Multipath Polarization

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Let's consider scene on Figure 1. The light source has an arbitrary polarization, all the reflections are specular.

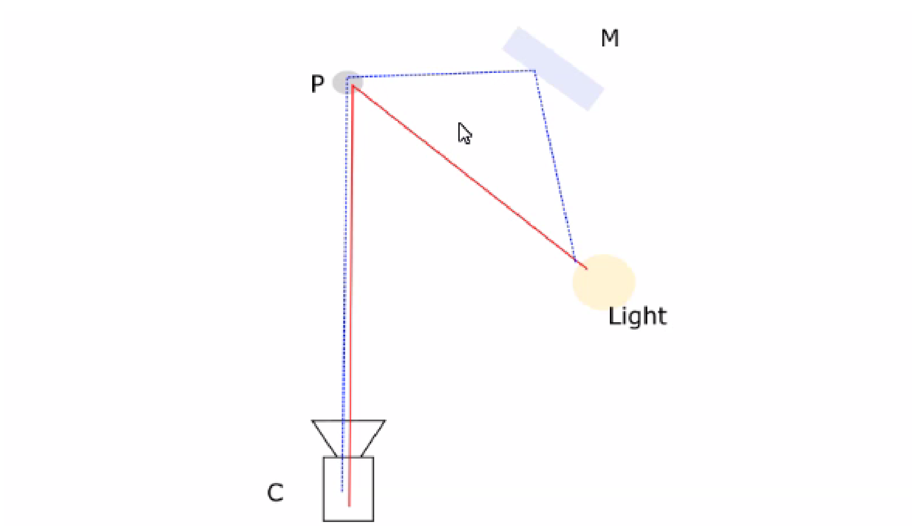

Figure 1: *Scene scheme.*

The only way to measure properties of the light coming to the camera is by rotating a polarizer. Therefore, let's consider light intensity variations for blue and red rays (I_2 and I_1 respectively) as well as their combination. As we now, the light intensity variation for rotating polarizer can be derived as

$$I = \frac{I_{max} + I_{min}}{2} + \frac{I_{max} - I_{min}}{2} \cos(2(\phi_{pol} - \varphi)) \tag{1}$$

Key unknowns here that describe light's properties are φ - angle of polarization and I_{max}, I_{min} characterize degree of polarization. Let's derive their values from the object and light source properties. To understand the properties of the reflected light let's write down reflected light properties derived from Fresnel equations on the surface of the object P. See Figure 2 for the notation.

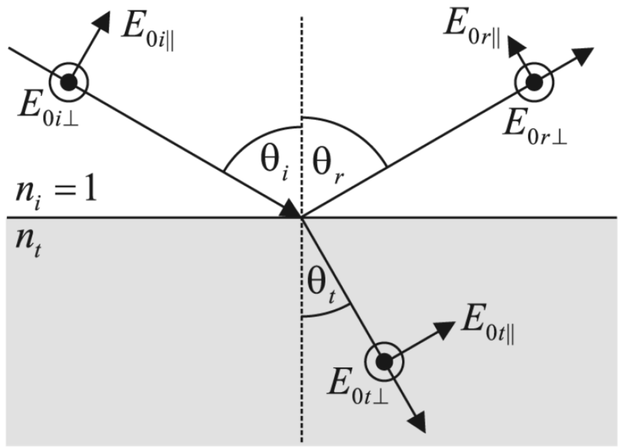

Figure 2: *Fresnel equations for the reflection on the surface of the object P.*

$$I_{rp} = R_p I_{ip} \tag{2}$$

$$I_{rs} = R_s I_{is} \tag{3}$$

If α is a polarization angle for incident light

$$\tan^2 \alpha = \frac{I_{ip}}{I_{is}} \tag{4}$$

$$I = I_{ip} + I_{is} \tag{5}$$

If β is a polarization angle for reflected light

$$\tan^2 \beta = \frac{I_{rp}}{I_{rs}} = \frac{R_p I_{ip}}{R_s I_{is}} = \tan^2 \alpha \frac{R_p}{R_s} \tag{6}$$

From the equations above we see that if the incident light is polarized then the reflection will change the polarization angle.

