10.4: Divergence, p-Series, and Integral Tests

Learning Objectives. Upon successful completion of Section 10.4, you will be able to...

- Answer conceptual questions involving Divergence or p-series Tests.
- Use the Divergence Test to determine whether series diverge.
- Use Divergence or p-series Tests to determine the convergence or divergence of a series.
- Determine if a series converges or diverges using the properties and tests introduced so far.
- Use the Integral Test to determine whether series converge or diverge.
- Use Divergence, Integral, or p-series Tests to determine the convergence or divergence of a series.
- Estimate the value of a series using Theorem 10.13.
- Determine if a series converges or diverges using the properties and tests introduced so far.

The Divergence Test

Divergence Test. Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

✍ **Example.** Determine if the series $\displaystyle\sum_{n=1}^{\infty} \arctan(n)$ converges or diverges.

✍ **Example.** Determine if the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{3n^3+n}{n^3+4}\right)$ converges or diverges.

✍ **Example.** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, called the **harmonic series**. How can we use what we know about $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges?

The Integral Test

Integral Test. Suppose for some positive integer $N \geq 1$, the function $f(x)$ is

(1) continuous for $x \geq N$,

(2) positive for $x \geq N$,

(3) decreasing for $x \geq N$,

and let $a_k = f(k)$ for $k \geq N$, where k is an integer. Then

$$\sum_{k=N}^{\infty} a_k \text{ converges if } \int_N^{\infty} f(x)\,dx \text{ converges, and}$$
$$\sum_{k=N}^{\infty} a_k \text{ diverges if } \int_N^{\infty} f(x)\,dx \text{ diverges.}$$

<u>Note 1</u>: The value of $\displaystyle\int_N^{\infty} f(x)\,dx$ is **NOT** (in general) the value of the series sum.

<u>Note 2</u>: If $\displaystyle\sum_{k=N}^{\infty} a_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ also converges, because convergence is not affected by a finite number of terms. (A similar idea holds for divergence.)

✍ **Example.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$

✍ **Example.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

The p-Series Test

p-Series Test. The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

✍ **Example.** Determine if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$

Estimating the Sum of a Series

Suppose we know $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the integral test (so $f(x)$, where $f(n) = a_n$, is positive, continuous, and decreasing for $x \geq 1$), and we want to find an approximation to the sum.

Definition. The **remainder** R_n is the error made when estimating a sum S by the n^{th} partial sum S_n.

The Integral Estimation Theorem

✍ **Example.** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(a) Find the partial sum S_5 of the series.

(b) Estimate the error in using S_5 as an approximation to the sum of the series.

(c) Use $n = 5$ and $S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_{n}^{\infty} f(x)\,dx$ to improve the estimate of S.

✍ **Example.** Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^6}$ so that the error in estimation is less than $\dfrac{1}{10^6}$.