4.9: Antiderivatives

Learning Objectives. Upon successful completion of Section 4.9, you will be able to...

- Answer conceptual questions involving antiderivatives.
- Find all antiderivatives of a function.
- Determine the indefinite integral of a function.
- Given a function, find the antiderivative satisfying a given condition.
- Given the derivative of a function, find the function satisfying an initial value.
- Graph the solutions to a differential equation, then the particular solution given an initial value.
- Find a position function given a velocity function and an initial position or an acceleration function and an initial velocity and an initial position.
- Solve applications involving derivatives.

Introduction

The two major concepts in calculus that we have learned thus far are the *limit* and the *derivative*. The third major concept, introduced in this section, is that of the *antiderivative*.

Definition. The function F is called an **antiderivative** of f if $F'(x) = f(x)$. In other words, an antiderivative F is a function whose derivative is a given function f.

✍ **Example.** Let's list some antiderivatives for the function $f(x) = 2x$.

The General Antiderivative

Theorem. Let F be an antiderivative of f. Then all antiderivatives of f have the form $F(x) + C$, where $C \in \mathbb{R}$ is an arbitrary constant.

✍ **Example.** Find the general antiderivative for $f(x) = 2x$.

✍ **Example.** Find the antiderivatives of the following functions.

1. $y' = 4x^3 + 3x^2 + 2x + 1$

2. $f(x) = \dfrac{2}{1+x^2}$

3. $f(x) = \cos x - \sin x + \sec^2 x$

4. $f(x) = 3e^x + 3^x \ln 3 + 3$

Indefinite Integrals

Recall that the differential operator $\dfrac{d}{dx}$ instructs us to take the derivative of the function under consideration.

$$\frac{d}{dx}(\arctan 2x) = \frac{2}{1+4x^2}$$

The operation of **antidifferentiation** is denoted by an **indefinite integral**.

$$\int f(x)\,dx = F(x) + C, \text{ where } F'(x) = f(x)$$

The integral symbol $\int$ must always be paired with the differential dx (or a differential that corresponds to another variable, such as dt or $d\theta$). We will explain what this differential represents in Chapter 5, but for now, know that this notation instructs us to find the general antiderivative for a function f.

$$\int f(x)\,dx \iff \text{“find the general antiderivative of } f$$

We can thus write $\displaystyle\int \frac{2}{1+4x^2}\,dx = \arctan 2x + C$.

✍ **Example.** Evaluate the following indefinite integrals.

① $\displaystyle\int 2x\,dx$

② $\displaystyle\int (3+e^x)\,dx$

③ $\displaystyle\int \left(\sec\theta\tan\theta + \sec^2\theta\right) d\theta$

Antiderivative Rules

We can find the antiderivative of a **constant** $k \in \mathbb{R}$.

$$\int k\,dx = kx + C$$

Antidifferentiation is a **linear operation**, i.e. for $a, b \in \mathbb{R}$,

$$\int \big(af(x) \pm bg(x)\big)dx = aF(x) \pm bG(x) + C.$$

To find the antiderivative of a **power term**, we reverse the power rule for derivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ where } n \neq -1$$

✍ **Example.** Evaluate the indefinite integrals.

1. $\displaystyle\int \big(x^5 + x^3 + x\big)\,dx$

2. $\displaystyle\int \left(\frac{3}{\sqrt{x}} + \frac{2}{\sqrt{x^3}}\right)dx$

To handle powers of negative one, we use the following rule.

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

The necessity of the absolute value will become clear later on.

✍ **Example.** Evaluate the indefinite integral $\int \left(\frac{1}{x^3} + \frac{2}{x^2} + \frac{3}{x} \right) dx$.

Common Antiderivative Mistakes. Be mindful of the following errors when evaluating indefinite integrals.

One mistake is to attempt to apply the "powers of negative one" rule from the previous page to other negative powers.

$$\int x^{-3}\, dx = \int \frac{1}{x^3}\, dx \neq \ln|x^3| + C$$

Another temptation is to attempt to find the antiderivative of a product or quotient of functions in the manner below. Unfortunately, antiderivatives of products and quotients do not "behave nicely." Unless the integrand can be simplified, we must use alternative methods to find the antiderivatives of such functions, which are taught in MATH 1080.

$$\int f(x)g(x)\, dx \neq F(x)G(x) + C$$

$$\int \frac{f(x)}{g(x)}\, dx \neq \frac{F(x)}{G(x)} + C$$

If presented with an indefinite integral of a product or quotient of functions in MATH 1060, you should **simplify** the integrand to find the antiderivative.

✍ **Example.** Evaluate the indefinite integral.

$$\int t^2 \left(1 - \frac{1}{t^2} + \frac{2}{t^3} \right) dt$$

✍ **Example.** Evaluate the indefinite integral.

$$\int \frac{x^3 - x^2 - 2x}{x^2 - 2x}\, dx$$

✍ **Example.** True or False: $\displaystyle\int \ln x \, dx = x \ln x + x + C$

✍ **Example.** Evaluate the indefinite integral.

$$\int \frac{3 + 2\sqrt{1 - x^2}}{\sqrt{1 - x^2}}\, dx$$

✍ **Example.** Evaluate the indefinite integral.

$$\int \left(t^2+1\right)(2t-5)\ dt$$

✍ **Example.** Evaluate the indefinite integral.

$$\int \frac{\cos x - 1}{\sin^2 x}\, dx$$

Introduction to Differential Equations

Antiderivatives are a critical element of solving differential equations.

Definition. An equation involving an unknown function and one or more of its derivatives is called a **differential equation**.

For example, the following is a differential equation.

$$4x^2y'' + 12xy' + 3y = 0$$

We can see that the equation on the previous page involves both the first and second derivatives of some function y, but we don't know which function y actually is. To solve a differential equation, we need to identify the function that satisfies such an equation. There are entire courses dedicated to differential equations, but we'll learn how to solve some basic ones in our course.

✍ **Example.** Solve the differential equation $2y' - 3 = \cos t$.

Initial Value Problems. If an initial condition is given (i.e. a particular value of the unknown function), it is possible to solve for the constant C. A differential equation with an initial condition is referred to as an **initial value problem**.

✍ **Example.** Solve the initial value problem.

$$e^x(y' - 1) = e^{2x} \qquad\qquad y(0) = 3$$

Motion Problems Revisited. Recall that, given a position function $s = f(t)$, we can find velocity and acceleration by taking the first and second derivatives of s, respectively. With the concept of antiderivatives, we can now see the following relationships.

$$v(t) = \frac{ds}{dt} = s'(t) \implies s(t) = \int v(t)\,dt$$

$$a(t) = \frac{d^2s}{dt^2} = s''(t) = v'(t) \implies v(t) = \int a(t)\,dt$$

Given initial conditions, we can "recover" the position function from an acceleration function.

✍ **Example.** If $a(t) = -32$ ft/s^2, find the velocity and position functions, s and v, given that $v(0) = 2$ ft/s and $s(0) = 10$ feet.