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**Abstract**

# 1 Camera geometry

Camera Matrix has the following form and depends only on the device itself.

$$K = \begin{pmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Pattern coordinates are defined as $\mathbf{C}^k$.

We perform $M$ measurements enumerated by index $\mu = 0, \dots, M-1$. For $\mu^{th}$ experiment we have the transformation matrix $\mathbf{R}_\mu$ and a translation vector $\mathbf{T}_\mu$ that transform the pattern points $\mathbf{C}$ to the camera coordinate system.

$$\mathbf{X}_\mu^k = \mathbf{R}_\mu \mathbf{C}^k + \mathbf{T}_\mu \tag{2}$$

The corresponding points are obtained via camera projection matrix as follows

$$\mathbf{U}_\mu^k = \mathbf{K}\mathbf{X}_\mu^k \equiv \mathbf{K}\left(\mathbf{R}_\mu \mathbf{C}^k + \mathbf{T}_\mu\right) \equiv \begin{pmatrix} u_\mu^k \\ v_\mu^k \\ w_\mu^k \end{pmatrix} \tag{3}$$

$$x_\mu^k = u_\mu^k / w_\mu^k, \quad y_\mu^k = v_\mu^k / w_\mu^k \tag{4}$$

Let us define $\tilde{x}, \tilde{y}$ - distorted points and

$$\rho^{(k)} = \sqrt{x^2 + y^2} \tag{5}$$

Radial distortion:

$$\begin{aligned} \tilde{x} &= x + x(k_1\rho^2 + k_2\rho^4 + k_3\rho^6) \\ \tilde{y} &= y + y(k_1\rho^2 + k_2\rho^4 + k_3\rho^6) \end{aligned} \tag{6}$$

Skew distortion:

$$\begin{aligned} \tilde{x} &= x + 2p_1 xy + p_2\left(\rho^2 + x^2\right) \\ \tilde{x} &= y + 2p_2 xy + p_1\left(\rho^2 + y^2\right) \end{aligned} \tag{7}$$

So, we have

$$\begin{aligned} \tilde{x}^k_\mu &\equiv \tilde{x}^k_\mu\left(R_\mu, T_\mu, C^k, K, k_1, k_2, k_3, p_1, p_2\right) \\ \tilde{y}^k_\mu &\equiv \tilde{y}^k_\mu\left(R_\mu, T_\mu, C^k, K, k_1, k_2, k_3, p_1, p_2\right) \end{aligned} \tag{8}$$

The error functions $\mathbf{e}^k_\mu$ read:

$$\mathbf{e}^k_\mu = \left( \begin{array}{c} \text{measured}(x^k_\mu) - \tilde{x}^k_\mu \\ \text{measured}(y^k_\mu) - \tilde{y}^k_\mu \end{array} \right) \tag{9}$$

And the function we have to minimize reads:

$$L = \sum_{k,\mu} \mathbf{e}^k_\mu \mathbf{e}^k_\mu \tag{10}$$

## 2 Calibration on a flat chessboard pattern

Let us consider a special case when the points $\mathbf{C}^k$ are the corners of a chessboard pattern:

$$\mathbf{C}^{pq} \equiv \mathbf{C}^k = a \begin{pmatrix} p \\ q \\ 0 \end{pmatrix} \tag{11}$$

where $p = 0, \ldots, N_x - 1$ and $q = 0, \ldots, N_y - 1$ enumerate corners in the pattern, $k = qN_x + p$ is unified index and $a$ is the length of the pattern's square size in any units of measure.

Since, on the pattern plane in its own coordinate system $Z = 0$, the last column of $\mathbf{R}$ does not play any role, and, the transformation (3) can be rewritten as:

$$\mathbf{U}^k_\mu = \mathbf{H}_\mu \tilde{\mathbf{C}}^k, \quad \tilde{\mathbf{C}}^k = \begin{pmatrix} c^k_1 \\ c^k_2 \\ 1 \end{pmatrix} \tag{12}$$

where

$$\mathbf{H} = \lambda \mathbf{K} \cdot \begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{pmatrix} \equiv \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} \tag{13}$$

is a homography matrix that maps the pattern plane onto the image plane. Here $\lambda$ is an arbitrary constant. Here we dropped the index $\mu$ for readability.

