11.3: Taylor Series

Learning Objectives. Upon successful completion of Section 11.3, you will be able to...

- Answer conceptual questions involving Taylor series.
- Find the Taylor series and interval of convergence for functions centered at a.
- Find the Taylor series for a given function centered at a.
- Manipulate a given Taylor series to find the Taylor series for a function centered at 0.
- Find remainders for Taylor series and show the remainder for certain Taylor series goes to 0 as n goes to infinity for all x in the interval of convergence.

Taylor and Maclaurin Series

In the previous sections, we learned about power series and nth-order Taylor polynomials. We'll now combine these two ideas by formally defining **Taylor series**, which are power series where the coefficients have a specific form.

Definition. Suppose the function f has derivatives of all orders on an interval containing the point a. The **Taylor series for f centered at a** is

Definition. A **Maclaurin series** for a function f is the Taylor series for f centered at the point $a = 0$.

Notes about Taylor Series.

- If f has a power series representation at a, then that power series must be the Taylor series of f centered at a.
- It is possible for a function to not equal its Taylor series.
- For Taylor series to be useful, we need to know...
  - The values of x for which the Taylor series converges (the interval of convergence).
  - The values of x or which the Taylor series for f equals f.

Finding the Interval of Convergence for a Taylor Series

✍ **Example.** Find the Maclaurin series for $f(x) = e^x$. Also find the interval of convergence.

✍ **Example.** Find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$. Also find the interval of convergence.

Determining When a Function Equals Its Taylor Series

Recall from Section 11.1 that the **remainder** when using a Taylor polynomial p_n to approximate f is

$$R_n(x) = f(x) - p_n(x).$$

We can use the remainder to to determine when a function f is equal to its Taylor series representation.

Theorem: Convergence of Taylor Series. Let f have derivatives of all orders on an open interval containing the point a. The Taylor series for f centered at a **converges to** f for all x in the interval if and only if

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is the remainder at x, with c between x and a.

Taylor's Inequality. If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then

$$\left|R_n(x)\right| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \text{ for } |x-a| < d.$$

✍ **Example.** Prove that e^x is equal to its Maclaurin series for all x.

✍ **Example.** Prove that $\cos x$ is equal to its Taylor series centered at $a = \pi$ for all x.

Manipulating Taylor Series

We can use the tools from the previous section to manipulate Taylor series like any other power series.

✍ **Example.** Use the Taylor series $\arctan(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, for $|x| \leq 1$, to find the first four nonzero terms of the Taylor series for the function $x \arctan\left(x^2\right)$ centered at 0.