Modular Existential Closedness with Derivatives

Vahagn Aslanyan

University of Manchester

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- The function $j : \mathbb{H} \to \mathbb{C}$ is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined and analytic on $\mathbb{H}$.
- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.

Fundamental domains of $SL_2(\mathbb{Z})$

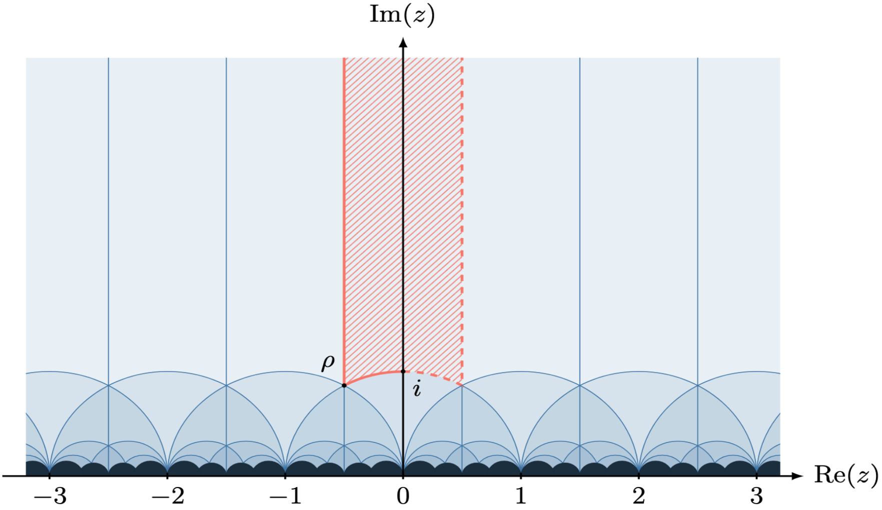

Figure: Fundamental domains of $SL_2(\mathbb{Z})$ (by V. Mantova)

Visual representation of j

Figure: Representation of j via domain colouring (from Wikipedia)

Derivatives of the j-function

- The j-function satisfies a third-order differential equation, namely,

$$\frac{j'''}{j'} - \frac{3}{2}\left(\frac{j''}{j'}\right)^2 + \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2} \cdot (j')^2 = 0.$$

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- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$j'(\gamma z) = (cz+d)^2 j'(z),$$

$$j''(\gamma z) = (cz+d)^4 j''(z) + 2c(cz+d)^3 j'(z).$$

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Geometrically, every ***free*** *and* ***broad*** *algebraic variety $V \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ intersects the graph $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ of the function $z \mapsto (j(z), j'(z), j''(z))$.*

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The above conjecture is based on Zilber's *Exponential Closedness* conjecture (c. 2002).

Zariski density

The Existential Closedness conjecture implies the following seemingly stronger version of itself.

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If $V \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ is free and broad then the intersection $V \cap \Gamma$ is Zariski dense in V.

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Let $V \subseteq \mathbb{H} \times \mathbb{C}^3$ be an algebraic variety of dimension 3 with no constant coordinates. Then V contains a Zariski dense subset of points of the form $(z, j(z), j'(z), j''(z)) \in \mathbb{H} \times \mathbb{C}^3$.

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In terms of equations this amounts to the following. Let $F(X, Y_0, Y_1, Y_2)$ be a polynomial over $\mathbb{C}$. We say the equation $F(z, j(z), j'(z), j''(z)) = 0$ has a *Zariski dense set of solutions* if for any polynomial $G(X, Y_0, Y_1, Y_2)$ which is not divisible by some irreducible factor of F, there is $z_0 \in \mathbb{H}$ such that $F(z_0, j(z_0), j'(z_0), j''(z_0)) = 0$ and $G(z_0, j(z_0), j'(z_0), j''(z_0)) \neq 0$.

