1 Manifold

In this post I would like to speak about an abstract mathematical construction called *manifold*. In the future we will work with a limited number of manifolds, however, I believe that introducing some fundamental concepts of the theory of manifolds is, at least, useful.

Let us call a *topological space* a set of points where there are defined all open subsets. We will not go any deeper into this definition. For more information about the topological spaces like what is an *open subset* check the wikipedia article [1].

On the other hand, we are not interested in the most general topological spaces rather those on which we can define *coordinate systems*. We are familiar with the Euclidean space. See Figure 1

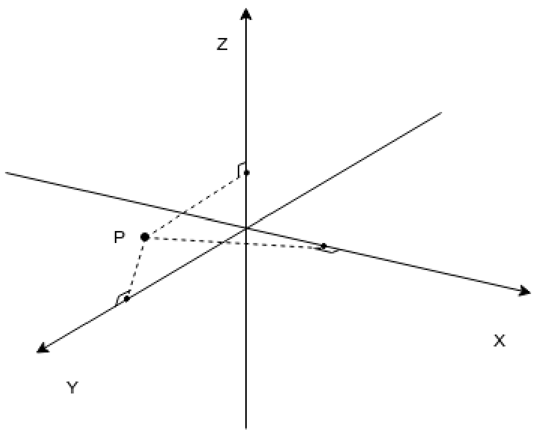

Figure 1: The 3-dimensional Euclidean space with Cartesian coordinate axis. The 3 coordinates (X, Y, Z) are determined by passing an pependiculars to the corresponding axis and measuring the distance between the intersection points and the origin.

To any point in the Euclidean space $\mathbb{R}^n$ we can uniquely assign Cartesian coordinates $(X_1, X_2, \ldots, X_n)$. Conversely, the set of Cartesian coordinates $(X_1, X_2, \ldots, X_n)$ uniquely determine a point in the Euclidean space.

Since we now, how to introduce the coordinate systems in $\mathbb{R}^n$ let us see how we can utilize it for defining a coordinate system on a topological space M. If we can define a continuous (maps open subsets to open subsets) invertable map φ from an open subset U of $\mathbb{R}^n$ onto an open subset of M then this will cover that subset with a coordinate system. Indeed, for any coordinates $(X_1, X_2, \ldots, X_n)$ we will have a unique point in $\mathbb{R}^n$, which in its turn has a unique point on the subset $\varphi U \subset M$.

This mapping, together with the open subset $U \subset \mathbb{R}^n$ is called a *chart* on M. If we can cover the entire M with finite or countable number of such charts so that any point on M will be covered by at least one of them, then we will have coordinate system on the entire M. In order to specify a point on M one has to specify the chart $(\varphi^i, U^{(i)})$ and coordinates $(X_1^{(i)}, \ldots, X_n^{(i)}) \in U^{(i)}$.

The set of the charts is called *atlas*. A topological space with an *atlas* is

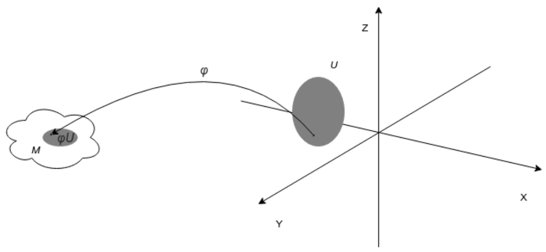

Figure 2: Mapping from an open subset of 3D Euclidean space onto an open subset of M

called *manifold*. In other words, a *manifold* is a topological space, which *locally* (in a neighborhood of a point) looks like an Euclidean space.

If some point of M is on two charts U and U' then there are some neighborhoods V and V' of that point on each of the charts such that all points in that neighborhoods are also pictured on both charts. Hence, we have a natural mapping $\phi'^{-1}\phi : V \mapsto V'$ of a part $V \subset U$ of one chart on a part of the other $V' \subset U'$ [2].

Let us consider a few examples.

2 Examples

Euclidean space

Obviously, the Euclidean space is a manifold by itself.

Sphere

A unit sphere in 3D-space is defined by the following constraint:

$$X^2 + Y^2 + Z^2 = 1 \tag{1}$$

Obviously, the sphere is a 2-dimensional manifold. To define a chart on the sphere, consider a 2D-plane $\mathbb{R}^2$. The Cartesian coordinate system which x and y axis coincide with X and Y respectively. Let us connect the point P' on (x, y) plane to the North Pole with a line. The corresponding point P on the sphere would be the point of the intersection of that line with the sphere. Figure 3.

In this way, we establish a correspondence between all points on $\mathbb{R}^{\nvDash} \equiv U$ with all points of sphere except the North Pole itself. To cover the whole sphere with charts we should repeat our construction using the South Pole instead. This second chart will cover the entire sphere except the South Pole.

For completeness, let us present the mapping from (X, Y, Z) coordinates of the ambient $\mathbb{R}^3$ to the chart coordinates (x, y).

$$x = \frac{X}{1 \pm Z}, \quad y = \frac{y}{1 \pm Z} \tag{2}$$

and the inverse

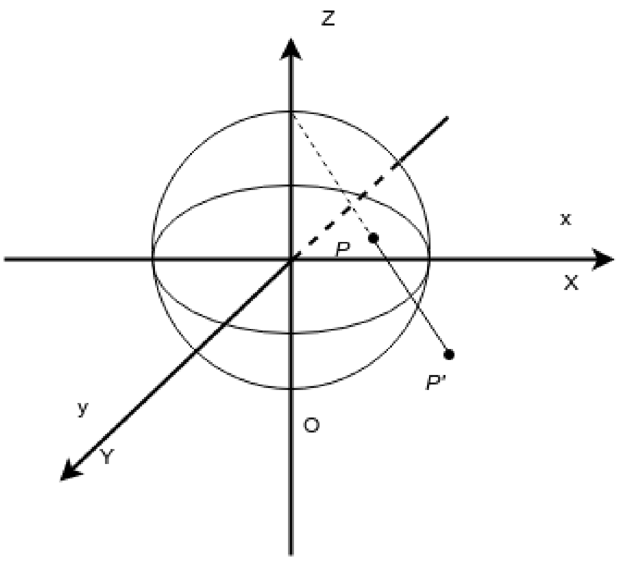

Figure 3: The projective coordinates on 2D-sphere.

$$X = \frac{2x}{1+\rho^2}, \quad Y = \frac{2y}{1+\rho^2}, \quad Z = \frac{1 \mp \rho^2}{1 \pm \rho^2}, \tag{3}$$

where $\rho^2 = x^2 + y^2$

A straightforward generalization of this construction exists for any dimensional spheres.

Projective space

Real projective space $\mathbb{RP}^n$[1] is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. A point $x \in \mathbb{R}^{n+1}, x \neq 0$ uniquely defines a line passing through the origin. On the other hand, the same line is defined by any other point $x' = \alpha x$.

To eliminate this arbitrariness, we can divide all coordinates by a single chosen one, thus introducing a chart on $\mathbb{RP}^n$.

$$\xi_i^{(k)} = \frac{x_i}{x_k}, \quad i \neq k, \quad i, k = 1, \dots, n+1 \tag{4}$$

This chart is not defined for points for which $x_k = 0$, therefore, those points are not present on it.

3 Examples (non-manifolds)

1-dimensional spaces with junctions are not manifolds. Intersecting 2-planes are not manifolds.

[1] In contrast to complex projective space $\mathbb{CP}^n$, which is out of our consideration.

References

[1] Wiki: Topological space

[2] Arnold, Mathematical Methods in Classical Mechanics