Real Projective Space

1 Definition

Real projective space $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. Given a Cartesian coordinate system $(x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ there are $n+1$ natural charts of $\mathbb{RP}^n$ defined as follows:

$$\xi_i^{(k)} = \frac{x_i}{x_k}, \quad x_k \neq 0, \tag{1}$$

with the transition functions between the j-th and k-th charts:

$$\xi_i^{(k)} = \frac{\xi_i^{(j)}}{\xi_k^{(j)}} \tag{2}$$

For the most dimensions the *atlas* of this charts is overcomplete [1].

2 Homogeneous coordinates

Two points x and αx are on the same line passing through the origin. Hence, from the perspective of the projective space, the coordinates $(x_1, \ldots, x_{n+1})$ are regarded as *homogeneous coordinates*, in contrast to *inhomogeneous coordinates* ξ (1). Clearly, the *homogeneous coordinates* are defined up to an arbitrary multiplier $\alpha \neq 0$. Throughout the post, we will use *homogeneous* and *inhomogeneous* coordinates interchangeably.

3 Hyperplanes as dual projective space

Consider an n-dimensional hyperplane of $\mathbb{R}^{n+1}$ passing through the origin. It is defined as follows:

$$\sum_{i=1}^{n+1} a_i x_i \equiv a \cdot x = 0 \tag{3}$$

Obviously, the parameters a are defined up to a scalar multiplier, i.e. $\forall \alpha \neq 0$ the parameters a and αa define the same hyperplane. This means, that the set of n-dimensional hyperplanes passing through the origin form a dual projective space $\mathbb{RP}^{n*}$ with a's being its *homogeneous coordinates*.

An invertable linear transformation $H : x \to Hx$ with respect to (3) transforms the dual space as follows:

$$x \to Hx, \quad a \to H^{-T}a. \tag{4}$$

In the language of vector analysis they say that the hyperplanes transform covariantly.

4 Conics on $\mathbb{RP}^n$

Let C be an $n + 1 \times n + 1$ dimensional matrix. Then, if C is not strictly positive, or strictly negative definite, then the equation

$$x^T C x = 0 \tag{5}$$

defines a conic surface.

5 $\mathbb{RP}^2$

In this letter I want to share an interesting fact about the parallel lines in photos and their intersection points.

To simplify things, the photos will be considered taken by a pinhole camera, which is true under certain very weak conditions for almost all modern cameras. The coordinates (ξ_1, ξ_2) on the photo are related to the world coordinates of the camera as follows:

$$\xi_1 = f \frac{x_1}{x_3}, \quad \xi_2 = f \frac{x_2}{x_3}, \tag{6}$$

where f is the focal length of the camera lens.

The directions of the axes ξ_1 and x_1 coincide in the world $\mathbb{R}^3$ as well as those for the axes ξ_2 and x_2. The axis x_3 is pointed from the viewer and coincides with the principal axis of the camera lens.

The form of the projection (6) hints that we can think about a photo as about a chart of $\mathbb{RP}^2$. Indeed, all points on a line passing through the center of the lens (the coordinate origin) are projected on a single point on the camera sensor. A line in the world $\mathbb{R}^3$ on the photo is indistinguishable from a plane passing through the lens center and containing that line.

In the section below I will prove the following statement. Let us have n pairs of parallel lines on a plane in the $\mathbb{R}^3$; we can draw them on the paper and put on a table. If we make a photo of those lines from an arbitrary perspective they will intersect. The interesting fact about the intersection points is that they all lie on a single line!

6 $\mathbb{RP}^2$

Real projective space $\mathbb{RP}^2$ is the space of lines in $\mathbb{R}^3$ passing through the origin. The coordinates of $\mathbb{R}^3$ (x, y, z) are homogeneous coordinates of $\mathbb{RP}^2$, i.e. a point (x, y, z) and all points $s(x, y, z)$, $s \neq 0$ correspond to a point of $\mathbb{RP}^2$.

Planes in $\mathbb{R}^3$ passing through the origin are defined as

$$\sum a_i x_i \equiv a \cdot x = 0. \tag{7}$$

Obviously, the quantities a_i are defined up to an arbitrary multiplier and hence, the planes form a dual $\mathbb{RP}^2$.

Consider a chart of $\mathbb{RP}^2$

$$\xi_1 = \frac{x_1}{x_3}, \quad \xi_2 = \frac{x_2}{x_3}, \quad x_3 \neq 0 \tag{8}$$

A plane a is projected on this chart as a line

$$a_1 \xi_1 + a_2 \xi_2 + a_3 = 0 \tag{9}$$

Here and further we will refer to planes a as "lines" meaning their projection on the certain chart. However, we should always keep in mind that not all planes have their projection on the chart. In particular, the plain

$$n^{(\infty)} = (0, 0, 1), \quad \text{or} \quad x_3 = 0 \tag{10}$$

does not project on the chart (8) and therefore cannot be referred as a "line". In the science of Computer Vision they call it a "line at infinity" (hence the superscript).

