12.1: Parametric Equations

Learning Objectives. Upon successful completion of Section 12.1, you will be able to...

- Graph and describe parametric equations and eliminate the parameter to find equations in x and y.
- Find parametric equations for a given description of a curve.
- Solve applications involving parametric equations.
- Differentiate parametric equations.
- Find the slopes and equations of tangent lines to a parametric equation.
- Find the arc length of a parametric curve.
- Answer conceptual questions involving parametric equations.
- Find the area under a parametric curve.
- Find the area of a surface of revolution of a parametric curve.

Introduction

A **parametric curve** C is a curve in the xy-plane described by three variables: x, y, and a *parametric variable* (often t or θ).

- We call $x = f(t)$ and $y = g(t)$ **parametric equations**, where t is the parameter.
- As t varies, the point $(x, y) = \big(f(t), g(t)\big)$ varies and traces out the curve C.

The curve C is traced out in a specific direction. When sketching parametric curves, we use arrows to indicate this direction, called the **orientation** of the curve.

✍ **Example.** Consider the parametric equations $x = 1 - t^2$ and $y = t - 2$ for $-2 \le t \le 2$.

(a) Eliminate the parameter to obtain an equation in x and y.

(b) Sketch the parametric curve, indicating the positive orientation (increasing t).

✍ **Example.** Consider the equations $x = \sin\left(\frac{\theta}{2}\right)$ and $y = \cos\left(\frac{\theta}{2}\right)$ for $-\pi \le \theta \le \pi$.

(a) Eliminate the parameter to obtain an equation in x and y.

(b) Sketch the parametric curve, indicating the positive orientation.

Finding Parametric Equations for a Curve

Parametric Equations of a Circle. The parametric equations

$$x = x_0 + a\cos(bt) \text{ and } y = y_0 + a\sin(bt)$$

describe all or part of the **circle** $(x - x_0)^2 + (y - y_0)^2 = a^2$ centered at (x_0, y_0) with radius $|a|$. If $b > 0$, then the circle is generated in the counterclockwise direction.

if $x = x_0 + a\sin(bt)$ and $y = y_0 + a\cos(bt)$ with $b > 0$, then the circle is generated in the clockwise direction.

✍ **Example.** Find parametric equations for a circle centered at $(2, 3)$ with a radius of 1, generated counterclockwise.

What parametric equations correspond to only the lower half of the circle?

Parametric Equations of a Line. The parametric equations

$$x = x_0 + at \text{ and } y = y_0 + bt, \text{ for } -\infty < t < \infty,$$

where x_0, y_0, a, and b are constants with $a \neq 0$, describe a **line** with a slope $\dfrac{b}{a}$ passing through the point (x_0, y_0). If $a = 0$ and $b \neq 0$, the line is vertical.

✍ **Example.** Find parametric equations for the line segment starting at $P(-1, -3)$ and ending at $Q(6, -16)$.

✍ **Example.** Find parametric equations for the complete parabola $y = 2x^2 - 4$.

What parametric equations correspond to the segment of the parabola where $-1 \leq x \leq 5$?

✍ **Example.** The tip of the 15-inch second hand of a clock completes one revolution in 60 seconds. Find the parametric equations that describe the circular path of the tip of the second hand. Assume (x, y) denotes the position relative to the origin at the center of the circle.

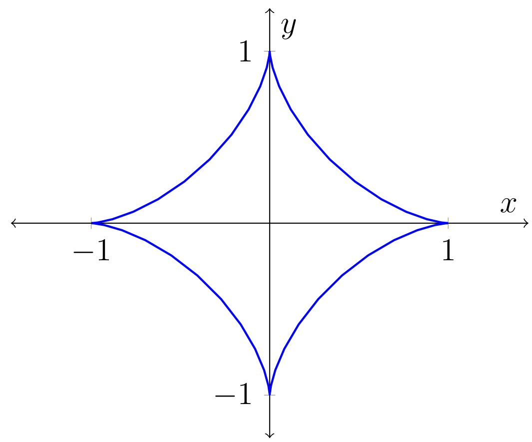
Calculus in Parametric Equations

Suppose we have a set of parametric equations $x = f(t)$ and $y = g(t)$ describing a curve C. If f and g are differentiable, how do we find the slope of the tangent line to a point on C?

