Continuum Mechanics

Chapter 3
Kinematics: Strains

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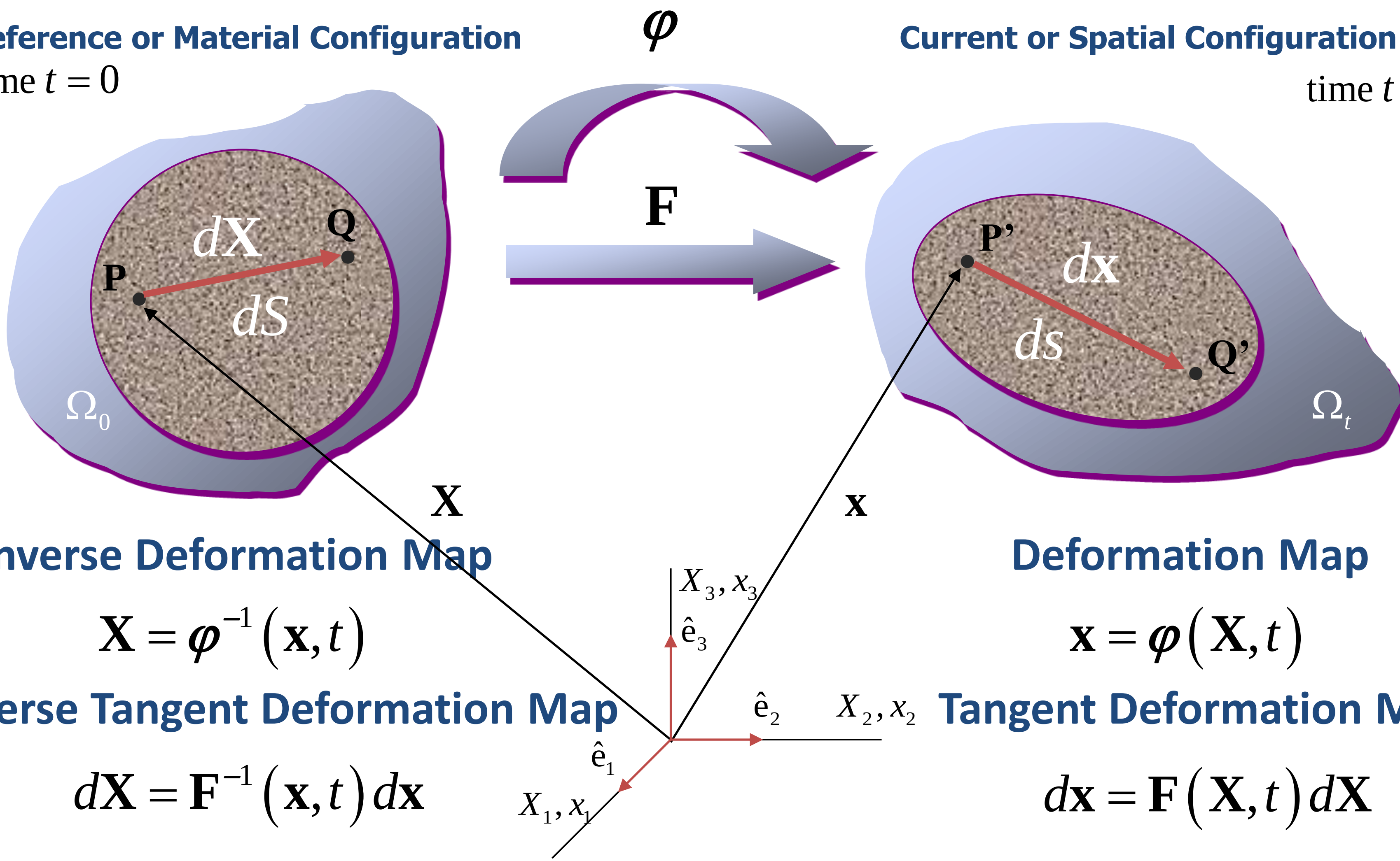
Reference or Material Configuration

time $t = 0$

Current or Spatial Configuration

time t



Inverse Deformation Map

$$\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

Deformation Map

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

Inverse Tangent Deformation Map

$$d\mathbf{X} = \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}$$

Tangent Deformation Map

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$$

Tangent Deformation Map

Deformation Gradient

Let us consider the **deformation map** given by,

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

Differentiating the deformation map, keeping constant the configuration at time *t*, using the chain rule, yields,

$$d\mathbf{x} = \left(\mathrm{GRAD}\,\boldsymbol{\varphi}(\mathbf{X}, t)\right) d\mathbf{X} = \left(\bar{\nabla} \otimes \boldsymbol{\varphi}(\mathbf{X}, t)\right) d\mathbf{X} := \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$$

where the *non-symmetric* second-order **deformation gradient** tensor, denoted as $\mathbf{F}(\mathbf{X}, t)$, has been introduced as,

$$\boxed{\mathbf{F}(\mathbf{X}, t) := \bar{\nabla} \otimes \boldsymbol{\varphi}(\mathbf{X}, t) = \mathrm{GRAD}\,\boldsymbol{\varphi}(\mathbf{X}, t), \quad F_{aA} := \varphi_{a,A}}$$

Tangent Deformation Map

Inverse Deformation Gradient

Let us consider the inverse **deformation map** given by,

$$\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

Differentiating the inverse deformation map, keeping constant the configuration at time *t*, using the chain rule, yields,

$$d\mathbf{X} = \left(\operatorname{grad} \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)\right) d\mathbf{x} = \left(\nabla \otimes \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)\right) d\mathbf{x} := \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}$$

where the *non-symmetric* second-order **inverse deformation gradient** tensor, denoted as $\mathbf{F}^{-1}(\mathbf{X}, t)$, has been introduced as,

$$\boxed{\mathbf{F}^{-1}(\mathbf{x}, t) := \nabla \otimes \boldsymbol{\varphi}^{-1}(\mathbf{x}, t) = \operatorname{grad} \boldsymbol{\varphi}^{-1}(\mathbf{x}, t), \quad F_{Aa}^{-1} := \varphi_{A,a}^{-1}}$$

Tangent Deformation Map

Uniform Deformation Gradient

Let us consider a *uniform* **deformation gradient** such that,

$$d\mathbf{x} = \mathbf{F}(t)\, d\mathbf{X}, \quad dx_a = F_{aA}(t)\, dX_A$$

As the deformation gradient is *uniform*, the **deformation map** is *linear* and it can be easily obtained, integrating, yielding,

$$\mathbf{x} = \mathbf{F}(t)\,\mathbf{X} + \mathbf{C}(t), \quad x_a = F_{aA}(t)\, X_A + C_a(t)$$

where $\mathbf{C}(t)$ is a vector of integration constants, such that, assuming the reference time is t=0, satisfies the condition $\mathbf{C}(0) = \mathbf{0}$.

Tangent Deformation Map

Deformation Gradient

$$d\mathbf{x} = \mathbf{F}(\mathbf{X},t)\,d\mathbf{X}, \quad dx_a = F_{aA}\,dX_A$$

$$\mathbf{F}(\mathbf{X},t) := \bar{\nabla} \otimes \boldsymbol{\varphi}(\mathbf{X},t) = \mathrm{GRAD}\,\boldsymbol{\varphi}(\mathbf{X},t), \quad F_{aA} = \varphi_{a,A}$$

$$J := \det \mathbf{F}(\mathbf{X},t) = \det\left(\mathrm{GRAD}\,\boldsymbol{\varphi}(\mathbf{X},t)\right) > 0$$

Inverse Deformation Gradient

$$d\mathbf{X} = \mathbf{F}^{-1}(\mathbf{x},t)\,d\mathbf{x}, \quad dX_A = F^{-1}_{Aa}\,dx_a$$

$$\mathbf{F}^{-1}(\mathbf{x},t) := \nabla \otimes \boldsymbol{\varphi}^{-1}(\mathbf{x},t) = \mathrm{grad}\,\boldsymbol{\varphi}^{-1}(\mathbf{x},t), \quad F^{-1}_{Aa} := \varphi^{-1}_{A,a}$$

$$J^{-1} := \det \mathbf{F}^{-1}(\mathbf{x},t) = \det\left(\mathrm{grad}\,\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right) > 0$$

Example 3.1

Example 3.1

Compute the *deformation gradient* and *inverse deformation gradient* for a motion equation with Cartesian components,

$$\left[\varphi\left(\mathbf{X},t\right)\right]=\left[X+Y^2t \quad Y\left(1+t\right) \quad Ze^t\right]^T$$

The Cartesian components of the **deformation gradient** are,

$$\left[\mathbf{F}\right]=\begin{bmatrix}1 & 2Yt & 0\\ 0 & 1+t & 0\\ 0 & 0 & e^t\end{bmatrix}$$

and the **jacobian** takes the value,

$$J=\det\mathbf{F}=\left(1+t\right)e^t>0$$

Example 3.1

The Cartesian components of the **inverse motion equation** are,

$$\left[\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right]=\left[\begin{matrix} x-y^2\dfrac{t}{(1+t)^2} & \dfrac{y}{1+t} & ze^{-t}\end{matrix}\right]^T$$

The Cartesian components of the **inverse deformation gradient** are,

$$\left[\mathbf{F}^{-1}\right]=\begin{bmatrix} 1 & -2y\dfrac{t}{(1+t)^2} & 0 \\ 0 & \dfrac{1}{1+t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Material Displacement Gradient

Material Displacement Gradient

Let us consider the material description of the **displacement** vector field, given by,

$$\mathbf{u} = \mathbf{U}\left(\mathbf{X},t\right)$$

Differentiating the material description of the displacements, keeping constant the time *t*, using the chain rule, yields,

$$d\mathbf{u} = \left(\mathrm{GRAD}\,\mathbf{U}\left(\mathbf{X},t\right)\right)d\mathbf{X} = \left(\bar{\nabla}\otimes\mathbf{U}\left(\mathbf{X},t\right)\right)d\mathbf{X} := \mathbf{J}\left(\mathbf{X},t\right)d\mathbf{X}$$

where the *non-symmetric* second-order **material displacement gradient** tensor, denoted as $\mathbf{J}\left(\mathbf{X},t\right)$, has been introduced as,

$$\boxed{\mathbf{J}\left(\mathbf{X},t\right) := \bar{\nabla}\otimes\mathbf{U}\left(\mathbf{X},t\right) = \mathrm{GRAD}\,\mathbf{U}\left(\mathbf{X},t\right), \quad J_{aA} = U_{a,A}}$$

Material Displacement Gradient

Material Displacement Gradient

The material description of the **displacement** vector field may be written as

$$\mathbf{U}\left(\mathbf{X},t\right)=\boldsymbol{\varphi}\left(\mathbf{X},t\right)-\mathbf{X}, \quad U_a=\varphi_a-X_a$$

Taking the *material gradient*, the **material displacement gradient** tensor may be related to the **deformation gradient** tensor, yielding,

$$\mathbf{J}\left(\mathbf{X},t\right)=\mathrm{GRAD}\,\mathbf{U}\left(\mathbf{X},t\right)=\mathrm{GRAD}\,\boldsymbol{\varphi}\left(\mathbf{X},t\right)-\mathbf{1}=\mathbf{F}\left(\mathbf{X},t\right)-\mathbf{1},$$

$$J_{aA}=U_{a,A}=\varphi_{a,A}-\delta_{aA}=F_{aA}-\delta_{aA}$$

Spatial Displacement Gradient

Spatial Displacement Gradient

Let us consider the spatial description of the **displacement** vector field, given by,

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$$

Differentiating the spatial description of the displacements, keeping constant the time *t*, using the chain rule, yields,

$$d\mathbf{u} = \left(\operatorname{grad}\mathbf{u}(\mathbf{x}, t)\right) d\mathbf{x} = \left(\nabla \otimes \mathbf{u}(\mathbf{x}, t)\right) d\mathbf{x} := \mathbf{j}(\mathbf{x}, t)\, d\mathbf{x}$$

where the *non-symmetric* second-order **spatial displacement gradient** tensor, denoted as $\mathbf{j}(\mathbf{x}, t)$, has been introduced as,

$$\boxed{\mathbf{j}(\mathbf{x}, t) := \nabla \otimes \mathbf{u}(\mathbf{x}, t) = \operatorname{grad}\mathbf{u}(\mathbf{x}, t), \quad j_{Aa} := u_{A,a}}$$

Spatial Displacement Gradient

Spatial Displacement Gradient

The spatial description of the **displacement** vector field may be written as

$$\mathbf{u}(\mathbf{x},t)=\mathbf{x}-\boldsymbol{\varphi}^{-1}(\mathbf{x},t), \quad u_A = x_A - \varphi_A^{-1}$$

Taking the *spatial gradient*, the **spatial displacement gradient** tensor may be related to the **inverse deformation gradient** tensor, yielding,

$$\mathbf{j}(\mathbf{x},t)=\operatorname{grad}\mathbf{u}(\mathbf{x},t)=\mathbf{1}-\operatorname{grad}\boldsymbol{\varphi}^{-1}(\mathbf{x},t)=\mathbf{1}-\mathbf{F}^{-1}(\mathbf{x},t),$$

$$j_{Aa}=u_{A,a}=\varphi_{A,a}^{-1}-\delta_{Aa}=F_{Aa}^{-1}-\delta_{Aa}$$

Displacement Gradient Tensors

Material Displacement Gradient

$$d\mathbf{U}\left(\mathbf{X},t\right)=\mathbf{J}\left(\mathbf{X},t\right)d\mathbf{X}, \quad dU_a = J_{aA}dX_A$$

$$\mathbf{J}\left(\mathbf{X},t\right)=\bar{\nabla}\otimes\mathbf{U}\left(\mathbf{X},t\right)=\mathrm{GRAD}\,\mathbf{U}\left(\mathbf{X},t\right), \quad J_{aA}=U_{a,A}$$

$$\mathbf{J}\left(\mathbf{X},t\right)=\mathbf{F}\left(\mathbf{X},t\right)-\mathbf{1}, \quad J_{aA}=F_{aA}-\delta_{aA}$$

Spatial Displacement Gradient

$$d\mathbf{u}\left(\mathbf{x},t\right)=\mathbf{j}\left(\mathbf{x},t\right)d\mathbf{x}, \quad du_A = j_{Aa}dx_a$$

$$\mathbf{j}\left(\mathbf{x},t\right)=\nabla\otimes\mathbf{u}\left(\mathbf{x},t\right)=\mathrm{grad}\,\mathbf{u}\left(\mathbf{x},t\right), \quad j_{Aa}=u_{A,a}$$

$$\mathbf{j}\left(\mathbf{x},t\right)=\mathbf{1}-\mathbf{F}^{-1}\left(\mathbf{x},t\right), \quad j_{Aa}=F^{-1}_{Aa}-\delta_{Aa}$$

Push-forward / Pull-back

Gradient of a Scalar

Let us consider an arbitrary scalar field *Θ* such that,

$$\theta = \theta(\mathbf{x},t) = \theta(\boldsymbol{\varphi}(\mathbf{X},t),t) = \Theta(\mathbf{X},t)$$

The *material gradient* of a scalar field can be written as the **pull-back** of the *spatial gradient* of the scalar field given by,

$$\left(\mathrm{GRAD}\,\Theta(\mathbf{X},t)\right)_A = \frac{\partial\Theta(\mathbf{X},t)}{\partial X_A} = \frac{\partial\theta(\mathbf{x},t)}{\partial x_a}\frac{\partial\varphi_a(\mathbf{X},t)}{\partial X_A}$$

$$= \left(\mathrm{grad}\,\theta(\mathbf{x},t)\right)_a F_{aA}(\mathbf{X},t) = F_{Aa}^T(\mathbf{X},t)\left(\mathrm{grad}\,\theta(\mathbf{x},t)\right)_a$$

$$\boxed{\mathrm{GRAD}\,\Theta = \mathbf{F}^T\,\mathrm{grad}\,\theta}$$

Push-forward / Pull-back

Gradient of a Scalar

Let us consider an arbitrary scalar field Θ such that,

$$\theta = \Theta(\mathbf{X},t) = \Theta\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t\right) = \theta(\mathbf{x},t)$$

The *spatial gradient* of a scalar field can be written as the **push-forward** of the *material gradient* of the scalar field given by,

$$\left(\operatorname{grad}\theta(\mathbf{x},t)\right)_a = \frac{\partial\theta(\mathbf{x},t)}{\partial x_a} = \frac{\partial\Theta(\mathbf{X},t)}{\partial X_A}\frac{\partial\varphi_A^{-1}(\mathbf{x},t)}{\partial x_a}$$

$$= \left(\mathrm{GRAD}\,\Theta(\mathbf{X},t)\right)_A F_{Aa}^{-1}(\mathbf{x},t) = F_{aA}^{-T}(\mathbf{x},t)\left(\mathrm{GRAD}\,\Theta(\mathbf{X},t)\right)_A$$

$$\boxed{\operatorname{grad}\theta = \mathbf{F}^{-T}\,\mathrm{GRAD}\,\Theta}$$

Push-forward / Pull-back

Gradient of a Vector

Let us consider an arbitrary vector field *u* such that,

$$\mathbf{u} = \mathbf{u}(\mathbf{x},t) = \mathbf{u}(\varphi(\mathbf{X},t),t) = \mathbf{U}(\mathbf{X},t)$$

The *material gradient* of a vector field can be written as the **pull-back** of the *spatial gradient* of the vector field given by,

$$\left(\mathrm{GRAD}\,\mathbf{U}(\mathbf{X},t)\right)_{aA} = \frac{\partial U_a(\mathbf{X},t)}{\partial X_A} = \frac{\partial u_a(\mathbf{x},t)}{\partial x_b}\frac{\partial \varphi_b(\mathbf{X},t)}{\partial X_A}$$

$$= \left(\mathrm{grad}\,\mathbf{u}(\mathbf{x},t)\right)_{ab} F_{bA}(\mathbf{X},t)$$

$$\boxed{\mathrm{GRAD}\,\mathbf{U} = (\mathrm{grad}\,\mathbf{u})\mathbf{F}}$$

Push-forward / Pull-back

Gradient of a Vector

Let us consider an arbitrary vector field *u* such that,

$$\mathbf{u} = \mathbf{U}\left(\mathbf{X},t\right) = \mathbf{U}\left(\boldsymbol{\varphi}^{-1}\left(\mathbf{x},t\right),t\right) = \mathbf{u}\left(\mathbf{x},t\right)$$

The *spatial gradient* of a vector field can be written as the **push-forward** of the *material gradient* of the vector field given by,

$$\left(\operatorname{grad}\mathbf{u}\left(\mathbf{x},t\right)\right)_{Aa} = \frac{\partial u_A\left(\mathbf{x},t\right)}{\partial x_a} = \frac{\partial U_A\left(\mathbf{X},t\right)}{\partial X_B}\frac{\partial \varphi_B^{-1}\left(\mathbf{x},t\right)}{\partial x_a}$$

$$= \left(\operatorname{GRAD}\mathbf{U}\left(\mathbf{X},t\right)\right)_{AB} F_{Ba}^{-1}\left(\mathbf{x},t\right)$$

$$\boxed{\operatorname{grad}\mathbf{u} = \left(\operatorname{GRAD}\mathbf{U}\right)\mathbf{F}^{-1}}$$

