Kalman Filter Basics

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August 31, 2021

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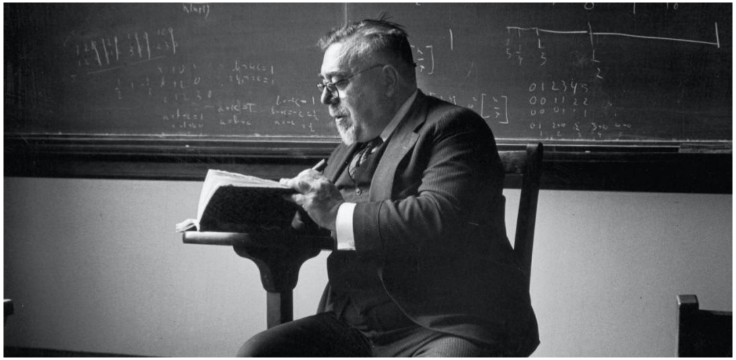
Prerequisites

- Basic Linear Algebra
- Python 3
- Jupyter lab/notebooks

numpy
scipy
sympy
matplotlib

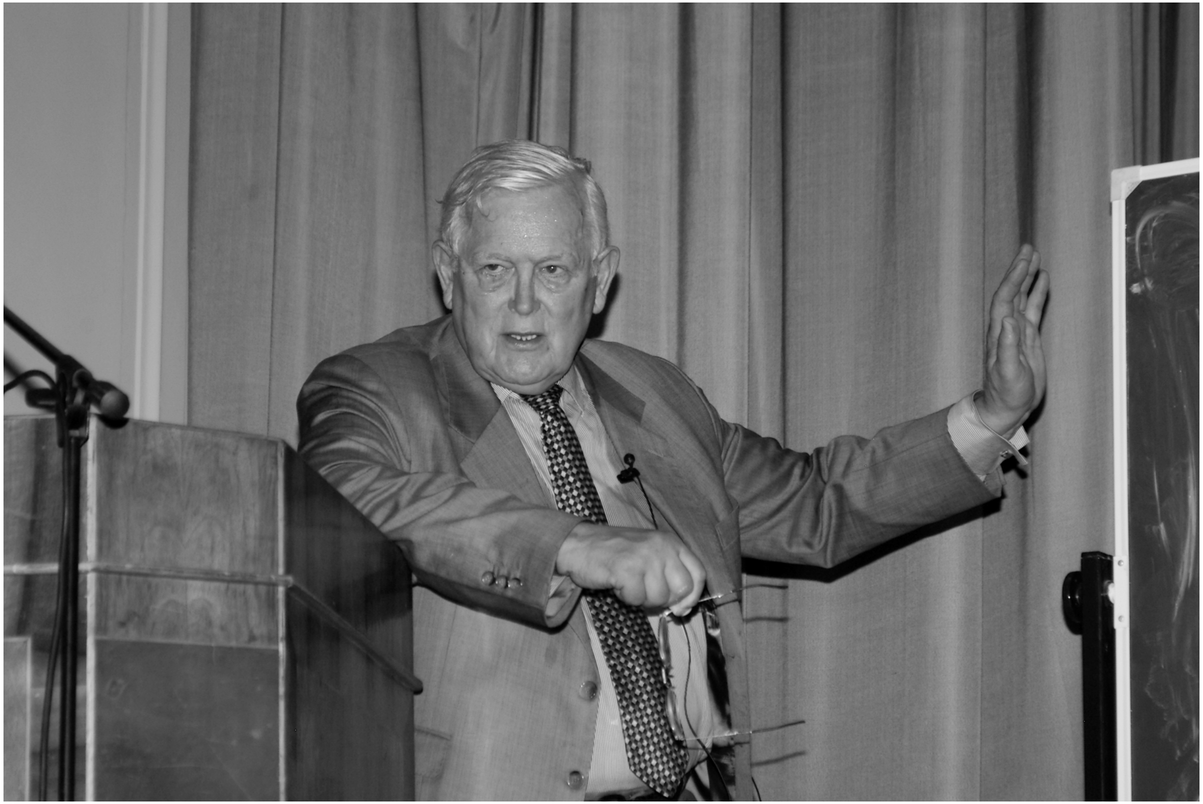
Background

"Given the spectral characteristics of an additive combination of signal and noise – what linear operation yield the **optimal** separation of the signal from the noise?"



