Blurrings of the j-function

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The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az + b}{cz + d}.$$

- Let $j : \mathbb{H} \to \mathbb{C}$ be the modular j-function.
- j is holomorphic on $\mathbb{H}$ and is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. $j(\gamma z) = j(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
- By means of j the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ ($N \geq 1$), called *modular polynomials*, such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.

Definition

A *special* subvariety of $\mathbb{C}^n$ (with coordinates $\bar{w}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(w_k, w_l) = 0$ for some $1 \leq k, l \leq n$ where Φ_N is a modular polynomial.

Modular Schanuel and EC

The following is a modular analogue of Schanuel's conjecture.

Conjecture (Modular Schanuel Conjecture)

Let $z_1, \ldots, z_n \in \mathbb{H}$ be non-quadratic numbers with distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits. Then $\mathrm{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n$.

By abuse of notation we will let j denote all Cartesian powers of itself. Similarly we let $\Gamma_j := \{(\bar{z}, j(\bar{z})) : \bar{z} \in \mathbb{H}^n\} \subseteq \mathbb{C}^{2n}$ be the graph of j in $\mathbb{H}^n \times \mathbb{C}^n$ for any n.

Conjecture (Existential Closedness for j)

*Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an irreducible j-**broad**, j-**free** and $\mathbb{H}$-**free** variety defined over $\mathbb{C}$. Then $V \cap \Gamma_j \neq \emptyset$.*

This is an analogue of Zilber's *Exponential Closedness* conjecture.

j-broad and j-free varieties

We will use the following notation.

- $(n) := (1, \ldots, n)$, and $\bar{k} \subseteq (n)$ means that $\bar{k} = (k_1, \ldots, k_l)$ for some $1 \leq k_1 < \ldots < k_l \leq n$.
- The coordinates of $\mathbb{C}^{2n}$ will be denoted by $(z_1, \ldots, z_n, w_1, \ldots, w_n)$.
- For $\bar{k} = (k_1, \ldots, k_l) \subseteq (n)$ define

$$\pi_{\bar{k}} : \mathbb{C}^n \to \mathbb{C}^l : (z_1, \ldots, z_n) \mapsto (z_{k_1}, \ldots, z_{k_l}),$$

$$\mathrm{pr}_{\bar{k}} : \mathbb{C}^{2n} \to \mathbb{C}^{2l} : (\bar{z}, \bar{w}) \mapsto (\pi_{\bar{k}}(\bar{z}), \pi_{\bar{k}}(\bar{w})).$$

Definition

Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety.

- V is *j-broad* if for any $\bar{k} \subseteq (n)$ of length l we have $\dim \mathrm{pr}_{\bar{k}} V \geq l$.
- V is *j-free* if no relation of the form $\Phi_N(w_i, w_k) = 0$ holds on V.
- V is $\mathbb{H}$-free if no relation of the form $z_k = g z_i$ holds on V where $g \in \mathrm{GL}_2(\mathbb{Q})$.

What is known

- A differential analogue of the EC conjecture for j (A.-Eterović-Kirby). In a differentially closed field j-broad (and j-free) varieties intersect the differential equation of the j-function.
- EC holds for varieties with dominant projection on $\mathbb{H}^n$ (Eterović-Herrero).
- We will show that EC holds for *blurrings* of Γ_j by certain subgroups of $\mathrm{GL}_2(\mathbb{C})$.
- These are analogous to some results on the exponential function (which in turn have been motivated by Zilber's Exponential Closedness conjecture), but there are important differences.

Blurred j-function

Definition

Given a subgroup $G \subseteq \mathrm{GL}_2(\mathbb{C})$, let $\mathrm{B}_j^G \subseteq \mathbb{C}^2$ be the relation $\{(z, j(gz)) : g \in G, gz \in \mathbb{H}\}$. By abuse of notation, for every n we also let B_j^G denote the set

$$\{(z_1, \ldots, z_n, j(g_1 z_1), \ldots, j(g_n z_n)) : g_k \in G, g_k z_k \in \mathbb{H} \text{ for all } k\}.$$

Example

- When $G \subseteq \mathrm{SL}_2(\mathbb{Z})$, we have $\mathrm{B}_j^G = \Gamma_j$.
- $\mathrm{B}_j^{\mathrm{GL}_2(\mathbb{C})} = \mathbb{C}^2$.
- $\mathrm{B}_j^{\mathrm{GL}_2^+(\mathbb{R})} = \mathbb{H} \times \mathbb{C}$.
- $\mathrm{A}_j := \mathrm{B}_j^{\mathrm{GL}_2^+(\mathbb{Q})}$ is the *approximate* j-function.

