Projective Spaces

1 Definition

Real/complex projective space $\mathbb{RP}^n/\mathbb{CP}^n$ is the set of lines in $\mathbb{R}^{n+1}/\mathbb{C}^{n+1}$ passing through the origin. Given a Cartesian coordinate system $(x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}/\mathbb{C}^{n+1}$ there are $n+1$ natural charts of $\mathbb{RP}^n/\mathbb{CP}^n$ defined as follows:

$$\xi_i^{(k)} = \frac{x_i}{x_k}, \quad x_k \neq 0, \tag{1}$$

with the transition functions between the j-th and k-th charts:

$$\xi_i^{(k)} = \frac{\xi_i^{(j)}}{\xi_k^{(j)}} \tag{2}$$

For the most dimensions the *atlas* of this charts is overcomplete [1].

Although they share a number of common properties, the real and complex projective spaces are different. To illustrate this, let us consider the relation of these spaces to spheres. An n-dimensional sphere can be defined as the set of rays (not lines!) starting from the origin. Indeed, a ray intersects the sphere with the center in the origin in a single point and hence uniquely specifies the point of the sphere. Contrary, a point on the sphere uniquely identifies a ray from the origin. On the other hand, a line passing through the origin intersects the sphere in two antipodal points. Hence, we can think about the real projective space as about a sphere, where we identified all antipodal points (all rays which are on the same line). In the language of topological fibrations this is written as follows:

$$\mathbb{RP}^n = S^n/\mathbb{Z}^2, \tag{3}$$

where $\mathbb{Z}^2$ is the cyclic group $(-1, 1)$ of order 2. Since $\mathbb{RP}^1 = S^1$, for $n = 1$ this is the famous Möbius strip.

A similar construction relates $2n+1$ dimensional spheres and complex projective spaces:

$$\mathbb{CP}^n = S^{2n+1}/S^1. \tag{4}$$

This is the so-called first Hopf's fibration.

This example illustrates that the complex projective spaces have richer geometry than real projective ones. On the other hand,

2 Homogeneous coordinates

Two points x and αx ($x \in \mathbb{R}, \mathbb{C}$) are on the same line passing through the origin. Hence, from the perspective of the projective space, the coordinates

$(x_1, \dots, x_{n+1})$ are regarded as *homogeneous coordinates*, in contrast to *inhomogeneous coordinates* ξ (1). Clearly, the *homogeneous coordinates* are defined up to an arbitrary multiplier $\alpha \neq 0$. Throughout the post, we will use *homogeneous* and *inhomogeneous* coordinates interchangeably.

3 Hyperplanes as dual projective space

Consider an n-dimensional hyperplane of $\mathbb{R}^{n+1}/\mathbb{C}^{n+1}$ passing through the origin. An n dimensional plane is defined via the following linear constraint:

$$\sum_{i=1}^{n+1} a_i x_i \equiv a \cdot x = 0 \tag{5}$$

Obviously, the parameters a are defined up to a scalar multiplier, i.e. $\forall \alpha \neq 0$ the parameters a and αa define the same hyperplane. This means, that the set of n-dimensional hyperplanes passing through the origin form a dual projective space $\mathbb{RP}^{n*}/\mathbb{CP}^{n*}$ with a's being its *homogeneous coordinates.*

An invertable linear transformation $H : x \to Hx$ with respect to (5) transforms the dual space as follows:

$$x \to Hx, \quad a \to H^{-T} a. \tag{6}$$

In the language of vector analysis they say that the hyperplanes transform covariantly[1]

4 Conics

Consider the following pure quadratic equation:

$$x^{\dagger} C x = 0, \tag{7}$$

where C is an $(n+1) \times (n+1)$ dimensional matrix and $x^{\dagger} \equiv \bar{x}^T$ is the so-called Hermitian conjugate of x - the complex conjugate transposed. Since the complex conjugate of a real number equals to itself, for real coordinates x the operation $\cdot^{\dagger}$ coincides with the transpose.

It is clear, that

$$C^{\dagger} \equiv \bar{C}^T = C, \tag{8}$$

because the anti-Hermitian part vanishes by itself.

Notice, that since the complex conjugate of the real numbers equal to themselves, for real matrices C the condition (8) coincides with (**??**). Because of this, below we will only use notations for complex coordinates and complex matrices having in mind, that for real coordinates we can re

This equation defines a conic surface in $\mathbb{F}^{n+1}$ which in its turn is projected on the corresponding $\mathbb{CF}^n$.