Norbert Wiener

Background



Rudolf E. Kalman

Kalman Filter Basics

- State-Space Notation
  - Frequency Domain (Wiener)
  - Time Domain (Lyapunov, Kalman)

time deriv of state vector

Dynamic matrix

state vector

w.n. coeff.

white noise

- Continuous
  - System Model

$$\dot{x} = Fx + Gu$$

  - Measurement Model

Design matrix

measurement noise

$$z = Hx + v$$

state vector

measurements

Kalman Filter Basics

- Discrete
  - System Model (ODE Solution)

Transition matrix

$$x_{k+1} = \phi_k x_k + w_k$$

  - Measurement Model

$$z_k = H x_k + v_k$$

The Analysis Problem

Definition

In any system satisfying a set of linear differential equations, the solution may be written as a superposition of an initial-condition part and another part due to the driving (or forcing) functions

Kalman Filter Basics (example)

- Simplified ODE Solution (i.e. $F = 0$, $G = 1$)

$$\dot{x} = u(t)$$

- Homogenous Solution

$$\dot{x} = 0 \implies x = x_k$$

- General Solution

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} u(\epsilon) \, \mathrm{d}\epsilon$$

Kalman Filter Basics

- Superposition Integral

$$w_k = \int_{t_k}^{t_{k+1}} u(\epsilon)\,\mathrm{d}\epsilon$$

  - Variance of Integral

$$E[w_k(\epsilon) w_k^T(\eta)] = 0 \quad \forall \quad \epsilon \neq \eta$$

- Process Noise Covariance

$$E[w_k w_k^T] = Q_k$$

Process noise

Kalman Filter Basics

- Measurement Model

$$z_k = Hx_k + v_k$$

measurement noise

- Measurement Noise Covariance

$$E[v_k v_k^T] = R_k$$

  - Note that Q and R are assumed to be uncorrelated

$$E[w_k v_j^T] = 0 \quad \forall \quad k, j$$

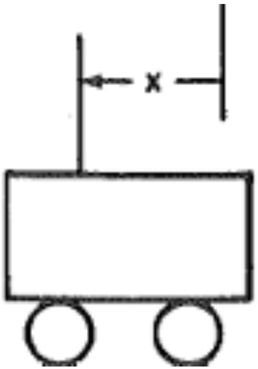
Kalman Filter Basics (example)

- Simple PV Model

$$\dot{x} = F \quad x + Gu$$

$$\ddot{x} = u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$



$$\dot{x} = \dot{x}$$

$$\ddot{x} = u$$

Kalman Filter Basics (example)

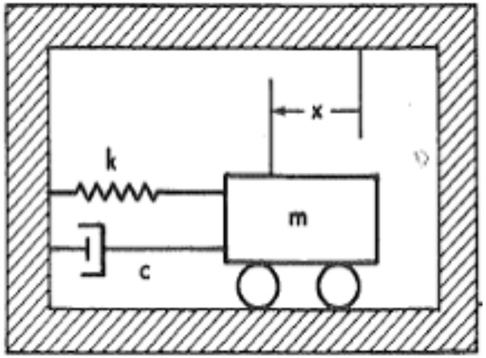
- Complex PV Model

$$\Sigma F = ma$$

$$m\ddot{x} + c\dot{x} + kx = mu$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