Push-forward / Pull-back

Divergence of a Vector

Let us consider an arbitrary vector field *u* such that,

$$\mathbf{u}=\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\varphi(\mathbf{X},t),t)=\mathbf{U}(\mathbf{X},t)$$

The *material divergence* of a vector field can be written in terms of the *material* or *spatial gradient* of the vector field as,

$$\mathrm{DIV}\,\mathbf{U}(\mathbf{X},t)=\frac{\partial U_A(\mathbf{X},t)}{\partial X_A}=\frac{\partial u_A(\mathbf{x},t)}{\partial x_a}\frac{\partial \varphi_a(\mathbf{X},t)}{\partial X_A}$$

$$=(\mathrm{grad}\,\mathbf{u}(\mathbf{x},t))_{Aa}\,F_{aA}(\mathbf{X},t)=(\mathrm{grad}\,\mathbf{u}(\mathbf{x},t))_{Aa}\,F_{Aa}^{T}(\mathbf{X},t)$$

$$\boxed{\mathrm{DIV}\,\mathbf{U}=(\mathrm{GRAD}\,\mathbf{U}):\mathbf{1}=(\mathrm{grad}\,\mathbf{u}):\mathbf{F}^{T}}$$

Push-forward / Pull-back

Divergence of a Vector

Let us consider an arbitrary vector field *u* such that,

$$\mathbf{u} = \mathbf{U}(\mathbf{X},t) = \mathbf{U}\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t\right) = \mathbf{u}(\mathbf{x},t)$$

The *spatial divergence* of a vector field can be written in terms of the *material* or *spatial gradient* of the vector field as,

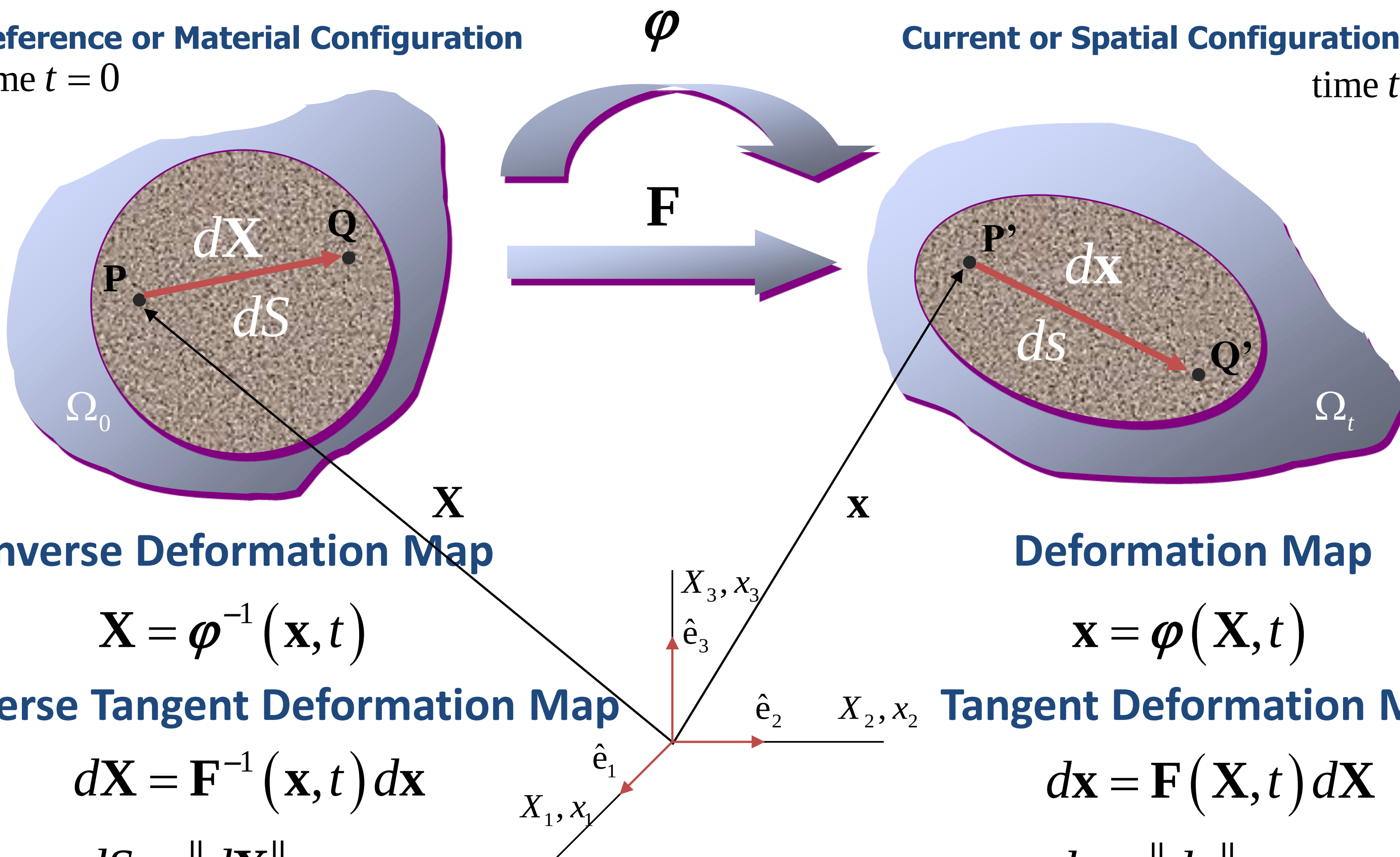
$$\operatorname{div}\mathbf{u}(\mathbf{x},t) = \frac{\partial u_a(\mathbf{x},t)}{\partial x_a} = \frac{\partial U_a(\mathbf{X},t)}{\partial X_A}\frac{\partial \varphi_A^{-1}(\mathbf{x},t)}{\partial x_a}$$
$$= \left(\mathrm{GRAD}\,\mathbf{U}(\mathbf{X},t)\right)_{aA} F_{Aa}^{-1}(\mathbf{x},t) = \left(\mathrm{GRAD}\,\mathbf{U}(\mathbf{X},t)\right)_{aA} F_{aA}^{-T}(\mathbf{x},t)$$

$$\boxed{\operatorname{div}\mathbf{u} = (\operatorname{grad}\mathbf{u}):\mathbf{1} = (\mathrm{GRAD}\,\mathbf{U}):\mathbf{F}^{-T}}$$

Deformation Tensors

Reference or Material Configuration

Current or Spatial Configuration



Inverse Deformation Map

$$\mathbf{X} = \varphi^{-1}(\mathbf{x}, t)$$

Deformation Map

$$\mathbf{x} = \varphi(\mathbf{X}, t)$$

Inverse Tangent Deformation Map

$$d\mathbf{X} = \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}$$

$$dS = \left\| d\mathbf{X} \right\|$$

Tangent Deformation Map

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$$

$$ds = \left\| d\mathbf{x} \right\|$$

Cauchy-Green Deformation Tensors

Right Cauchy-Green Deformation Tensor

The square of the norm of the differential vector $d\mathbf{x}$ may be written as,

$$ds^2 = \|d\mathbf{x}\|^2 = d\mathbf{x}\cdot d\mathbf{x} = d\mathbf{X}\cdot\mathbf{F}^T\mathbf{F}d\mathbf{X} := d\mathbf{X}\cdot\mathbf{C}d\mathbf{X},$$

$$ds^2 = dx_a dx_a = dX_A F^T_{Aa} F_{aB} dX_B := dX_A C_{AB} dX_B$$

where the *symmetric positive-definite* second-order **right Cauchy-Green deformation** tensor, denoted as $\mathbf{C}$, has been defined as,

$$\boxed{\mathbf{C} := \mathbf{F}^T\mathbf{F}, \quad C_{AB} := F^T_{Aa} F_{aB} = F_{aA} F_{aB}}$$

with

$$\boxed{\det\mathbf{C} := \left(\det\mathbf{F}\right)^2 = J^2 > 0}$$

Cauchy-Green Deformation Tensors

Left Cauchy-Green Deformation Tensor

The square of the norm of the differential vector $d\mathbf{X}$ may be written as,

$$dS^2 = \|d\mathbf{X}\|^2 = d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot \mathbf{F}^{-T}\mathbf{F}^{-1} d\mathbf{x} := d\mathbf{x} \cdot \mathbf{b}^{-1} d\mathbf{x},$$

$$dS^2 = dX_A dX_A = dx_a F_{aA}^{-T} F_{Ab}^{-1} dx_b := dx_a b_{ab}^{-1} dx_b$$

where the *symmetric positive-definite* second-order **left Cauchy-Green deformation** tensor, denoted as $\mathbf{b}$, has been defined as,

$$\boxed{\mathbf{b} := \mathbf{F}\mathbf{F}^T, \quad b_{ab} := F_{aA} F_{Ab}^T = F_{aA} F_{bA}}$$

with

$$\boxed{\det \mathbf{b} := \left(\det \mathbf{F}\right)^2 = J^2 > 0}$$

Green-Lagrange Strain Tensor

Green-Lagrange Strain Tensor

Let us consider the following scalar quantity as *strain measure*,

$$\begin{aligned} ds^2 - dS^2 &= d\mathbf{x}\cdot d\mathbf{x} - d\mathbf{X}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot\left(\mathbf{C}-\mathbf{1}\right)d\mathbf{X} := 2d\mathbf{X}\cdot\mathbf{E}d\mathbf{X} \end{aligned}$$

where the *symmetric* second-order **Green-Lagrange** (or **material**) **strain** tensor, denoted as $\mathbf{E}$, has been defined as,

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{C}-\mathbf{1}\right) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F}-\mathbf{1}\right) = \frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T+\mathbf{J}^T\mathbf{J}\right),$$

$$E_{AB} := \frac{1}{2}\left(C_{AB}-\delta_{AB}\right) = \frac{1}{2}\left(F_{aA}F_{aB}-\delta_{AB}\right) = \frac{1}{2}\left(J_{AB}+J_{BA}+J_{CA}J_{CB}\right)$$

Almansi Strain Tensor

Almansi Strain Tensor

Let us consider the following scalar quantity as *strain measure*,

$$ds^2 - dS^2 = d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}$$
$$= d\mathbf{x} \cdot \left(\mathbf{1} - \mathbf{b}^{-1}\right) d\mathbf{x} := 2 d\mathbf{x} \cdot \mathbf{e} d\mathbf{x}$$

where the *symmetric* second-order **Almansi** (or **spatial**) **strain** tensor, denoted as $\mathbf{e}$, has been defined as,

$$\mathbf{e} := \frac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^T - \mathbf{j}^T\mathbf{j}\right),$$
$$e_{ab} := \frac{1}{2}\left(\delta_{ab} - b_{ab}^{-1}\right) = \frac{1}{2}\left(\delta_{ab} - F_{Aa}^{-1}F_{Ab}^{-1}\right) = \frac{1}{2}\left(j_{ab} + j_{ba} - j_{ca}\, j_{cb}\right)$$

Strain Tensors

Green-Lagrange Strain Tensor

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{C} - \mathbf{1}\right) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T\mathbf{J}\right),$$

$$E_{AB} := \frac{1}{2}\left(C_{AB} - \delta_{AB}\right) = \frac{1}{2}\left(F_{aA}F_{aB} - \delta_{AB}\right) = \frac{1}{2}\left(J_{AB} + J_{BA} + J_{CA}J_{CB}\right)$$

Almansi Strain Tensor

$$\mathbf{e} := \frac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^T - \mathbf{j}^T\mathbf{j}\right),$$

$$e_{ab} := \frac{1}{2}\left(\delta_{ab} - b_{ab}^{-1}\right) = \frac{1}{2}\left(\delta_{ab} - F_{Aa}^{-1}F_{Ab}^{-1}\right) = \frac{1}{2}\left(j_{ab} + j_{ba} - j_{ca}j_{cb}\right)$$

Push-forward / Pull-back Maps

Push-forward of a Covariant Second-order Tensor

The **push-forward** of a *covariant* second-order tensor is defined as,

$$\varphi_*\left(\circ\right) := \mathbf{F}^{-T}\left(\circ\right)\mathbf{F}^{-1}$$

The **Almansi strain**, spatial second-order **unit** and **inverse of the left Cauchy-Green deformation** tensors can be viewed as the *push-forward* of the **Green-Lagrange strain, right Cauchy-Green deformation** and material second-order **unit** tensors, respecti-vely, such that,

$$\mathbf{e} = \varphi_*\left(\mathbf{E}\right) := \mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1},$$

$$\mathbf{1} = \varphi_*\left(\mathbf{C}\right) := \mathbf{F}^{-T}\mathbf{C}\mathbf{F}^{-1},$$

$$\mathbf{b}^{-1} = \varphi_*\left(\mathbf{1}\right) := \mathbf{F}^{-T}\mathbf{1}\mathbf{F}^{-1}$$

Push-forward / Pull-back Maps

Pull-back of a Covariant Second-order Tensor

The **pull-back** of a *covariant* second-order tensor is defined as,

$$\varphi_*^{-1}(\circ) := \mathbf{F}^T(\circ)\mathbf{F}$$

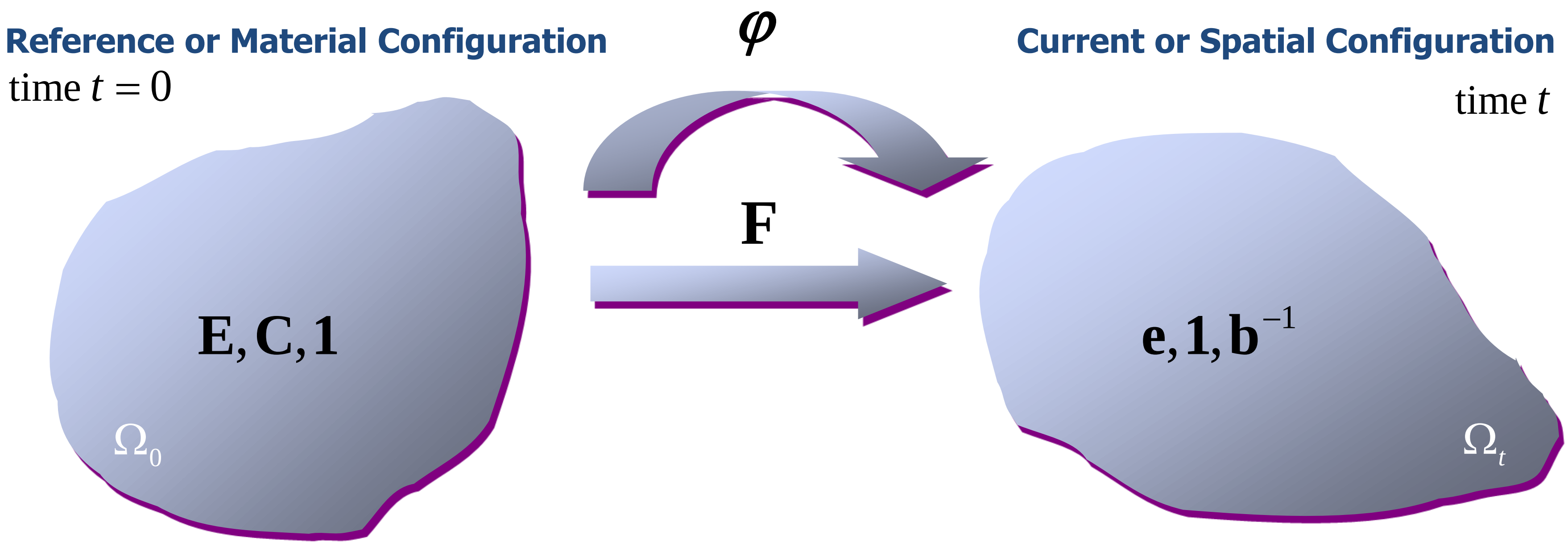
The **Green-Lagrange strain**, **right Cauchy-Green deformation** and material second-order **unit** tensors can be viewed as the *pull-back* of the **Almansi strain**, spatial second-order **unit** and **inverse of the left Cauchy-Green deformation** tensors, respectively, such that,

$$\mathbf{E} = \varphi_*^{-1}(\mathbf{e}) := \mathbf{F}^T\mathbf{e}\mathbf{F},$$

$$\mathbf{C} = \varphi_*^{-1}(\mathbf{1}) := \mathbf{F}^T\mathbf{1}\mathbf{F},$$

$$\mathbf{1} = \varphi_*^{-1}(\mathbf{b}^{-1}) := \mathbf{F}^T\mathbf{b}^{-1}\mathbf{F}$$

Push-forward / Pull-back Maps

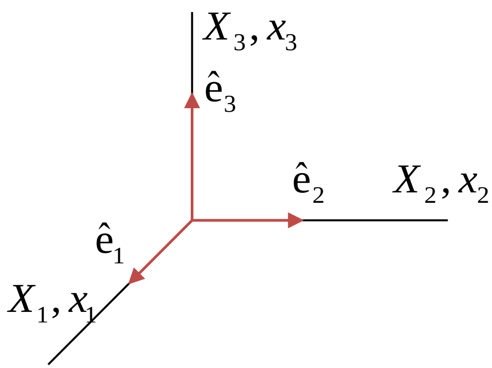


Pull-back Maps

$$\mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F}$$

$$\mathbf{C} = \mathbf{F}^T \mathbf{1} \mathbf{F}$$

$$\mathbf{1} = \mathbf{F}^T \mathbf{b}^{-1} \mathbf{F}$$



Push-forward Maps

$$\mathbf{e} = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1}$$

$$\mathbf{1} = \mathbf{F}^{-T} \mathbf{C} \mathbf{F}^{-1}$$

$$\mathbf{b}^{-1} = \mathbf{F}^{-T} \mathbf{1} \mathbf{F}^{-1}$$

Example 3.2

Example 3.2

Compute the *Green-Lagrange* and *Almansi* strain tensors for a motion equation given by,

$$[\mathbf{x}] = [\boldsymbol{\varphi}(\mathbf{X},t)] = \left[X + Yt, Ye^{-t}, Ze^{t} \right]^{T}$$

Example 3.2

Example 3.2

Compute the *Green-Lagrange* and *Almansi* strain tensors for a motion equation given by,

$$[\mathbf{x}] = [\boldsymbol{\varphi}(\mathbf{X},t)] = \left[X + Yt, Ye^{-t}, Ze^{t}\right]^T$$

The components of the **inverse motion**, **deformation gradient** and **inverse deformation gradient** are given by,

$$[\mathbf{X}] = \left[\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right] = \left[x - yte^{t}, ye^{t}, ze^{-t}\right]^T$$

$$[\mathbf{F}] = \begin{bmatrix} 1 & t & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}, \quad \left[\mathbf{F}^{-1}\right] = \begin{bmatrix} 1 & -te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Example 3.2

The components of the **right Cauchy-Green deformation tensor** and **Green-Lagrange strain tensor** take the form,

$$[\mathbf{C}] = \left[\mathbf{F}^T\mathbf{F}\right] = \begin{bmatrix} 1 & 0 & 0 \\ t & e^{-t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{bmatrix} = \begin{bmatrix} 1 & t & 0 \\ t & t^2 + e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$[\mathbf{E}] = \left[\frac{1}{2}(\mathbf{C} - \mathbf{1})\right] = \frac{1}{2}\begin{bmatrix} 0 & t & 0 \\ t & t^2 + e^{-2t} - 1 & 0 \\ 0 & 0 & e^{2t} - 1 \end{bmatrix}$$

