2.1: The Idea of Limits

Learning Objectives. Upon successful completion of Section 2.1, you will be able to...

- Answer conceptual questions involving average velocity or secant and tangent lines.
- Calculate average and instantaneous velocities.
- Calculate slopes of secant and tangent lines.
- Solve applications involving average and instantaneous velocities.

The Idea of a Limit

Given a function f and some input $x = c$ in the domain of f, we are able to calculate the y-value, i.e. $f(c)$. For instance, if $f(x) = 3x - 1$, then...

However, in some cases, things might not work out as nicely. Consider instead the function $f(x) = \dfrac{x^2 - 9}{x - 3}$. Because $x = 3$ is not in the domain of f, we cannot calculate $f(3)$.

Very informally, we might say that the "limit" of $f(x)$ at $x = 3$ is what $f(3)$ "should" be. As you learn about how to work with limits, you will be able to show the following.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

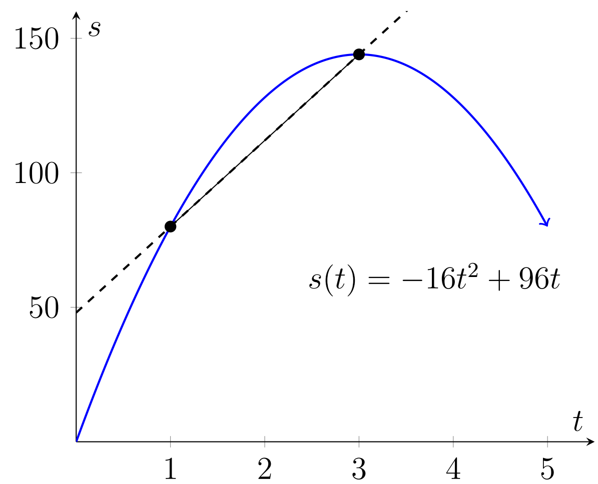
Exploring Limits

We will explore the idea of a limit through the concept of velocity. The **average velocity** on some time interval $[t_1, t_2]$ is as follows.

$$v_{\text{avg}} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

✍ **Example.** The position of a rock t seconds after being launched at a speed of 96 ft/s is given by $s(t) = -16t^2 + 96t$ feet. The average velocity between $t = 1$ and $t = 3$ is...

Geometric Interpretation. Geometrically, this average velocity is the slope of the line passing through the points $(1, 80)$ and $(3, 144)$.



We call such a line a **secant line**.

We can generalize the average velocity function to any function f.

> **Average Rate of Change.** Given a general function f defined on $[a, b]$, the quantity
>
> is called the **average rate of change** of f on $[a, b]$.

What if, instead of computing the average velocity between two time points, we want to compute the **instantaneous velocity** at a particular time? We cannot do this through standard means. . .

✍ **Example.** Let's see if we can determine what the instantaneous velocity "should be" at a particular time. Suppose we want the instantaneous velocity at $t = 1$ second after the rock was launched.

$[1, t_1]$	$\dfrac{s(t_1) - s(1)}{t_1 - 1}$
$[1, 1.500]$	
$[1, 1.100]$	
$[1, 1.010]$	
$[1, 1.001]$	

What is the pattern as t_1 approaches 1, i.e. $t_1 \to 1$?

✍ **Example.** Suppose $f(x) = \dfrac{2x-8}{\sqrt{x}-2}$. The following table was constructed for different values of x.

x	3.7	3.8	3.9	4.0	4.1	4.2	4.3
$f(x)$	7.847	7.899	7.950	0/0	8.050	8.099	8.147

How can we express the limit of $f(x)$ using limit notation?

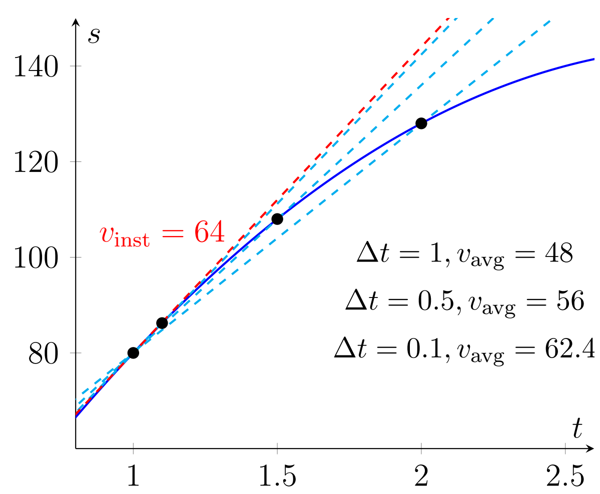
As we progress through the course, we will see that every fundamental task in calculus involves the calculation of a **limit**!

Tangent Lines

Definition. A **tangent line** to a curve at a point $P(x, y)$ is a line that "just touches" the curve at P and has the same direction as the curve.

Tangent lines can help us see the geometric connection between **average** and **instantaneous** velocity. As the time interval for an average velocity shrinks, the slope of the associated *secant* line approaches the slope of the *tangent* line at the point of instantaneous velocity.

$s(t) = -16t^2 + 96t$



View an interactive demo here: https://www.desmos.com/calculator/gfpvpjeksy

Summary

- Informally, a **limit** is the "expected output" of a function or expression that cannot be directly calculated. We write this limit as

$$\lim_{x \to c} f(x) = L.$$

- The **instantaneous velocity** is the limit of the average velocity as the time interval shrinks to zero.

$$v_{\text{inst}} = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

- Geometrically, for two points P and Q...
  - the **average** velocity is the slope of the **secant** line passing through PQ, and
  - the **instantaneous** velocity is the slope of the **tangent** line passing through P.

- Calculating the slope of a tangent line is not possible through standard algebraic means. Our study of **calculus** with introduce techniques for finding the slope of a tangent line.

Helpful Equations. The following equations from an algebra course will be useful when working with secant and tangent lines.

- **Slope of a Line.**

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

- **Slope Intercept Form.**

$$y = mx + b$$

- **Point-Slope Form.**

$$y - y_1 = m(x - x_1)$$