EC for blurred j

Theorem

If $V \subseteq \mathbb{C}^{2n}$ is a j-broad and j-free variety and $G \subseteq \mathrm{GL}_2(\mathbb{C})$ is a dense subgroup in the complex topology, then $V \cap \mathrm{B}_j^G$ is dense in V, and hence it is non-empty.

This is an analogue of Kirby's theorem for blurred complex exponentiation. For the j-function we can do better.

Theorem

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an irreducible j-broad and j-free variety and let $G \subseteq \mathrm{GL}_2^+(\mathbb{R})$ be a dense subgroup (in the Euclidean topology). Then $V \cap \mathrm{B}_j^G$ is dense in V in the complex topology. In particular, $V \cap \mathrm{A}_j \neq \emptyset$.

EC for j with derivatives

Let $J : \mathbb{H} \to \mathbb{C}^3$ be given by

$$J : z \mapsto (j(z), j'(z), j''(z)).$$

We extend J to $\mathbb{H}^n$ by defining

$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$

where $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2$.
Let $\Gamma_J \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of J for any n.
We consider only the first two derivatives of j, for the higher derivatives are algebraic over those.

Conjecture (Existential Closedness for J)

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be an irreducible J-broad, J-free and $\mathbb{H}$-free variety defined over $\mathbb{C}$. Then $V \cap \Gamma_J \neq \emptyset$.

J-broad and J-free varieties

- The coordinates of $\mathbb{C}^{4n}$ will be denoted by $(\bar{z}, \bar{w}, \bar{w}_1, \bar{w}_2)$.
- For a tuple $\bar{k} = (k_1, \ldots, k_l) \subseteq (n)$ define a map

$$\Pr_{\bar{k}} : \mathbb{C}^{4n} \to \mathbb{C}^{4l} : (\bar{z}, \bar{w}, \bar{w}_1, \bar{w}_2) \mapsto (\pi_{\bar{k}}(\bar{z}), \pi_{\bar{k}}(\bar{w}), \pi_{\bar{k}}(\bar{w}_1), \pi_{\bar{k}}(\bar{w}_2)).$$

Definition

- An algebraic variety $V \subseteq \mathbb{C}^{4n}$ is *J-broad* if for any $\bar{k} \subseteq (n)$ of length l we have $\dim \Pr_{\bar{k}} V \geq 3l$.
- An algebraic variety $V \subseteq \mathbb{C}^{4n}$ is *J-free* if no relation of the form $\Phi_N(w_i, w_k) = 0$ holds on V.

Blurred J-function

Definition

For a subgroup $G \subseteq \mathrm{GL}_2(\mathbb{C})$ define a relation

$$\mathrm{B}_J^G := \left\{ \left(z, j(gz), \frac{d}{dz} j(gz), \frac{d^2}{dz^2} j(gz) \right) : g \in G, gz \in \mathbb{H} \right\} \subseteq \mathbb{C}^4 .$$

By abuse of notation for each n we let B_J^G denote the set

$$\{(\bar{z}, \bar{w}, \bar{w}_1, \bar{w}_2) : (z_k, w_k, w_{1,k}, w_{2,k}) \in \mathrm{B}_J^G \text{ for all } k\} \subseteq \mathbb{C}^{4n} .$$

Theorem

Let $V \subseteq \mathbb{C}^{4n}$ be an irreducible J-broad and J-free variety, and let $G \subseteq \mathrm{GL}_2(\mathbb{C})$ be a subgroup which is dense in the complex topology. Then $V \cap \mathrm{B}_J^G$ is dense in V in the complex topology.