$$\dot{x} = F x + G u$$



Kalman Filter Basics

- Discrete Filter Loop
  - Time Update

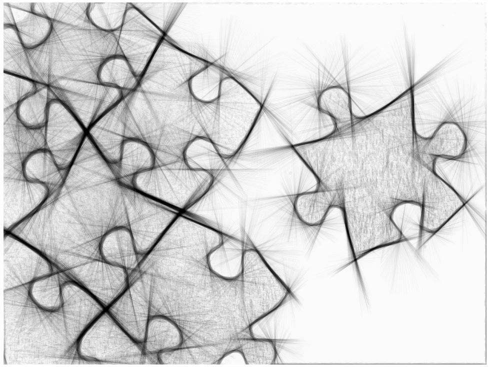
$$
\begin{aligned}
\tilde{x}_k =& \phi_{k-1}\hat{x}_{k-1} \\
\tilde{P}_k =& \phi_{k-1}\hat{P}_{k-1}\phi_{k-1}^T + Q_{k-1}
\end{aligned}
$$

  - Measurement Update

$$
\begin{aligned}
K_k =& \tilde{P}_k H_k^T (H_k \tilde{P}_k H_k^T + R_k)^{-1} \\
\hat{x}_k =& \tilde{x}_k + K_k(z_k - H_k \tilde{x}_k) \\
\hat{P}_k =& (I - K_k H_k)\tilde{P}_k
\end{aligned}
$$

Kalman Filter Basics

- Parameters to be determined
  - Transition Matrix ϕ
  - Process Noise Q
  - Design Matrix H
  - Measurement Noise R
  - Initial Conditions x_0, P_0



Then comes the hard work...

Transition Matrix

- System Model ODE

$$\dot{x} = Fx + Gu$$

- General Solution

$$\phi = \mathcal{L}^{-1}\{(sI - F)^{-1}\}_{t=\Delta t}$$

- Approximate Solution (F is constant)

$$\phi = e^{F\Delta t} = I + F\Delta t + \frac{(F\Delta t)^2}{2!} + \dots$$

Transition Matrix (example)

- Simple PV Model

$$\dot{x} = Fx + Gu$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{q} \end{bmatrix} u$$

- Transition Matrix

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \Delta t \\ 0 & 0 \end{bmatrix} + \cdots \implies \phi = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

Your Turn

- Given the following system model, compute the corresponding transition matrix ϕ using the general solution.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{q} \end{bmatrix} u$$

Transition Matrix

$\dot{x} = Fx + Gu$

- Numerical Solution (Van Loan)

$$A = \begin{bmatrix} -F & GG^T \\ 0 & F^T \end{bmatrix} \Delta t$$

upper right

$$B = e^A = \begin{bmatrix} \cdots & \phi_k^{-1} Q_k \\ 0 & \phi_k^T \end{bmatrix}$$

"mexp"

Lower right

$$\phi_k = B[\text{"Lower right"}]^T$$
$$Q_k = \phi_k[\text{"Upper right"}]$$

Process Noise

- Random Process

$$x_{k+1} = \phi_k x_k + w_k$$

- Process Noise

$$w_k = \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \epsilon)\, G(\epsilon) u(\epsilon)\, \mathrm{d}\epsilon$$

$$Q_k = E[w_k w_k^T]$$

Process Noise (example)

- Solution

$$Q_k = \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \epsilon) G(\epsilon) G^T(\epsilon) \phi^T(t_{k+1}, \epsilon) \mathrm{d}\epsilon$$

$$Q_k = \begin{bmatrix} \frac{1}{3} q \Delta t^3 & \frac{1}{2} q \Delta t^2 \\ \frac{1}{2} q \Delta t^2 & q \Delta t \end{bmatrix}$$

Your Turn

- Given the following system model, compute the corresponding transition matrix ϕ and process noise matrix Q using the Van Loan numerical solution.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{q} \end{bmatrix} u$$

Let $\Delta t = 5\,\mathrm{s}$ and $q = 0.3\,\mathrm{m}^2\,\mathrm{s}^{-1}$.

