# 12.1: Parametric Equations

**Learning Objectives.** Upon successful completion of Section 12.1, you will be able to...

- Graph and describe parametric equations and eliminate the parameter to find equations in $x$ and $y$.
- Find parametric equations for a given description of a curve.
- Solve applications involving parametric equations.
- Differentiate parametric equations.
- Find the slopes and equations of tangent lines to a parametric equation.
- Find the arc length of a parametric curve.
- Answer conceptual questions involving parametric equations.
- Find the area under a parametric curve.
- Find the area of a surface of revolution of a parametric curve.

## Introduction

A **parametric curve** $C$ is a curve in the $xy$-plane described by three variables: $x$, $y$, and a *parametric variable* (often $t$ or $\theta$).

- We call $x = f(t)$ and $y = g(t)$ **parametric equations**, where $t$ is the parameter.
- As $t$ varies, the point $(x, y) = \big(f(t), g(t)\big)$ varies and traces out the curve $C$.

The curve $C$ is traced out in a specific direction. When sketching parametric curves, we use arrows to indicate this direction, called the **orientation** of the curve.

✍ **Example.** Consider the parametric equations $x = 1 - t^2$ and $y = t - 2$ for $-2 \leq t \leq 2$.

(a) Eliminate the parameter to obtain an equation in $x$ and $y$.

(b) Sketch the parametric curve, indicating the positive orientation (increasing $t$).

✍ **Example.** Consider the equations $x = \sin\left(\frac{\theta}{2}\right)$ and $y = \cos\left(\frac{\theta}{2}\right)$ for $-\pi \leq \theta \leq \pi$.

(a) Eliminate the parameter to obtain an equation in $x$ and $y$.

(b) Sketch the parametric curve, indicating the positive orientation.

## Finding Parametric Equations for a Curve

**Parametric Equations of a Circle.** The parametric equations

$$x = x_0 + a\cos(bt) \text{ and } y = y_0 + a\sin(bt)$$

describe all or part of the **circle** $(x - x_0)^2 + (y - y_0)^2 = a^2$ centered at $(x_0, y_0)$ with radius $|a|$. If $b > 0$, then the circle is generated in the counterclockwise direction.

if $x = x_0 + a\sin(bt)$ and $y = y_0 + a\cos(bt)$ with $b > 0$, then the circle is generated in the clockwise direction.

✍ **Example.** Find parametric equations for a circle centered at $(2, 3)$ with a radius of 1, generated counterclockwise.

What parametric equations correspond to only the lower half of the circle?

**Parametric Equations of a Line.** The parametric equations

$$x = x_0 + at \text{ and } y = y_0 + bt, \text{ for } -\infty < t < \infty,$$

where $x_0$, $y_0$, $a$, and $b$ are constants with $a \neq 0$, describe a **line** with a slope $\dfrac{b}{a}$ passing through the point $(x_0, y_0)$. If $a = 0$ and $b \neq 0$, the line is vertical.

✍ **Example.** Find parametric equations for the line segment starting at $P(-1, -3)$ and ending at $Q(6, -16)$.

✍ **Example.** Find parametric equations for the complete parabola $y = 2x^2 - 4$.

What parametric equations correspond to the segment of the parabola where $-1 \leq x \leq 5$?

✍ **Example.** The tip of the 15-inch second hand of a clock completes one revolution in 60 seconds. Find the parametric equations that describe the circular path of the tip of the second hand. Assume $(x, y)$ denotes the position relative to the origin at the center of the circle.

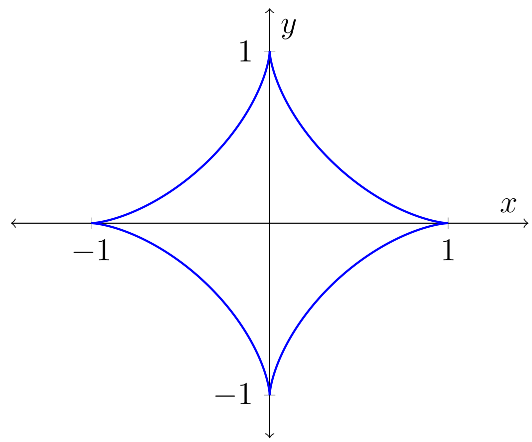
## Calculus in Parametric Equations

Suppose we have a set of parametric equations $x = f(t)$ and $y = g(t)$ describing a curve $C$. If $f$ and $g$ are differentiable, how do we find the slope of the tangent line to a point on $C$?

