8.5: Partial Fractions

Learning Objectives. Upon successful completion of Section 8.5, you will be able to...

- Answer conceptual questions involving partial fractions.
- Set up or find a partial fraction decomposition.
- Evaluate integrals involving partial fractions with only simple linear factors.
- Evaluate integrals involving partial fractions with repeated linear factors.
- Evaluate integrals involving partial fractions with irreducible quadratic factors (simple or repeated).
- Evaluate integrals involving improper rational functions (where long division is needed first).
- Use partial fractions to find areas, volumes, or arc lengths.
- Evaluate integrals involving partial fractions that require a preliminary step (such as a change of variables).

Introduction

Consider the integral $\int \frac{x+5}{x^2+x-2}\,dx$. Will any of our previous integration techniques work?

- Antiderivative Rules
- U-Substitution
- Algebraic Manipulation
- Integration by Parts
- Trig Integral
- Trig Substitution

We can factor the denominator and write $\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$, for some $A, B \in \mathbb{R}$.

This will result in two "simpler" integrals that we can evaluate.

The Idea of Partial Fraction Decomposition

The technique of **partial fraction decomposition** allows us to rewrite a rational function as a sum of two simpler rational functions. Recall that a rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials.

When can we use partial fraction decomposition (PFD) for a rational function? To determine this, we look at the degree of the numerator (n) and degree of the denominator (k).

- PFD can be used when we have a **proper** rational function ($n < k$).
- If the rational function is **improper** ($n \geq k$), perform long division before PFD.

✍ **Example.** Consider the improper rational function $\dfrac{x^4 + x + 5}{x^3 + 3}$.

Steps for Partial Fraction Decomposition

Given a rational function...

1. Determine if the function is proper or improper. If improper, do long division.
2. If proper, factor the denominator as much as possible.
3. Determine the *form* of the proper rational function based on the denominator.
    - Simple linear form
    - Repeated linear form
    - Irreducible quadratic form
    - Repeated irreducible quadratic
4. Solve for the constants needed.
5. Rewrite the integral and evaluate.

Forms in Partial Fraction Decomposition

1 Simple Linear

✍ **Example.** $f(x) = \dfrac{3}{x^3 - x^2 - 12x}$

2 Repeated Linear

✍ **Example.** $f(x) = \dfrac{x^2 - 5x + 16}{(2x+1)(x-2)^2}$

✍ **Example.** $f(x) = \dfrac{x+2}{x^2(x+1)^3(2x+3)}$

3 Irreducible Quadratic

✍ **Example.** $f(x) = \dfrac{10}{(x-1)(x^2+9)}$

✍ **Example.** $f(x) = \dfrac{3+x}{x^2(2x^2-x-1)}$

④ **Repeated Irreducible Quadratic**

✍ **Example.** $f(x) = \dfrac{1}{x(x^2+4)^2}$

✍ **Example.** Write out the form of the partial fraction decomposition of the function:

1. $f(x) = \dfrac{x^4}{(x+3)^3(x^2-x+1)}$

2. $f(x) = \dfrac{3x}{(x-1)(x^4+4x^2+4)}$

Integrating Rational Functions Using Partial Fractions

✍ **Example.** $\displaystyle\int \frac{3}{x^3 - x^2 - 12x}\,dx$

(1) Is the integrand a proper rational function? How do we know?

(2) Factor the denominator as fully as possible.

(3) Identify the form of the partial fraction decomposition.

(4) Find the constants.

✍ **Example (continued).** $\displaystyle\int \frac{3}{x^3 - x^2 - 12x}\,dx$

4 Find the constants (continued from previous page).

5 Rewrite the integral and evaluate.

✍ **Example.** $\displaystyle\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2}\,dx$

✍ **Example.** $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

✍ **Example.** $\displaystyle\int \frac{1}{x(x^2+4)^2}\,dx$

How can **u-substitution** be used to convert an integrand into a rational function so we can apply partial fraction decomposition?

✍ **Example.** $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx$

✍ **Example.** $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$