11.1: Approximating Functions with Polynomials

Learning Objectives. Upon successful completion of Section 11.1, you will be able to...

- Answer conceptual questions about Taylor polynomials.
- Use linear and quadratic polynomials to approximate functions.
- Find the Taylor polynomial for a function centered at a specified number.
- Use Taylor polynomials to approximate functions.
- Compare the graph of a function and its Taylor polynomials.
- Find the remainder of an n^{th} order Taylor polynomial for a given function.
- Find the remainder term of a Taylor approximation and use it to estimate error.

Power Series

Chapter 11 focuses on a particular type of infinite series called a **power series**.

Definition. A **power series** is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + c_{n+1} x^{n+1} \cdots ,$$

or, more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + c_{n+1}(x-a)^{n+1} \cdots ,$$

where a and c_k are constants. The values of c_k are called the **coefficients**.

Motivation. Why are power series important?

- Functions can be represented by power series.
- Functions can be defined as power series.
- Power series are like infinitely long polynomials, and polynomials have a lot of nice properties that make them straightforward to work with.
- Power series can be used to solve differential equations. (Differential equations were introduced in Section 4.9 in MATH 1060.)

Question: What should the form of the coefficients c_k be to provide a good approximation of a function? Let's explore! We will consider the function $f(x) = e^x$ centered at $x = 0$.

1. First, we will find a *linear approximation* for $f(x)$ at $x = 0$. In other words, we will find the equation $p_1(x) = c_0 + c_1 x$ of the line tangent to $f(x)$ at $x = 0$.

   Note: $p_1(x)$ and $f(x)$ have the same y-value and same first derivative at $x = 0$.

2. To obtain a better approximation, let's consider a *quadratic approximation* that adds a quadratic term to $p_1(x)$:

$$p_2(x) = p_1(x) + c_2 x^2 = c_0 + c_1 x + c_2 x^2$$

   We want a value for c_2 that results in a good approximation of $f(x)$ near $x = 0$. $p_1(x)$ was a good linear approximation to $f(x)$ because the y-values and first derivatives matched. It would be reasonable to want a value c_2 so that the second derivatives match as well.

   (a) Calculate $p_2(0)$ to show that the y-values match.

   (b) Calculate $p_2'(0)$ to show that the first derivatives match.

   (c) Calculate $p_2''(x)$. Then find a value for c_2 so that the second derivatives match.

(3) We can keep adding higher-order terms to get a better approximation to $f(x)$ at $x = 0$. Let

$$p_3(x) = c_0 + c_1x + c_2x^2 + c_3x^3.$$

Based on our work from the previous page, we know what c_0, c_1, and c_2 are. Let's find c_3 so that the third derivatives match.

(4) Let's look at one more approximation:

$$p_4(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4.$$

Find the value of c_4 so that the fourth derivatives are equal.

Taylor Polynomials

The polynomial approximations we explored on the previous two pages are referred to as **Taylor polynomials**.

Definition. The ***n*th-order Taylor polynomial** $p_n(x)$ for f centered at $x = a$ is

✍ **Example.** Consider the function $f(x) = \sqrt{x}$.

(a) Find the Taylor polynomial p_3 at $a = 1$ for $f(x) = \sqrt{x}$.

(b) Now use p_3 to approximate $\sqrt{1.06}$.

Approximations with Taylor Polynomials

Taylor polynomials provide a good approximation of f near the center a, and the approximation gets better as the order of the Taylor polynomial increases. Similar to our work when approximating the sum of a series, we can determine *how accurate* an approximation is through the idea of looking at remainders and error bounds.

Definition. Let p_n be the nth-order Taylor polynomial for f. The **remainder** in using p_n to approximate f at the point x is

$\left|R_n(x)\right|$ is the **error** made in approximating f by p_n.

There are two theorems that will allow us to estimate bounds on the error in approximation with Taylor polynomials.

Taylor's Theorem (Remainder Theorem). Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval containing a. For all x-values in the interval,

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point x between x and a.

Theorem: Estimate of the Remainder. Let n be a fixed positive integer. Suppose there exists a number M such that $\left|f^{(n+1)}(c)\right| \leq M$ for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$\left|R_n(x)\right| = \left|f(x) = p_n(x)\right| \leq M\frac{\left|x-a\right|^{n+1}}{(n+1)!}$$

As a result of the two theorems above, we can get a **bound on the error** by…

- finding $f^{(n+1)}(x)$,
- finding a number M so that $\left|f^{(n+1)}(c)\right| \leq M$ for all $c \in [a, x]$, and then
- plugging M into the formula from the second Theorem above.

Finding Error Bounds. For a Taylor polynomial used to approximate f...

1. Find $f^{(n+1)}(x)$.
2. Find a number M so that $\left|f^{(n+1)}(c)\right| \leq M$ for all $c \in [a, x]$.
3. Plug M into the estimation theorem to find the error bound:

$$\left|R_n(x)\right| \leq M \frac{\left|x-a\right|^{n+1}}{(n+1)!}$$

✍ **Example.** Consider the function $f(x) = \cos x$ centered at $a = 0$.

(a) Find the Taylor polynomials of order $n = 2$ and $n = 3$ for f.

(b) Use the remainder term to find a bound on the error in the approximation $p_3(x)$ to $f(x)$ on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

✍ **Example.** Let $\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ on $[4, 4.2]$. Use the remainder term to find a bound on the error in the approximation on the given interval.

✍ **Example.** What is the minimum order of Taylor polynomial required to approximate $e^{-0.5}$ with an absolute error no greater than 10^{-3}?