10.3: Infinite Series

Learning Objectives. Upon successful completion of Section 10.3, you will be able to...

- Answer conceptual questions involving infinite series.
- Evaluate geometric series.
- Solve applications involving series.
- Write repeating decimals as series and fractions.
- Evaluate telescoping series.
- Evaluate an infinite series.

Introduction

In Section 10.1, an infinite series was defined as a sum of infinitely many terms. More formally, given a sequence $\{a_1, a_2, a_3, \dots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**.

The **sequence of partial sums** $\{S_n\}$ associated with this series has the terms...

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{ for } n = 1, 2, 3, \dots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\lim\limits_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

✍ **Example.** Determine if the series $\displaystyle\sum_{i=2}^{\infty} \frac{2}{i^2-1}$ converges or diverges.

If it converges, determine the value to which it converges. (Note: This is a special type of series called a **telescoping series**.)

Geometric Series Test

The geometric series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots, a \neq 0 \ldots$

converges if ________________________.

diverges if ________________________.

✍ **Examples.** For each of the following, determine if the series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

✍ **Examples (continued).** For each of the following, determine if the series converges or diverges. If it converges, find its sum.

$0.3 + 0.03 + 0.003 + 0.0003 + \cdots$

$3 - 4 + \frac{16}{3} - \frac{64}{9} + \ldots$

$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. Then the series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.

2. Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges, for any real number $c \neq 0$.

3. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. Then the series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.

4. Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum(a_k \pm b_k)$ diverges.

5. If M is a positive integer, then for $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=M}^{\infty} a_k$, either both converge or both diverge.

   - In general, *whether* a series converges or diverges does not depend on a finite number of terms added to or removed from the series.
   - The *value* of a convergent series, however, does change if nonzero terms are added or removed.

✍ **Example.** Evaluate the following series or state that it diverges.

$$\sum_{k=0}^{\infty}\left(3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right)$$