Two lines on $\mathbb{RP}^2$ a and b are parallel on the chart (8) if

$$(a \times b) \cdot n^{(\infty)} = 0, \tag{11}$$

Alternatively, the lines on $\mathbb{RP}^2$ intersect in the point

$$x = a \times b \tag{12}$$

In the language of lines and planes in $\mathbb{R}^3$ this can be interpreted as follows. Obviously, any two planes passing through the origin intersect. The intersection of the planes is the line (12) passing through the origin. If the intersection line lies in the plane (10) then the projections of the planes on the chart (8) are parallel.

The same result in the language of lines on the charts of $\mathbb{RP}^2$ is interpreted as follows. The invariant form of (12) shows that, generally speaking, all lines on $\mathbb{RP}^2$ intersect, however if the condition (11) holds, then the intersection point is not present on the considered chart.

Now let us consider the most generic invertable transformations

$$x' = Hx, \quad \det H \neq 0. \tag{13}$$

Obviously a plane a under this transformation is "moving in the opposite direction"

$$a' = H^{-T} a \tag{14}$$

These kind of transformations are called homographies and clearly form a group.

Now, let two lines a and b be parallel, i.e. satisfy (11), and let H be a homography such that

$$H^{-T} \cdot a, b, n^{(\infty)} \neq \alpha n^{(\infty)}, \quad \forall \alpha \in \mathbb{R} \tag{15}$$

It is clear that the condition (11) is not invariant under the transformation (14). Namely,

$$(a' \times b') \cdot n^{(\infty)} \neq 0 \tag{16}$$

and, hence, the transformed lines a' and b' intersect in the point (see(12)):

$$x' = a' \times b' \tag{17}$$

It is easy to show that

$$(H^{-T}n^{(\infty)}) \cdot x' = 0 \tag{18}$$

and, therefore, the point of $\mathbb{RP}^2$ x' on the chart (8) lies on the line $H^{-T}n^{(\infty)}$.

Note, that while the plane $n^{(\infty)}$ does not have a projection on the chart (8) the projection of the transformed line $H^{-T}n^{(\infty)}$ is well defined. In other words, a generic homography moves the line at infinity.

7 Result
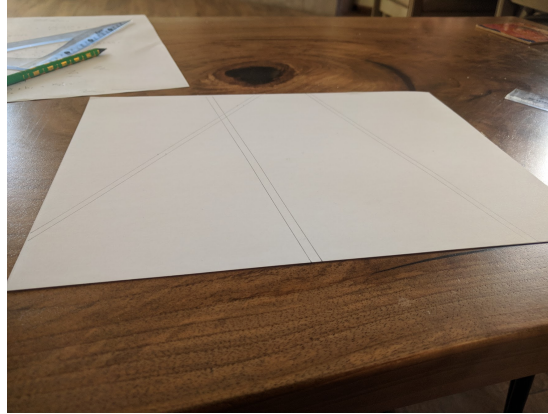
The result is illustrated in the Fig 2

Figure 1: Parallel lines taken from an arbitrary perspective.

References

[1] Hopkins M.J. (1989) Minimal atlases of real projective spaces. In: Carlsson G., Cohen R., Miller H., Ravenel D. (eds) Algebraic Topology. Lecture Notes in Mathematics, vol 1370. Springer, Berlin, Heidelberg
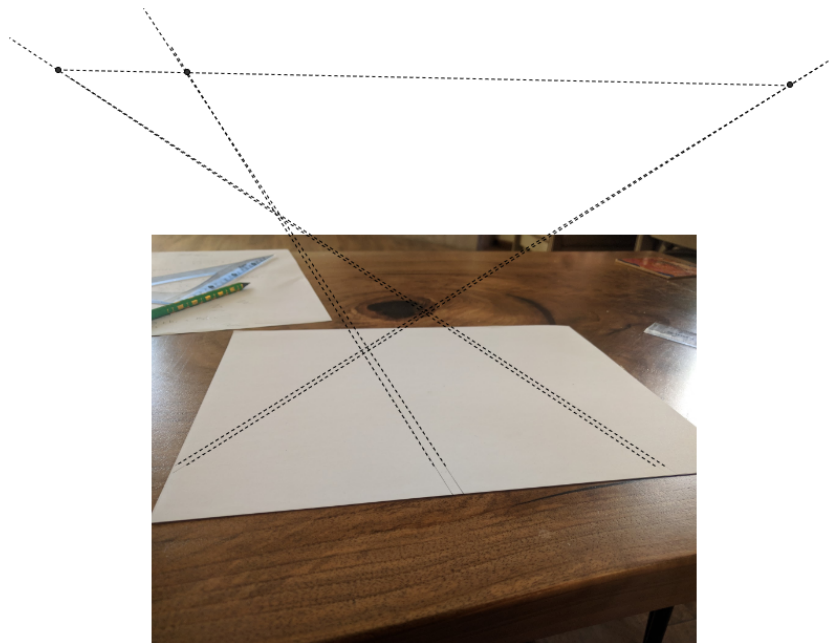
Figure 2: The parallel lines intersect on the line at infinity.