Ax-Schanuel

Theorem (Pila-Tsimerman)

Let $V \subseteq \mathbb{C}^{4n}$ be an algebraic variety and let U be an analytic component of the intersection $V \cap \Gamma_J$. If $\dim U > \dim V - 3n$ and no coordinate is constant on $\operatorname{Pr}_w U$ then $\operatorname{Pr}_w U$ is contained in a proper special subvariety of $\mathbb{C}^n$.

Here Pr_w is the projection $(\bar{z}, \bar{w}, \bar{w}_1, \bar{w}_2) \mapsto \bar{w}$.

Theorem (Ax-Schanuel without derivatives)

Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety and let U be an analytic component of the intersection $V \cap \Gamma_j$. If $\dim U > \dim V - n$ and no coordinate is constant on $\operatorname{pr}_w U$ then $\operatorname{pr}_w U$ is contained in a proper special subvariety of $\mathbb{C}^n$.

Here pr_w is the projection $(\bar{z}, \bar{w}) \mapsto \bar{w}$.

Uniform Ax-Schanuel

For $g \in \mathrm{GL}_2(\mathbb{C})$ let $\mathbb{H}^g := g^{-1}\mathbb{H}$ and let $j_g : \mathbb{H}^g \to \mathbb{C}$ be the function $j_g(z) = j(gz)$. For a tuple $\bar{g} = (g_1, \ldots, g_n) \in \mathrm{GL}_2(\mathbb{C})^n$ let $\mathbb{H}^{\bar{g}} := \mathbb{H}^{g_1} \times \cdots \times \mathbb{H}^{g_n}$ and consider the function

$$j_{\bar{g}} : \mathbb{H}^{\bar{g}} \to \mathbb{C}^n : (z_1, \ldots, z_n) \mapsto (j_{g_1}(z_1), \ldots, j_{g_n}(z_n)).$$

We let $\Gamma_j^{\bar{g}} \subseteq \mathbb{H}^{\bar{g}} \times \mathbb{C}^n$ denote the graph of $j_{\bar{g}}$.
Then $\mathrm{B}_j^G = \bigcup_{\bar{g} \in G^n} \Gamma_j^{\bar{g}}$.

Theorem (Uniform Ax-Schanuel for j)

Let $(V_{\bar{s}})_{\bar{s} \in Q}$ be a parametric family of algebraic varieties in $\mathbb{C}^{2n}$. Then there is a finite collection Σ of proper special subvarieties of $\mathbb{C}^n$ such that for every $\bar{s} \in Q(\mathbb{C})$ and every $\bar{g} \in \mathrm{GL}_2(\mathbb{C})^n$, if U is an analytic component of the intersection $V_{\bar{s}} \cap \Gamma_j^{\bar{g}}$ with $\dim U > \dim V_{\bar{s}} - n$, and no coordinate is constant on $\mathrm{pr}_w U$, then $\mathrm{pr}_w U$ is contained in some $T \in \Sigma$.

This is equivalent to a differential algebraic statement which follows from (differential) Ax-Schanuel by a compactness argument.

EC for blurred j – proof

Let $\mathcal{G} := \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{C})$.

Theorem

If $V \subseteq \mathbb{C}^{2n}$ is a j-broad and j-free variety and G is a dense subgroup of $\mathcal{G}$ in the complex topology then $V \cap \mathrm{B}_j^G \neq \emptyset$.

- By j-broadness $\dim V \geq n$. We may assume $\dim V = n$ by intersecting V with generic hyperplanes and reducing its dimension.
- Pick a fundamental domain $\mathbb{F} \subseteq \mathbb{H}$ and let $j^{-1} : \mathbb{C} \to \mathbb{F}$ be the inverse of j. It is holomorphic on $\mathbb{C}' := j(\mathbb{F}^0)$.
- Define a map $\theta : \mathbb{C}^{2n} \to \mathcal{G}^n : (\bar{z}, \bar{w}) \mapsto (g_1, \ldots, g_n)$, where

$$g_k := \begin{pmatrix} 1 & j^{-1}(w_k) - z_k \\ 0 & 1 \end{pmatrix} \in \mathcal{G}.$$

- Clearly, $j(g_k z_k) = w_k$, so $(z_k, w_k) \in \Gamma_j^{g_k}$.