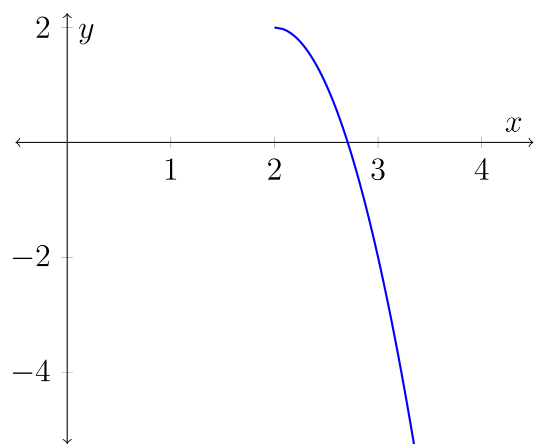
Derivatives for Parametric Equations.

✍ **Example.** A curve is defined by the parametric equations $x = e^t$, $y = te^{-t}$. Find $\dfrac{dy}{dx}$.

✍ **Example.** A curve is defined by the parametric equations $x = \sin^3 t$, $y = \cos^3 t$, $0 \le t \le 2\pi$. Find an equation of the tangent line at the point where $t = \pi/6$.



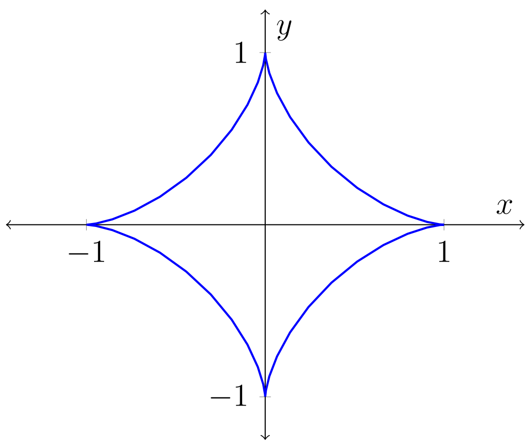
✍ **Example.** Find all points at which the curve $x = 2 + \sqrt{t}$, $y = 2 - 4t$ has the slope -8.



Arc Length for Parametric Equations. Suppose a curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $\dfrac{dx}{dt} > 0$, f' and g' are continuous, and C is transversed once as t increases from α to β.

Then the **arc length $\boldsymbol{L}$** of the curve C is

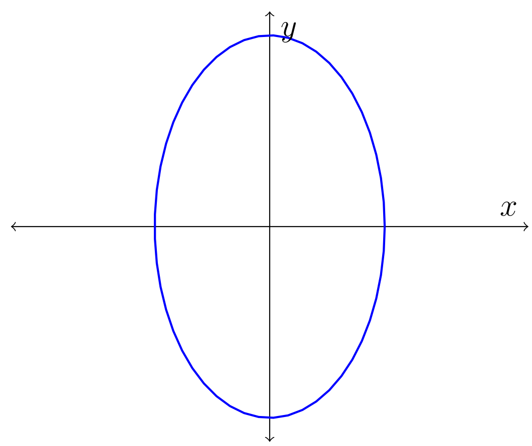
✍ **Example.** Find the total length of the astroid curve $x = \cos^3\theta$, $y = \sin^3\theta$, which is traced out once for $0 \leq \theta \leq 2\pi$.



Area and Parametric Equations. Suppose $y = h(x)$ is nonnegative and continuous on $[a, b]$, implying that the area bounded by the graph of h and the x-axis on $[a, b]$ equals $\displaystyle\int_a^b h(x)\,dx = \int_a^b y\,dx$.

If the graph of $y = h(x)$ on $[a, b]$ is traced exactly once by the parametric equations $x = f(t)$, $y = g(t)$, for $\alpha \le t \le \beta$, then (by substitution) the area bounded by h is

✍ **Example.** Use parametric requations of an ellipse centered at the origin $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.



Surface Area and Parametric Equations. Let C be the curve $x = f(t)$, $y = g(t)$, for $\alpha \leq t \leq \beta$, where f' and g' are continuous, and C does not intersect itself (except possibly at its endpoints).

If g is nonnegative on $[\alpha, \beta]$, then the area of the surface obtained by revolving C around the x-axis is

Likewise, if f is nonnegative on $[\alpha, \beta]$, then the area of the surface obtained by revolving C about the y-axis is

✍ **Example.** Find the exact area of the surface obtained by rotating the curve given by $x = \cos^3\theta$, $y = \sin^3\theta$, $0 \leq \theta \leq \pi/2$ about the x-axis.