Note that at the reference configuration for t=0,

$$\mathbf{F} = \mathbf{1}, \quad \mathbf{C} = \mathbf{1}, \quad \mathbf{E} = \mathbf{0}$$

Example 3.2

The components of the **left Cauchy-Green deformation tensor** and **Almansi strain tensor** take the form,

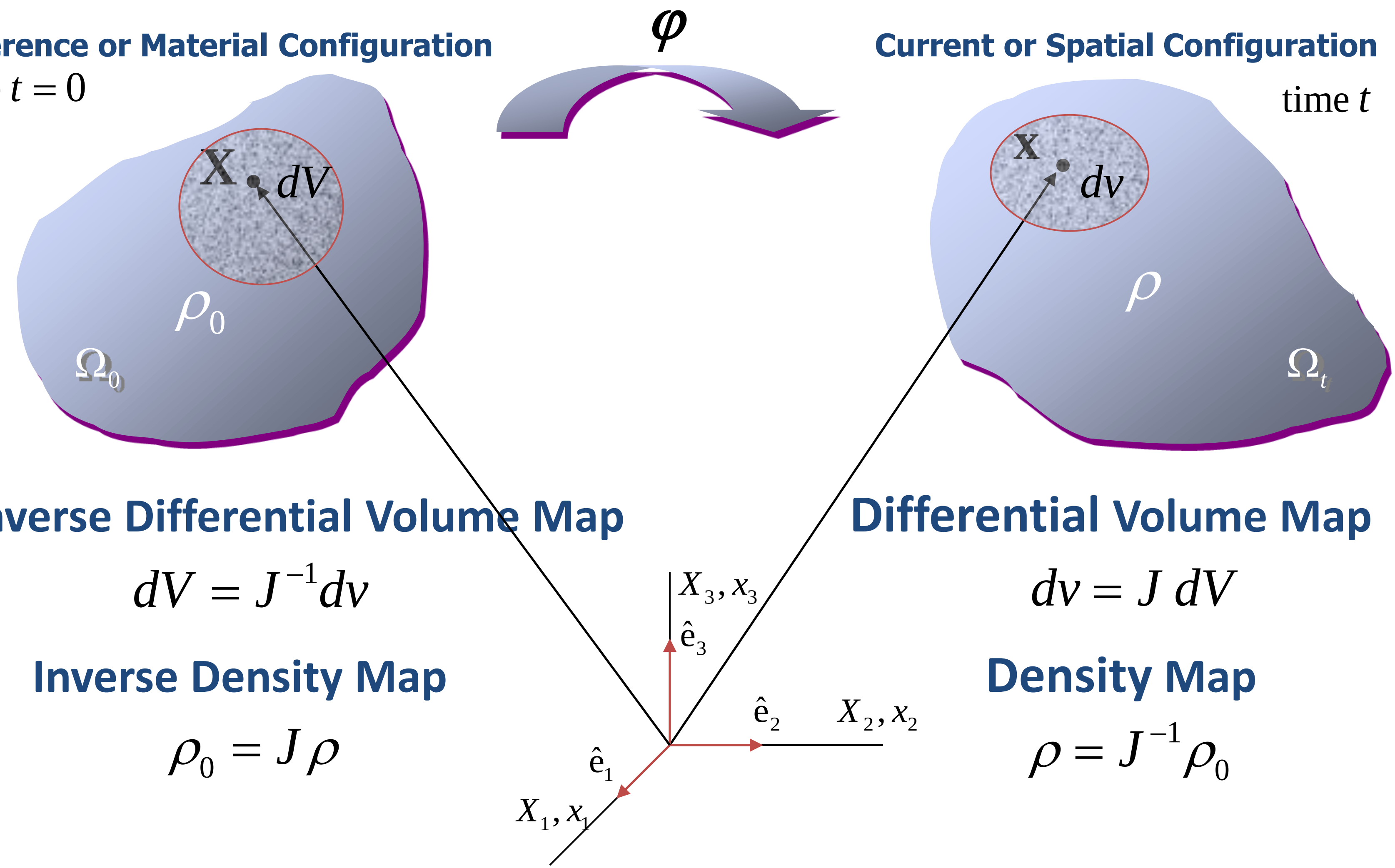
$$\left[\mathbf{b}^{-1}\right]=\left[\mathbf{F}^{-T}\mathbf{F}^{-1}\right]=\begin{bmatrix} 1 & 0 & 0 \\ -te^{t} & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}=\begin{bmatrix} 1 & -te^{t} & 0 \\ -te^{t} & \left(t^{2}+1\right)e^{2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

$$\left[\mathbf{e}\right]=\left[\frac{1}{2}\left(\mathbf{1}-\mathbf{b}^{-1}\right)\right]=\frac{1}{2}\begin{bmatrix} 0 & te^{t} & 0 \\ te^{t} & 1-\left(t^{2}+1\right)e^{2t} & 0 \\ 0 & 0 & 1-e^{-2t} \end{bmatrix}$$

Note that at the reference configuration for t=0,

$$\mathbf{F}=\mathbf{1},\quad \mathbf{b}^{-1}=\mathbf{1},\quad \mathbf{e}=\mathbf{0} \quad \blacksquare$$

Volumetric Deformation Map



Inverse Differential Volume Map

$$dV = J^{-1} dv$$

Inverse Density Map

$$\rho_0 = J\rho$$

Differential Volume Map

$$dv = J\, dV$$

Density Map

$$\rho = J^{-1}\rho_0$$

Volumetric Deformation

Volumetric Deformation

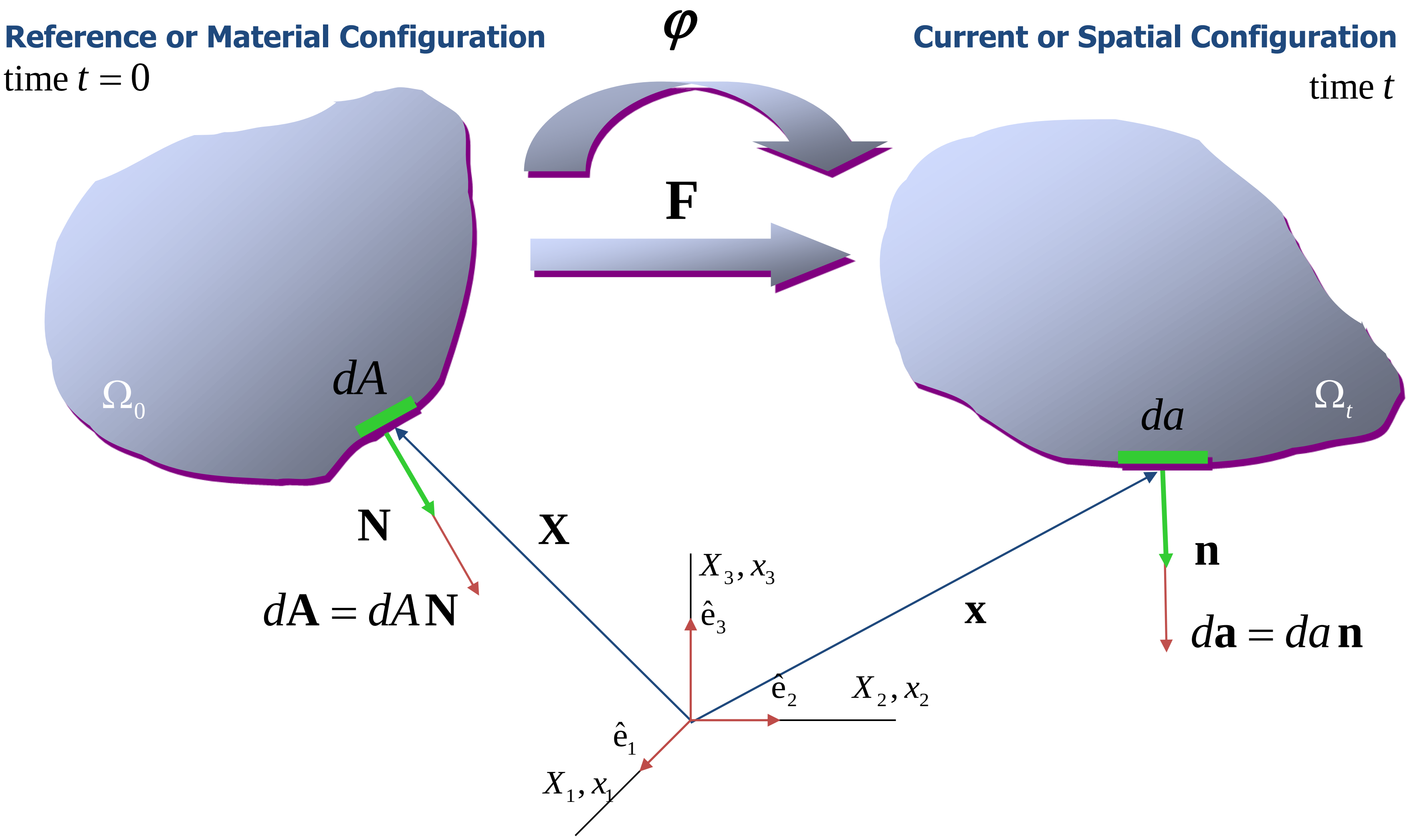
The **volumetric deformation**, denoted as *e*, is a scalar quantity defined as,

$$e = \frac{dv - dV}{dV} = \frac{dv}{dV} - 1 = J - 1$$

The **incompressibility** condition, i.e. zero volumetric deformation, takes the form,

$$\boxed{J = 1}$$

Area Deformation



Area Deformation

Area Deformation

Let us consider a **differential of area vector** on the reference and spatial configurations written in terms of the **unit outward normal** to the surface on the material and spatial configurations, respectively, given by,

$$d\mathbf{A} = dA\,\mathbf{N}, \quad d\mathbf{a} = da\,\mathbf{n}$$

Taking an *arbitrary* vector $d\mathbf{X}$, associated **differential of volumes** in the material and spatial configurations take the form,

$$dV = d\mathbf{X} \cdot d\mathbf{A} = d\mathbf{X} \cdot dA\,\mathbf{N}, \quad dv = d\mathbf{x} \cdot d\mathbf{a} = d\mathbf{x} \cdot da\,\mathbf{n}$$

where

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \quad dv = JdV$$

Area Deformation

The differential of volumes satisfy the following expression,

$$dv = d\mathbf{a} \cdot d\mathbf{x} = d\mathbf{a} \cdot \mathbf{F} d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T d\mathbf{a}$$

$$= J\, d\mathbf{A} \cdot d\mathbf{X} = d\mathbf{X} \cdot J\, d\mathbf{A} \quad \forall d\mathbf{X} \quad \Rightarrow \quad \mathbf{F}^T d\mathbf{a} = J\, d\mathbf{A}$$

yielding the relation, known as **Nanson's formula**, given by,

$$\boxed{d\mathbf{a} = J\, \mathbf{F}^{-T} d\mathbf{A}, \quad da\, \mathbf{n} = dA\, J\, \mathbf{F}^{-T}\, \mathbf{N}}$$

Polar Decomposition

Polar Decomposition

For any *non-singular* second-order tensor, denoted as $\mathbf{F}$, there exist two unique *symmetric positive-definite* second-order tensors, denoted as $\mathbf{U}$ and $\mathbf{v}$, and a unique *proper orthogonal* second-order tensor, denoted as $\mathbf{R}$, such that,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \quad F_{aA} = R_{aB}U_{BA} = v_{ab}R_{bA}$$

where,

$$\mathbf{U} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2}, \quad \mathbf{U} = \mathbf{U}^T, \quad \mathbf{x}\cdot\mathbf{U}\mathbf{x} > 0 \;\; \forall \mathbf{x} \neq 0$$

$$\mathbf{v} = \left(\mathbf{F}\mathbf{F}^T\right)^{1/2}, \quad \mathbf{v} = \mathbf{v}^T, \quad \mathbf{x}\cdot\mathbf{v}\mathbf{x} > 0 \;\; \forall \mathbf{x} \neq 0$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{v}^{-1}\mathbf{F}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det\mathbf{R} = 1$$

Polar Decomposition

Polar Decomposition

The **polar decomposition** of the **deformation gradient** tensor $\mathbf{F}$, reads,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \quad F_{aA} = R_{aB} U_{BA} = v_{ab} R_{bA}$$

where $\mathbf{U}$ is the **right** (or **material**) **stretch tensor**, $\mathbf{v}$ is the **left** (or **spatial**) **stretch tensor** and $\mathbf{R}$ is the **rotation tensor**, such that,

$$\mathbf{U} = \left(\mathbf{F}^T \mathbf{F}\right)^{1/2} = \mathbf{C}^{1/2}, \quad \mathbf{U} = \mathbf{U}^T, \quad \mathbf{x} \cdot \mathbf{U}\mathbf{x} > 0 \ \ \forall \mathbf{x} \neq 0, \quad \det \mathbf{U} = J$$

$$\mathbf{v} = \left(\mathbf{F}\mathbf{F}^T\right)^{1/2} = \mathbf{b}^{1/2}, \quad \mathbf{v} = \mathbf{v}^T, \quad \mathbf{x} \cdot \mathbf{v}\mathbf{x} > 0 \ \ \forall \mathbf{x} \neq 0, \quad \det \mathbf{v} = J$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{v}^{-1}\mathbf{F}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = 1$$

Polar Decomposition

Polar Decomposition

The **rotation tensor** $\mathbf{R}$ rotates a material line segment $d\mathbf{X}$ onto a unique spatial line segment $d\mathbf{x} = \mathbf{R}\,d\mathbf{X}$, such that the *norm* of the line segment is *preserved*.

$$\|d\mathbf{x}\|^2 = d\mathbf{x}\cdot d\mathbf{x} = \left(\mathbf{R}\,d\mathbf{X}\right)\cdot\left(\mathbf{R}\,d\mathbf{X}\right) = d\mathbf{X}\cdot\mathbf{R}^T\mathbf{R}\,d\mathbf{X} = d\mathbf{X}\cdot d\mathbf{X} = \|d\mathbf{X}\|^2$$

The **rotation tensor** $\mathbf{R}$ rotates material line segments $d\mathbf{X}$ and $d\mathbf{Y}$ onto unique spatial line segments $d\mathbf{x} = \mathbf{R}\,d\mathbf{X}$ and $d\mathbf{y} = \mathbf{R}\,d\mathbf{Y}$, such that the *angle* between the line segments is *preserved*.

$$\cos\theta = \frac{d\mathbf{x}\cdot d\mathbf{y}}{\|d\mathbf{x}\|\|d\mathbf{y}\|} = \frac{d\mathbf{X}\cdot\mathbf{R}^T\mathbf{R}\,d\mathbf{Y}}{\|d\mathbf{x}\|\|d\mathbf{y}\|} = \frac{d\mathbf{X}\cdot d\mathbf{Y}}{\|d\mathbf{X}\|\|d\mathbf{Y}\|} = \cos\Theta$$

Polar Decomposition

Polar Decomposition

The **right** (or **material**) **stretch tensor** $\mathbf{U}$ and the **left** (or **spatial**) **stretch tensor** $\mathbf{v}$ satisfy the following *pull-back* and *push-forward* relations with the *rotation tensor*,

$$\mathbf{U} = \mathbf{R}^{-1}\mathbf{F} = \mathbf{R}^{T}\mathbf{F} = \mathbf{R}^{T}\mathbf{v}\mathbf{R}, \quad U_{AB} = R_{aA} v_{ab} R_{bB}$$

$$\mathbf{v} = \mathbf{F}\mathbf{R}^{-1} = \mathbf{F}\mathbf{R}^{T} = \mathbf{R}\mathbf{U}\mathbf{R}^{T}, \quad v_{ab} = R_{aA} U_{AB} R_{bB}$$

The **right Cauchy-Green tensor** $\mathbf{C}$ and the **left Cauchy-Green tensor** $\mathbf{b}$ satisfy the following *pull-back* and *push-forward* relations with the *rotation tensor*,

$$\mathbf{C} = \mathbf{U}^{2} = \mathbf{R}^{T}\mathbf{v}\mathbf{R}\mathbf{R}^{T}\mathbf{v}\mathbf{R} = \mathbf{R}^{T}\mathbf{v}^{2}\mathbf{R} = \mathbf{R}^{T}\mathbf{b}\mathbf{R}, \quad C_{AB} = R_{aA} b_{ab} R_{bB}$$

$$\mathbf{b} = \mathbf{v}^{2} = \mathbf{R}\mathbf{U}\mathbf{R}^{T}\mathbf{R}\mathbf{U}\mathbf{R}^{T} = \mathbf{R}\mathbf{U}^{2}\mathbf{R}^{T} = \mathbf{R}\mathbf{C}\mathbf{R}^{T}, \quad b_{ab} = R_{aA} C_{AB} R_{bB}$$

Polar Decomposition

Polar Decomposition

A **rigid body motion** satisfies the following relations,

$$\mathbf{F} = \mathbf{R} \quad \Leftrightarrow \quad \mathbf{U} = \mathbf{v} = \mathbf{1} \quad \Leftrightarrow \quad \mathbf{E} = \mathbf{e} = \mathbf{0}$$

A **pure stretch** deformation satisfies the following relations,

$$\mathbf{R} = \mathbf{1} \quad \Leftrightarrow \quad \mathbf{F} = \mathbf{U} = \mathbf{v}$$

Polar Decomposition

Polar Decomposition

Any deformation can be seen either as a *composition* of a **right** (or **material**) **stretch**, characterized by $\mathbf{U}$, with a **rotation**, characterized by $\mathbf{R}$, given by the *right polar decomposition*,

