10.2: Sequences

Learning Objectives. Upon successful completion of Section 10.2, you will be able to...

- Answer conceptual questions involving sequences.
- Find whether sequences are monotonic or whether they oscillate and give the limit if the sequence converges.
- Use properties and theorems to determine limits of sequences.
  - Note 1: It is useful to review L'Hôpital's Rule (Section 4.7).
  - Note 2: The fact that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$ may be used without proof.
- Use the growth rate of sequences to determine limits of sequences that converge.

Computing Limits of Sequences

In Section 10.1, we introduced the general idea of what it means for a **sequence** to converge or diverge. We said that if the terms of a sequence $\{a_n\}$ approach some number L, then $\lim_{n\to\infty} a_n = L$ exists and the sequence **converges** to L.

If the terms of the sequence do not approach a single number as n increases, then the sequence has no limit and we say that it **diverges**.

Theorem: Limits of Sequences from Limits of Functions. Suppose that f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L $\left(\lim_{n\to\infty} a_n = L\right)$, where L may be $\pm\infty$.

Limit Laws for Sequences. Assume the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively (that is, both sequences converge), and c is a constant.

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, where $c \in \mathbb{R}$
3. $\lim_{n\to\infty} (a_n b_n) = AB$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$

✍ **Examples.** Determine if each of the following sequences converges or diverges. If the sequence converges, find the value to which it converges.

$$a_n = \frac{3 + 5n^2}{n + n^2}$$

$$\left\{ \frac{n^3 + 2n}{n + 1} \right\}$$

$$\left\{ \tan\left(\frac{2n\pi}{1 + 8n} \right) \right\}$$

Definition. Let r be a real number ($r \in \mathbb{R}$). Then $\{r^n\}$ is a **geometric sequence**.

For what value of r does a geometric sequence converge?

The Squeeze Theorem. If $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

✍ **Examples.** Determine if each of the following sequences converges or diverges. If the sequence converges, find the value to which it converges.

$$\left\{\frac{\cos^2 n}{2^n}\right\}$$

$$\{2^{n+1}3^{-n}\}$$

$$a_n = \frac{(-1)^n}{2\sqrt{n}}$$

$$\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$$

✍ **Example.** $\left\{ \left(\dfrac{n}{n+5} \right)^n \right\}$ Hint: Recall from Section 4.7 that $\displaystyle\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$.

Terminology for Sequences

- $\{a_n\}$ is **increasing** if
- $\{a_n\}$ is **nondecreasing** if
- $\{a_n\}$ is **decreasing** if
- $\{a_n\}$ is **nonincreasing** if
- $\{a_n\}$ is **monotonic** if it is either **nonincreasing** or **nondecreasing**.
- $\{a_n\}$ is **bounded above** if
- $\{a_n\}$ is **bounded below** if
- If $\{a_n\}$ is bounded above and below, then we say that $\{a_n\}$ is a **bounded** sequence.

✍ **Example.** Determine whether the sequence $\{(-2)^{n+1}\}$ converges or diverges and state whether it is monotonic or whether it oscillates. Give the limit if the sequence converges.

> **Monotonic Sequence Theorem.** Every bounded monotonic sequence is convergent.

Notes on this theorem:

Growth Rates of Sequences

The relative growth rates of functions (established in Section 4.7: L'Hôpital's Rule) are now applied to sequences. A few notes:

- To compare growth rates of two nondecreasing sequences of positive terms $\{a_n\}$ and $\{b_n\}$, evaluate $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$.
  - If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then $\{b_n\}$ grows faster than $\{a_n\}$.
  - If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, then $\{a_n\}$ grows faster than $\{b_n\}$.
- The notation $\{a_n\} << \{b_n\}$ means that $\{b_n\}$ grows faster than $\{a_n\}$.

Theorem: Growth Rates of Sequences. The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} << \{n^p\} << \{n^p \ln^r n\} << \left\{n^{p+s}\right\} << \{b^n\} << \{n!\} << \{n^n\}$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

✍ **Examples.** Use the theorem on growth rates to find the limit of the following sequences or state that they diverge.

$$\left\{\frac{n^{10}}{\ln^{1000} n}\right\}$$

$$a_n = \frac{6^n + 3^n}{6^n + n^{1000}}$$