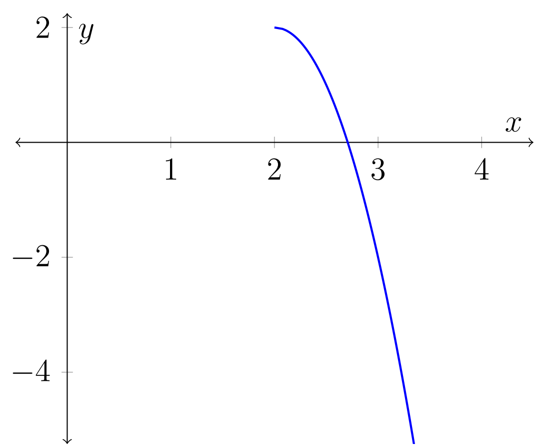
**Derivatives for Parametric Equations.**

✍ **Example.** A curve is defined by the parametric equations $x = e^t$, $y = te^{-t}$. Find $\dfrac{dy}{dx}$.

✍ **Example.** A curve is defined by the parametric equations $x = \sin^3 t$, $y = \cos^3 t$, $0 \leq t \leq 2\pi$. Find an equation of the tangent line at the point where $t = \pi/6$.



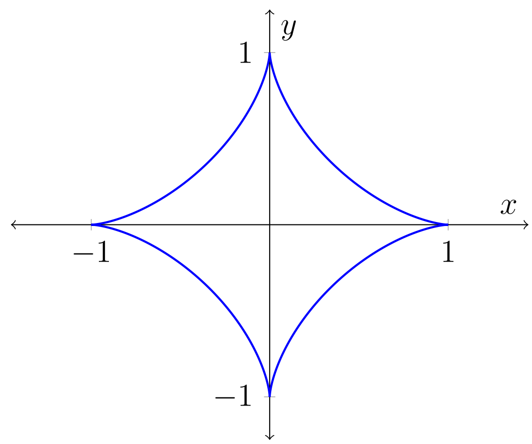

✍ **Example.** Find all points at which the curve $x = 2 + \sqrt{t}$, $y = 2 - 4t$ has the slope $-8$.



**Arc Length for Parametric Equations.** Suppose a curve $C$ is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $\frac{dx}{dt} > 0$, $f'$ and $g'$ are continuous, and $C$ is transversed once as $t$ increases from $\alpha$ to $\beta$.

Then the **arc length *L*** of the curve $C$ is

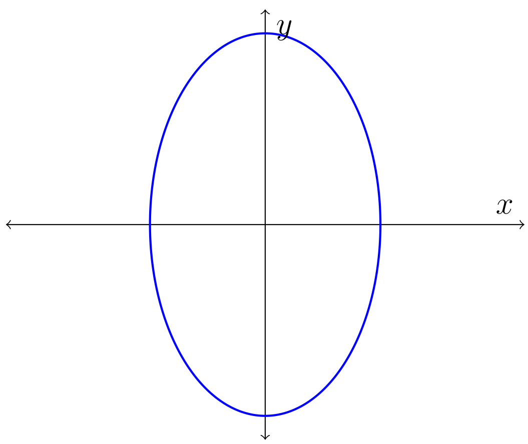
✍ **Example.** Find the total length of the astroid curve $x = \cos^3\theta$, $y = \sin^3\theta$, which is traced out once for $0 \leq \theta \leq 2\pi$.



**Area and Parametric Equations.** Suppose $y = h(x)$ is nonnegative and continuous on $[a, b]$, implying that the area bounded by the graph of $h$ and the $x$-axis on $[a, b]$ equals $\displaystyle\int_a^b h(x)\,dx = \int_a^b y\,dx.$

If the graph of $y = h(x)$ on $[a, b]$ is traced exactly once by the parametric equations $x = f(t)$, $y = g(t)$, for $\alpha \le t \le \beta$, then (by substitution) the area bounded by $h$ is

✍ **Example.** Use parametric requations of an ellipse centered at the origin $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.



**Surface Area and Parametric Equations.** Let $C$ be the curve $x = f(t)$, $y = g(t)$, for $\alpha \leq t \leq \beta$, where $f'$ and $g'$ are continuous, and $C$ does not intersect itself (except possibly at its endpoints).

If $g$ is nonnegative on $[\alpha, \beta]$, then the area of the surface obtained by revolving $C$ around the $x$-axis is

Likewise, if $f$ is nonnegative on $[\alpha, \beta]$, then the area of the surface obtained by revolving $C$ about the $y$-axis is

✍ **Example.** Find the exact area of the surface obtained by rotating the curve given by $x = \cos^3\theta$, $y = \sin^3\theta$, $0 \leq \theta \leq \pi/2$ about the $x$-axis.