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad F_{aA} = R_{aB} U_{BA}$$

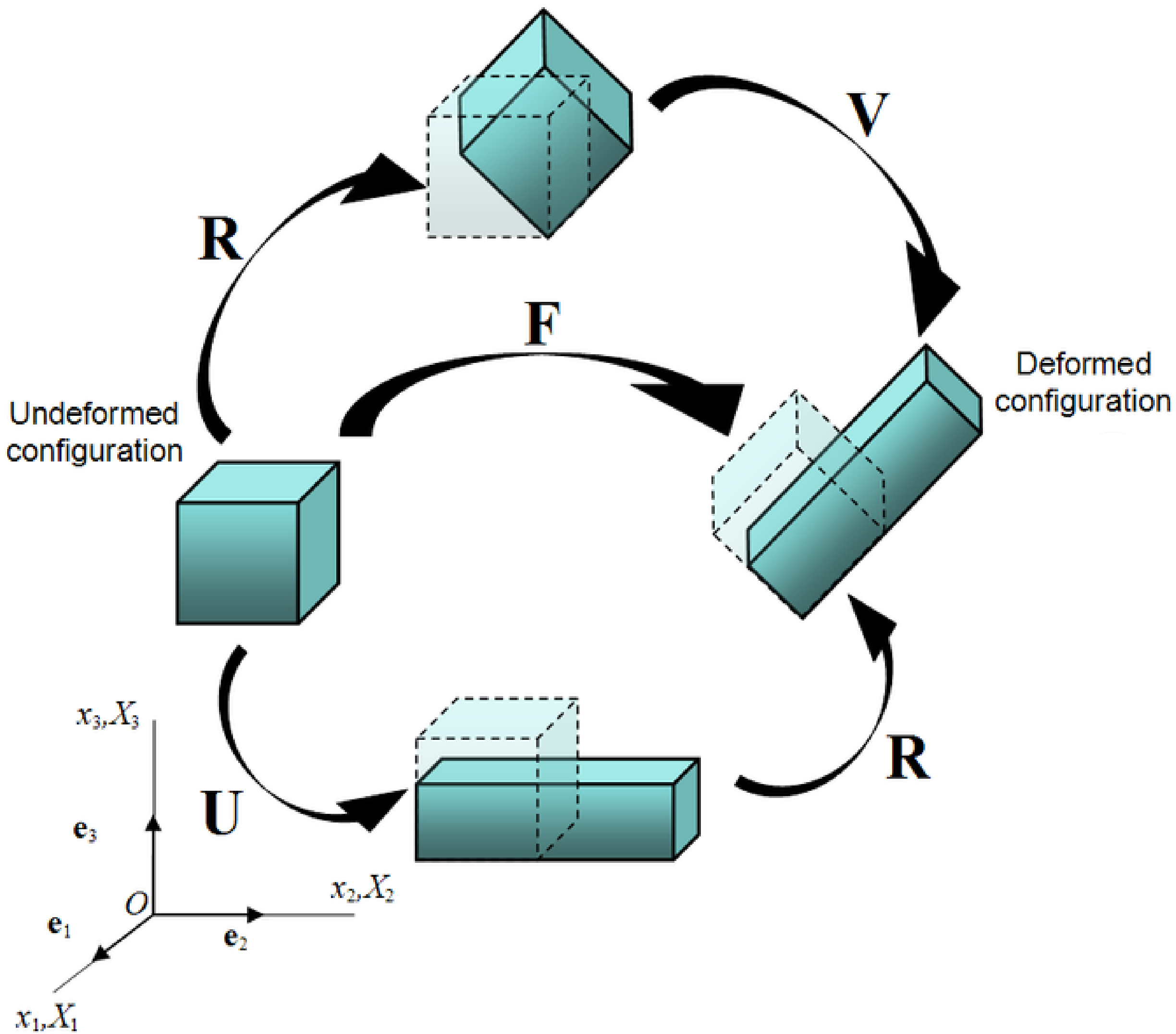
$$d\mathbf{x} = \mathbf{F} d\mathbf{X} = \mathbf{R}\mathbf{U} d\mathbf{X} = \mathbf{R}\left(\mathbf{U} d\mathbf{X}\right), \quad dx_a = F_{aA} dX_A = R_{aA} U_{AB} dX_B$$

or as a *composition* of a **rotation**, characterized by $\mathbf{R}$, with a **left** (or **spatial**) **stretch**, characterized by $\mathbf{v}$, given by the *left polar decomposition*,

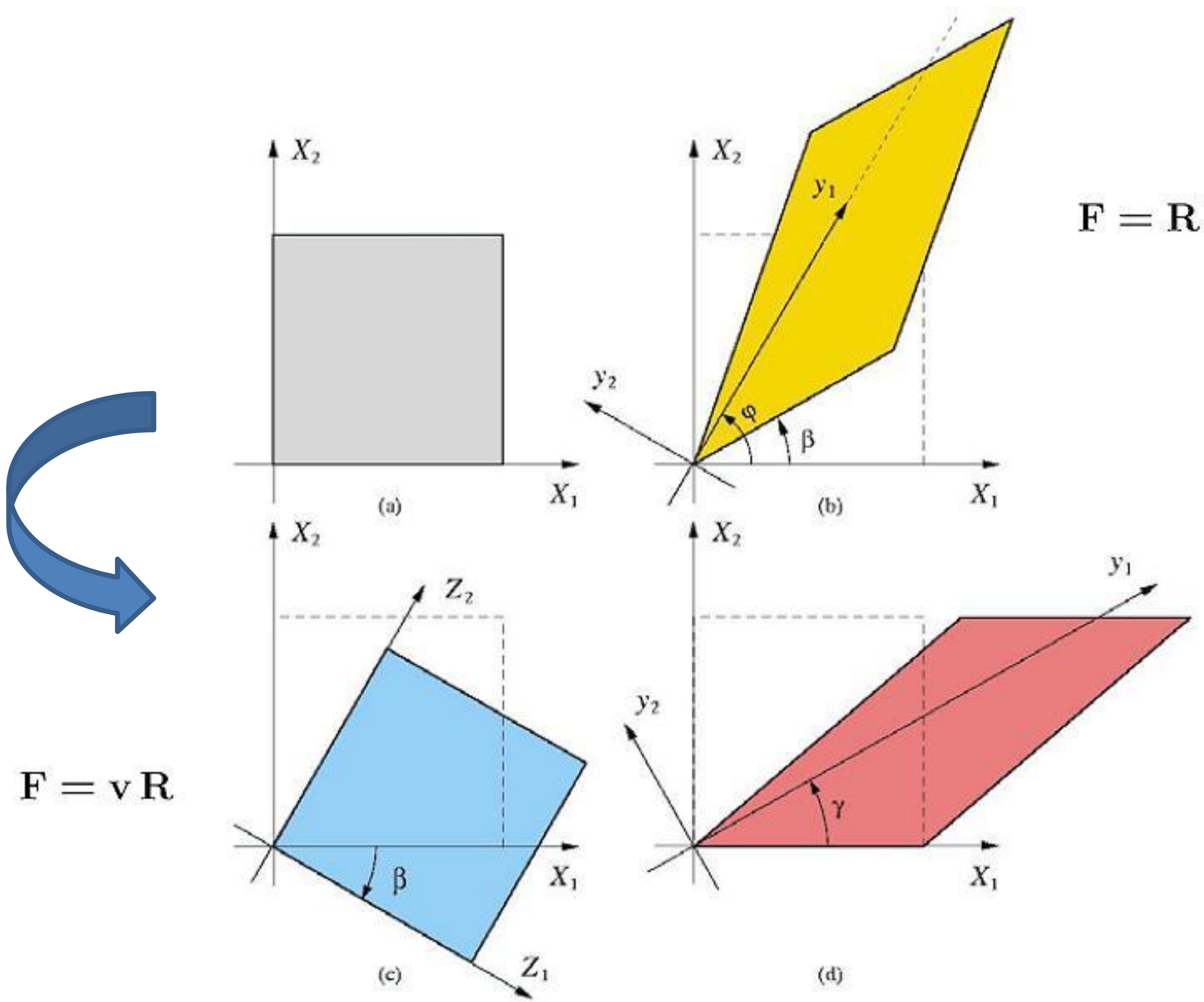
$$\mathbf{F} = \mathbf{v}\mathbf{R}, \quad F_{aA} = v_{ab} R_{bA}$$

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} = \mathbf{v}\mathbf{R} d\mathbf{X} = \mathbf{v}\left(\mathbf{R} d\mathbf{X}\right), \quad dx_a = F_{aA} dX_A = v_{ab} R_{bA} dX_A$$

Polar Decomposition



Polar Decomposition



$$\mathbf{F} = \mathbf{R}\,\mathbf{U}$$

$$\mathbf{F} = \mathbf{v}\,\mathbf{R}$$

Polar Decomposition

Polar Decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \quad F_{aA} = R_{aB} U_{BA} = v_{ab} R_{bA}$$

Material Right and Spatial Left Stretch Tensors

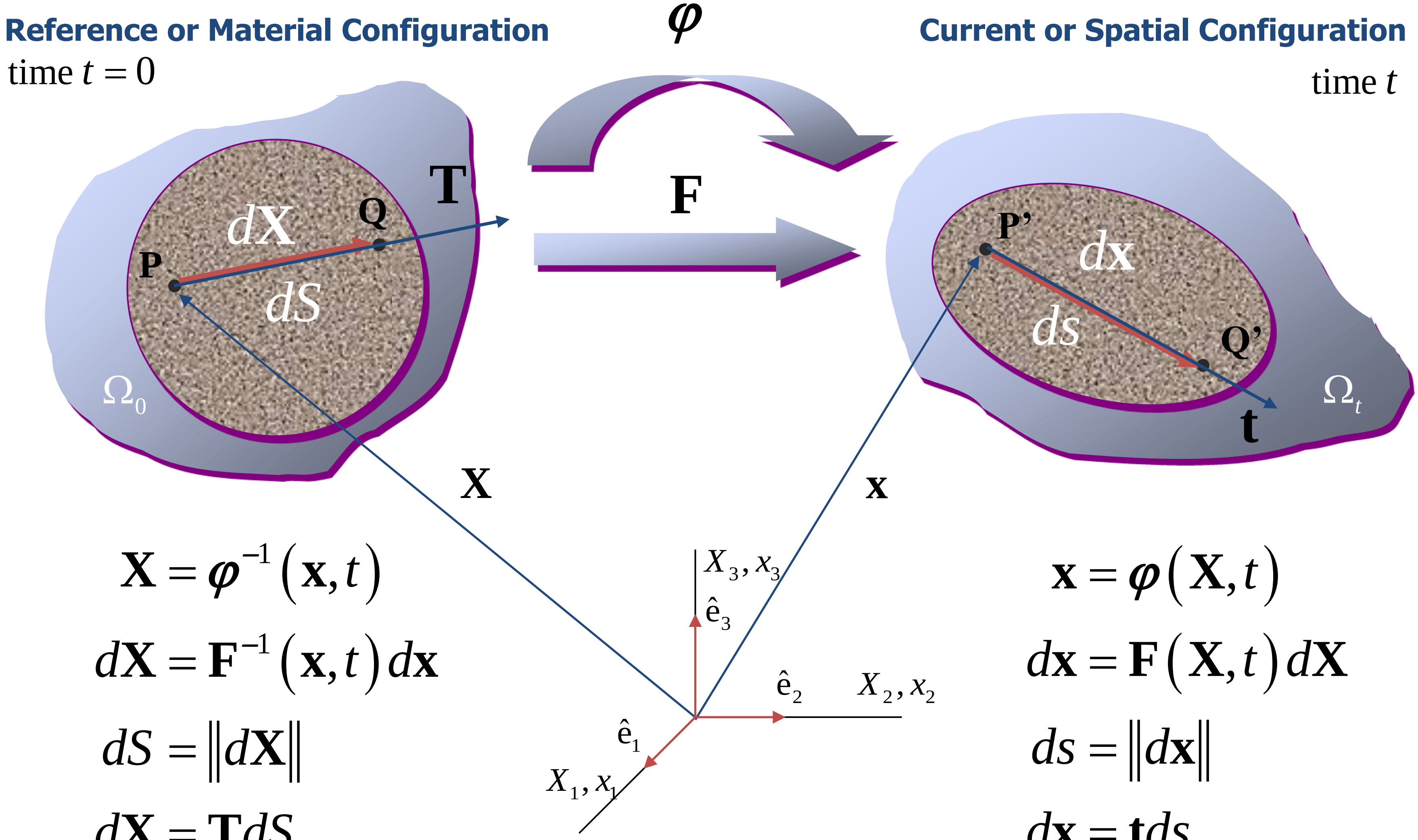
$$\mathbf{U} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2} = \mathbf{C}^{1/2}, \quad U_{AB} = \left(F_{aA} F_{aB}\right)^{1/2} = C_{AB}^{1/2}$$

$$\mathbf{v} = \left(\mathbf{F}\mathbf{F}^T\right)^{1/2} = \mathbf{b}^{1/2}, \quad v_{ab} = \left(F_{aA} F_{bA}\right)^{1/2} = b_{ab}^{1/2}$$

Rotation Tensor

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{v}^{-1}\mathbf{F}, \quad R_{aA} = F_{aB} U_{BA}^{-1} = v_{ab}^{-1} F_{bA}$$

Stretches



$$\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

$$d\mathbf{X} = \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}$$

$$dS = \|d\mathbf{X}\|$$

$$d\mathbf{X} = \mathbf{T}\, dS$$

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$$

$$ds = \|d\mathbf{x}\|$$

$$d\mathbf{x} = \mathbf{t}\, ds$$

Stretch Vectors

Material Stretch Vector

Let us denote as $\boldsymbol{\lambda}_T$ the **material stretch vector** at a *material point* $\mathbf{X}$ at time *t*, along a *material direction* given by the unit vector $\mathbf{T}$ on the material configuration,

$$\boldsymbol{\lambda}_T\left(\mathbf{X},t\right)=\mathbf{F}\left(\mathbf{X},t\right)\mathbf{T}, \quad \lambda_{T_a}=F_{aA}T_A$$

Multiplying by dS yields,

$$d\mathbf{x}=\boldsymbol{\lambda}_T\left(\mathbf{X},t\right)dS=\mathbf{F}\left(\mathbf{X},t\right)\mathbf{T}dS=\mathbf{F}\left(\mathbf{X},t\right)d\mathbf{X}$$

Taking norms, the **stretch**, denoted as λ, is defined as,

$$ds=\left\|\boldsymbol{\lambda}_T\left(\mathbf{X},t\right)\right\|dS:=\lambda dS,$$

$$\lambda:=\left\|\boldsymbol{\lambda}_T\left(\mathbf{X},t\right)\right\|=\left(\mathbf{T}\cdot\mathbf{F}^T\mathbf{F}\mathbf{T}\right)^{1/2}=\left(\mathbf{T}\cdot\mathbf{C}\mathbf{T}\right)^{1/2}=\left(1+2\mathbf{T}\cdot\mathbf{E}\mathbf{T}\right)^{1/2}$$

Stretch Vectors

Material Stretch Vector

Let us denote as $\boldsymbol{\lambda}_T$ the **material stretch vector** at a *material point* $\mathbf{X}$ at time *t*, along a *material direction* given by the unit vector $\mathbf{T}$ on the material configuration. The following situations may arise

$$\lambda := \left\| \boldsymbol{\lambda}_T \left(\mathbf{X}, t \right) \right\| > 1 \quad \ldots \quad \text{extension, length increases}$$

$$\lambda := \left\| \boldsymbol{\lambda}_T \left(\mathbf{X}, t \right) \right\| = 1 \quad \ldots \quad \text{length does not changes}$$

$$\lambda := \left\| \boldsymbol{\lambda}_T \left(\mathbf{X}, t \right) \right\| < 1 \quad \ldots \quad \text{compression, length decreases}$$

Stretch Vectors

Spatial Stretch Vector

Let us denote as $\boldsymbol{\lambda}_t$ the **spatial stretch vector** at a *spatial point* $\mathbf{x}$ at time *t*, along a *spatial direction* given by the unit vector $\mathbf{t}$ on the spatial configuration,

$$\boldsymbol{\lambda}_t(\mathbf{x},t) = \mathbf{F}^{-1}(\mathbf{x},t)\,\mathbf{t}, \quad \lambda_{t_A} = F^{-1}_{Aa} t_a$$

Multiplying by ds yields,

$$d\mathbf{X} = \boldsymbol{\lambda}_t(\mathbf{x},t)\,ds = \mathbf{F}^{-1}(\mathbf{x},t)\,\mathbf{t}\,ds = \mathbf{F}^{-1}(\mathbf{x},t)\,d\mathbf{x}$$

Taking norms, the **inverse stretch**, denoted as λ^{-1}, is defined as,

$$dS = \left\|\boldsymbol{\lambda}_t(\mathbf{x},t)\right\| ds := \lambda^{-1} ds,$$

$$\lambda^{-1} := \left\|\boldsymbol{\lambda}_t(\mathbf{x},t)\right\| = \left(\mathbf{t}\cdot\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{t}\right)^{1/2} = \left(\mathbf{t}\cdot\mathbf{b}^{-1}\mathbf{t}\right)^{1/2} = \left(1 - 2\mathbf{t}\cdot\mathbf{e}\mathbf{t}\right)^{1/2}$$

Stretches

Material Stretch Vector

$$\boldsymbol{\lambda}_T(\mathbf{X},t) := \mathbf{F}(\mathbf{X},t)\mathbf{T}, \quad \lambda_{T_a} := F_{aA}T_A$$

$$ds := \lambda dS, \quad \lambda := \left\|\boldsymbol{\lambda}_T(\mathbf{X},t)\right\|$$

$$\lambda := \left(\mathbf{T}\cdot\mathbf{F}^T\mathbf{F}\mathbf{T}\right)^{1/2} = \left(\mathbf{T}\cdot\mathbf{C}\mathbf{T}\right)^{1/2} = \left(1+2\mathbf{T}\cdot\mathbf{E}\mathbf{T}\right)^{1/2}$$

Spatial Stretch Vector

$$\boldsymbol{\lambda}_t(\mathbf{x},t) := \mathbf{F}^{-1}(\mathbf{x},t)\mathbf{t}, \quad \lambda_{t_A} := F_{Aa}^{-1}t_a$$

$$dS := \lambda^{-1}ds, \quad \lambda^{-1} := \left\|\boldsymbol{\lambda}_t(\mathbf{x},t)\right\|$$

$$\lambda^{-1} := \left(\mathbf{t}\cdot\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{t}\right)^{1/2} = \left(\mathbf{t}\cdot\mathbf{b}^{-1}\mathbf{t}\right)^{1/2} = \left(1-2\mathbf{t}\cdot\mathbf{e}\mathbf{t}\right)^{1/2}$$

Physical Interpretation of E11 Component

Green-Lagrange Strain Component E11

Let us consider a material segment $d\mathbf{X}^{(1)} = \mathbf{T}^{(1)} dS$ along the X1-axis on the material configuration.

The material stretch along the X1 direction will be given by,

$$\lambda_1 := \left(1 + 2\mathbf{T}^{(1)} \cdot \mathbf{E}\mathbf{T}^{(1)}\right)^{1/2} = 1 + 2E_{11}$$

The length of the deformed segment will be given by,

$$ds := \lambda_1 dS = \left(1 + 2E_{11}\right) dS$$

and the *Green-Lagrange component E11* may be interpreted as,

$$E_{11} = \frac{1}{2}\left(\frac{ds}{dS} - 1\right) = \frac{1}{2}\left(\frac{ds - dS}{dS}\right)$$

Physical Interpretation of e11 Component

Almansi Strain Component e11

Let us consider a spatial segment $d\mathbf{x}^{(1)} = \mathbf{t}^{(1)} ds$ along the x1-axis on the spatial configuration.

The inverse stretch along the x1 direction will be given by,

$$\lambda_1^{-1} := \left(1 - 2\mathbf{t}^{(1)} \cdot \mathbf{e}\mathbf{t}^{(1)}\right)^{1/2} = 1 - 2e_{11}$$

The length of the material segment will be given by,

$$dS := \lambda_1^{-1} ds = \left(1 - 2e_{11}\right) ds$$

and the *Almansi strain component e11* may be interpreted as,

$$e_{11} = \frac{1}{2}\left(1 - \frac{dS}{ds}\right) = \frac{1}{2}\left(\frac{ds - dS}{ds}\right)$$

Assignment 3.1

Assignment 3.1

The components of the *Almansi strain tensor*, with reference time t=0, are given by,

$$[\mathbf{e}] = \begin{bmatrix} 0 & 0 & -te^{tz} \\ 0 & 0 & 0 \\ -te^{tz} & 0 & t\left(2e^{tz} - e^{t}\right) \end{bmatrix}$$

Compute at the reference time t=0, the *length* of a material curve that at time t=2 is a straight line going from point a with coordinates (0,0,0) to point b with coordinates (1,1,1).