[1] Strictly speaking, the co/contra-variant transformations are defined for special geometric objects called tensors(vectors) only. However, since we restrict ourselves to considering only linear transformations of the coordinates, we can use those terms with regard to the coordinates and their duals.

Firstly, let us observe, that the matrix C is defined up to an arbitrary multiplier. Indeed, the transformation $C \to \alpha C$, $\alpha \neq 0$ does not affect the equation (7).

Secondly, an arbitrary linear transformation $x \to Hx$ transforms the matrix C as follows:

$$C \to H^{-\dagger} C H^{-1}. \tag{9}$$

On the other hand, according to the Spectral Theorem, the matrix which diagonalizes a Hermitian matrix C is a unitary matrix, i.e.

$$U^{-1} C U = \mathrm{diag}\{\alpha_1, \dots, \lambda_{n+1}\}, \quad U^\dagger U = \mathbb{1}. \tag{10}$$

This means, that if a matrix U diagonlaizes C, then we can choose $H = U^{-1}$ and that will transform C to a diagonal matrix according to (9).

The unitary matrices are complex analogues of the orthogonal matrices R, which, instead satify

$$R^T R = \mathbb{1}. \tag{11}$$

Now, let the matrix H diagonalizes C. Then, in the new coordinate system $\tilde{x} = Hx$ we have:

$$\sum_{i=1}^{n+1} \lambda_i \tilde{x}_i^2 = 0 \tag{12}$$

If the matrix C is positive definite, then all solutions of (7) are pure imaginary. Indeed, let the matrix A diagonalize C, i.e.

$$d \tag{13}$$

5 $\mathbb{RP}^2$

The directions of the axes ξ_1 and x_1 coincide in the world $\mathbb{R}^3$ as well as those for the axes ξ_2 and x_2. The axis x_3 is pointed from the viewer and coincides with the principal axis of the camera lens.

The form of the projection (**??**) hints that we can think about a photo as about a chart of $\mathbb{RP}^2$. Indeed, all points on a line passing through the center of the lens (the coordinate origin) are projected on a single point on the camera sensor. A line in the world $\mathbb{R}^3$ on the photo is indistinguishable from a plane passing through the lens center and containing that line.

In the section below I will prove the following statement. Let us have n pairs of parallel lines on a plane in the $\mathbb{R}^3$; we can draw them on the paper and put on a table. If we make a photo of those lines from an arbitrary perspective they will intersect. The interesting fact about the intersection points is that they all lie on a single line!

6 $\mathbb{RP}^2$

Real projective space $\mathbb{RP}^2$ is the space of lines in $\mathbb{R}^3$ passing through the origin. The coordinates of $\mathbb{R}^3$ (x, y, z) are homogeneous coordinates of $\mathbb{RP}^2$, i.e. a point (x, y, z) and all points $s(x, y, z)$, $s \neq 0$ correspond to a point of $\mathbb{RP}^2$.

Planes in $\mathbb{R}^3$ passing through the origin are defined as

$$\sum a_i x_i \equiv a \cdot x = 0. \tag{14}$$

Obviously, the quantities a_i are defined up to an arbitrary multiplier and hence, the planes form a dual $\mathbb{RP}^2$.

Consider a chart of $\mathbb{RP}^2$

$$\xi_1 = \frac{x_1}{x_3}, \quad \xi_2 = \frac{x_2}{x_3}, \quad x_3 \neq 0 \tag{15}$$

A plane a is projected on this chart as a line

$$a_1\xi_1 + a_2\xi_2 + a_3 = 0 \tag{16}$$

Here and further we will refer to planes a as "lines" meaning their projection on the certain chart. However, we should always keep in mind that not all planes have their projection on the chart. In particular, the plain

$$n^{(\infty)} = (0, 0, 1), \quad \text{or} \quad x_3 = 0 \tag{17}$$

does not project on the chart (15) and therefore cannot be referred as a "line". In the science of Computer Vision they call it a "line at infinity" (hence the superscript).

Two lines on $\mathbb{RP}^2$ a and b are parallel on the chart (15) if

$$(a \times b) \cdot n^{(\infty)} = 0, \tag{18}$$

Alternatively, the lines on $\mathbb{RP}^2$ intersect in the point

$$x = a \times b \tag{19}$$

In the language of lines and planes in $\mathbb{R}^3$ this can be interpreted as follows. Obviously, any two planes passing through the origin intersect. The intersection of the planes is the line (19) passing through the origin. If the intersection line lies in the plane (17) then the projections of the planes on the chart (15) are parallel.

