8.4: Trigonometric Substitutions

Learning Objectives. Upon successful completion of Section 8.4, you will be able to...

- Answer conceptual questions involving trigonometric substitutions.
- Evaluate indefinite integrals involving trigonometric substitution.
- Evaluate definite integrals involving trigonometric substitution.
- Find area and volume for regions involving integrals requiring trigonometric substitution.
- Complete the square to solve integrals involving trigonometric substitution.

Introduction

The integration technique we will learn in this section, **trigonometric substitution**, is useful for integrating functions containing expressions of the form $a^2 - x^2$, $x^2 + a^2$, and $x^2 - a^2$, often under a root, where a is a constant real number.

Examples

Expressions of the Form " $a^2 - x^2$ "

Expressions of the Form " $x^2 + a^2$ "

Expressions of the Form " $x^2 - a^2$ "

In trig substitution, we introduce a trig function into our problem in order to use the Pythagorean identities. This will allow us to simplify the integral into something that we can evaluate using other techniques, often those from Section 8.3.

✍ **Example.** Consider $\int x\sqrt{1-x^2}\,dx$ versus $\int \sqrt{1-x^2}\,dx$.

How do our approaches to evaluating these two integrals differ?

Key Ideas for Trig Substitution

We can think about trig substitution as an "inverse substitution."

Note that, in the previous example, if we let $x = \sin\theta$, then $\theta = \arcsin(x)$. For our trig substitution to result in an equivalent integral, we need for our function to be **one-to-one**, which means restricting the domain of the trig function (i.e., the θ-values).

Form	**Substitution**	**Identity**
" $a^2 - x^2$ "		
" $x^2 + a^2$ "		
" $x^2 - a^2$ "		

✍ **Example.** $\displaystyle\int \frac{dx}{x\sqrt{5-x^2}}$

✍ **Example.** $\displaystyle\int \frac{dt}{\sqrt{t^2 - 6t + 13}}$

✍ **Example.** $\displaystyle\int \frac{x}{\sqrt{x^2+10x+16}}\,dx$

✍ **Example.** $\displaystyle\int_0^{2/3} \frac{x^2}{\sqrt{4-9x^2}}\,dx$