Now let's consider the case of reflection on object's surface if the light is just partly polarized. This case is especially relevant for I_2. Partly polarized light can be written as a combination of polarized and unpolarized components which are completely independent from each other.

$$I_2 = I_p + I_{up} \tag{7}$$

I_p at reflection behaves as discussed above, i.e. it will change it's polarization angle. In its turn I_{up} will get partly polarized after reflection, and that polarization will be incoherent to I_p. In order to tell what should be the polarization of the resulting light (both angle and DOP) we need to answer on the following question: what would be the polarization of the combination of two polarized incoherent light rays with polarization angles α and β (please refer to Figure 3)?

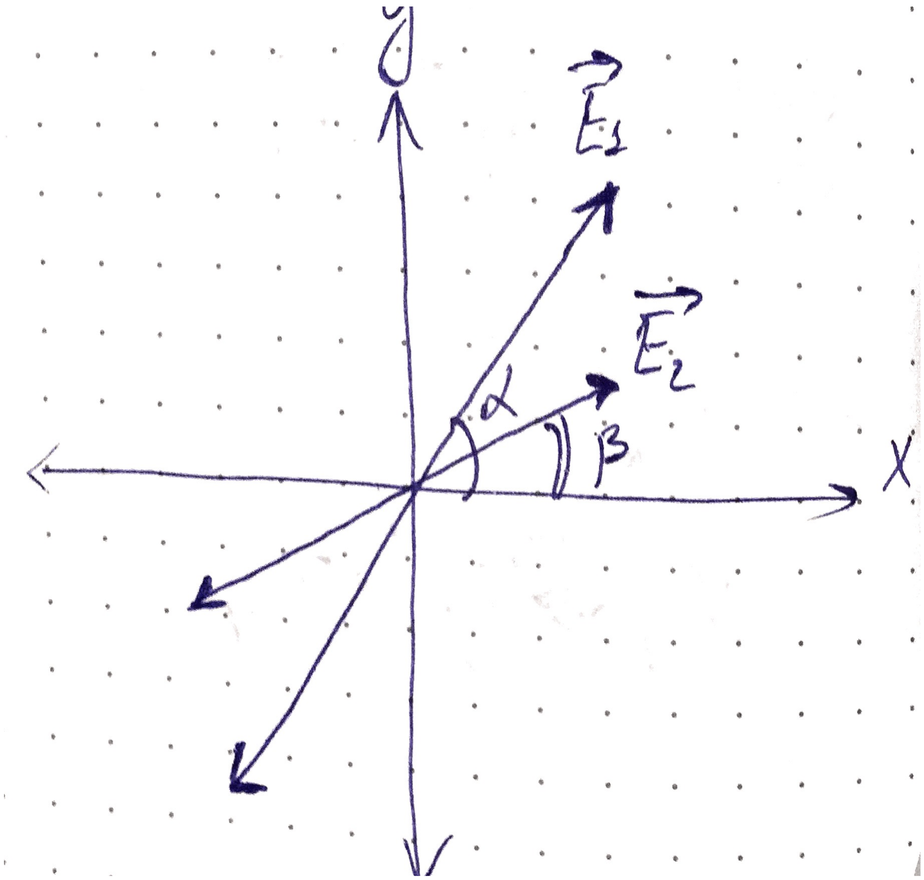

Figure 3: *Summation of two incoherent polarized rays.*

If these two incoherent polarized rays will get through a polarizer, the intensity of the resulting light will be the following:

$$I = |\vec{E_1}|^2 \cos^2(\alpha + \phi - \phi_{pol}) + |\vec{E_2}|^2 \cos^2(\beta + \phi - \phi_{pol}) \tag{8}$$

Or, in more general case of multiple polarized components:

$$I = I_0 + \sum I_i \cos^2(\alpha_i + \phi - \phi_{pol}) \tag{9}$$

ϕ in these equations is the same for all components of the sum, since it is the azimuth angle of surface normal in the point of an object, at which light is reflected. This sum is a sine wave with arbitrary phase shift and amplitude, plus constant. Therefore, just by measuring properties of this sine wave, it is not possible to resolve different components. Let's consider the options that can help resolving polarization multipath.

The case when the incident light is partly polarized Let's consider the case, when the incident light coming from each direction is partly polarized. This is a real world case, since all of the light around us is partly polarized. And, as we've seen from equation 8, this affects both azimuth angle and zenith angle measurements.

In order to resolve this simple case of multipath between single polarized and unpolarized component, lets consider its reflection from a known object, as depicted on Figure 4.

