2.4: Infinite Limits

Learning Objectives. Upon successful completion of Section 2.4, you will be able to...

- Answer conceptual questions involving infinite limits and vertical asymptotes.
- Find infinite limits numerically or graphically.
- Sketch graphs or functions involving infinite limits.
- Evaluate limits analytically.
- Find vertical asymptotes.

Introduction to Infinite Limits

Definition. In an **infinite limit**, the dependent variable (y-value) becomes "boundless" in the positive or negative direction as the dependent variable (x-value) approaches some finite value.

For example, if $f(x) \to \infty$ as $x \to c$, we will write the following.

$$\lim_{x \to c} f(x) = \infty$$

Remark. Technically, such limits do not exist. However, this particular case of the limit not existing provides valuable information about the behavior of a function, so we make a minor exception in our notation to indicate this special case.

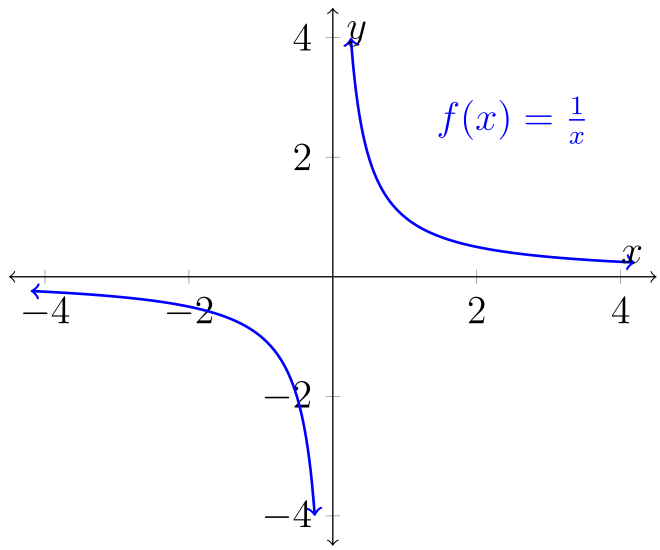
Infinite limits will occur in cases when attempting direct substitution gives a zero in the *denominator only.*

Example. Consider $f(x) = \dfrac{1}{x}$. What happens as $x \to 0^+$?

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$1/x$	1	10	100	1,000	10,000	100,000

In limit terms, we would say that $\displaystyle\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Similarly, we could numerically show that $\displaystyle\lim_{x \to 0^-} \frac{1}{x} = -\infty$.



Unfortunately, there is no "good" and "compact" way of expressing our way through such limits. In MATH 1060, we use the following notational conventions.

Remark. Let $c \in \mathbb{R}$ with $c > 0$. We use the following notation.

$$\frac{c}{\text{small } +} = +\infty$$

$$\frac{c}{\text{small } -} = -\infty$$

Summary. Direct substitution is always a reasonable first step when trying to evaluate a limit. So far, we know that if direct substitution...

- gives a real number, then no further work is needed to evaluate a limit.
- gives the $\frac{0}{0}$ indeterminate form, the limit usually exists, and we can apply techniques from Section 2.3 to evaluate it.
- gives 0 in the denominator only, we have an infinite limit. This limit technically does not exist, but we will investigate it anyway.

Strategy for Evaluating Infinite Limits

If we know we have an infinite limit, we can use the following general strategy to determine whether the $f(x)$ values approach $+\infty$ or $-\infty$. For a limit $\lim_{x\to c} f(x)$...

1. Choose a test point close to c based on the direction from which x is approaching. (Choose a value smaller than c if $x \to c^-$ and a value larger than c if $x \to c^+$.)

2. Plug the test point into the denominator to determine if the denominator approaches zero "through the positives" (small $+$) or "through the negatives" (small $-$).

✍ **Example.** Evaluate the following limits.

$$\lim_{x \to 2^-} \frac{x+3}{x-2} \qquad \text{and} \qquad \lim_{x \to 2^+} \frac{x+3}{x-2}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x \to 0^+} \frac{x^2 - 3x + 2}{x^3 - 2x^2}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x \to 0^+} \frac{1 + 2x}{x^2}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{j \to 0^+} \frac{8^j}{1 - 3^j}$$

Vertical Asymptotes

Infinite limits provide information about vertical asymptotes.

Definition. The line $x = c$ is called a **vertical asymptote** (V.A.) of a function f if any one of the limits as x approaches c is infinite.

For example, we saw earlier in this section that $\lim_{x \to 0^+} \frac{1}{x} = \infty$. We can thus say that $x = 0$ is a vertical asymptote of $f(x) = \frac{1}{x}$.

✍ **Example.** Find (and verify) the vertical asymptotes of $f(x) = \frac{x+1}{2x^2 + x - 3}$.

✍ **Example.** Prove that $x = \frac{\pi}{2}$ is a vertical asymptote for $y = \tan x$.

✍ **Example.** Evaluate each of the following limits.

① $\displaystyle\lim_{x \to 0^-} \frac{x^2+1}{\sin x}$

② $\displaystyle\lim_{x \to 2^+} \frac{-x}{5^x - 25}$

③ $\displaystyle\lim_{t \to \frac{3}{2}^+} \frac{t^2+6t-7}{2t-3}$

④ $\displaystyle\lim_{x \to e^-} \frac{-\ln x}{x-e}$