3.7: The Chain Rule

Learning Objectives. Upon successful completion of Section 3.7, you will be able to...

- Answer conceptual questions involving the chain rule.
- Identify inner and outer functions for composite functions, then apply the chain rule.
- Find derivatives of functions involving the chain rule by using tables and graphs.
- Find derivatives of basic functions using the chain rule.
- Use the chain rule multiple times and use the chain rule with the product and quotient rules.
- Find higher order derivatives using the chain rule.
- Find slopes of curves and equations of tangent lines by using the chain rule.
- Solve applications involving the chain rule.

Derivatives of Composite Functions

The **chain rule** allows us to take derivatives of composite functions.

The Chain Rule. Let f and g be two functions and let $h(x) = f\big(g(x)\big)$ be their composition. The derivative of h is

$$\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big) \cdot g'(x).$$

In words, when applying the chain rule to a composite function, we take the derivative of the *outside* function, keeping the inside function the same, then multiplying by the derivative of the *inside* function.

✍ **Example.** Find the derivative of $y = \tan(x^2 - 3x)$.

Alternative Definition. Another formulation of the chain rule says that if $y = f(u)$ and $u = g(x)$, then the derivative $\frac{dy}{dx}$ is found as follows.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We will usually use the first form shown, but you should be familiar with this form as well.

✍ **Example.** If $f(x) = \cos x$ and $g(x) = x^2 - 2x$, find the derivative of $h(x) = f\big(g(x)\big)$.

✍ **Example.** Find $\dfrac{dy}{dx}$ for $y = \sqrt{4 + 3x}$.

✍ **Example.** Find the derivative of $y = \sec^2 x$.

✍ **Example.** Find the derivative of $f(x) = e^{x^2+x} - \cos 2x$.

✍ **Example.** Find the derivative of $y = \dfrac{e^{\pi x}}{e^x + e^{-x}}$.

✍ **Example.** Find $\dfrac{dy}{dx}$ for $y = (2x^3 + x^2)^{10}$.

✍ **Example.** Find the derivative of $g(x) = x^2 \left(x^3 + 2\right)^4$.

✍ **Example.** Find the derivative of $f(x) = \sqrt{x^4 + x^3 + x^2 + x}$.

✍ **Example.** Find $\dfrac{dy}{dx}$ for $y = \sin\big(\cos(\tan x)\big)$.

✍ **Example.** Find f' given that $f(x) = x^3 g(2x+1)$, where g is differentiable.

✍ **Example.** Find $\dfrac{dy}{dx}$ for $y = \tan(\sin 2x)$.

✍ **Example.** Differentiate $f(x) = e^{k \tan \sqrt{x}}$ for $k \in \mathbb{R}$, $k \neq 0$.

✍ **Example.** Find the derivative of $g(t) = \sin^3(3t)$ and $f(t) = 3 \sin t^3$.

✍ **Example.** Find all values of x where $y = (3x-1)^2(5-x)^4$ is horizontal.

✍ **Example.** Suppose f is differentiable with $f'(0) = 3$ and $f'\left(\frac{\sqrt{2}}{2}\right) = 2$.

If $g(x) = f(\sin x) + f\left(x - \frac{\pi}{4}\right)$, calculate $g'\left(\frac{\pi}{4}\right)$.

✍ **Example.** Suppose f and g are two lines with slopes m and $\frac{1}{m}$, respectively, where $m \neq 0$. If $y = f\big(g(x)\big)$, what is the value of $y'(c)$ where $c \in \mathbb{R}$?