MATH 1080 Vagnozzi

## 10.2: Sequences

**Learning Objectives.** Upon successful completion of Section 10.2, you will be able to...

- Answer conceptual questions involving sequences.
- Find whether sequences are monotonic or whether they oscillate and give the limit if the sequence converges.
- Use properties and theorems to determine limits of sequences.
  - Note 1: It is useful to review L'Hôpital's Rule (Section 4.7).
  - Note 2: The fact that  $\lim_{x\to\infty} \left(1+\frac{a}{r}\right)^x = e^a$  may be used without proof.
- Use the growth rate of sequences to determine limits of sequences that converge.

## Computing Limits of Sequences

In Section 10.1, we introduced the general idea of what it means for a **sequence** to converge or diverge. We said that if the terms of a sequence  $\{a_n\}$  approach some number L, then  $\lim_{n\to\infty} a_n = L$  exists and the sequence **converges** to L.

If the terms of the sequence do not approach a single number as n increases, then the sequence has no limit and we say that it **diverges**.

Theorem: Limits of Sequences from Limits of Functions. Suppose that f is a function such that  $f(n) = a_n$ , for positive integers n. If  $\lim_{x \to \infty} f(x) = L$ , then the limit of the sequence  $\{a_n\}$  is also  $L\left(\lim_{n \to \infty} a_n = L\right)$ , where L may be  $\pm \infty$ .

**Limit Laws for Sequences.** Assume the sequences  $\{a_n\}$  and  $\{b_n\}$  have limits A and B, respectively (that is, both sequences converge), and c is a constant.

- $(1) \lim_{n \to \infty} (a_n \pm b_n) = A \pm B$
- (2)  $\lim_{n\to\infty} ca_n = cA$ , where  $c \in \mathbb{R}$
- $(3) \lim_{n \to \infty} (a_n b_n) = AB$
- $\underbrace{4} \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B}, \text{ provided } B \neq 0$

**Examples.** Determine if each of the following sequences converges or diverges. If the sequence converges, find the value to which it converges.

$$a_n = \frac{3 + 5n^2}{n + n^2}$$

$$\left\{\frac{n^3 + 2n}{n+1}\right\}$$

$$\left\{ \tan \left( \frac{2n\pi}{1+8n} \right) \right\}$$

**Definition.** Let r be a real number  $(r \in \mathbb{R})$ . Then  $\{r^n\}$  is a **geometric sequence**.

For what value of r does a geometric sequence converge?

The Squeeze Theorem. If  $a_n \leq b_n \leq c_n$  for all  $n \geq N$  and  $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$ , then  $\lim_{n \to \infty} b_n = L$ .

**Examples.** Determine if each of the following sequences converges or diverges. If the sequence converges, find the value to which it converges.

$$\left\{\frac{\cos^2 n}{2^n}\right\}$$

$$\{2^{n+1}3^{-n}\}$$

$$a_n = \frac{(-1)^n}{2\sqrt{n}}$$

$$\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$$

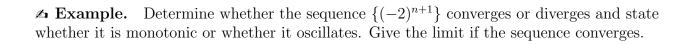
 $\triangle$  Example.  $\left\{ \left( \frac{n}{n+5} \right)^n \right\}$ 

Hint: Recall from Section 4.7 that  $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$ .

## Terminology for Sequences

- $\{a_n\}$  is increasing if
- $\{a_n\}$  is nondecreasing if
- $\{a_n\}$  is decreasing if
- $\{a_n\}$  is **nonincreasing** if
- $\{a_n\}$  is monotonic if it is either nonincreasing or nondecreasing.
- $\{a_n\}$  is bounded above if
- $\{a_n\}$  is **bounded below** if
- If  $\{a_n\}$  is bounded above and below, then we say that  $\{a_n\}$  is a **bounded** sequence.

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Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

Notes on this theorem:

## Growth Rates of Sequences

The relative growth rates of functions (established in Section 4.7: L'Hôpital's Rule) are now applied to sequences. A few notes:

- To compare growth rates of two nondecreasing sequences of positive terms  $\{a_n\}$  and  $\{b_n\}$ , evaluate  $\lim_{n\to\infty} \frac{a_n}{b_n}$ .
  - If  $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ , then  $\{b_n\}$  grows faster than  $\{a_n\}$ .
  - If  $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ , then  $\{a_n\}$  grows faster than  $\{b_n\}$ .
- The notation  $\{a_n\} \ll \{b_n\}$  means that  $\{b_n\}$  grows faster than  $\{a_n\}$ .

**Theorem: Growth Rates of Sequences.** The following sequences are ordered according to increasing growth rates as  $n \to \infty$ ; that is, if  $\{a_n\}$  appears before  $\{b_n\}$  in the list, then  $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$  and  $\lim_{n \to \infty} \frac{b_n}{a_n} = \infty$ :

$$\{\ln^q n\} << \{n^p\} << \{n^p \ln^r n\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$$

The ordering applies for positive real numbers p, q, r, s, and b > 1.

**Examples.** Use the theorem on growth rates to find the limit of the following sequences or state that they diverge.

$$\left\{\frac{n^{10}}{\ln^{1000} n}\right\}$$

$$a_n = \frac{6^n + 3^n}{6^n + n^{1000}}$$