The same result in the language of lines on the charts of $\mathbb{RP}^2$ is interpreted as follows. The invariant form of (19) shows that, generally speaking, all lines on $\mathbb{RP}^2$ intersect, however if the condition (18) holds, then the intersection point is not present on the considered chart.

Now let us consider the most generic invertable transformations

$$x' = Hx, \quad \det H \neq 0. \tag{20}$$

Obviously a plane a under this transformation is "moving in the opposite direction"

$$a' = H^{-T} a \tag{21}$$

These kind of transformations are called homographies and clearly form a group.

Now, let two lines a and b be parallel, i.e. satisfy (18), and let H be a homography such that

$$H^{-T} \cdot a, b, n^{(\infty)} \neq \alpha n^{(\infty)}, \quad \forall \alpha \in \mathbb{R} \tag{22}$$

It is clear that the condition (18) is not invariant under the transformation (21). Namely,

$$(a' \times b') \cdot n^{(\infty)} \neq 0 \tag{23}$$

and, hence, the transformed lines a' and b' intersect in the point (see(19)):

$$x' = a' \times b' \tag{24}$$

It is easy to show that

$$(H^{-T} n^{(\infty)}) \cdot x' = 0 \tag{25}$$

and, therefore, the point of $\mathbb{RP}^2$ x' on the chart (15) lies on the line $H^{-T} n^{(\infty)}$.

Note, that while the plane $n^{(\infty)}$ does not have a projection on the chart (15) the projection of the transformed line $H^{-T} n^{(\infty)}$ is well defined. In other words, a generic homography moves the line at infinity.

7 Result

The result is illustrated in the Fig 2

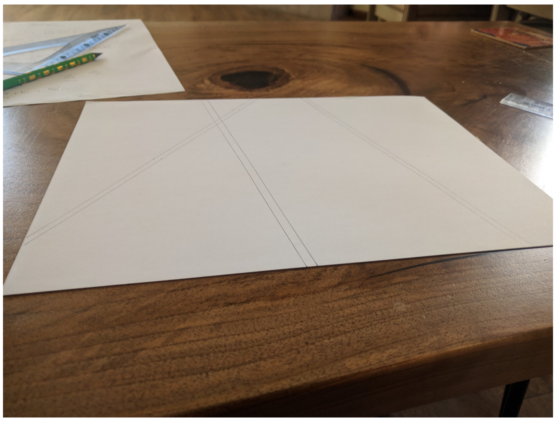

Figure 1: Parallel lines taken from an arbitrary perspective.

References

[1] Hopkins M.J. (1989) Minimal atlases of real projective spaces. In: Carlsson G., Cohen R., Miller H., Ravenel D. (eds) Algebraic Topology. Lecture Notes in Mathematics, vol 1370. Springer, Berlin, Heidelberg

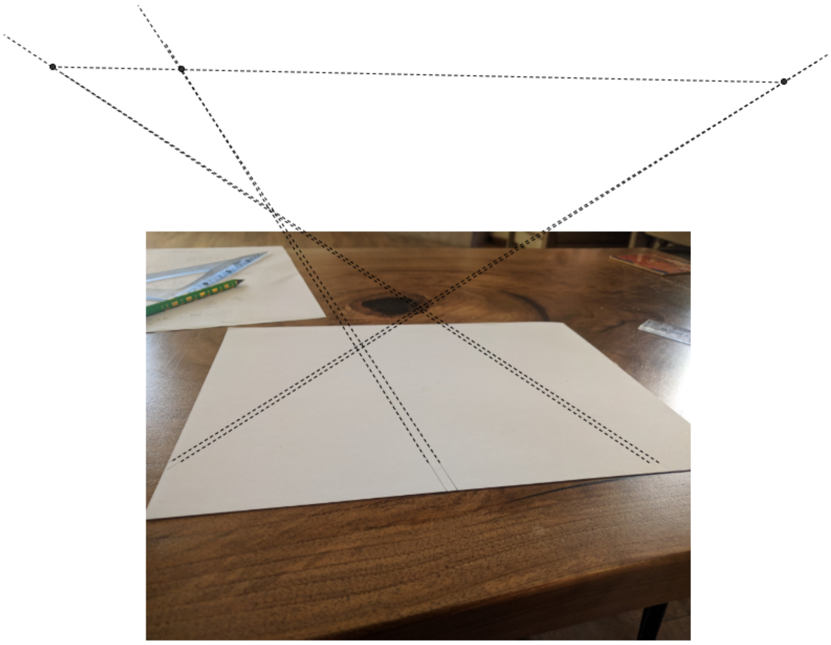

Figure 2: The parallel lines intersect on the line at infinity.