2.3: Limit Techniques

Learning Objectives. Upon successful completion of Section 2.3, you will be able to...

- Answer conceptual questions involving techniques to compute limits.
- Compute limits, stating the limit laws used.
- Evaluate two-sided limits using limit laws and theorems.
- Evaluate one-sided limits using limit laws and theorems.

Limit Laws

Limits have several essential properties. Assume $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist where $k \in \mathbb{R}$.

1. $\lim\limits_{x\to c} (f(x) \pm g(x)) =$

2. $\lim\limits_{x\to c} kf(x) =$

3. $\lim\limits_{x\to c} f(x)g(x) =$

4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} =$

✍ **Example.** Suppose $\lim\limits_{x\to 1} f(x) = 8$, $\lim\limits_{x\to 1} g(x) = 3$, and $\lim\limits_{x\to 1} h(x) = 2$. Use the limit laws to evaluate the following limit, if it exists.

$$\lim_{x\to 1} \frac{f(x)g(x)}{g(x) - 3h(x)} =$$

Common Techniques for Evaluating Limits

Direct Substitution. Direct substitution is performed by plugging the "target" of a limit into the function whose limit you are evaluating. If direct substitution yields a real number, then no further work is needed to evaluate a limit. Direct substitution will always work for the following types of functions with domain $\mathbb{R}$.

- Polynomials
- Exponentials
- Absolute Value
- Sine and Cosine

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 0} \frac{2x-1}{x-2}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 1} \left(2x^2+3x+1\right)$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 2} \left(e^{\frac{1}{2}x-1}+\cos(x\pi)\right)$$

Direct substitution is convenient, but it will not always work. For example, direct substitution may yield the $\frac{0}{0}$ indeterminate form. This does not, however, mean that the limit does not exist — it means there is more work to be done! The following techniques will introduce some approaches to dealing with these cases.

Factoring and Cancellation. We can employ our knowledge of algebra to manipulate a function by factoring and canceling certain expressions.

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 2} \frac{x^2 - 7x + 10}{x - 2}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 1} \frac{2x^2 - 3x + 1}{x^2 + 2x - 3}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x\to 0} \frac{3x^2 + 5x}{x^2 + 2x}$$

Multiplying by the Conjugate. Sometimes it is useful to multiply a function above and below by the conjugate of either the numerator or denominator.

> **Definition.** The expressions $a+b$ and $a-b$ are called **conjugates**.

When conjugates are multiplied, the cross terms always cancel.

$$(a-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

This technique often occurs when square roots are involved because the multiplication of two square roots has the effect of "eliminating" the root.

$$\sqrt{x}\sqrt{x} = x^{\frac{1}{2}}x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$$

✍ **Example.** Multiply $\sqrt{x-1}+\sqrt{x}$ by its conjugate.

✍ **Example.** Evaluate the following limit.

$$\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{z\to 4} \frac{z-4}{\sqrt{z}-2}$$

Finding a Common Denominator. Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, we may add or subtract them so long as they have a common denominator.

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \pm \frac{c}{d} \cdot \frac{b}{b} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$

✍ **Example.** Evaluate the following limit.

$$\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{2 - x}$$

✍ **Example.** Evaluate the following limit. (Hint: It is often necessary to combine multiple techniques.)

$$\lim_{x \to 9} \frac{x^{-\frac{1}{2}} - \frac{1}{3}}{x - 9}$$

Limits of Piecewise Functions

Limit evaluation techniques can be applied to piecewise functions.

✍ **Example.** Consider the following piecewise function.

$$f(x) = \begin{cases} 0 & x \leqslant -5 \\ \sqrt{25 - x^2} & -5 < x < 5 \\ 3x & x \geqslant 5 \end{cases}$$

1. Evaluate $\lim\limits_{x \to -5} f(x)$.

2. Evaluate $\lim\limits_{x \to 5} f(x)$.

✍ **Example.** Determine a value for $k \in \mathbb{R}$ for which $\lim\limits_{x \to 2} f(x)$ exists and state the value of the limit.

$$f(x) = \begin{cases} 3x + k & x \leqslant 2 \\ x - 2 & x > 2 \end{cases}$$

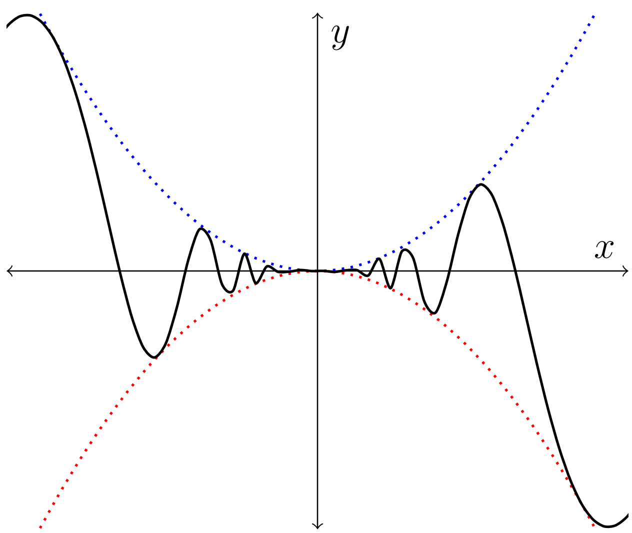
The Squeeze Theorem

Sometimes, when evaluating a limit using the techniques described so far is not possible, it is necessary to apply this useful result. This theorem is known as the "Squeeze Theorem," but is referred to in some texts as the "Sandwich Theorem" or "Pinching Theorem."

Theorem. Suppose $f(x) \le g(x) \le h(x)$ for all x near c except possibly at $x = c$ itself. Also, suppose that

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L.$$

Then, $\lim_{x\to c} g(x) = L$.



$f(x) = x^2$ $\qquad$ $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ $\qquad$ $h(x) = -x^2$

✍ **Example.** Suppose that $2 - x^2 \le g(x) \le 2\cos x$ for all values of x. Find $\lim_{x\to 0} g(x)$.

Applying Properties of Trig Functions. Recall that the range of the sine and cosine functions is $[-1, 1]$. This means that the following inequalities are true...

$$-1 \leq \sin x \leq 1$$

$$-1 \leq \cos x \leq 1$$

This is true regardless of the function's argument. More generally, if f is an arbitrary function, then...

$$-1 \leq \sin\left(f(x)\right) \leq 1$$

$$-1 \leq \cos\left(f(x)\right) \leq 1$$

These will often be starting points for a Squeeze Theorem argument.

✍ **Example.** Evaluate $\lim\limits_{x\to 0^+} x\cos\left(\dfrac{1}{x}\right)$.

Strictly Increasing Functions. It is often helpful to leverage other properties of certain functions in order to construct a Squeeze Theorem argument.

Definition. A function f is said to be **strictly increasing** if

$$a < b \text{ means that } f(a) < f(b)$$

for all a, b in the domain of f.

Remark. Exponential functions with base $b > 1$ are strictly increasing. This means that we can apply such a function "across an inequality" and the inequality remains true.

For example, the function e^x is strictly increasing, so we know that $e^{\frac{1}{2}} < e^{\frac{3}{4}}$ because $\frac{1}{2} < \frac{3}{4}$.

✍ **Example.** Prove that $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} = 0$.