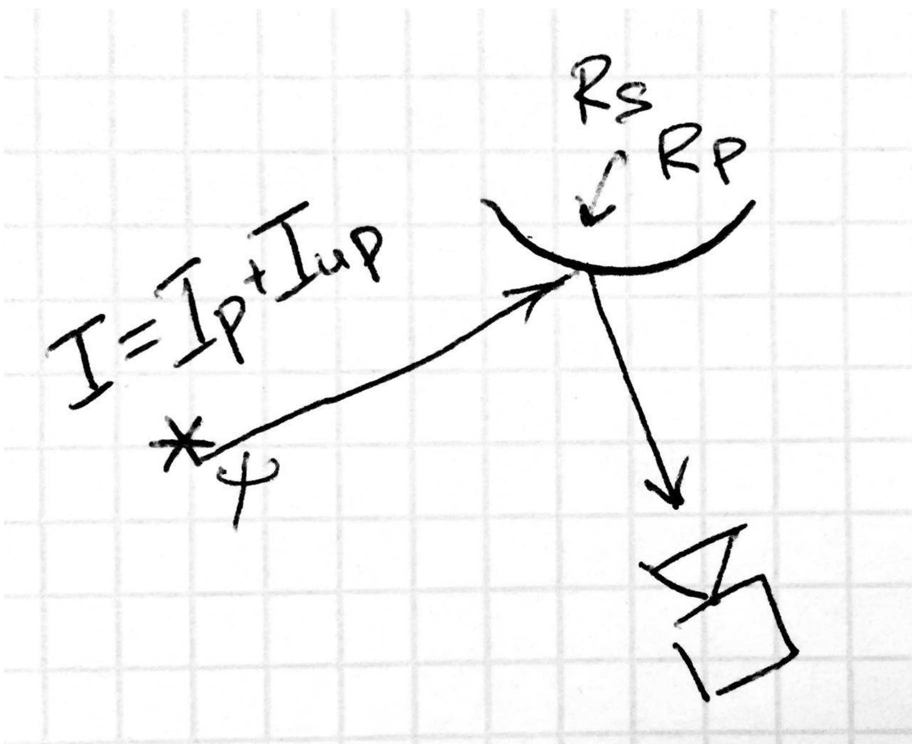

Figure 4: *Reflection of partly polarized light from the object with some known properties.*

The incident light that gets reflected from the object has two independent components:

$$I = I_{up} + I_p \tag{10}$$

where I_{up} is unpolarized component, and I_p is polarized component with polarization angle ψ_0. After reflection from the surface I_{up} gets partly polarized:

$$I_{up} \to I_{up1} + I_{p1} \tag{11}$$

and polarized component changes its polarization angle to ψ and intensity:

$$I_p \to I_{p2} \tag{12}$$

Now let's consider each of these components:

$$I_{up} \to R_p I_{upp} + R_s I_{ups} = R_p \frac{I_{up}}{2} + R_s \frac{I_{up}}{2} \tag{13}$$

and because $R_s > R_p$

$$I_{p1} = R_s \frac{I_{up}}{2} - R_p \frac{I_{up}}{2} \tag{14}$$

$$I_{up1} = R_p I_{up} \tag{15}$$

(unpolarized component is equal to doubled minimal polarized component out of two, and polarized component is equal to their difference).

$$I_{p2} = R_p I_{pp} + R_s I_{ps} = R_p I_p \cos^2 \psi + R_s I_p \sin^2 \psi \tag{16}$$

Where ψ is the incident light polarization angle. According to 8 and 9, the intensity of light after the polarizer will be the following:

$$I = I_0 + I_{p1} \cos^2(\alpha + \phi - \phi_{pol}) + I_{p2} \cos^2(\beta + \phi - \phi_{pol}) \tag{17}$$

$$\begin{aligned} I = I_0 + \frac{R_s I_{up} - R_p I_{up}}{2} \cos^2(\alpha + \phi - \phi_{pol}) + \\ + I_p (R_p \cos^2 \psi + R_s \sin^2 \psi) \cos^2(\beta + \phi - \phi_{pol}) \end{aligned} \tag{18}$$