Design Matrix

- Design Matrix
  - The connection between the measurements and the elements of the state vector
  - Linearization (typical non-linear function)

$$
\begin{aligned}
z =& h(x) \\
z =& h(x_0) + \frac{\partial h}{\partial x} \mathrm{d}x + \dots
\end{aligned}
$$

Design Matrix (example)

- Geometric range

$$\rho = \sqrt{(X^i - X)^2 + (Y^i - Y)^2 + (Z^i - Z)^2}$$

- Linearization

$$\mathrm{d}\rho = \frac{\partial \rho}{\partial X}\mathrm{d}X + \frac{\partial \rho}{\partial Y}\mathrm{d}Y + \frac{\partial \rho}{\partial Z}\mathrm{d}Z$$

Preliminary values

$$H = \begin{bmatrix} -\frac{(X^i - X_0)}{\rho_0} & -\frac{(Y^i - Y_0)}{\rho_0} & -\frac{(Z^i - Z_0)}{\rho_0} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Your Turn

- Given the following non-linear measurement equations, derive the corresponding linearized set of measurement equations

$$
\begin{aligned}
x_1 =& x_0 + s \cos \alpha \\
y_1 =& y_0 + s \sin \alpha
\end{aligned}
$$

The Analysis Problem

- In the deterministic case we seek an explicit expression for the response or output
- In the random process case we the most convenient descriptors are:
    1. Autocorrelation
    2. Power Spectral Density
    3. Mean Square Value

Stationary (Steady-state) Analysis

- Laplace transform ($s = \sigma + j\omega$)

$$S_x(s) = G(s)G(-s)S_f(s)$$

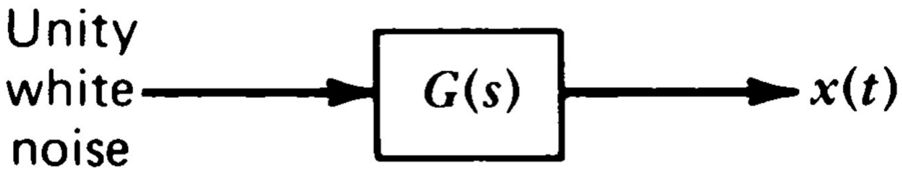
- Fourier transform

$$\begin{aligned} S_x(j\omega) =& G(j\omega)G(-j\omega)S_f(j\omega) \\ =& |G(j\omega)|^2 S_f(j\omega) \end{aligned}$$

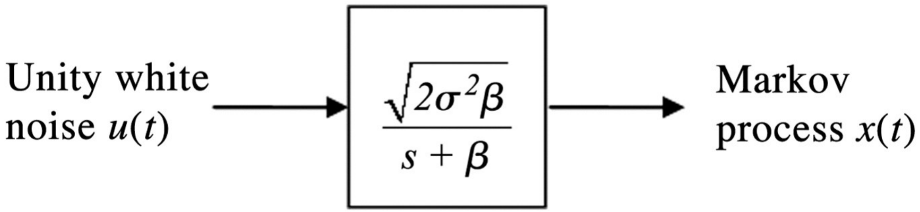
Laplace vs Fourier Transforms

- The Fourier transform of a function is a complex function of a real variable (frequency)
- The Laplace transform of a function is a complex function of a complex variable.
- Fourier is used primarily for *steady state* signal analysis, while Laplace is used for *transient* signal analysis.
- The Laplace transform of a function is a holomorphic function of the variable s

Shaping Filter



Markov Process



Spectral Factorization

- Given the second order Markov process as follows:

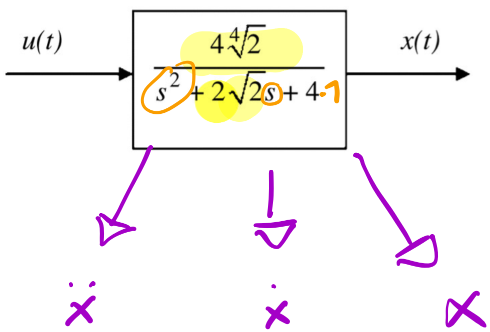
$$S_x(s) = \frac{16\sqrt{2}}{s^4 + 16}$$