Main theorem

Theorem (A.-Eterović-Mantova, 2023)

For any polynomial $F(X, Y_0, Y_1, Y_2) \in \mathbb{C}[X, Y_0, Y_1, Y_2] \setminus \mathbb{C}[X]$ *which is coprime to* $Y_0(Y_0 - 1728)Y_1$, *the equation* $F(z, j(z), j'(z), j''(z)) = 0$ *has a Zariski dense set of solutions, i.e. the set* $\{(z, j(z), j'(z), j''(z)) \in \mathbb{H} \times \mathbb{C}^3 : F(z, j(z), j'(z), j''(z)) = 0\}$ *is Zariski dense in the hypersurface* $F(X, Y_0, Y_1, Y_2) = 0$.

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- Observe that

$$\forall z \in \mathbb{H} \left[j(z)(j(z) - 1728) = 0 \Leftrightarrow j'(z) = 0 \right].$$

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  This immediately gives that the three equations $j(z) = 0$, $j(z) - 1728 = 0$, and $j'(z) = 0$ do not have Zariski dense sets of solutions.
- As an immediate consequence of the theorem we get that $j''(z)$ has zeroes outside $\mathrm{SL}_2(\mathbb{Z})\rho$.

Argument principle and Rouché's theorem

Theorem (Argument Principle)

Let f be a meromorphic function on a complex domain Ω. Let C be a simple closed curve (positively oriented) which is homologous to 0 in Ω and such that f has no zeroes or poles on C. Let Z and P respectively denote the number of zeroes and poles of f in the interior of C. Then

$$2\pi i(Z-P) = \oint_C \frac{f'(z)}{f(z)} dz = \oint_{f\circ C} \frac{dz}{z}.$$

Theorem (Rouché's theorem)

Let f, g be holomorphic functions on a complex domain Ω. Let C denote a simple closed curve which is homologous to 0 in Ω. If the inequality

$$|g(z)| < |f(z)|$$

holds for all z on C, then f and g have the same number of zeroes in the interior of C.

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- So we apply the argument principle in a suitable region – the standard fundamental domain cut by a horizontal line from above.

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- Unlike the previous case, here all zeroes of j' are also zeroes of $j(j - 1728)$ (even if we count multiplicities). So the previous argument doesn't go through.
- However, at $i\infty$ the functions j, j', j'' all have exponential growth, and $j(j - 1728)$ grows faster than j'. In other words, $\frac{j(j-1728)}{j'}$ has a pole at $i\infty$.
- So we apply the argument principle in a suitable region – the standard fundamental domain cut by a horizontal line from above.
- We can show that the winding number of the image of the boundary of that region under the function $\sqrt{\frac{j(j-1728)}{j'}} - z$ around large integers is non-zero.

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- Then $z_m + m$ is a solution to $(\dagger)$, and Zariski density follows as before.

A result on 1-periodic functions

As a by-product, we establish the following theorem.

Proposition (A.-Eterović-Mantova, 2023)

Let $f_0, \ldots, f_n : \mathbb{H} \to \mathbb{C}$ be 1-periodic meromorphic functions. Suppose that for some k one of the following conditions is satisfied:

- *there is $\tau \in \mathbb{H}$ such that $\frac{f_k}{f_n}(z) \to \infty$ as $z \to \tau \in \mathbb{H}$, or*
- *$\frac{f_k}{f_n}(z) \to \infty$ as $\mathrm{Im}(z) \to +\infty$.*

Then there is a sequence of points $\{z_m\}_{m\in\mathbb{N}} \subseteq \mathbb{H}$ with $z_m \neq \tau$ and $z_m \to \tau$ in the first case, or $\mathrm{Im}(z_m) \to +\infty$ and $0 \leq \mathrm{Re}(z_m) \leq 1$ in the second case, such that for all sufficiently large m the point $z_m + m$ is a solution to the equation

$$f_n(z)z^n + f_{n-1}(z)z^{n-1} + \ldots + f_0(z) = 0.$$