where

$$I_0 = \frac{R_p I_{up}}{2} \tag{19}$$

$$\alpha = \frac{\pi}{2} \tag{20}$$

where ϕ is azimuth angle. And according to 6

$$\tan \beta = \tan \psi \sqrt{\frac{R_p}{R_s}} \tag{21}$$

therefore

$$\beta = \arctan{(\tan \psi \sqrt{\frac{R_p}{R_s}})} \tag{22}$$

Therefore, for a calibrated object with known shape and reflection properties

$$I = f(I_{up}, I_p, \psi) \tag{23}$$

Finally,

$$\begin{aligned} I = \frac{I_{up}}{2}(R_s \sin^2(\phi - \phi_{pol}) + R_p(1 - \sin^2(\phi - \phi_{pol})))+ \\ + I_p(R_p \cos^2\psi + R_s \sin^2\psi)\cos^2(\arctan(\tan\psi\sqrt{\frac{R_p}{R_s}}) + \phi - \phi_{pol}) \end{aligned} \tag{24}$$

And therefore partly polarized incident light mixture can be resolved.

Using the method, described above, one can collect a polarization map of the light around using specular calibration object (similar to Figure 5 and do robust shape from polarization reconstruction in the wild under partly polarized light (for example, outside during the daylight).

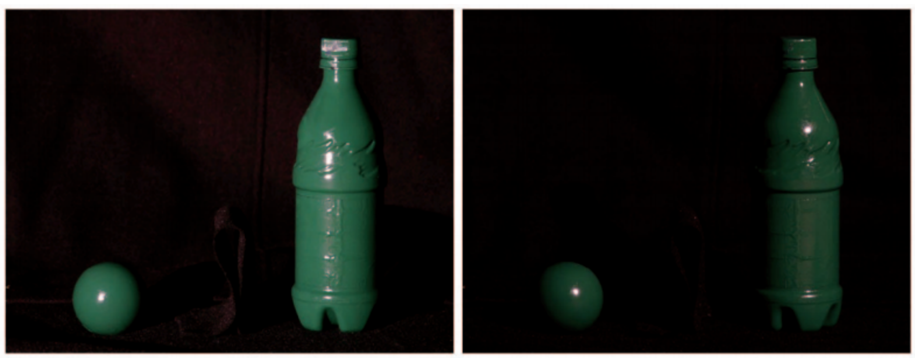

Figure 5: *Reflection of partly polarized light from the object with some known properties.*

Specular object, ideally - metallic ball, should be used as a calibration object. For the metallic ball R_s and R_p can be defined as following:

$$R_s = r_s r_s^* \tag{25}$$

$$R_p = r_p r_p^* \tag{26}$$

where

$$r_s = \frac{\cos\theta_i - \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}} \tag{27}$$

$$r_p = \frac{-\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}} \tag{28}$$

and

$$\tilde{n} = n_R + i n_I \tag{29}$$

Also, θ_i is equal to surface normal's zenith angle.

Now, let's consider, how the "polarization light field" - the information about the polarization state of the light around the object can be used to restore an arbitrary object's shape. The light reflected from a point P of an arbitrary object gets partly polarized, as discussed above. Equation 24 describes how the intensity of the reflected light will depend on polarization angle. Only this time R_s, R_p and ϕ are unknown, R_s, R_p are functions of θ and n, and I_{up}, I_p, ψ are also functions of ϕ and θ. Therefore, capturing three different intensities under three different polarizer angles and by solving a nonlinear system of equations with ϕ, θ and n as unknowns we will be able to find surface normal and material properties at the point of the object.

The key point of this concept is that polarization ambiguity can be resolved using inhomogenity of the light around the object. Therefore, it is important for the light to be indeed inhomogenious. Otherwise, the system of equations becomes ill-posed.

Let's write down the final set of equations

$$\begin{aligned} I_k = {} & \frac{I_{up}}{2}(R_s \sin^2(\phi - \phi_{pol,k}) + R_p(1 - \sin^2(\phi - \phi_{pol,k})))+ \\ & + I_p(R_p \cos^2\psi + R_s \sin^2\psi)\cos^2(\arctan(\tan\psi\sqrt{\frac{R_p}{R_s}}) + \phi - \phi_{pol,k}) \end{aligned} \tag{30}$$

where

$$\begin{aligned} I_{up} &= I_{up}(\theta, \phi) \\ I_p &= I_p(\theta, \phi) \\ \psi &= \psi(\theta, \phi) \end{aligned} \tag{31}$$

$$\begin{aligned} R_s &= \left(\frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)^2 \\ R_p &= \left(\frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)^2 \end{aligned} \tag{32}$$

By having at least three measurements of 30 and by solving the following set of nonlinear equations one can find θ, ϕ and n.