Assignment 3.1

Assignment 3.1

The **length** of the curve at the reference time t=0 may be written as,

$$L = \int_{\Gamma} dS = \int_{a}^{b} \lambda^{-1}\left(\mathbf{x},t\right) ds$$

The **inverse of the stretch** at any spatial point of the straight line, along the (constant) direction of the straight line is given by,

$$\lambda^{-1}\left(\mathbf{x},t\right) = \sqrt{\mathbf{1} - 2\mathbf{t}\cdot\mathbf{e}\left(\mathbf{x},t\right)\mathbf{t}}$$

where the (constant) **unit vector** along the direction of the straight line is given by,

$$\left[\mathbf{t}\right] = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^{T}$$

Assignment 3.1

Substituting the expressions for the Almansi strain tensor and the unit direction vector into the expression of the inverse stretch, and particularizing for *t*=2, yields a *uniform* **inverse stretch** given by,

$$\lambda^{-1}(\mathbf{x},t)\Big|_{t=2} = \sqrt{1+\frac{2}{3}te^{t}}\,\Bigg|_{t=2} = \sqrt{1+\frac{4}{3}e^{2}}$$

Substituting into the integral expression for the **length** yields,

$$L = \int_{\Gamma} dS = \int_a^b \lambda^{-1}(\mathbf{x},t)\Big|_{t=2}\, ds = \int_a^b \sqrt{1+\frac{4}{3}e^{2}}\, ds = \sqrt{1+\frac{4}{3}e^{2}}\,\sqrt{3}$$

$$L = \sqrt{3+4e^{2}} \quad \blacksquare$$

Assignment 3.2

Assignment 3.2 [Classwork]

Consider the *equations of motion* given by,

$$x = X, \quad y = Y + Z^2 t, \quad z = Z + Y^2 t$$

Compute at time t=1 the *length* of a material curve that at the reference time t=0 was a straight line going from point A with coordinates (0,0,0) to point B with coordinates (0,1,1).

Assignment 3.2

Assignment 3.2 [Classwork]

Consider the *equations of motion* given by,

$$x = X, \quad y = Y + Z^2 t, \quad z = Z + Y^2 t$$

Compute at time *t*=1 the *length* of a material curve that at the reference time *t*=0 was a straight line going from point A with coordinates (0,0,0) to point B with coordinates (0,1,1).

The **length** of the material curve at time *t*=1 may be computed as,

$$l = \int_a^b ds = \int_A^B \lambda \, dS = \int_A^B \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}} \, dS$$

Assignment 3.2

The **unit vector** along the straight line is given by,

$$[\mathbf{T}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$$

The **Green-Lagrange strain tensor** is given by,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) = \frac{1}{2}\left(\mathbf{C} - \mathbf{1}\right)$$

The components of the **deformation gradient** are,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2Zt \\ 0 & 2Yt & 1 \end{bmatrix}$$

Assignment 3.2

The components of the **right Cauchy-Green deformation tensor** are,

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2Yt \\ 0 & 2Zt & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2Zt \\ 0 & 2Yt & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+4Y^2t^2 & 2Zt+2Yt \\ 0 & 2Zt+2Yt & 1+4Z^2t^2 \end{bmatrix}$$

The components of the **Green-Lagrange strain tensor** are,

$$[\mathbf{E}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2Y^2t^2 & Zt+Yt \\ 0 & Zt+Yt & 2Z^2t^2 \end{bmatrix}$$

Assignment 3.2

The components of the **Green-Lagrange strain tensor** have to be *particularized* for the points of the straight line, i.e. *points that satisfy the equation* Y=Z, yielding,

$$[\mathbf{E}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2Y^2t^2 & 2Zt \\ 0 & 2Zt & 2Z^2t^2 \end{bmatrix}$$

Substituting into the expression for the **stretch** yields,

$$\lambda = \sqrt{1 + \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2Z^2t^2 & 2Zt \\ 0 & 2Zt & 2Z^2t^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} = 1 + 2Zt$$

Assignment 3.2

The **length** of the material curve at time t=1 is given by,

$$l = \int_A^B (1 + 2Z)\, dS$$

Points along the straigth line *satisfy the following equations,*

$$X = 0, \quad Y = Z$$

$$dX = 0, \quad dY = dZ, \quad dS = \sqrt{2} dZ$$

The **length** of the material curve at time *t*=1 is given by,

$$l = \int_0^1 (1 + 2Z) \sqrt{2} dZ = 2\sqrt{2} \quad \blacksquare$$

Assignment 3.2

Alternatively, the length of the material curve at time t=1 may be computed as follows.

Let us consider a **material differential vector** at an arbitrary point of the straight line AB, along the direction AB, given by,

$$[d\mathbf{X}] = [dX \quad dY \quad dZ]^T = [0 \quad dZ \quad dZ]^T$$

Using the **deformation gradient** computed at the *points of the straight line AB*, i.e. setting X=0, Y=Z, the **deformed differential vector** at the spatial configuration takes the form,

$$d\mathbf{x} = \mathbf{F}\,d\mathbf{X}, \quad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2Zt \\ 0 & 2Zt & 1 \end{bmatrix} \begin{bmatrix} 0 \\ dZ \\ dZ \end{bmatrix} = (1+2Zt) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} dZ$$

Assignment 3.2

The **differential of length** at the spatial configuration may be computed as,

$$ds = \|d\mathbf{x}\| = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = (1 + 2Zt)\sqrt{2}\,dZ$$

and *particularizing* for t=1, yields

$$ds\big|_{t=1} = (1 + 2Z)\sqrt{2}\,dZ$$

The **length** of the material curve at time t=1 may be computed as,

$$l = \int_a^b ds\big|_{t=1} = \int_0^1 (1 + 2Z)\sqrt{2}\,dZ = 2\sqrt{2} \quad \blacksquare$$

Assignment 3.3

Assignment 3.3 [Homework]

Consider the *equations of motion* given by,

$$x = X + Yt, \quad y = Y, \quad z = Z$$

Compute at time *t*=2 the *length* of a material curve that at time *t*=1 was a curve parametrized as,

$$x(\alpha) = 0, \quad y(\alpha) = \alpha^2, \quad z(\alpha) = \alpha \quad 0 \leq \alpha \leq 1$$

Assignment 3.3

Assignment 3.3 [Homework]

Consider the *equations of motion* given by,

$$x = X + Yt, \quad y = Y, \quad z = Z$$

Compute at time *t*=2 the *length* of a material curve that at time *t*=1 was a curve parametrized as,

$$x(\alpha) = 0, \quad y(\alpha) = \alpha^2, \quad z(\alpha) = \alpha \quad 0 \leq \alpha \leq 1$$

For time *t*=1 the equations of motion read,

$$x^* = X + Y, \quad y^* = Y, \quad z^* = Z$$

Assignment 3.3

The material points that at time *t*=1 are on the given parametrized curve satisfy the following equations,

$$x^* = X + Y = 0, \quad y^* = Y = \alpha^2, \quad z^* = Z = \alpha \quad 0 \le \alpha \le 1$$

The inverse of the equations of motion takes the form,

$$X = x - yt, \quad Y = y, \quad Z = z$$

Using the inverse of the equations of motion, the spatial position of those material points is given by,

$$x^* = x - yt + y = 0, \quad y^* = y = \alpha^2, \quad z^* = z = \alpha \quad 0 \le \alpha \le 1$$

And the parametrized curve at any time *t*>0 is given by,

$$x = y\left(t-1\right) = \alpha^2\left(t-1\right), \quad y = \alpha^2, \quad z = \alpha \quad 0 \le \alpha \le 1$$

Assignment 3.3

The tangent to the spatial parametrized curve at any time t>0 is given by,

$$dx = 2\alpha\left(t-1\right)d\alpha, \quad y = 2\alpha d\alpha, \quad z = d\alpha \quad 0 \le \alpha \le 1$$

The differential of length reads,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{4\alpha^2\left(\left(t-1\right)^2 + 1\right) + 1}d\alpha$$

and at time t=2 reads,

$$ds = \sqrt{8\alpha^2 + 1}d\alpha$$

The **length** of the material curve at time t=2 reads,

$$l = \int ds = \int_0^1 \sqrt{8\alpha^2 + 1}d\alpha$$

Assignment 3.3

Alternatively, we could change the reference configuration, taking *t*=1 as new reference time. Imposing the consistency condition at *t*=1 yields,

$$\mathbf{X}^* := \varphi(\mathbf{x}, t=1)$$

$$X^* = X + Y, \quad Y^* = Y, \quad Z^* = Z$$

Then,

$$X = X^* - Y^*, \quad Y = Y^*, \quad Z = Z^*$$

Substituting into the equations of motion,

$$x = X + Yt, \quad y = Y, \quad z = Z$$

the new equations of motion with reference time *t*=1 take the form,

$$x = X^* + (t-1)Y^*, \quad y = Y^*, \quad z = Z^*$$

Assignment 3.3

The deformation gradient with respect to the new reference configuration takes the form,

$$[\mathbf{F}^*] = \begin{bmatrix} 1 & t-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The parametrized curve at *t*=1 and the differential tangent vector to the curve are given by,

$$[\mathbf{X}^*] = \begin{bmatrix} 0 & \alpha^2 & \alpha \end{bmatrix}^T, \quad [d\mathbf{X}^*] = \begin{bmatrix} 0 & 2\alpha & 1 \end{bmatrix}^T d\alpha$$

Assignment 3.3

The differential vector at the deformed configuration takes the form,

$$d\mathbf{x} = \mathbf{F}^* d\mathbf{X}^*, \quad [d\mathbf{x}] = \begin{bmatrix} 2\alpha(t-1) & 2\alpha & 1 \end{bmatrix}^T d\alpha$$

The differential length is given by,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{4\alpha^2\left((t-1)^2 + 1\right) + 1d\alpha}$$

and at time t=2 reads,

$$ds = \sqrt{8\alpha^2 + 1d\alpha}$$

The **length** of the material curve at time t=2 reads,

$$l = \int ds = \int_0^1 \sqrt{8\alpha^2 + 1d\alpha}$$

Assignment 3.3

Alternatively, using also t=1 as reference time, the **length** of the curve may be computed as,

$$l = \int ds = \int \lambda^* \Big|_{t=2} dS^*$$

with,

$$\lambda^* = \sqrt{1 + 2\mathbf{T}^* \cdot \mathbf{E}^* \mathbf{T}^*}, \quad dS^* = \sqrt{1 + 4\alpha^2}\, d\alpha$$

$$\mathbf{T}^* = \frac{d\mathbf{X}^*}{dS^*}, \quad \left[d\mathbf{X}^* \right] = \frac{1}{\sqrt{1 + 4\alpha^2}} \left[0, 2\alpha, 1 \right]^T$$

$$\mathbf{E}^* = \frac{1}{2}\left(\mathbf{F}^{*T} \mathbf{F}^* - \mathbf{1} \right), \quad \mathbf{F}^* = \bar{\nabla}^* \otimes \mathbf{x} \quad \blacksquare$$

Assignment 3.4

Assignment 3.4

The components of the *Green-Lagrange strain tensor*, with reference time t=0, are given by,

$$[\mathbf{E}] = \begin{bmatrix} 0 & te^{tX} & 0 \\ te^{tX} & 0 & 0 \\ 0 & 0 & te^{tY} \end{bmatrix}$$

Compute at time t=1 the *length* of a material curve that at the reference time t=0 was a straight line going from point A (1,1,1) to point B (2,2,2).

Assignment 3.4

Assignment 3.4

The components of the *Green-Lagrange strain tensor*, with reference time t=0, are given by,

$$[\mathbf{E}] = \begin{bmatrix} 0 & te^{tX} & 0 \\ te^{tX} & 0 & 0 \\ 0 & 0 & te^{tY} \end{bmatrix}$$

The **length** of the material curve at time t=1 may be computed as,

$$l = \int_a^b ds = \int_A^B \lambda \, dS = \int_A^B \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}} \, dS$$

Assignment 3.4

The **unit vector** is given by,

$$[\mathbf{T}] = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

The material points of the straight line AB satisfy X=Y=Z and dX=dY=dZ. Then, the **Green-Lagrange strain** tensor and the **differential of length**, particularized at the points of the line AB, may be written as,

$$[\mathbf{E}] = \begin{bmatrix} 0 & te^{tX} & 0 \\ te^{tX} & 0 & 0 \\ 0 & 0 & te^{tX} \end{bmatrix}$$

$$dS = \sqrt{dX^2 + dY^2 + dZ^2} = \sqrt{3dX^2} = \sqrt{3}dX$$

Assignment 3.4

The **stretch** at the points of the line AB along the line AB may be written as,

$$\lambda = \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}}$$

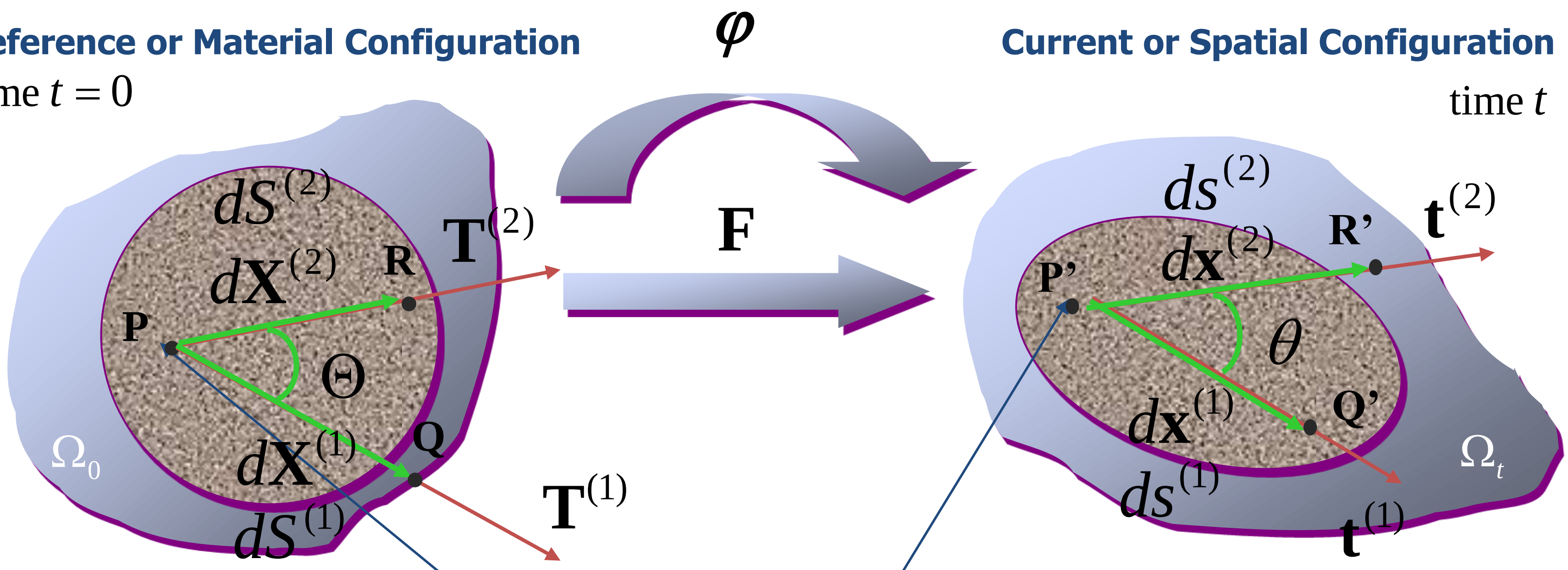
$$= \sqrt{1 + \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & te^{tX} & 0 \\ te^{tX} & 0 & 0 \\ 0 & 0 & te^{tX} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$= \sqrt{1 + 2te^{tX}}$$

The **length** of the material curve at time t=1 reads,

$$l = \int_A^B \sqrt{1 + 2e^X}\, dS = \int_1^2 \sqrt{1 + 2e^X}\, \sqrt{3}\, dX \quad \blacksquare$$

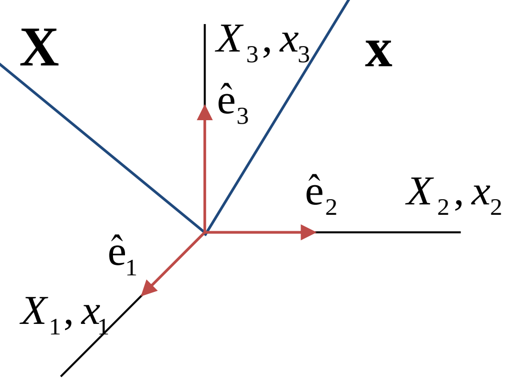
Variation of Angles



$$d\mathbf{X}^{(a)} = \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}^{(a)}$$

$$d\mathbf{X}^{(a)} = \mathbf{T}^{(a)} dS^{(a)}$$

$$\cos\Theta = \frac{d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)}}{dS^{(1)} dS^{(2)}}$$



$$d\mathbf{x}^{(a)} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}^{(a)}$$

$$d\mathbf{x}^{(a)} = \mathbf{t}^{(a)} ds^{(a)}$$

$$\cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{ds^{(1)} ds^{(2)}}$$

Variation of Angles

Spatial Angle

The dot product of the two differential vectors at the *spatial configuration* reads,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = ds^{(1)} ds^{(2)} \cos\theta$$

Alternatively, it may be written in terms of the differential vectors at the *material configuration* and using the unit vectors and the stretches yields,

$$\begin{aligned} d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\mathbf{X}^{(1)} \cdot \mathbf{F}^T \mathbf{F} d\mathbf{X}^{(2)} = dS^{(1)} dS^{(2)} \mathbf{T}^{(1)} \cdot \mathbf{F}^T \mathbf{F} \mathbf{T}^{(2)} \\ &= ds^{(1)} ds^{(2)} \lambda^{-1(1)} \lambda^{-1(2)} \mathbf{T}^{(1)} \cdot \mathbf{F}^T \mathbf{F} \mathbf{T}^{(2)} \\ &= ds^{(1)} ds^{(2)} \lambda^{-1(1)} \lambda^{-1(2)} \mathbf{T}^{(1)} \cdot \mathbf{C} \mathbf{T}^{(2)} \\ &= ds^{(1)} ds^{(2)} \lambda^{-1(1)} \lambda^{-1(2)} \mathbf{T}^{(1)} \cdot \left(\mathbf{1} + 2\mathbf{E}\right) \mathbf{T}^{(2)} \end{aligned}$$