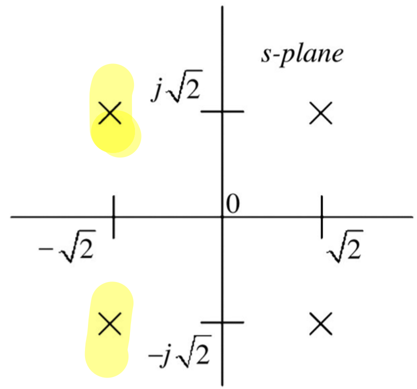
- Spectral factorization gives:

$$S_x(s) = \frac{4\sqrt[4]{2}}{s^2 + 2\sqrt{2}s + 4} \cdot \frac{4\sqrt[4]{2}}{(-s)^2 + 2\sqrt{2}(-s) + 4}$$

$$G(s) \cdot G(-s)$$

Spectral Factorization





Spectral Factorization

- The scalar differential equation can be determined from the transfer function:

$$\ddot{x} + 2\sqrt{2}\dot{x} + 4x = 4\sqrt[4]{2}u$$

- Choose x and $\dot{x}$ as the phase variables x_1 and x_2. The corresponding matrix equation reads:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 4\sqrt[4]{2} \end{bmatrix} u$$

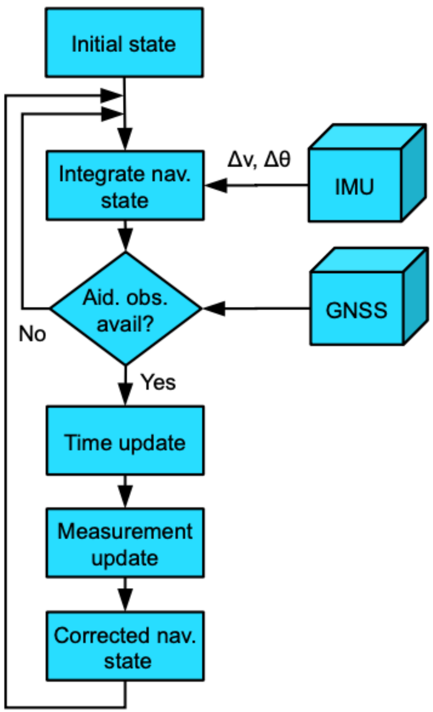
$$\dot{x} \quad F \quad x + G u$$

Your Turn

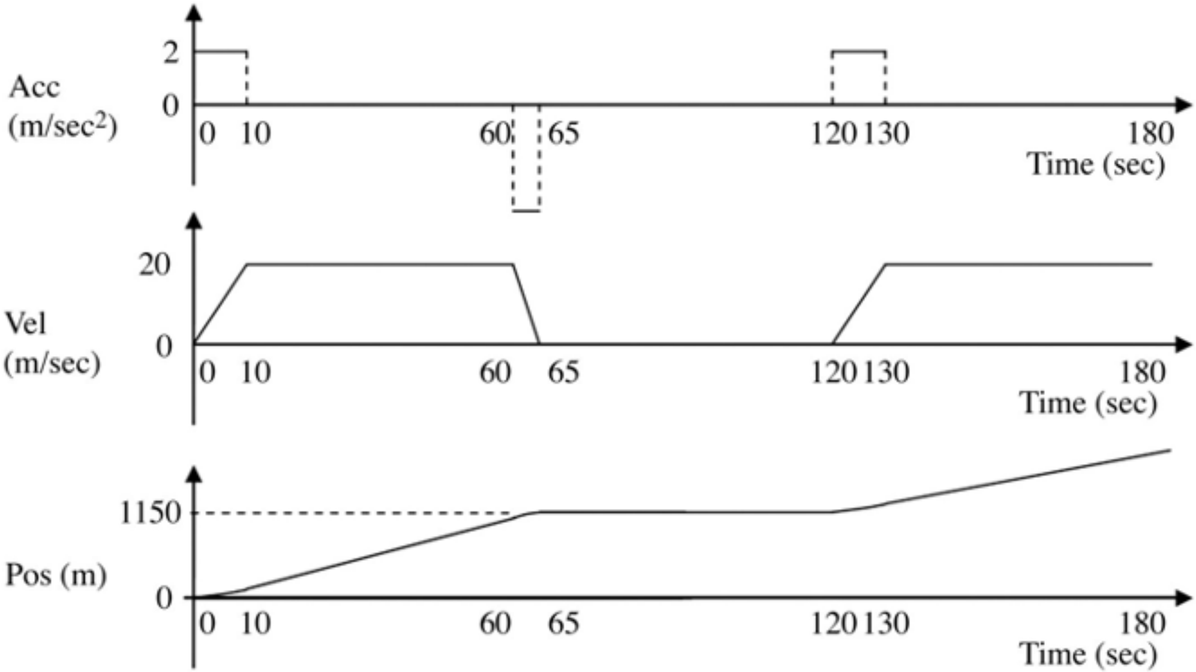
- Given the Power Spectral Density S_x of a second order Gauss-Markov process x, derive the corresponding dynamic equations

