2.2: Definitions of Limits

Learning Objectives. Upon successful completion of Section 2.2, you will be able to...

- Answer conceptual questions involving definitions of limits.
- Find limits from a graph.
- Estimate limits from a table.
- Solve applications involving the evaluation of limits by graphing.
- Estimate limits using a graphing utility.
- Sketch graphs of functions given information about limits and function values.

The Limit: An Informal Definition

Definition. Suppose f is defined when x is near the number c except possibly at c. For instance, $f(c)$ may be undefined. Then we write

$$\lim_{x \to c} f(x) = L$$

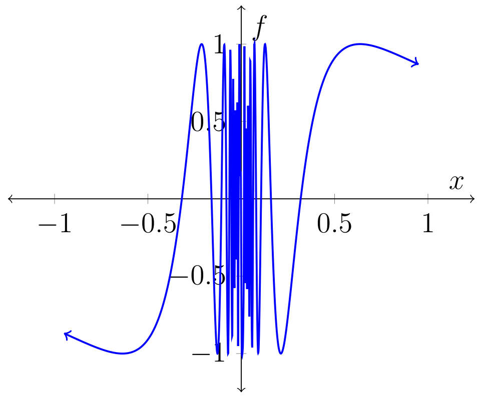
and say "the limit of f as x approaches c is L" if the values of f are arbitrarily close to L for all x sufficiently close to c.

Remark. This definition is formalized in Section 2.7 of the textbook.

Existence of Limits. When does a limit exist? It is easier to consider the cases when limits do **not** exist. There are three typical cases:

1. The function *jumps* or *approaches different values* depending on the direction of approach.
2. The function *grows too large* (or *too small*) and never approaches a particular value.
3. The function *oscillates* and never approaches a particular value.

Oscillation. Below is a graph of $f(x) = \sin\left(\frac{1}{x}\right)$.



Here we would say that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist (DNE). We will not encounter many functions that behave this way in our course.

One-Sided Limits.

Definition. The limit of f as x approaches c **from the left** is written as

and is read "the limit of f as x approaches c from the left is L."

Definition. The limit of f as x approaches c **from the right** is written as

and is read "the limit of f as x approaches c from the right is L."

Remark. We interpret $x \to c^-$ to mean $x \to c$ and $x \to c^+$ to mean $x > c$.

Theorem. Suppose $L \in \mathbb{R}$. We say that $\lim_{x \to c} f(x) = L$ if and only if the one-sided limits exist and are equal. In other words...

Otherwise, we say that $\lim_{x \to c} f(x)$ does not exist (DNE).

We can now also say that a limit will not exist **at an endpoint** because an endpoint will not have both left and right limits.

Limits are not affected by what happens at a particular value. For $f(x) = \dfrac{x^2-9}{x-3}$,

$$f(3) \text{ is undefined, but } \lim_{x\to 3} \frac{x^2-9}{x-3} = 6.$$

Informally, a limit is more interested in the "journey" of a function than the "destination."

✍ **Example.** Use the table of values to make a conjecture about $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$.

x	1.9	1.99	1.999	1.9999
$\frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999

x	2.1	2.01	2.001	2.0001
$\frac{x^2-4}{x-2}$	4.1	4.01	4.001	4.0001

Finding Limits Graphically.