Proof (continued)

- For $\bar{k} = (k_1, \ldots, k_l) \subseteq (1, \ldots, n)$ and $\bar{s} \in \mathrm{pr}_{\bar{k}} V \subseteq \mathbb{C}^{2l}$ consider the fibre $V_{\bar{s}} \subseteq \mathbb{C}^{2(n-l)}$ above $\bar{s}$. This gives a parametric family of algebraic varieties. Let $\Sigma_{\bar{k}}$ be the collection of special subvarieties of $\mathbb{C}^{n-l}$ given by uniform Ax-Schanuel for this family.
- By the fibre dimension theorem there is a proper Zariski closed subset $W_{\bar{k}}$ of $\mathrm{pr}_{\bar{k}} V$ such that if $\bar{s} \notin W_{\bar{k}}$ then $\dim V_{\bar{s}} = \dim V - \dim \mathrm{pr}_{\bar{k}} V \leq n - l$ where the last inequality follows from the assumption that V is j-broad.
- Consider the set

$$V' := V^{\mathrm{reg}} \cap \left\{ \bar{e} \in V : \mathrm{pr}_{\bar{k}} \bar{e} \notin W_{\bar{k}}, \ \ \mathrm{pr}_w \mathrm{pr}_{\bar{k}} \bar{e} \notin \bigcup_{S \in \Sigma_{\bar{k}}} S, \ \text{ for all } \bar{k} \right\}.$$

  Then V' is a Zariski open subset of V and $V' \neq \emptyset$ as V is j-free.
- This allows us to apply Ax-Schanuel and the fibre dimension theorem.

Proof (continued)

- **Claim.** The fibres of the restriction $\zeta := \theta|_{V'}$ are discrete.
  Proof. Indeed, $(\zeta^{-1})(\bar{g}) \subseteq V' \cap \Gamma_j^{\bar{g}}$. Let U be an analytic component. Assume no coordinate is constant on U. Then, by uniform Ax-Schanuel, $\dim U = \dim V' - n = 0$. If U has constant coordinates then we work with a fibre of U above those constants.
- By Remmert's open mapping theorem the map $\zeta : V' \to \mathcal{G}^n$ is open (since $\dim V' = \dim \mathcal{G}^n = n$).
- Therefore $\zeta(V') \cap G^n \neq \emptyset$ and $V' \cap \mathrm{B}_j^G \neq \emptyset$.

EC for j – real version

Consider the group $\mathcal{G} := \begin{pmatrix} \mathbb{R}^{>0} & \mathbb{R} \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2^+(\mathbb{R})$.

Theorem

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an irreducible j-broad and j-free variety defined over $\mathbb{C}$, and let $G \subseteq \mathcal{G}$ be a dense subgroup (in the Euclidean topology). Then $V \cap \mathrm{B}_j^G \neq \emptyset$. In particular, this holds for $G = \mathrm{GL}_2(\mathbb{Q}) \cap \mathcal{G}$.

- Assume $\dim V = n$.
- Let $V' \subseteq V$ be a non-empty Zariski open subset defined as above. Recall that restricting to V' allows us to apply Ax-Schanuel and the fibre dimension theorem.

Proof (continued)

- Pick a fundamental domain $\mathbb{F} \subseteq \mathbb{H}$ and let $j^{-1} : \mathbb{C} \to \mathbb{F}$ be the inverse of j. It is definable in $\mathbb{R}_{\text{an,exp}}$.
- $\mathcal{G}$ acts transitively on $\mathbb{H}$. Let $z_1 = x + iy$ and $z_2 = u + iv$ where $x, u \in \mathbb{R}$, $y, v \in \mathbb{R}^{>0}$. Then

$$g(z_1, z_2) := \begin{pmatrix} \frac{v}{y} & u - \frac{xv}{y} \\ 0 & 1 \end{pmatrix} \in \mathcal{G}$$

  maps z_1 to z_2, and it is the only element of $\mathcal{G}$ with that property.
- Define a map

$$\theta : \mathbb{H}^n \times \mathbb{C}^n \to \mathcal{G}^n,$$
$$\theta : (\bar{z}, \bar{w}) \mapsto (g(z_1, j^{-1}(w_1)), \dots, g(z_n, j^{-1}(w_n))),$$

  and let $\zeta := \theta|_{V'}$ be the restriction of θ to V'.
- ζ is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof (continued)

