2.6: Continuity

Learning Objectives. Upon successful completion of Section 2.6, you will be able to...

- Answer conceptual questions involving continuity.
- Find points of discontinuity or intervals of continuity.
- Determine if functions are continuous at given values.
- Evaluate limits using continuity principles.
- Use the Intermediate Value Theorem to show equations have solutions on given intervals.
- Sketch graphs of continuous functions given information about their points of discontinuity.
- Solve applications involving continuity principles.
- Classify discontinuities.

Introduction to Continuity

Intuitively, a function is continuous if it is "well-connected" – that is, it has no holes, breaks, gaps, or jumps. To begin our discussion of continuity, we introduce the idea of continuity at a point.

Definition. A function f is **continuous at a point** $x = c$ if

If f is not continuous at $x = c$, then we call $x = c$ a **point of discontinuity** and say that f is **discontinuous** at $x = c$.

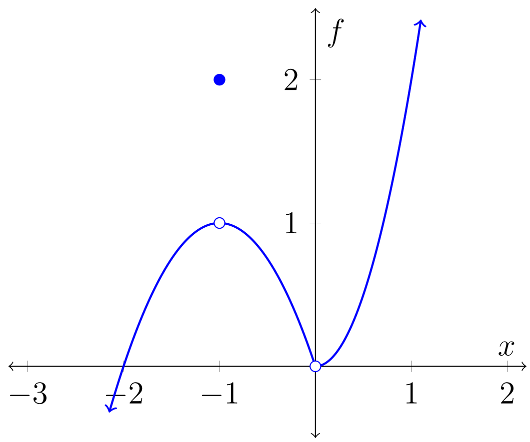
This definition of continuity at a point implies three conditions that must simultaneously hold for a point c in the domain of f.

1. $f(c) \in \mathbb{R}$
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

If any one of these conditions fails to hold, the function fails to be continuous at $x = c$.

Types of Discontinuities

Removable Discontinuities. Informally, we can think of removable continuities as minor holes in the graph that can be "removed" or "fixed."

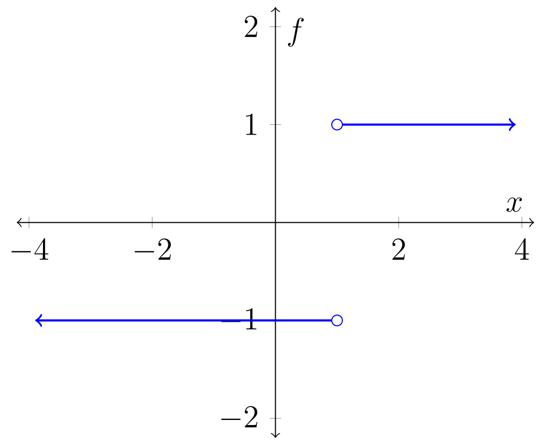


If we do not have a graph available, we can still identify this type of discontinuity. A removable discontinuity will occur at $x = c$ when $\lim_{x \to c} f(x)$ exists, but it is not equal to $f(c)$.

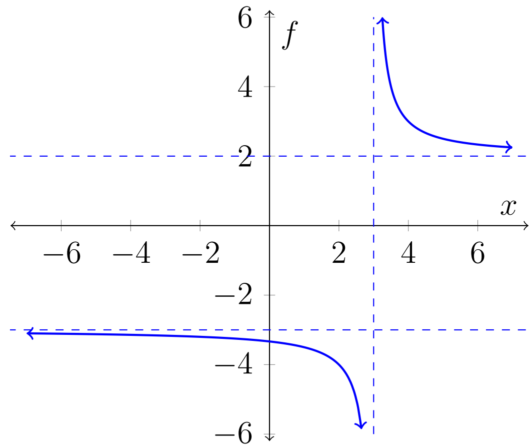
✍ **Example.** What type of discontinuity exists at $x = 0$ for $f(x) = \dfrac{x^2 \cos x}{x^2 e^x + x^2}$?

✍ **Example.** Redefine $f(x) = \dfrac{x-1}{x^2-1}$ so it is continuous at $x = 1$.

Jump Discontinuities. Jump discontinuities occur when a function approaches two different y-values as x approaches a certain point. In other words, we can identify this type of discontinuity where the left- and right-hand limits are not equal to one another.



Infinite Discontinuities. We have actually worked with infinite discontinuities before — these are our vertical asymptotes.

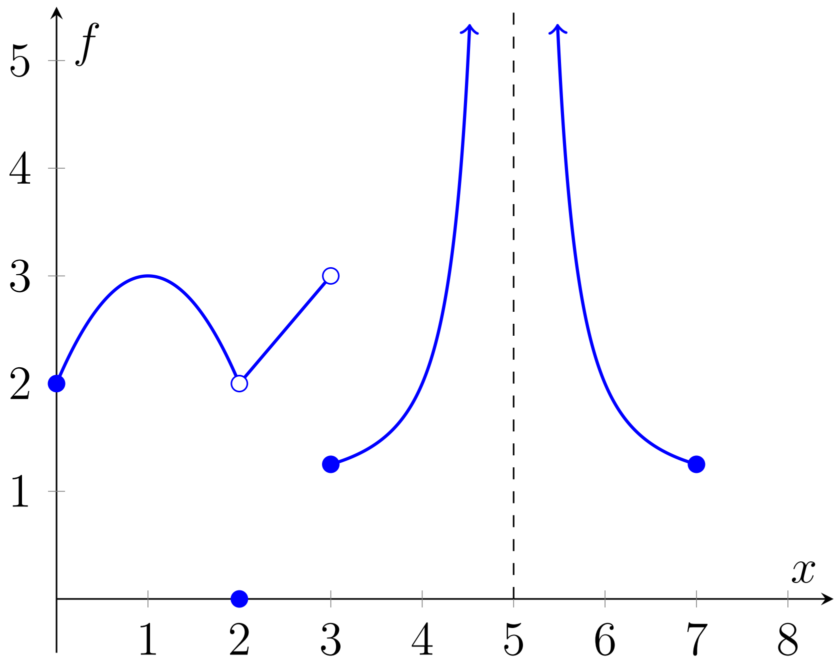


Continuity on an Interval

Definition. A function f is **right continuous** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.
A function f is **left continuous** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$.

