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Abstract

1 Camera geometry

Camera Matrix has the following form and depends only on the device itself.

$$K = \begin{pmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Pattern coordinates are defined as $\mathbf{C}^k$.

We perform M measurements enumerated by index $\mu = 0, \ldots, M-1$. For μ^{th} experiment we have the transformation matrix $\mathbf{R}_\mu$ and a translation vector $\mathbf{T}_\mu$ that transform the pattern points $\mathbf{C}$ to the camera coordinate system.

$$\mathbf{X}^k_\mu = \mathbf{R}_\mu \mathbf{C}^k + \mathbf{T}_\mu \tag{2}$$

The corresponding points are obtained via camera projection matrix as follows

$$\mathbf{U}^k_\mu = \mathbf{K}\mathbf{X}^k_\mu \equiv \mathbf{K}\left(\mathbf{R}_\mu \mathbf{C}^k + \mathbf{T}_\mu\right) \equiv \begin{pmatrix} u^k_\mu \\ v^k_\mu \\ w^k_\mu \end{pmatrix} \tag{3}$$

$$x^k_\mu = u^k_\mu / w^k_\mu, \quad y^k_\mu = v^k_\mu / w^k_\mu \tag{4}$$

Let us define $\tilde{x}, \tilde{y}$ - distorted points and

$$\rho^{(k)} = \sqrt{x^2 + y^2} \tag{5}$$

Radial distortion:

$$\begin{aligned} \tilde{x} &= x + x(k_1\rho^2 + k_2\rho^4 + k_3\rho^6) \\ \tilde{y} &= y + y(k_1\rho^2 + k_2\rho^4 + k_3\rho^6) \end{aligned} \tag{6}$$

Skew distortion:

$$
\begin{aligned}
\tilde{x} &= x + 2p_1 xy + p_2\left(\rho^2 + x^2\right) \\
\tilde{x} &= y + 2p_2 xy + p_1\left(\rho^2 + y^2\right)
\end{aligned}
\tag{7}
$$

So, we have

$$
\begin{aligned}
\tilde{x}^k_\mu &\equiv \tilde{x}^k_\mu\left(R_\mu, T_\mu, C^k, K, k_1, k_2, k_3, p_1, p_2\right) \\
\tilde{y}^k_\mu &\equiv \tilde{y}^k_\mu\left(R_\mu, T_\mu, C^k, K, k_1, k_2, k_3, p_1, p_2\right)
\end{aligned}
\tag{8}
$$

The error functions $\mathbf{e}^k_\mu$ read:

$$
\mathbf{e}^k_\mu = \begin{pmatrix} \text{measured}(x^k_\mu) - \tilde{x}^k_\mu \\ \text{measured}(y^k_\mu) - \tilde{y}^k_\mu \end{pmatrix}
\tag{9}
$$

And the function we have to minimize reads:

$$
L = \sum_{k,\mu} \mathbf{e}^k_\mu \mathbf{e}^k_\mu
\tag{10}
$$

2 Calibration on a flat chessboard pattern

Let us consider a special case when the points $\mathbf{C}^k$ are the corners of a chessboard pattern:

$$
\mathbf{C}^{pq} \equiv \mathbf{C}^k = a \begin{pmatrix} p \\ q \\ 0 \end{pmatrix}
\tag{11}
$$

where $p = 0, \ldots, N_x - 1$ and $q = 0, \ldots, N_y - 1$ enumerate corners in the pattern, $k = qN_x + p$ is unified index and a is the length of the pattern's square size in any units of measure.

Since, on the pattern plane in its own coordinate system $Z = 0$, the last column of $\mathbf{R}$ does not play any role, and, the transformation (3) can be rewritten as:

$$
\mathbf{U}^k_\mu = \mathbf{H}_\mu \tilde{\mathbf{C}}^k, \quad \tilde{\mathbf{C}}^k = \begin{pmatrix} c^k_1 \\ c^k_2 \\ 1 \end{pmatrix}
\tag{12}
$$

where

$$
\mathbf{H} = \lambda \mathbf{K} \cdot \begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{pmatrix} \equiv \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix}
\tag{13}
$$

is a homography matrix that maps the pattern plane onto the image plane. Here λ is an arbitrary constant. Here we dropped the index μ for readability.

Each point correspondence $U_\mu^k \rightarrow C_\mu^k$ gives us a constraint on the elements of $\mathbf{H}$

Multiplying (13) from the left by $\mathbf{K}^-1$ and imposing that $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthonormal gives us 2 constraints on the homography matrix:

$$\begin{aligned} \mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 &= 0 \\ \mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 &= \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 \end{aligned} \tag{14}$$

$$\mathbf{B} = \mathbf{K}^{-T} \mathbf{K}^{-1} = \begin{pmatrix} \frac{1}{f_x^2} & -\frac{s}{f_x^2 f_y} & \frac{c_y s - c_x f_y}{f_x^2 f_y} \\ -\frac{s}{f_x^2 f_y} & \frac{s^2}{f_x^2 f_y^2} + \frac{1}{f_y^2} & -\frac{s(c_y s - c_x f_y)}{f_x^2 f_y^2} - \frac{c_y}{f_y^2} \\ \frac{c_y s - c_x f_y}{f_x^2 f_y} & -\frac{s(c_y s - c_x f_y)}{f_x^2 f_y^2} - \frac{c_y}{f_y^2} & \frac{(c_y s - c_x f_y)^2}{f_x^2 f_y^2} + \frac{c_y^2}{f_y^2} + 1 \end{pmatrix} \tag{15}$$

3 Homography estimation by 4 point correspondences

Let us have 4 points $U^k = (u_1, u_2, 1)$ and $X^k = (X_1^k, X_2^k, 1)$ that are related via a homography matrix $\mathbf{H}$ as follows:

$$u_\alpha^k = \frac{\left(\mathbf{H}\mathbf{X}^k\right)_\alpha}{\left(\mathbf{H}\mathbf{X}^k\right)_3} \tag{16}$$

Taking into account the notation (13), this equation can be rewritten as follows:

$$\begin{pmatrix} X_1^k & X_2^k & 1 & 0 & 0 & 0 & -u_1^k X_1^k & -u_1^k X_2^k & -u_1^k \\ 0 & 0 & 0 & X_1^k & X_2^k & 1 & -u_2^k X_1^k & -u_2^k X_2^k & -u_2^k \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{pmatrix} = 0 \tag{17}$$

Stacking 4 such equations together we get an 8×9 matrix $\mathbf{L}$ and the combined equations can be rewritten as follows:

$$\mathbf{L}\mathbf{h} = 0 \tag{18}$$