- **Claim.** The fibres of the restriction $\zeta := \theta|_{V'}$ are finite.
  Proof. As above, $(\zeta^{-1})(\bar{g}) \subseteq V' \cap \Gamma_j^{\bar{g}}$. By Ax-Schanuel, $(\zeta^{-1})(\bar{g})$ is discrete. By o-minimality it must be finite.
- Thus, $\zeta : V' \to \mathcal{G}^n$ has finite fibres and $\dim_{\mathbb{R}} V' = 2n$.
- Hence $\dim_{\mathbb{R}} \zeta(V') = 2n = \dim_{\mathbb{R}} \mathcal{G}^n$ and so $\zeta(V') \subseteq \mathcal{G}^n$ has non-empty interior.
- Since $G \subseteq \mathcal{G}$ is dense, $G^n \cap \zeta(V') \neq \emptyset$ and and $V' \cap \mathrm{B}_j^G \neq \emptyset$.

j-derivations

Definition

A j-derivation on the field of complex numbers is a derivation $\delta : \mathbb{C} \to \mathbb{C}$ such that for any $z \in \mathbb{H}$ we have

$$\delta j(z) = j'(z)\delta(z), \ \ \delta j'(z) = j''(z)\delta(z), \ \ \delta j''(z) = j'''(z)\delta(z).$$

The space of j-derivations is denoted by $j\mathrm{Der}(\mathbb{C})$.

Let

$$C := \bigcap_{\delta \in j\mathrm{Der}(\mathbb{C})} \ker \delta.$$

Then C is a countable algebraically closed subfield of $\mathbb{C}$ and $j(C \cap \mathbb{H}) = C$. This fact and the above definition are due to Eterović.

Stability

Theorem

Let C be as above and $G = \mathrm{GL}_2(C)$. Then $\mathbb{C}_{\mathrm{B}_j^G}$ is elementarily equivalent to a reduct of a differentially closed field. In particular, $\mathrm{Th}(\mathbb{C}_{\mathrm{B}_j^G})$ is ω-stable of Morley rank ω and is near model complete.

We also get an axiomatisation of $\mathrm{Th}(\mathbb{C}_{\mathrm{B}_j^G})$. It consists of basic axioms, functional equations of j, Ax-Schanuel over C (follows from Ax-Schanuel, and also from a theorem of Eterović), and Existential Closedness.
A similar theorem holds for $\mathbb{C}_{\mathrm{B}_j^G}$.

Quasiminimality

Theorem

Let C be as above and $G = \mathrm{GL}_2(C)$. Then the structures $\mathbb{C}_{\mathrm{B}_j^G}$ and $\mathbb{C}_{\mathrm{B}_J^G}$ are quasiminimal (every definable set is countable or co-countable).

Question

For which proper subgroups G of $\mathrm{PGL}_2(\mathbb{C})$ is $\mathbb{C}_{\mathrm{B}_j^G}$ quasiminimal?

- When G is uncountable, the fibres of B_j^G above the second coordinate are uncountable.
- If $G \subseteq \mathrm{PGL}_2(\mathbb{R})$, then the projection of B_j^G on the first coordinate is $\mathbb{H}$.
- When G is finite then the fibres of B_j^G above the first coordinate may be finite and of different cardinalities which allows one to define an uncountable co-uncountable set.
- It seems plausible that $\mathbb{C}_{\mathrm{B}_j^G}$ is quasiminimal if and only if $G \not\subseteq \mathrm{PGL}_2(\mathbb{R})$ and G is countably infinite.

Thank you