Each point correspondence $U_\mu^k \to C_\mu^k$ gives us a constraint on the elements of $\mathbf{H}$

Multiplying (13) from the left by $\mathbf{K}^-1$ and imposing that $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthonormal gives us 2 constraints on the homography matrix:

$$\begin{aligned} \mathbf{h}_1^T\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{h}_2 &= 0 \\ \mathbf{h}_1^T\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{h}_1 &= \mathbf{h}_2^T\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{h}_2 \end{aligned} \tag{14}$$

Let us define

$$\mathbf{B} = \mathbf{K}^{-T}\mathbf{K}^{-1} = \begin{pmatrix} \frac{1}{f_x^2} & -\frac{s}{f_x^2 f_y} & \frac{c_y s - c_x f_y}{f_x^2 f_y} \\ -\frac{s}{f_x^2 f_y} & \frac{s^2}{f_x^2 f_y^2} + \frac{1}{f_y^2} & -\frac{s(c_y s - c_x f_y)}{f_x^2 f_y^2} - \frac{c_y}{f_y^2} \\ \frac{c_y s - c_x f_y}{f_x^2 f_y} & -\frac{s(c_y s - c_x f_y)}{f_x^2 f_y^2} - \frac{c_y}{f_y^2} & \frac{(c_y s - c_x f_y)^2}{f_x^2 f_y^2} + \frac{c_y^2}{f_y^2} + 1 \end{pmatrix} \tag{15}$$

$$\mathbf{b} = (B_{11}, B_{12}, B_{22}, B_13, B_{23}, B_{33})^T \tag{16}$$

With this notations the constraints (14) can be rewritten as follows:

$$\mathbf{V}\mathbf{b} = 0, \tag{17}$$

where

$$\mathbf{V}^T = \begin{pmatrix} h_{11}h_{12} & h_{11}^2 - h_{12}^2 \\ h_{12}h_{21} + h_{11}h_{22} & 2(h_{11}h_{21} - h_{12}h_{22}) \\ h_{21}h_{22} & h_{21}^2 - h_{22}^2 \\ h_{12}h_{31} + h_{11}h_{32} & 2(h_{31}h_{11} - h_{12}h_{32}) \\ h_{31}h_{22} + h_{21}h_{32} & 2(h_{21}h_{31} - h_{22}h_{32}) \\ h_{31}h_{32} & h_{31}^2 - h_{32}^2 \end{pmatrix} \tag{18}$$

Stacking $n \geq 3$ equations we can obtain the vector $\mathbf{b}$ up to an arbitrary scalar.

# 3 Homography estimation by 4 point correspondences

Let us have 4 points $U^k = (u_1, u_2, 1)$ and $X^k = (X_1^k, X_2^k, 1)$ that are related via a homography matrix $\mathbf{H}$ as follows:

$$u_\alpha^k = \frac{\left(\mathbf{H}\mathbf{X}^k\right)_\alpha}{\left(\mathbf{H}\mathbf{X}^k\right)_3} \tag{19}$$

Taking into account the notation (13), this equation can be rewritten as follows:

$$\begin{pmatrix} X_1^k & X_2^k & 1 & 0 & 0 & 0 & -u_1^k X_1^k & -u_1^k X_2^k & -u_1^k \\ 0 & 0 & 0 & X_1^k & X_2^k & 1 & -u_2^k X_1^k & -u_2^k X_2^k & -u_2^k \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{pmatrix} = 0 \tag{20}$$

Stacking 4 such equations together we get an $8 \times 9$ matrix $\mathbf{L}$ and the combined equations can be rewritten as follows:

$$\mathbf{L}\mathbf{h} = 0 \tag{21}$$