$$S_x(s) = \frac{16}{s^4 + 64}$$

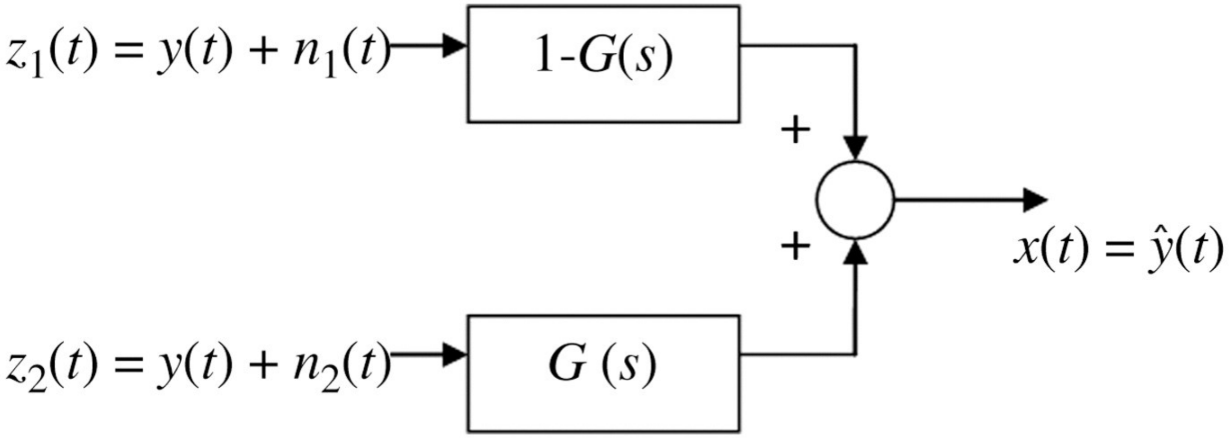
GNSS/INS Integration



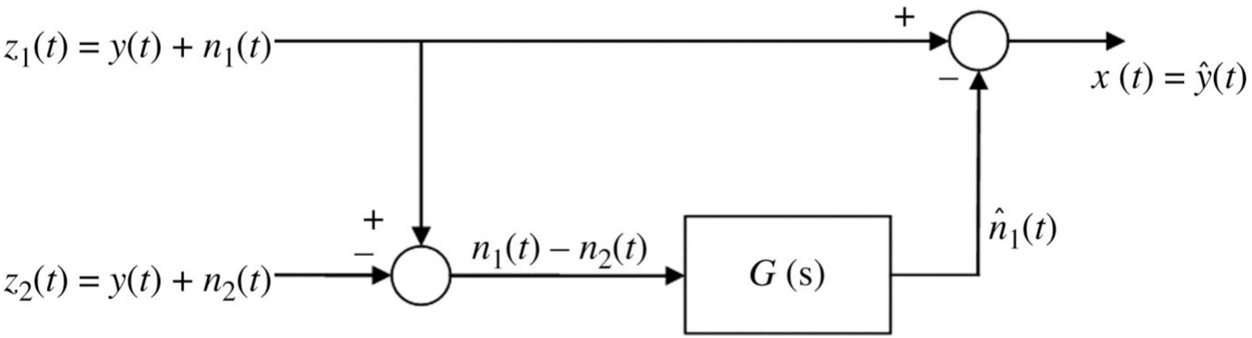
