2}\,dx$

✍ **Example.** Solve the equation for x: $\displaystyle\int_0^x (4t-9)\ dt = -4.$

✍ **Example.** If $x > 0$, solve for x: $\displaystyle\int_x^{x^2} \frac{1}{t}\, dt = \pi$.

✍ **Example.** Evaluate the definite integrals.

① $\displaystyle\int_1^3 \left(x^2 + \frac{1}{x^2}\right) dx$

② $\displaystyle\int_1^5 2\sqrt{x-1}\, dx$

③ $\displaystyle\int_1^4 t\left(\sqrt{t} + t^{-2}\right) dt$

✍ **Example (continued).** Evaluate the definite integrals.

④ $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2}{\sqrt{1-j^2}}\,dj$

⑤ $\displaystyle\int_{2}^{1} \left(\frac{1}{x^4} - \frac{1}{\sqrt{x^5}}\right) dx$

✍ **Example.** Find a linear function f passing through $(0,-3)$ with $\displaystyle\int_0^1 f(x)\,dx = 1$.

✍ **Example.** Find the maximum and minimum values of $\int_0^x \cos(t)\, dt$ on the interval $(0, 2\pi)$.

Recall that, for FTC2 to apply, the function f must be continuous on the interval $[a, b]$. Because of this condition, note that we cannot use FTC2 to evaluate the following.

$$\int_{-1}^{2} \frac{1}{x}\, dx \neq \ln|x| \Big|_{-1}^{2} = \ln(2) - \ln(1) = \ln(2)$$

Why not?

Connection Back to Limits. The two major thematic elements of calculus, the *derivative* and the *integral*, share a common element: both are limits. In fact, calculus does not exist without the limit!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

Approximating Integrals

Broadly speaking, the main challenge of integration is finding an antiderivative. Sometimes, if the antiderivative is difficult to find, we may approximate a definite integral using the following method.

Integral Approximation. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx \approx f(a)(b-a) \text{ so long as } b \text{ is "close" to } a.$$

✍ **Example.** Calculate an approximation to $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x^2\cos(x)\,dx$.

Note: This integral requires a double application of "integration by parts" (MATH 1080).