4.2: Mean Value and Rolle's Theorems

Learning Objectives. Upon successful completion of Section 4.2, you will be able to...

- Answer conceptual questions involving Rolle's Theorem and the Mean Value Theorem.
- Determine if Rolle's Theorem applies and find the point(s) guaranteed to exist by Rolle's Theorem.
- Find the point(s) guaranteed to exist by the Mean Value Theorem.
- Find functions with the same derivative of a given function.
- Use graphs to answer questions involving the Mean Value Theorem.
- Solve applications involving the Mean Value Theorem.

Rolle's Theorem

How can we determine whether a function has a zero derivative? Rolle's Theorem is a useful result of calculus that allows us to guarantee the existence of at least one horizontal tangent line if certain conditions hold.

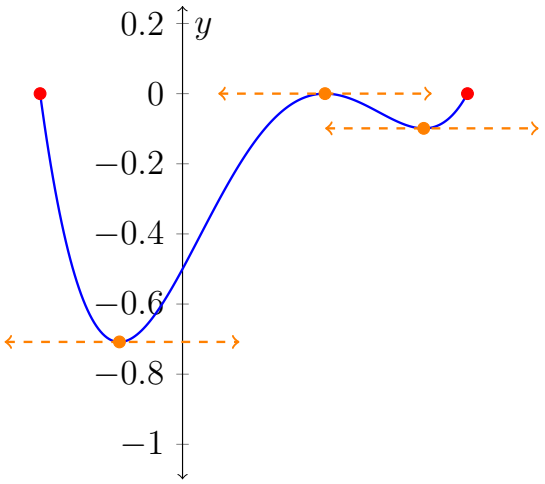
Rolle's Theorem. If f is a function that is

(1) continuous on $[a, b]$,

(2) differentiable on (a, b), and

(3) $f(a) = f(b)$,

then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.



The following functions are continuous on their domains, which is a fact you can cite when making Rolle's Theorem arguments.

- Polynomials
- Rational Functions
- Root Functions
- Trig and Inverse Trig Functions
- Exponential Functions
- Logarithmic Functions

✍ **Example.** Consider $f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$.

(1) Verify that Rolle's Theorem applies.

(2) Find all points c guaranteed to exist by the theorem.

Mean Value Theorem

Rolle's Theorem is a special case of a more general theorem known as Mean Value Theorem.

Mean Value Theorem (MVT). If f is a function that is

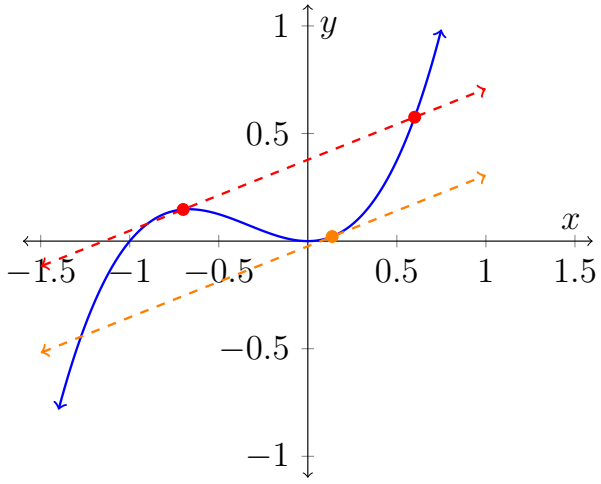
(1) continuous on $[a, b]$ and

(2) differentiable on (a, b),

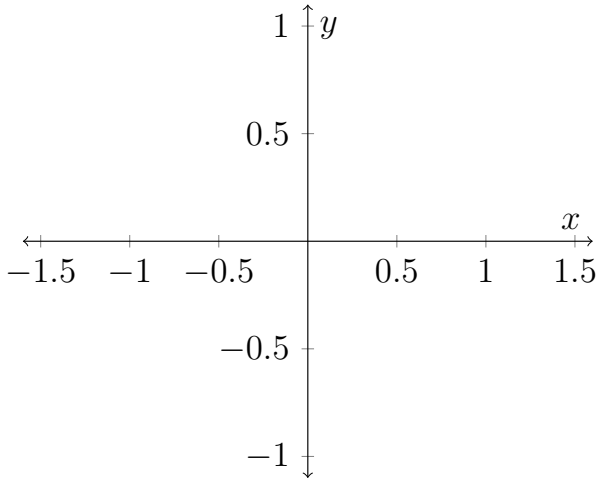
then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem guarantees that there is a point c where the slope of the tangent line at c matches the slope of the secant line through the endpoints of the closed interval.



Let's further explore the connection between the MVT and Rolle's.



✍ **Example.** Consider $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$.

(1) Verify that the MVT applies to f on this interval.

(2) Find all points c guaranteed to exist by the theorem.

✍ **Example.** You enter a six-mile-long stretch of toll road at 1:30 PM and exit at 1:35 PM with no stops in between. Suppose that the road does not have any bends and that the toll road has a speed limit of 65 mph. Two weeks later, you get a speeding ticket in the mail. How can the toll authority prove you were speeding?