Typical Vehicle Dynamics



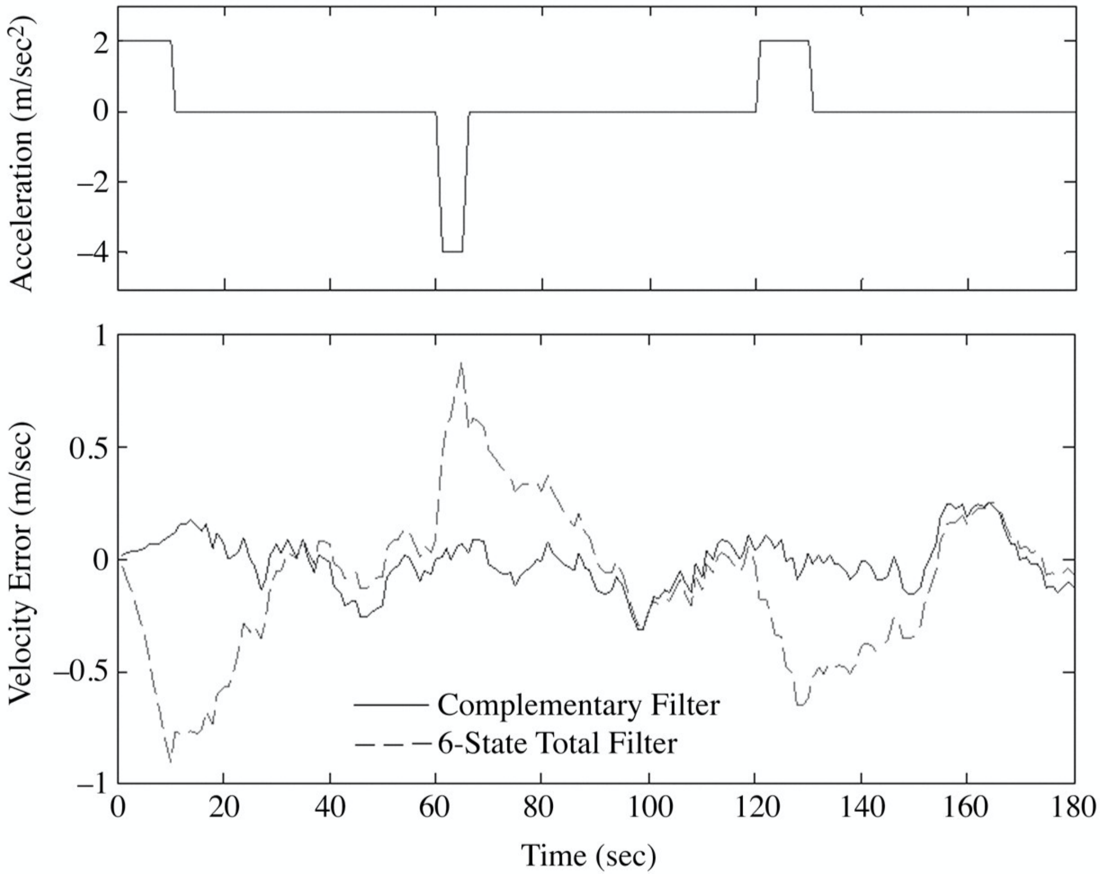
Complementary Filter



Error State Formulation



Performance



GNSS/IMU Complementary Filter