Definition. A function f is **continuous on an interval** $\mathcal{I}$ if it is continuous (right, left, or both) for all $x \in \mathcal{I}$.

✍ **Example.** Identify the locations of discontinuities and classify them. Then determine the intervals for which the given function is continuous.



Continuous Functions

Definition. A function is called **continuous** if it is continuous (right, left, or both) for every x-value in its domain.

You've worked with continuous functions before! The following functions are continuous.

- Polynomials
- Rational Functions
- Root functions
- Trig and Inverse Trig Functions
- Exponential and Logarithmic Functions

Being familiar with types of functions that are continuous is valuable because you can use it as an argument to apply some of the tools and theorems from calculus.

✍ **Example.** Determine the intervals on which $f(x) = \dfrac{x}{2x-5}$ is continuous. Classify any discontinuities.

✍ **Example.** Determine the intervals on which $f(x) = \sqrt[4]{\ln x - 2}$ is continuous.

Remark. Note the following two properties of continuous functions.

- If f is continuous and one-to-one, then f^{-1} is continuous.
- If f and g are continuous, then $f \circ g$ and $g \circ f$ are continuous.

This leads us to the following result for taking the limit of composite functions.

Theorem. If g is continuous at $x = c$ and f is continuous at $g(c)$, then

$$\lim_{x\to c} f\left(g(x)\right) = f\left(\lim_{x\to c} g(x)\right).$$

✍ **Example.** Evaluate the following limit.

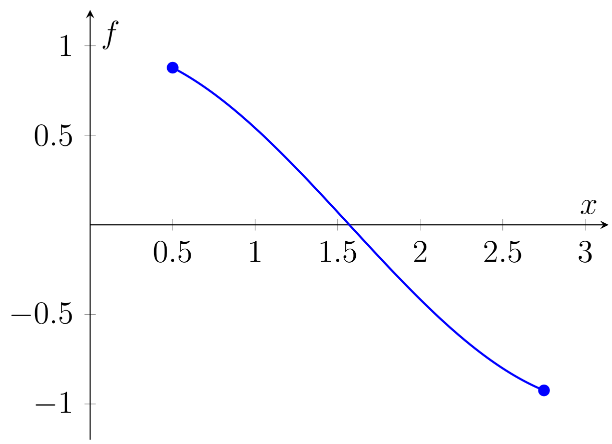
$$\lim_{x\to\infty} 2\cos\left(\frac{1}{2}\arctan x\right)$$

Intermediate Value Theorem (IVT)

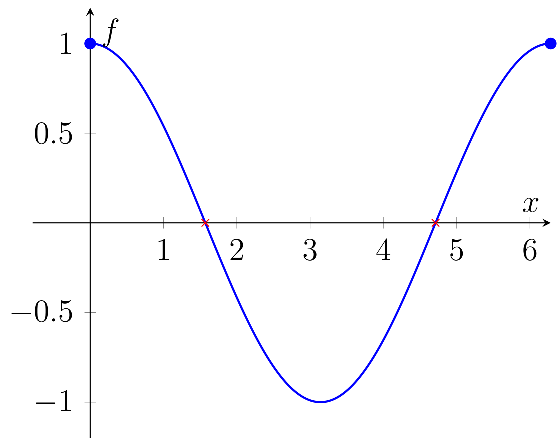
Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ with

$$f(a) \cdot f(b) < 0 \iff f(a) \text{ and } f(b) \text{ differ in sign.}$$

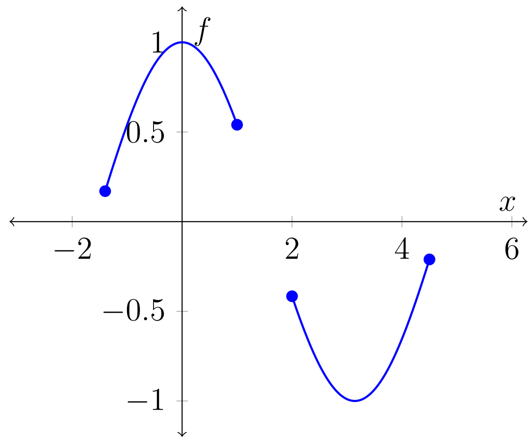
Then there exists $x \in (a, b)$ such that $f(x) = 0$.



Note that the converse of the Intermediate Value Theorem is not true. In other words, $f(a) \cdot f(b) > 0$ does <u>not</u> imply that there is <u>not</u> a zero on $[a, b]$.



The graph below illustrates why continuity is critical to the Intermediate Value Theorem.



✍ **Example.** Show that $f(x) = x^4 + x - 3$ has a zero between $x = 1$ and $x = 2$.