Variation of Angles

Spatial Angle

Comparing the two expressions, the angle between the two segments at the spatial configuration is given by,

$$\cos\theta = \lambda^{-1(1)}\lambda^{-1(2)}\mathbf{T}^{(1)}\cdot\left(\mathbf{1}+2\mathbf{E}\right)\mathbf{T}^{(2)}$$

$$= \left(1+2\mathbf{T}^{(1)}\cdot\mathbf{E}\mathbf{T}^{(1)}\right)^{-1/2}\left(1+2\mathbf{T}^{(2)}\cdot\mathbf{E}\mathbf{T}^{(2)}\right)^{-1/2}\mathbf{T}^{(1)}\cdot\left(\mathbf{1}+2\mathbf{E}\right)\mathbf{T}^{(2)}$$

Variation of Angles

Material Angle

The dot product of the two differential vectors at the *material configuration* reads,

$$d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = dS^{(1)} dS^{(2)} \cos\Theta$$

Alternatively, it may be written in terms of the differential vectors at the *spatial configuration* and using the unit vectors and the stretches yields,

$$\begin{aligned}
d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} &= d\mathbf{x}^{(1)} \cdot \mathbf{F}^{-T}\mathbf{F}^{-1} d\mathbf{x}^{(2)} = ds^{(1)} ds^{(2)} \mathbf{t}^{(1)} \cdot \mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{t}^{(2)} \\
&= dS^{(1)} dS^{(2)} \lambda^{(1)} \lambda^{(2)} \mathbf{t}^{(1)} \cdot \mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{t}^{(2)} \\
&= dS^{(1)} dS^{(2)} \lambda^{(1)} \lambda^{(2)} \mathbf{t}^{(1)} \cdot \mathbf{b}^{-1}\mathbf{t}^{(2)} \\
&= dS^{(1)} dS^{(2)} \lambda^{(1)} \lambda^{(2)} \mathbf{t}^{(1)} \cdot \left(\mathbf{1} - 2\mathbf{e}\right)\mathbf{t}^{(2)}
\end{aligned}$$

Variation of Angles

Material Angle

Comparing the two expressions the angle between the two segments at the material configuration is given by,

$$\cos\Theta = \lambda^{(1)}\lambda^{(2)}\mathbf{t}^{(1)}\cdot(\mathbf{1}-2\mathbf{e})\mathbf{t}^{(2)}$$

$$=\left(1-2\mathbf{t}^{(1)}\cdot\mathbf{e}\mathbf{t}^{(1)}\right)^{-1/2}\left(1-2\mathbf{t}^{(2)}\cdot\mathbf{e}\mathbf{t}^{(2)}\right)^{-1/2}\mathbf{t}^{(1)}\cdot(\mathbf{1}-2\mathbf{e})\mathbf{t}^{(2)}$$

Variation of Angles

Spatial Angle

$$\cos\Theta = \mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)}, \quad \cos\theta = \frac{\mathbf{T}^{(1)} \cdot \left(\mathbf{1} + 2\mathbf{E}\right)\mathbf{T}^{(2)}}{\sqrt{1 + 2\,\mathbf{T}^{(1)} \cdot \mathbf{E}\,\mathbf{T}^{(1)}}\sqrt{1 + 2\,\mathbf{T}^{(2)} \cdot \mathbf{E}\,\mathbf{T}^{(2)}}}$$

Material Angle

$$\cos\Theta = \frac{\mathbf{t}^{(1)} \cdot \left(\mathbf{1} - 2\mathbf{e}\right)\mathbf{t}^{(2)}}{\sqrt{1 - 2\,\mathbf{t}^{(1)} \cdot \mathbf{e}\,\mathbf{t}^{(1)}}\sqrt{1 - 2\,\mathbf{t}^{(2)} \cdot \mathbf{e}\,\mathbf{t}^{(2)}}}, \quad \cos\theta = \mathbf{t}^{(1)} \cdot \mathbf{t}^{(2)}$$

Physical Interpretation of E12 Component

Green-Lagrange Strain Component E12

Let us consider material segments $d\mathbf{X}^{(1)} = \mathbf{T}^{(1)} dS^{(1)}$ and $d\mathbf{X}^{(2)} = \mathbf{T}^{(2)} dS^{(2)}$ along the X1- and X2-axis, respectively, on the material configuration.

The *angle* between the two segments at the spatial configuration is given by,

$$\cos\theta_{12} = \frac{\mathbf{T}^{(1)} \cdot \left(\mathbf{1} + 2\mathbf{E}\right)\mathbf{T}^{(2)}}{\sqrt{1 + 2\mathbf{T}^{(1)} \cdot \mathbf{E}\mathbf{T}^{(1)}}\sqrt{1 + 2\mathbf{T}^{(2)} \cdot \mathbf{E}\mathbf{T}^{(2)}}} = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}}$$

The *Green-Lagrange component E12* may be interpreted as,

$$E_{12} = \frac{1}{2}\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}\cos\theta_{12} = \frac{1}{2}\frac{ds^{(1)}}{dS^{(1)}}\frac{ds^{(2)}}{dS^{(2)}}\cos\theta_{12}$$

Physical Interpretation of E12 Component

Green-Lagrange Strain Component E12

Taking into account that the initial angle between the two segments at the reference configuration is 90º, the *angle increment* may be written as,

$$\Delta\theta_{12} := \theta_{12} - \frac{\pi}{2} = -\arcsin \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

The *Green-Lagrange component E12* may be interpreted as,

$$E_{12} = -\frac{1}{2}\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}\,\sin\Delta\theta_{12} = -\frac{1}{2}\frac{ds^{(1)}}{dS^{(1)}}\frac{ds^{(2)}}{dS^{(2)}}\sin\Delta\theta_{12}$$

Physical Interpretation of e12 Component

Almansi Strain Component e12

Let us consider spatial segments $d\mathbf{x}^{(1)} = \mathbf{t}^{(1)} ds$ and $d\mathbf{x}^{(2)} = \mathbf{t}^{(2)} ds$ along the x1- and x2-axis, respectively, on the spatial configuration.

The *angle* between the two segments at the material configuration is given by,

$$\cos\Theta_{12} = \frac{\mathbf{t}^{(1)} \cdot \left(\mathbf{1} - 2\mathbf{e}\right)\mathbf{t}^{(2)}}{\sqrt{1 - 2\mathbf{t}^{(1)} \cdot \mathbf{e}\mathbf{t}^{(1)}}\sqrt{1 - 2\mathbf{t}^{(2)} \cdot \mathbf{e}\mathbf{t}^{(2)}}} = \frac{-2e_{12}}{\sqrt{1 - 2e_{11}}\sqrt{1 - 2e_{22}}}$$

The *Almansi strain component e12* may be interpreted as,

$$e_{12} = -\frac{1}{2}\sqrt{1 - 2e_{11}}\sqrt{1 - 2e_{22}}\cos\Theta_{12} = -\frac{1}{2}\frac{dS^{(1)}}{ds^{(1)}}\frac{dS^{(2)}}{ds^{(2)}}\cos\Theta_{12}$$

Physical Interpretation of e12 Component

Almansi Strain Component e12

Taking into account that the deformed angle between the two segments at the spatial configuration is 90º, the *angle increment* may be written as,

$$\Delta\theta_{12} := \frac{\pi}{2} - \Theta_{12} = -\arcsin \frac{2e_{12}}{\sqrt{1-2e_{11}}\sqrt{1-2e_{22}}}$$

The *Almansi strain component e12* may be interpreted as,

$$e_{12} = -\frac{1}{2}\sqrt{1-2e_{11}}\sqrt{1-2e_{22}}\sin\Delta\theta_{12} = -\frac{1}{2}\frac{dS^{(1)}}{ds^{(1)}}\frac{dS^{(2)}}{ds^{(2)}}\sin\Delta\theta_{12}$$

Assignment 3.5

Assignment 3.5

The *equations of motion* are given by,

$$x = X, \quad y = Y, \quad z = Z - Xt$$

Consider two differential segments which at time t=1 are parallel to the Cartesian axes x and z. Compute which was the *angle* formed by those two segments at the reference time t=0.

Assignment 3.5

Assignment 3.5

The *equations of motion* are given by,

$$x = X, \quad y = Y, \quad z = Z - Xt$$

Consider two differential segments which at time t=1 are parallel to the Cartesian axes x and z. Compute which was the *angle* formed by those two segments at the reference time t=0.

Let us consider at the spatial configuration t=1, unit vectors along the Cartesian axes x and z given by,

$$\left[\mathbf{t}^{(1)}\right] = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, \quad \left[\mathbf{t}^{(2)}\right] = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

Assignment 3.5

The **angle** between those two unit vectors at the *reference configuration* may be written as,

$$\cos\Theta_{13} = \frac{\mathbf{t}^{(1)} \cdot \left(\mathbf{1} - 2\mathbf{e}\right)\mathbf{t}^{(2)}}{\sqrt{1-2\mathbf{t}^{(1)}\cdot\mathbf{e}\mathbf{t}^{(1)}}\sqrt{1-2\mathbf{t}^{(2)}\cdot\mathbf{e}\mathbf{t}^{(2)}}} = \frac{-2e_{13}}{\sqrt{1-2e_{11}}\sqrt{1-2e_{33}}}$$

The **inverse motion equations** are given by,

$$X = x, \quad Y = y, \quad Z = z + xt$$

The **invese deformation gradient** is given by,

$$\left[\mathbf{F}^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{bmatrix}$$

Assignment 3.5

The **Almansi strain tensor** is given by,

$$\mathbf{e} := \frac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

$$[\mathbf{e}] := \frac{1}{2}\left([\mathbf{1}] - \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix} t^2 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{bmatrix}$$

Then the **angle** at the *reference configuration* is given by,

$$\cos\Theta_{13} = \frac{-2e_{13}}{\sqrt{1-2e_{11}}\sqrt{1-2e_{33}}} = \left.\frac{t}{\sqrt{1+t^2}}\right|_{t=1} = \frac{1}{\sqrt{2}} \quad \blacksquare$$

Assignment 3.5

Alternatively, we could obtain the angle at the reference configuration as follows. Let us consider at the spatial configuration t=1, two differential vectors along the Cartesian axes x and z given by,

$$\left[d\mathbf{x}^{(1)}\right]=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T dx, \quad \left[d\mathbf{x}^{(2)}\right]=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T dz$$

The corresponding differential vectors at the reference configuration will be given by,

$$d\mathbf{X}^{(1)}=\mathbf{F}^{-1}d\mathbf{x}^{(1)}, \quad d\mathbf{X}^{(2)}=\mathbf{F}^{-1}d\mathbf{x}^{(2)}$$

$$\left[d\mathbf{X}^{(1)}\right]=\begin{bmatrix}1 & 0 & 1\end{bmatrix}dx, \quad \left[d\mathbf{X}^{(2)}\right]=\begin{bmatrix}0 & 0 & 1\end{bmatrix}dz$$

Assignment 3.5

The norms of the differential vectors at the reference configuration are given by,

$$\left\| d\mathbf{X}^{(1)} \right\| = \sqrt{1+t^2}\, dx, \quad \left\| d\mathbf{X}^{(2)} \right\| = dz$$

The angle between the two segments at the reference configuration reads,

$$\cos\Theta = \frac{d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)}}{\left\| d\mathbf{X}^{(1)} \right\| \cdot \left\| d\mathbf{X}^{(2)} \right\|} = \left. \frac{t}{\sqrt{1+t^2}} \right|_{t=1} = \frac{1}{\sqrt{2}} \quad \blacksquare$$

Assignment 3.6

Assignment 3.6

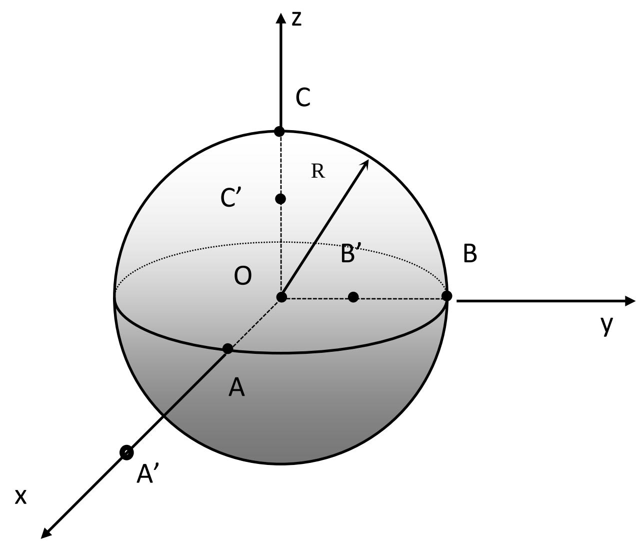
The sphere of the figure is subjected to a finite *uniform* deformation, with *uniform* deformation gradient. The motion is such that,

i. The origin O does not moves

ii. Material points A, B and C move to spatial positions A', B' and C', where AA'=p>0, BB'=CC'=q>0.

1) Obtain the deformation gradient, Green-Lagrange and Almansi strain tensors and the displacement vector field.

2) Obtain the relation between p and q if the material is incompressible.



Assignment 3.6

Assignment 3.6

As the deformation gradient is *uniform*,

$$\mathbf{F} = \mathbf{F}\left(\mathbf{X}, t\right) = \mathbf{F}\left(t\right) \quad \Rightarrow \quad d\mathbf{x} = \mathbf{F}\left(t\right) d\mathbf{X}$$

the equation of motion is *linear* and may be easily obtained yielding,

$$\mathbf{x} = \mathbf{F}\left(t\right)\mathbf{X} + \mathbf{C}\left(t\right)$$

Condition 1. The material point O does not moves,

$$\mathbf{x}_O = \mathbf{F}\left(t\right)\mathbf{X}_O + \mathbf{C}\left(t\right)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad \Rightarrow \quad C_1 = C_2 = C_3 = 0$$

Assignment 3.6

Condition 2. The material point A moves to the position A',

$$\mathbf{x}_A = \mathbf{F}(t)\mathbf{X}_A$$

$$\begin{bmatrix} R+p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad F_{11} = 1 + p/R, \quad F_{21} = F_{31} = 0$$

Condition 3. The material point B moves to the position B',

$$\mathbf{x}_B = \mathbf{F}(t)\mathbf{X}_B$$

$$\begin{bmatrix} 0 \\ R-q \\ 0 \end{bmatrix} = \begin{bmatrix} 1+p/R & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} \Rightarrow \quad F_{22} = 1 - \frac{q}{R}, \quad F_{12} = F_{32} = 0$$

Assignment 3.6

Condition 4. The material point C moves to the position C',

$$\mathbf{x}_C = \mathbf{F}(t)\,\mathbf{X}_C$$

$$\begin{bmatrix} 0 \\ 0 \\ R-q \end{bmatrix} = \begin{bmatrix} 1+p/R & 0 & F_{13} \\ 0 & 1-q/R & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} \Rightarrow F_{33} = 1-q/R, \quad F_{13} = F_{23} = 0$$

The **deformation gradient** takes the form,

$$[\mathbf{F}] = \begin{bmatrix} 1+p/R & 0 & 0 \\ 0 & 1-q/R & 0 \\ 0 & 0 & 1-q/R \end{bmatrix}$$

Assignment 3.6

The **motion equation** takes the form,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+p/R & 0 & 0 \\ 0 & 1-q/R & 0 \\ 0 & 0 & 1-q/R \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The **displacement vector field** takes the form,

$$\mathbf{u} = \mathbf{U}(\mathbf{X},t) = \mathbf{F}(t)\mathbf{X} - \mathbf{X} = \left(\mathbf{F}(t) - \mathbf{1}\right)\mathbf{X} = \mathbf{J}(t)\mathbf{X}$$

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} p/R & 0 & 0 \\ 0 & -q/R & 0 \\ 0 & 0 & -q/R \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Assignment 3.6

The **Green-Lagrange strain tensor** takes the form,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} (1+p/R)^2 - 1 & 0 & 0 \\ 0 & (1-q/R)^2 - 1 & 0 \\ 0 & 0 & (1-q/R)^2 - 1 \end{bmatrix}$$

Assignment 3.6

The **Almansi strain tensor** takes the form,

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

$$[\mathbf{e}] = \frac{1}{2}\begin{bmatrix} 1 - 1\big/\left(1 + p/R\right)^2 & 0 & 0 \\ 0 & 1 - 1\big/\left(1 - q/R\right)^2 & 0 \\ 0 & 0 & 1 - 1\big/\left(1 - q/R\right)^2 \end{bmatrix}$$

Assignment 3.6

If the material is **incompressible**, the following condition has to be verified,

$$J = \det \mathbf{F}(t) = 1$$

Then,

$$\det \mathbf{F}(t) = \begin{vmatrix} 1+p/R & 0 & 0 \\ 0 & 1-q/R & 0 \\ 0 & 0 & 1-q/R \end{vmatrix} = (1+p/R)(1-q/R)^2 = 1$$

$$p = R\big/(1-q/R)^2 - R \quad \blacksquare$$

Assignment 3.7

Assignment 3.7 [Classwork]

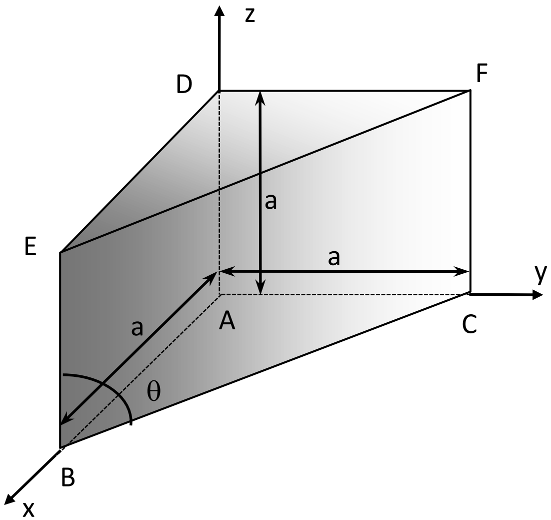
The solid of the figure is subjected to a finite *linear* displacement field, yielding a *uniform* deformation gradient, such that,

i. The *displacements* of the material points A, B and C are zero.

ii. The *volume* of the solid becomes $p\sqrt{2}$ times the initial one.

iii. The *length* of the material segment AE becomes p times the initial one.

Obtain the deformation gradient, and the material and spatial descriptions of the displacement vector field.



