NKG Summer School

Exercise 2 - External Forces

Given a 2D coordinate system where the x-axis indicates the horizontal direction and the y-axis indicates the vertical direction. A cannonball is fired with the velocity v at the angle θ relative to the x-axis. Only gravity acts on the cannonball and any air resistance can be neglected.

The dynamic equation is given as:

$$\dot{x} = Fx + Lw + Gu \tag{1}$$

, where F, L and G are matrices. The vector w is the control vector and the vector u is white noise. The state vector x is given as:

$$x = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} \tag{2}$$

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1. Derive the dynamic equation that describes the movement of the cannonball.

   (a) Show that the transition matrix can be written as:

$$\phi_k = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

(b) Show that the control input can be written as:

$$\Lambda_k = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}g\Delta t^2 \\ -g\Delta t \end{bmatrix} \tag{4}$$

2. Assume that the cannonball is fired at a speed of $s = 100\,\mathrm{m\,s^{-1}}$ at an angle $\theta = 45°$ relative to the x-axis. Select $\Delta t = 0.1\,\mathrm{s}$ and the following initial values:

$$x_0 = \begin{bmatrix} 0 \\ s\cos\theta \\ 0 \\ s\sin\theta \end{bmatrix} \tag{5}$$

$$P_0 = \begin{bmatrix} (10\,\mathrm{m})^2 & 0 & 0 & 0 \\ 0 & (3\,\mathrm{m\,s^{-1}})^2 & 0 & 0 \\ 0 & 0 & (10\,\mathrm{m})^2 & 0 \\ 0 & 0 & 0 & (3\,\mathrm{m\,s^{-1}})^2 \end{bmatrix} \tag{6}$$

(a) Compute the trajectory of the cannonball based on the dynamic equation.

(b) Generate a set of measurements z_k of the position of the cannonball with measurement noise corresponding to $\sigma_z = 10\,\mathrm{m}$.

(c) Estimate the trajectory of the cannonball using a discrete Kalman filter based on the generated measurements z_k. Assume process noise $q = (1\,\mathrm{m^2/s^2})/\mathrm{s}$.

(d) Modify the dynamic equation so that it no longer takes into account gravity and estimate the trajectory of the cannonball using a discrete Kalman filter under the same conditions.

Discrete Kalman Filter Loop

Time update

$$
\begin{aligned}
\tilde{x}_k =& \phi_{k-1}\hat{x}_{k-1} + \Lambda_{k-1} \\
\tilde{P}_k =& \phi_{k-1}\hat{P}_{k-1}\phi_{k-1}^T + Q_{k-1}
\end{aligned}
$$

Measurement update

$$
\begin{aligned}
K_k =& \tilde{P}_k H_k^T (H_k \tilde{P}_k H_k^T + R_k)^{-1} \\
\hat{x}_k =& \tilde{x}_k + K_k(z_k - H_k\tilde{x}_k) \\
\hat{P}_k =& (I - K_k H_k)\tilde{P}_k
\end{aligned}
$$