

10.6: Alternating Series

Learning Objectives. Upon successful completion of Section 10.6, you will be able to...

- Answer conceptual questions involving alternate series.
- Apply the alternating series test if possible. Otherwise, apply a different appropriate test.
- Use Theorem 10.18 to find an upper bound for the error in using the n^{th} partial sum to estimate the value of the series.
- Determine the number of terms needed to ensure a given error.
- Estimate the value of an alternating series.
- Determine if a series converges absolutely, converges conditionally, or diverges.

Introduction

Both the Integral Test and the Comparison Tests apply only to series with *positive terms*. An important type of series with positive and negative terms is called an **alternating** series.

 **Examples.** Consider the terms of each of the following series.

$$\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} e^{2/n}$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Definition. An **alternating series** is an infinite sum of the form

The terms of an alternating series will *alternate* between positive and negative values.

Alternating Series Test. Given an alternating series defined as above, if both of the following are true. . .

Thinking about the conditions of the Alternating Series Test. Is it possible to have a sequence that decreases but does not converge to 0? Give an example or explain why not.

Is it possible to have a sequence that converges to 0 but doesn't decrease? Give an example or explain why not.

Idea behind the proof of the Alternating Series Test. Why do the conditions of the test imply convergence?

▮ **Examples.** Determine whether the following series converge or diverge.

① The **alternating harmonic series**: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

▮ Examples (continued).

$$\textcircled{2} \sum_{n=1}^{\infty} (-1)^{n+1} e^{2/n}$$

$$\textcircled{3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$$

✚ **Examples (continued).**

$$\textcircled{4} \sum_{n=0}^{\infty} \frac{\sin \left(\left(n + \frac{1}{2} \right) \pi \right)}{1 + \sqrt{n}}$$

Estimating Sums

Recall that the error in estimating the sum S of a convergent series by S_n is the **remainder** $R_n = S - S_n$. The Integral Estimation Theorem allows us to find the bounds on the remainder,

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$ and $f(n) = a_n$. This theorem will not work for alternating series because the terms are, by definition, not always positive, so we'll need a different theorem.

Alternating Series Estimation Theorem. If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies (1) $b_{n+1} \leq b_n$ and (2) $\lim_{n \rightarrow \infty} b_n = 0$, then...

✎ **Example.** Consider the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$.

- (a) If we use the sum of the first three terms to estimate the sum of the series, what is the bound on the error?

- (b) How many terms of the series do we need to add in order to estimate the sum so that $|\text{error}| < 1/10^6$?

Absolute Convergence and Conditional Convergence

So far, we have only considered *whether* a series converges or diverges, but now we will also consider the *type* of convergence. The types of convergence we're interested in has to do with the relationship between the convergence/divergence of $\sum a_n$ and $\sum |a_n|$.

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Important Notes:

- If a series $\sum a_n$ is absolutely convergent, then it is convergent.
- This means that if $\sum |a_n|$ converges, then $\sum a_n$ converges.
- So if $\sum a_n$ is absolutely convergent, then both $\sum |a_n|$ and $\sum a_n$ converge.
- **Caution:** If $\sum a_n$ converges, $\sum |a_n|$ may converge or diverge.

Definition. A series $\sum a_n$ is called **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Steps to Determine if a Series is Absolutely Convergent, Conditionally Convergent, or Divergent.

- ① Look at $\sum |a_n|$. Determine if this series converges or diverges.
 - (a) If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent and we are done.
 - (b) If $\sum |a_n|$ diverges, then go to Step 2.
- ② Look at a_n . Determine if this series converges or diverges.
 - (a) If $\sum a_n$ converges, then $\sum a_n$ is conditionally convergent.
 - (b) If $\sum a_n$ diverges, then $\sum a_n$ is divergent.

✚ **Examples.** Determine if each of the following series is absolutely convergent, conditionally convergent, or divergent.

$$\textcircled{1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

$$\textcircled{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

▮ **Examples (continued).** Determine if each of the following series is absolutely convergent, conditionally convergent, or divergent.

$$\textcircled{3} \sum_{n=1}^{\infty} (-1)^n \arctan(n)$$

$$\textcircled{4} \sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$$