Assignment 3.7

Assignment 3.7 [Classwork]

As the deformation gradient is *uniform*,

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \mathbf{F}(t) \quad \Rightarrow \quad d\mathbf{x} = \mathbf{F}(t)\, d\mathbf{X}$$

the equation of motion is *linear* and may be easily obtained yielding,

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X} + \mathbf{C}(t)$$

Condition 1. The material point A does not moves,

$$\mathbf{x}_A = \mathbf{F}(t)\mathbf{X}_A + \mathbf{C}(t)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad \Rightarrow \quad C_1 = C_2 = C_3 = 0$$

Assignment 3.7

Condition 2. The material point B does not moves,

$$\mathbf{x}_B = \mathbf{F}(t)\,\mathbf{X}_B$$

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad F_{11} = 1, \quad F_{21} = F_{31} = 0$$

Condition 3. The material point C does not moves,

$$\mathbf{x}_C = \mathbf{F}(t)\,\mathbf{X}_C$$

$$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \quad \Rightarrow \quad F_{22} = 1, \quad F_{12} = F_{32} = 0$$

Assignment 3.7

The **deformation gradient** takes the form,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix}$$

Condition 4. The *volume* of the solid becomes p√2 times the initial one.

$$dV = \left(\det \mathbf{F}(t)\right) dV_0 \quad \Rightarrow \quad V = \left(\det \mathbf{F}(t)\right) V_0 = p\sqrt{2} V_0$$

$$\det \mathbf{F}(t) = F_{33} = p\sqrt{2}$$

Assignment 3.7

Condition 5. The *length* of the material segment AE becomes p times the initial one.

$$[d\mathbf{X}] = [dX \quad 0 \quad dX]^T$$

$$d\mathbf{x} = \mathbf{F}(t)\, d\mathbf{X} \quad \Rightarrow \quad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & p\sqrt{2} \end{bmatrix} \begin{bmatrix} dX \\ 0 \\ dX \end{bmatrix} = \begin{bmatrix} 1+F_{13} \\ F_{23} \\ p\sqrt{2} \end{bmatrix} dX$$

$$ds = \|d\mathbf{x}\| = \sqrt{(1+F_{13})^2 + (F_{23})^2 + 2p^2}\, dX$$

$$\overline{ae} = \int_a^e ds = \int_0^a \sqrt{(1+F_{13})^2 + (F_{23})^2 + 2p^2}\, dX$$

Assignment 3.7

$$pa\sqrt{2} = \sqrt{\left(1+F_{13}\right)^2 + \left(F_{23}\right)^2 + 2p^2a}$$

$$2p^2 = \left(1+F_{13}\right)^2 + \left(F_{23}\right)^2 + 2p^2$$

$$\left(1+F_{13}\right)^2 + \left(F_{23}\right)^2 = 0 \quad \Rightarrow \quad F_{13} = -1, \quad F_{23} = 0$$

The **deformation gradient** takes the form,

$$\left[\mathbf{F}(t)\right] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & p\sqrt{2} \end{bmatrix}$$

Assignment 3.7

The **motion equation** takes the form,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & p\sqrt{2} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The material description of the **displacement vector field** takes the form,

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & p\sqrt{2} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & p\sqrt{2}-1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Assignment 3.7

The inverse of the **motion equation** takes the form,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/p\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1/p\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The spatial description of the **displacement vector field** takes the form,

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/p\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1-1/p\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ ■

Assignment 3.8

Assignment 3.8 [Homework]

The solid of the figure is subjected to a finite *linear* displacement field, yielding a *uniform* deformation gradient, such that,

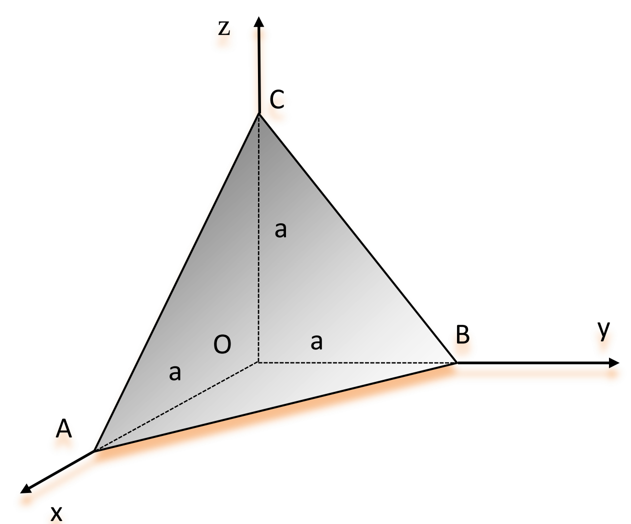
i. The *displacements* of the material points O, A and B are zero.

ii. The *volume* of the solid becomes p times the initial one.

iii. The *length* of the material segment AC becomes $p/\sqrt{2}$ times the initial one.

iv. The *deformed angle* formed by OA and OC is 45º

Obtain the deformation gradient, and the material and spatial descriptions of the displacement vector field.



Assignment 3.8

Assignment 3.8 [Homework]

As the deformation gradient is *uniform*,

$$\mathbf{F} = \mathbf{F}(\mathbf{X},t) = \mathbf{F}(t) \quad \Rightarrow \quad d\mathbf{x} = \mathbf{F}(t)\, d\mathbf{X}$$

the equation of motion is *linear* and may be easily obtained yielding,

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X} + \mathbf{C}(t)$$

Condition 1. The material point O does not moves,

$$\mathbf{x}_O = \mathbf{F}(t)\mathbf{X}_O + \mathbf{C}(t)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad \Rightarrow \quad C_1 = C_2 = C_3 = 0$$

Assignment 3.8

Condition 2. The material point A does not moves,

$$\mathbf{x}_A = \mathbf{F}(t)\mathbf{X}_A$$

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad F_{11} = 1, \quad F_{21} = F_{31} = 0$$

Condition 3. The material point B does not moves,

$$\mathbf{x}_B = \mathbf{F}(t)\mathbf{X}_B$$

$$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \quad \Rightarrow \quad F_{22} = 1, \quad F_{12} = F_{32} = 0$$

Assignment 3.8

The **deformation gradient** takes the form,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix}$$

Condition 4. The *volume* of the solid becomes p times the initial one.

$$dV = \left(\det \mathbf{F}(t)\right) dV_0 \quad \Rightarrow \quad V = \left(\det \mathbf{F}(t)\right) V_0 = pV_0$$

$$\det \mathbf{F}(t) = F_{33} = p$$

Assignment 3.8

Condition 5. The *length* of the material segment AE becomes p/√2 times the initial one.

$$l_{ac} = \int_A^C \lambda_{AC} dS = \int_A^C \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}} dS = \frac{p}{\sqrt{2}} L_{AC}$$

As the stretch is *uniform*,

$$l_{ac} = \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}}\, L_{AC} = \frac{p}{\sqrt{2}} L_{AC}$$

yielding,

$$1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T} = \frac{p^2}{2} \quad \Rightarrow \quad 2\mathbf{T} \cdot \mathbf{E}\mathbf{T} = \frac{p^2}{2} - 1$$

The unit vector is given by,

$$[\mathbf{T}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$$

Assignment 3.8

The **Green-Lagrange strain tensor** takes the form,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$$

$$[\mathbf{E}] = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_{13} & F_{23} & p \end{bmatrix}\begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & p \end{bmatrix} - [\mathbf{1}]\right)$$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & F_{13} \\ 0 & 0 & F_{23} \\ F_{13} & F_{23} & F_{13}^2 + F_{23}^2 + p^2 - 1 \end{bmatrix}$$

Assignment 3.8

Substituting yields,

$$2\mathbf{T}\cdot\mathbf{E}\mathbf{T}=\frac{1}{2}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & F_{13}\\ 0 & 0 & F_{23}\\ F_{13} & F_{23} & F_{13}^2+F_{23}^2+p^2-1\end{bmatrix}\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

$$=\frac{1}{2}\left(-2F_{13}+F_{13}^2+F_{23}^2+p^2-1\right)=\frac{p^2}{2}-1$$

$$-2F_{13}+F_{13}^2+F_{23}^2+1=\left(F_{13}-1\right)^2+F_{23}^2=0 \quad\Rightarrow\quad F_{13}=1, \quad F_{23}=0$$

Assignment 3.8

The **deformation gradient** takes the form,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}$$

The **Green-Lagrange strain tensor** takes the form,

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & p^2 \end{bmatrix}$$

Assignment 3.8

Condition 6. The *deformed angle* between OA and OC is 45º.

$$\cos\theta_{xz} = \frac{2E_{xz}}{\sqrt{1+2E_{xx}}\sqrt{1+2E_{zz}}} = \frac{\sqrt{2}}{2}$$

Substituting,

$$\cos\theta_{xz} = \frac{2E_{xz}}{\sqrt{1+2E_{xx}}\sqrt{1+2E_{zz}}} = \frac{1}{\sqrt{1+p^2}} = \frac{\sqrt{2}}{2}$$

$$\frac{1}{1+p^2} = \frac{1}{2} \quad \Rightarrow \quad p^2 = 1 \quad \Rightarrow \quad p = \pm 1$$

$$J = \det \mathbf{F} = p > 0 \quad \Rightarrow \quad p = 1$$

Assignment 3.8

The **deformation gradient** takes the form,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The **Green-Lagrange strain tensor** takes the form,

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Assignment 3.8

The **displacement vector field** is given by,

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = (\mathbf{F} - \mathbf{1})\mathbf{X} = \mathbf{J}\mathbf{X}$$

The *material* and *spatial descriptions* of the **displacement vector field** are given by,

$$[\mathbf{u}] = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{u}] = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$$

Assignment 3.9

Assignment 3.9

The solid of the figure is subjected to a deformation, such that,

i. The *displacement* is linear on X, Y and Z and skew-symmetric with respect to the plane Y=0, such that U(X,Y,Z)=-U(X,-Y,Z) for any X, Y, Z.

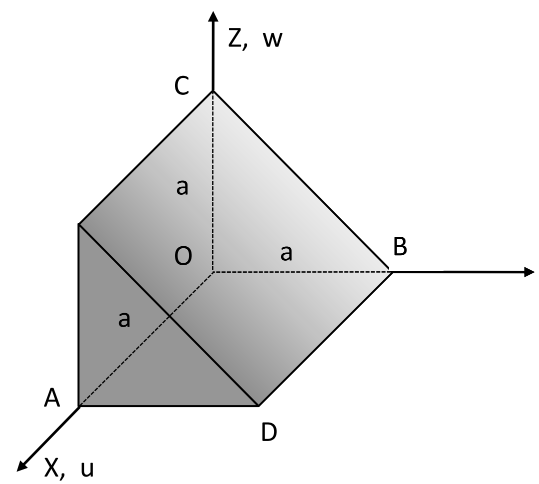
ii. The *volume* of the solid becomes remains constant.

iii. The angle given by OA, OB remains constant, equal to 90º.

iv. The length of the material segment OB becomes √2 times the initial one.

v. The z-displacement of point B is positive.

Obtain the *deformation gradient*, the *Green-Lagrange strain* tensor and the *displacement* vector field.



Assignment 3.9

Assignment 3.9

The solid is subjected to *finite displacements*.

Condition 1. The *displacement field* is *linear* on X, Y and Z, hence the **material displacement gradient** is *uniform* and the displacement vector field may be written as,

$$\mathbf{U}\left(\mathbf{X},t\right)=\mathbf{J}\left(t\right)\mathbf{X}+\mathbf{C}\left(t\right)$$

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Assignment 3.9

Condition 2. The *displacement field* is *skew-symmetric* with respect to the plane Y=0.

$$\mathbf{U}(X,Y,Z) = -\mathbf{U}(X,-Y,Z) \quad \forall\ X,Y,Z$$

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = -\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} X \\ -Y \\ Z \end{bmatrix} - \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$\begin{cases} C_1 = C_2 = C_3 = 0 \\ J_{11} = J_{21} = J_{31} = 0 \\ J_{13} = J_{23} = J_{33} = 0 \end{cases}$$

Assignment 3.9

Then the **displacement field** takes the form,

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & J_{12} & 0 \\ 0 & J_{22} & 0 \\ 0 & J_{32} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Condition 3. The *volume* of the solid remains constant.

$$dV = \left(\det \mathbf{F}\left(t\right)\right) dV_0 \quad \Rightarrow \quad V = \left(\det \mathbf{F}\left(t\right)\right) V_0 = V_0$$

$$\det \mathbf{F}\left(t\right) = \det\left(\mathbf{1} + \mathbf{J}\left(t\right)\right) = 1 + J_{22} = 1 \quad \Rightarrow \quad J_{22} = 0$$

Assignment 3.9

Then the **displacement field** takes the form,

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & J_{12} & 0 \\ 0 & 0 & 0 \\ 0 & J_{32} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Condition 4. The *angle* given by OA and OB remains constant, i.e. the deformed angle is 90º.

$$\cos\theta_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}} = 0 \quad \Rightarrow \quad E_{12} = 0$$

Assignment 3.9

Then **Green-Lagrange strain** tensor takes the form,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T\mathbf{J}\right),$$

$$[\mathbf{E}] = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ J_{12} & 1 & J_{32} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & J_{12} & 0 \\ 0 & 1 & 0 \\ 0 & J_{32} & 1 \end{bmatrix} - [\mathbf{1}]\right)$$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & J_{12} & 0 \\ J_{12} & J_{12}^2 + J_{32}^2 & J_{32} \\ 0 & J_{32} & 0 \end{bmatrix} \quad \Rightarrow \quad E_{12} = J_{12} = 0$$

Assignment 3.9

Then the **displacement field** takes the form,

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & J_{32} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Condition 4. The *length* of the segment OB becomes √2 times the initial one, i.e. the length of the deformed segment is a√2.

$$l_{ob} = \int_o^b ds = \int_O^B \lambda \, dS = \int_O^B \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}} \, dS$$

The unit vector along the line OB is given by,

$$[\mathbf{T}] = [0 \quad 1 \quad 0]^T$$

Assignment 3.9

Then the **Green-Lagrange strain** tensor takes the form,

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & J_{32}^2 & J_{32} \\ 0 & J_{32} & 0 \end{bmatrix}$$

Substituting into the **stretch** and the integral expression yields,

$$l_{ob} = \int_O^B \sqrt{1+2E_{22}}\, dS = a\sqrt{1+2E_{22}} = a\sqrt{2} \quad \Rightarrow \quad E_{22} = 1/2$$

$$E_{22} = \frac{1}{2} J_{32}^2 = \frac{1}{2} \quad \Rightarrow \quad J_{32} = \pm 1$$

Assignment 3.9

Then the **displacement field** takes the form,

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

and the **Green-Lagrange strain** tensor takes the form,

$$[\mathbf{E}] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix}$$

Assignment 3.9

Condition 6. The *z-displacement* of the point B is positive.

$$U_z\big|_B = \pm Y\big|_B = \pm a > 0 \quad \Rightarrow \quad U_z = Y$$

Then the **diplacement field** takes the form,

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Assignment 3.9

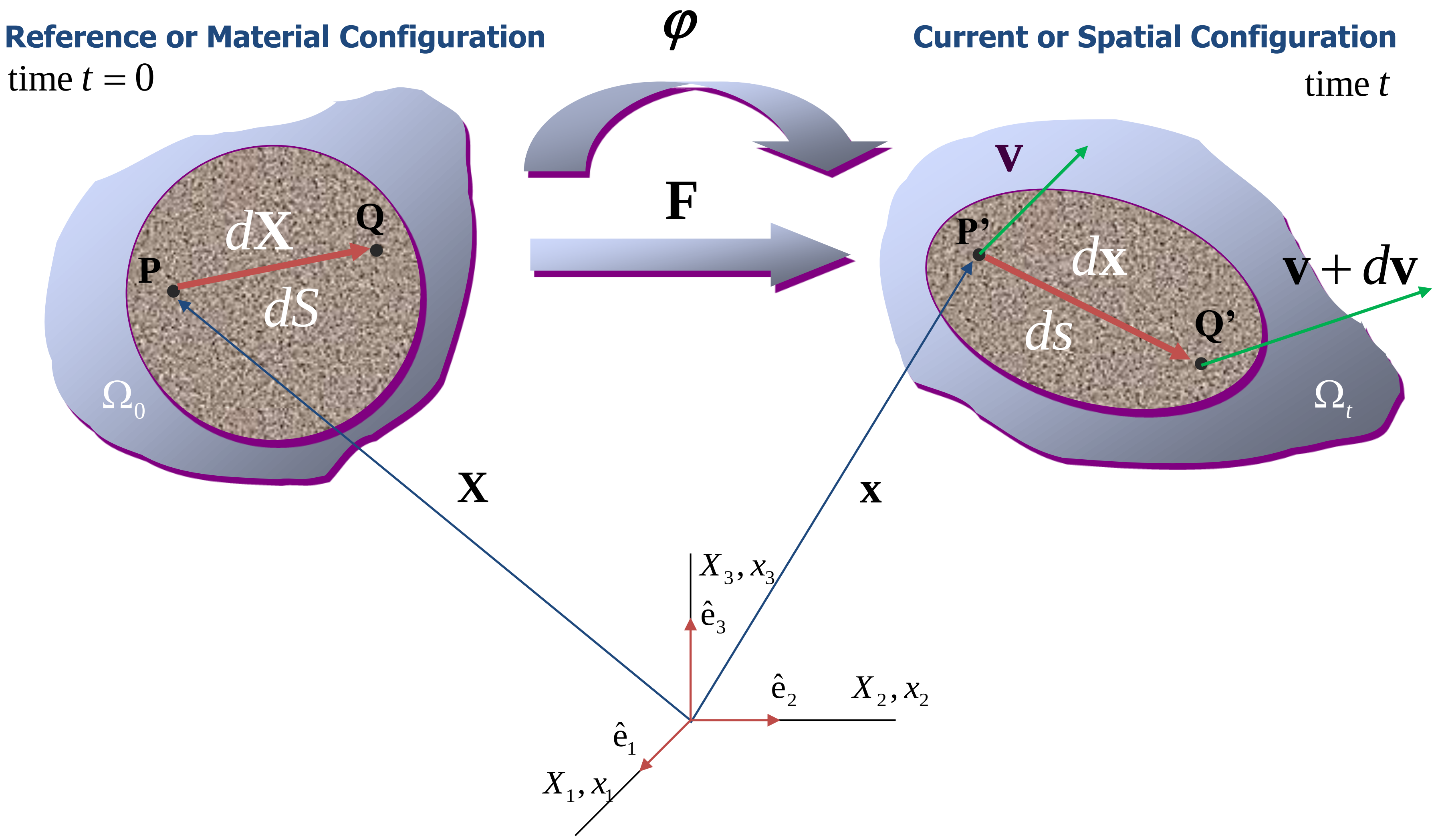
The **material diplacement gradient** and the **deformation gradient** tensors take the form,

$$[\mathbf{J}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad [\mathbf{F}] = [\mathbf{1} + \mathbf{J}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The **Green-Lagrange strain** tensor takes the form,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right), \quad [\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Spatial Velocity Gradient



Spatial Velocity Gradient

Spatial Velocity Gradient Tensor

Let us consider the **spacial velocity** vector field at a spatial point and time *t*, given by,

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$

Differentiating the spatial velocity field, keeping constant the configuration at time *t*, using the chain rule, yields,

$$d\mathbf{v} = (\operatorname{grad} \mathbf{v}(\mathbf{x}, t))\, d\mathbf{x} = (\nabla \otimes \mathbf{v}(\mathbf{x}, t))\, d\mathbf{x} := \mathbf{l}(\mathbf{x}, t)\, d\mathbf{x}$$

where the *non-symmetric* second-order **spatial velocity gradient** tensor, denoted as $\mathbf{l}(\mathbf{x}, t)$, has been introduced as,

$$\boxed{\mathbf{l}(\mathbf{x}, t) := \nabla \otimes \mathbf{v}(\mathbf{x}, t) = \operatorname{grad} \mathbf{v}(\mathbf{x}, t), \quad l_{ab} = v_{a,b}}$$

