5.4: Applications of Integration

Learning Objectives. Upon successful completion of Section 5.4, you will be able to...

- Answer conceptual questions involving symmetry and average value of functions.

- Use symmetry to evaluate definite integrals.

- Find average value of functions over given intervals.

- Solve applications involving average value of functions.

- Solve problems involving the Mean Value Theorem for Integrals.

Average of a Function

Recall from previous courses that the **average** of a finite set $\mathcal{S}$ can be found by summing the values in the set and dividing by the number of values in the set.

$$\mathcal{S} = \{1, 2, 3, 4, 5\} \implies \mathcal{S}_{\text{avg}} = \frac{1}{|\mathcal{S}|} \sum_{i=1}^{5} s_i = \frac{1 + 2 + 3 + 4 + 5}{5} = 3$$

How could we calculate the average of a *function* f (i.e. the average y-value) on a closed interval $[a, b]$?

Definition.   The **average value** of a continuous function f on the interval $[a, b]$ is

Geometric Interpretation.   If $f > 0$ on $[a, b]$, then $\overline{f}$ gives the height of a rectangle with base $b - a$ whose area matches the area under the curve.
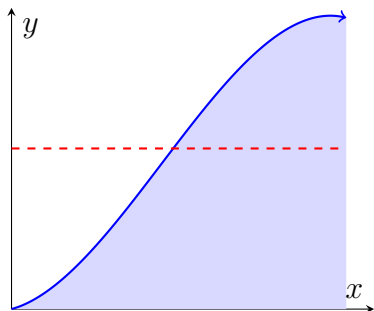
✍ **Example.**   Find the average value of $f(x) = \cos x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

✍ **Example.**   If f is linear on $[a, b]$, show that $\overline{f} = f\left(\dfrac{a + b}{2}\right)$. Use this result to find the average of $f(x) = 8x - 3$ on $[0, 1]$.

✍ **Example.** If $f(x) = ax^2 + 1$, find $a \in \mathbb{R}$ so that $\overline{f} = \pi$ on $[-1, 1]$.

Mean Value Theorem for Integrals

In Section 4.2, we learned about the Mean Value Theorem for functions. We can extend this theorem to integrals now that we have the definition of the average value of a function.

Mean Value Theorem for Integrals (MVTI). Let f be continuous on $[a, b]$. Then there exists a point $c \in (a, b)$ such that $f(x) = \overline{f}$.

✍ **Example.** Apply the MVTI to $f(x) = 4 - x^2$ on the interval $[-2, 0]$.

Integrating Functions with Symmetry

A function is called **even** if it has y-axis symmetry.

A function is called **odd** if it has origin symmetry.

We can take advantage of symmetry to simplify integration.

Parity of f and F. Suppose F is an antiderivative of f. If f is even, then F is odd. Alternatively, if f is odd, then F is even.

Even and Odd Integrands. Let $a > 0$ and let f be integrable on $[-a, a]$.

- If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

- If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Remark. The sine function is odd while the cosine function is even.

$$\sin(-x) = -\sin x \qquad\qquad \cos(-x) = \cos x$$

✍ **Example.**   Evaluate the following definite integrals.

① $\displaystyle\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sin x \, dx$

② $\displaystyle\int_{-4}^{4} (4 - x^2) \, dx$