10.7: The Ratio and Root Tests

Learning Objectives. Upon successful completion of Section 10.7, you will be able to. . .

- Answer conceptual questions involving the root and ratio tests.
- Apply the root and ratio tests.
- Determine if a series converges absolutely, converges conditionally, or diverges.
- Determine values for which a series converges.

The Ratio Test

Ratio Test. Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum a_n$ is divergent.
3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the Ratio Test is inconclusive.

Notes about the Ratio Test.

- The Ratio test is useful for series whose terms contain exponential functions, factorials, or exponentials times powers of n.
- The Ratio Test is **not** useful for p-series type series (only powers of n involved).
- A series cannot converge conditionally by the Ratio Test.

✍ **Examples.** Determine if each of the following series converges conditionally, absolutely, or diverges.

① $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 5^{n-1}}{(-2)^n}$

✍ **Example.** Determine if the series converges conditionally, absolutely, or diverges.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+4}$$

Extra Examples. Determine if the series converges conditionally, absolutely, or diverges.

1. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^{2k}}{k!k!}$

2. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{0.99}}$

Note. For the first example, recall that $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

The Root Test

Ratio Test. Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

① If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.

② If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum a_n$ is divergent.

③ If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

Notes about the Root Test.

- If the Ratio Test is inconclusive, then the Root Test will be, too (and vice versa).
- A series cannot be conditionally convergent by the Root Test.

✍ **Examples.** Determine if each of the following series converges conditionally, absolutely, or diverges.

① $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^{2n}$

② $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{3}{k}\right)^{k^2}$