Deformation and Rotation Rate

Deformation and Rotation Rate Tensors

The **spatial velocity gradient** tensor can be split into *symmetric* and *skew-symmetric* parts, yielding,

$$\mathbf{l}(\mathbf{x},t) = \operatorname{symm}\left[\mathbf{l}(\mathbf{x},t)\right] + \operatorname{skew}\left[\mathbf{l}(\mathbf{x},t)\right] := \mathbf{d}(\mathbf{x},t) + \mathbf{w}(\mathbf{x},t)$$

where the *symmetric* spatial **deformation rate** tensor, denoted as $\mathbf{d}(\mathbf{x},t)$, has been defined as,

$$\mathbf{d} := \operatorname{symm}[\mathbf{l}] = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) = \frac{1}{2}\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\right)$$

and the *skew-symmetric* spatial **rotation rate** tensor, denoted as $\mathbf{w}(\mathbf{x},t)$, has been defined as,

$$\mathbf{w} := \operatorname{skew}[\mathbf{l}] = \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^T\right) = \frac{1}{2}\left(\operatorname{grad}\mathbf{v} - (\operatorname{grad}\mathbf{v})^T\right)$$

Deformation Rate

Deformation Rate Tensor

The *symmetric* spatial **deformation rate** tensor, denoted as $\mathbf{d}(\mathbf{x},t)$, may be viewed as a measure of the rate of deformation given by,

$$\frac{d}{dt}\left(ds^2 - dS^2\right) = \frac{d}{dt}\left(ds^2\right) = \frac{d}{dt}\left(d\mathbf{x}\cdot d\mathbf{x}\right) = d\mathbf{v}\cdot d\mathbf{x} + d\mathbf{x}\cdot d\mathbf{v}$$

Using the expression $d\mathbf{v} = \mathbf{l}\, d\mathbf{x}$ yields,

$$\frac{d}{dt}\left(ds^2 - dS^2\right) = \frac{d}{dt}\left(ds^2\right) = 2\, d\mathbf{x}\cdot\frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) d\mathbf{x} = 2\, d\mathbf{x}\cdot\mathbf{d}\, d\mathbf{x}$$

Deformation Rate

Deformation Rate Tensor

The *symmetric* spatial **deformation rate** tensor, denoted as $\mathbf{d}(\mathbf{x},t)$, may be related to the **material time derivative** of the **Green-Lagrange strain** tensor as,

$$\left.\begin{aligned} \frac{d}{dt}\left(ds^2 - dS^2\right) &= \frac{d}{dt}\left(2d\mathbf{X}\cdot\mathbf{E}d\mathbf{X}\right) = 2d\mathbf{X}\cdot\dot{\mathbf{E}}d\mathbf{X} \\ \frac{d}{dt}\left(ds^2 - dS^2\right) &= 2\,d\mathbf{x}\cdot\mathbf{d}\,d\mathbf{x} = 2\,d\mathbf{X}\cdot\mathbf{F}^T\mathbf{d}\mathbf{F}d\mathbf{X} \end{aligned}\right\}$$

$$\boxed{\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{d}\mathbf{F}, \quad \mathbf{d} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}}$$

Deformation Rate

Deformation Rate Tensor

The *symmetric* spatial **deformation rate** tensor, denoted as $\mathbf{d}(\mathbf{x},t)$, may be related to the **material time derivative** of the **Almansi strain** tensor as,

$$\left.\begin{aligned}\frac{d}{dt}\left(ds^2 - dS^2\right) &= \frac{d}{dt}\left(2d\mathbf{x}\cdot\mathbf{e}d\mathbf{x}\right)\\ &= 2\left(d\mathbf{x}\cdot\dot{\mathbf{e}}d\mathbf{x} + d\mathbf{v}\cdot\mathbf{e}d\mathbf{x} + d\mathbf{x}\cdot\mathbf{e}d\mathbf{v}\right)\\ &= 2d\mathbf{x}\cdot\left(\dot{\mathbf{e}} + \mathbf{l}^T\mathbf{e} + \mathbf{e}\mathbf{l}\right)d\mathbf{x}\\ \frac{d}{dt}\left(ds^2 - dS^2\right) &= 2\,d\mathbf{x}\cdot\mathbf{d}\,d\mathbf{x}\end{aligned}\right\}$$

$$\boxed{\mathbf{d} = \dot{\mathbf{e}} + \mathbf{l}^T\mathbf{e} + \mathbf{e}\mathbf{l}, \quad \dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^T\mathbf{e} - \mathbf{e}\mathbf{l}}$$

Rotation Rate

Rotation Rate Tensor

The *skew-symmetric* spatial **rotation rate** tensor, denoted as $\mathbf{w}(\mathbf{x},t)$, satisfies the following expressions,

$$\mathbf{w}\,d\mathbf{x} = \boldsymbol{\omega}\times d\mathbf{x} \quad \forall d\mathbf{x}, \quad w_{ab}dx_b = \varepsilon_{abc}\omega_b dx_c$$

$$\boldsymbol{\omega} = \frac{1}{2}\operatorname{curl}\mathbf{v} = \frac{1}{2}\nabla\times\mathbf{v}, \quad \omega_a = \frac{1}{2}\varepsilon_{abc}\frac{\partial v_c}{\partial x_b} = \frac{1}{2}\varepsilon_{abc}v_{c,b}$$

where $\boldsymbol{\omega}$ is the **axial** (or **dual**) **rotation rate** vector.

Rotation Rate

Rotation Rate Tensor

The components of the *skew-symmetric* spatial **rotation rate** tensor $\mathbf{w}$ and the components of the **axial** (or **dual**) **rotation rate** vector $\boldsymbol{\omega}$, are such that,

$$[\mathbf{w}] = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$[\boldsymbol{\omega}] = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -w_{23} \\ w_{13} \\ -w_{12} \end{bmatrix}$$

Spatial Velocity Gradient

Spatial Velocity Gradient Tensor

$$d\mathbf{v} = \mathbf{l}\, d\mathbf{x}, \quad dv_a = l_{ab} dx_b$$

$$\mathbf{l} := \operatorname{grad} \mathbf{v} = \nabla \otimes \mathbf{v}, \quad l_{ab} := v_{a,b}$$

Deformation and Rotation Rate Tensors

$$\mathbf{l} = \operatorname{sym}[\mathbf{l}] + \operatorname{skew}[\mathbf{l}] := \mathbf{d} + \mathbf{w}$$

$$\mathbf{d} := \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) = \frac{1}{2}\left(\nabla \otimes \mathbf{v} + \left(\nabla \otimes \mathbf{v}\right)^T\right), \quad d_{ab} := \frac{1}{2}\left(v_{a,b} + v_{b,a}\right)$$

$$\mathbf{w} := \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^T\right) = \frac{1}{2}\left(\nabla \otimes \mathbf{v} - \left(\nabla \otimes \mathbf{v}\right)^T\right), \quad w_{ab} := \frac{1}{2}\left(v_{a,b} - v_{b,a}\right)$$

Assignment 3.10

Assignment 3.10 [Classwork]

Consider two different motions with velocity vector fields given by,

$$\left[\mathbf{v}^{I}\right]=\left[\mathbf{V}^{I}\left(X,Y,Z\right)\right]=\left[Z \quad X \quad Z\right]^{T}$$

$$\left[\mathbf{v}^{II}\right]=\left[\mathbf{v}^{II}\left(x,y,z\right)\right]=\left[z \quad x \quad z\right]^{T}$$

Assuming that the reference time is t=0, obtain for each one of the motions,

1) The *motion equation* and the *deformation gradient*
2) The *Green-Lagrange* and the *Almansi* strain tensors
3) The *deformation rate* tensor

Assignment 3.10

Assignment 3.10 [Classwork]

Setting the differential equations of motion for the field (I), integrating in time and imposing the consistency condition for a reference time t=0 yields the **motion equations** given by,

$$\begin{cases} \dfrac{dx}{dt} = V_x^I = Z \\ \dfrac{dy}{dt} = V_y^I = X \\ \dfrac{dz}{dt} = V_z^I = Z \end{cases} \Rightarrow \begin{cases} x = X + Zt \\ y = Y + Xt \\ z = Z(1+t) \end{cases}$$

Assignment 3.10

The **deformation gradient** for the motion field (I) reads,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & t \\ t & 1 & 0 \\ 0 & 0 & 1+t \end{bmatrix}$$

The **Green-Lagrange strain** tensor for the motion field (I) reads,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} t^2 & t & t \\ t & 0 & 0 \\ t & 0 & 2t(1+t) \end{bmatrix}$$

Assignment 3.10

The **inverse motion equation** for the motion field (I) reads,

$$\begin{cases} X = x - zt/(1+t) \\ Y = y - xt - zt^2/(1+t) \\ Z = z/(1+t) \end{cases}$$

The **inverse deformation gradient** for the motion field (I) reads,

$$\left[\mathbf{F}^{-1}\right] = \begin{bmatrix} 1 & 0 & -t/(1+t) \\ -t & 1 & -t^2/(1+t) \\ 0 & 0 & 1/(1+t) \end{bmatrix}$$

Assignment 3.10

The **Almansi strain** tensor for the motion field (I) reads,

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

$$[\mathbf{e}] = \frac{1}{2}\begin{bmatrix} -t^2 & t & t(1-t) \\ t & 0 & t^2/(1+t) \\ t(1-t) & t^2/(1+t) & 1-\left(t^4+t^2+1\right)/(1+t)^2 \end{bmatrix}$$

Assignment 3.10

The **spatial velocity** vector field for the motion field (I) reads,

$$\begin{cases} v_x = z/(1+t) \\ v_y = x - zt/(1+t) \\ v_z = z/(1+t) \end{cases}$$

The **spatial velocity gradient** tensor for the motion field (I) reads,

$$[\mathbf{l}] = [\nabla \otimes \mathbf{v}] = \begin{bmatrix} 0 & 0 & 1/(1+t) \\ 1 & 0 & -t/(1+t) \\ 0 & 0 & 1/(1+t) \end{bmatrix}$$

Assignment 3.10

The **deformation rate** tensor for the motion field (I) reads,

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right)$$

$$[\mathbf{d}] = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1/(1+t) \\ 1 & 0 & -t/(1+t) \\ 1/(1+t) & -t/(1+t) & 2/(1+t) \end{bmatrix}$$

Assignment 3.10

The differential equations of motion for the field (II) read,

$$\begin{cases} \dfrac{dx}{dt} = v_x^{II} = z \\[2ex] \dfrac{dy}{dt} = v_y^{II} = x \\[2ex] \dfrac{dz}{dt} = v_z^{II} = z \end{cases}$$

Assignment 3.10

Integrating the differential equations of motion for the field (II) yields,

$$\frac{dz}{dt} = z \quad \Rightarrow \quad \frac{dz}{z} = dt \quad \Rightarrow \quad \log\frac{z}{C_3} = t \quad \Rightarrow \quad z = C_3 e^t$$

$$\frac{dx}{dt} = z = C_3 e^t \quad \Rightarrow \quad x = C_1 + C_3 e^t$$

$$\frac{dy}{dt} = x = C_1 + C_3 e^t \quad \Rightarrow \quad y = C_1 t + C_2 + C_3 e^t$$

Imposing the consistency condition, taking t=0 as reference time, yields,

$$C_1 = X - Z, \quad C_2 = Y - Z, \quad C_3 = Z$$

Assignment 3.10

The *canonical form* of the **equations of motion** for the field (II) read,

$$\begin{cases} x = X + Z\left(e^t - 1\right) \\ y = Y + Xt + Z\left(e^t - t - 1\right) \\ z = Ze^t \end{cases}$$

The **deformation gradient** for the motion field (II) reads,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & e^t - 1 \\ t & 1 & e^t - t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

Assignment 3.10

The **Green-Lagrange strain** tensor for the field (II) reads,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} t^2 & t & (1+t)\left(e^t - t\right) - 1 \\ t & 0 & e^t - t - 1 \\ (1+t)\left(e^t - t\right) - 1 & e^t - t - 1 & 3e^{2t} - (2t+4)e^t + (1+t)^2 \end{bmatrix}$$

Assignment 3.10

The **inverse of the equations of motion** for the field (II) reads,

$$X = x - z\left(1 - e^{-t}\right)$$

$$Y = y - xt + zt\left(t - 1 + e^{-t}\right)$$

$$Z = ze^{-t}$$

The **inverse deformation gradient** for the motion field (II) reads,

$$\left[\mathbf{F}^{-1}\right] = \begin{bmatrix} 1 & 0 & e^{-t} - 1 \\ -t & 1 & t - 1 + e^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Assignment 3.10

The **Almansi strain** tensor for the field (II) reads,

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

$$[\mathbf{e}] = \frac{1}{2}\begin{bmatrix} -t^2 & t & 1-(1-t)\left(t+e^{-t}\right) \\ t & 0 & 1-t-e^{-t} \\ 1-(1-t)\left(t+e^{-t}\right) & 1-t-e^{-t} & -e^{-2t}+(1-t)\left(2e^{-t}+t-1\right) \end{bmatrix}$$

Assignment 3.10

The **spatial velocity gradient** for the field (II) reads,

$$[\mathbf{l}] = [\nabla \otimes \mathbf{v}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The **deformation rate** tensor for the field (II) reads,

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right)$$

$$[\mathbf{d}] = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Material Time Derivatives

Material Time Derivative of the Deformation Gradient

The material time derivative of the **deformation gradient** reads,

$$\dot{F}_{aA} = \frac{d}{dt}\left(\frac{\partial \varphi_a}{\partial X_A}\right) = \frac{\partial}{\partial X_A}\left(\frac{\partial \varphi_a}{\partial t}\right) = \frac{\partial v_a}{\partial x_b}\frac{\partial \varphi_b}{\partial X_A} = l_{ab} F_{bA}$$

$$\boxed{\dot{\mathbf{F}} = \mathbf{l}\,\mathbf{F}, \quad \dot{F}_{aA} = l_{ab} F_{bA}}$$

The material time derivative of the **inverse deformation gradient** may be computed as follows,

$$\frac{d}{dt}\left(\mathbf{F}\mathbf{F}^{-1}\right) = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\frac{d}{dt}\left(\mathbf{F}^{-1}\right) = \mathbf{l} + \mathbf{F}\frac{d}{dt}\left(\mathbf{F}^{-1}\right) = \mathbf{0}$$

$$\boxed{\frac{d}{dt}\left(\mathbf{F}^{-1}\right) = -\mathbf{F}^{-1}\mathbf{l}}$$

Material Time Derivatives

Material Time Derivative of the Green-Lagrange Strain

The material time derivative of the **Green-Lagrange strain** tensor reads,

$$
\begin{aligned}
\dot{\mathbf{E}} &= \frac{1}{2}\dot{\mathbf{C}} = \frac{1}{2}\frac{d}{dt}\left(\mathbf{F}^T\mathbf{F}\right) \\
&= \frac{1}{2}\left(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}\right) = \frac{1}{2}\mathbf{F}^T\left(\mathbf{l} + \mathbf{l}^T\right)\mathbf{F} \\
&= \mathbf{F}^T\mathbf{d}\mathbf{F}
\end{aligned}
$$

Material Time Derivatives

Material Time Derivative of the Green-Lagrange Strain

The material time derivative of the **Green-Lagrange strain** tensor may be viewed as the *pull-back* of the spatial **deformation rate** tensor,

$$\dot{\mathbf{E}} = \varphi_*^{-1}(\mathbf{d}) = \mathbf{F}^T \mathbf{d} \mathbf{F}$$

The spatial **deformation rate** tensor may be viewed as the *push-forward* of the material time derivative of the **Green-Lagrange strain** tensor

$$\mathbf{d} = \varphi_*(\dot{\mathbf{E}}) = \mathbf{F}^{-T} \dot{\mathbf{E}} \mathbf{F}^{-1}$$

Material Time Derivatives

Material Time Derivative of the Almansi Strain

The material time derivative of the **Almansi strain** tensor reads,

$$\dot{\mathbf{e}} = -\frac{1}{2}\frac{d}{dt}\left(\mathbf{b}^{-1}\right) = -\frac{1}{2}\frac{d}{dt}\left(\mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

$$= -\frac{1}{2}\left(\frac{d}{dt}\left(\mathbf{F}^{-T}\right)\mathbf{F}^{-1} + \mathbf{F}^{-T}\frac{d}{dt}\left(\mathbf{F}^{-1}\right)\right)$$

$$= \frac{1}{2}\left(\mathbf{l}^{T}\mathbf{F}^{-T}\mathbf{F}^{-1} + \mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{l}^{T}\right)$$

$$= \frac{1}{2}\left(\mathbf{l}^{T}\mathbf{b}^{-1} + \mathbf{b}^{-1}\mathbf{l}\right)$$

Material Time Derivatives

Material Time Derivative of the Jacobian

The material time derivative of the **Jacobian** reads,

$$\dot{J} = \frac{d}{dt}\left|\mathbf{F}\right| = \frac{d\left|\mathbf{F}\right|}{d\mathbf{F}} : \frac{d\mathbf{F}}{dt} = \frac{d\left|\mathbf{F}\right|}{d\mathbf{F}} : \dot{\mathbf{F}} = \frac{d\left|\mathbf{F}\right|}{d\mathbf{F}} : \left(\mathbf{l}\mathbf{F}\right)$$

$$\frac{d\left|\mathbf{F}\right|}{d\mathbf{F}} = J\,\mathbf{F}^{-T}, \qquad \frac{d\left|\mathbf{F}\right|}{dF_{aA}} = J\,F_{aA}^{-T}$$

$$\dot{J} = \frac{d}{dt}\left|\mathbf{F}\right| = \frac{d\left|\mathbf{F}\right|}{dF_{aA}}\frac{dF_{aA}}{dt} = J\,F_{aA}^{-T} l_{ab} F_{bA} = J\,F_{bA} F_{Aa}^{-1} l_{ab} = J\,l_{aa} = J v_{a,a}$$

$$\boxed{\dot{J} = J\,\mathrm{div}\,\mathbf{v}}$$

Material Time Derivatives

Material Time Derivative of the Differential of Volume

The material time derivative of the **differential of volume** reads,

$$\frac{d}{dt}(dv) = \frac{d}{dt}\left(J\,dV\right) = \dot{J}\,dV = J\,\mathrm{div}\,\mathbf{v}\,dV = \mathrm{div}\,\mathbf{v}\,dv$$

Material Time Derivative of the Differential of Area

The material time derivative of the **differential of area** reads,

$$\frac{d}{dt}(d\mathbf{a}) = \frac{d}{dt}\left(J\,\mathbf{F}^{-T}d\mathbf{A}\right) = \left((\mathrm{div}\,\mathbf{v})\mathbf{1} - \mathbf{l}^{T}\right)J\,\mathbf{F}^{-T}d\mathbf{A}$$

$$= \left((\mathrm{div}\,\mathbf{v})\mathbf{1} - \mathbf{l}^{T}\right)d\mathbf{a}$$

Material Time Derivatives

Deformation Gradient

$$\dot{\mathbf{F}} = \mathbf{l}\,\mathbf{F}$$

Strain Tensors

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{d} \mathbf{F}$$

$$\dot{\mathbf{e}} = \frac{1}{2}\left(\mathbf{l}^T \mathbf{b}^{-1} + \mathbf{b}^{-1}\mathbf{l}\right) = \mathbf{d} - \mathbf{l}^T \mathbf{e} - \mathbf{e}\,\mathbf{l}$$

Jacobian

$$\dot{J} = J \operatorname{div} \mathbf{v}$$