Initiation and Extension of Hydraulic Fractures in Rocks

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ABSTRACT

A criterion is proposed for the initiation of vertical hydraulic fracturing taking into consideration the three stress fields around the wellbore. These fields arise from (1) nonhydrostatic regional stresses in earth, (2) the difference between the fluid pressure in the wellbore and the formation fluid pressure and (3) the radial fluid flow through porous rock from the wellbore into the formation due to this pressure difference. The wellbore fluid pressure required to initiate a fracture (assuming elastic rock and a smooth wellbore wall) is a function of the porous elastic constants of the rock, the two unequal horizontal principal regional stresses, the tensile strength of the rock and the formation fluid pressure. A constant injection rate will extend the fracture to a point where equilibrium is reached and then, to keep the fracture open, the pressure required is a function of the porous elastic constants of the rock, the component of the regional stress normal to the plane of the fracture, the formation fluid pressure and the dimensions of the crack. The same expression may also be used to estimate the vertical fracture width, provided all other variables are known. The derived equations for the initiation and extension pressures in vertical fracturing may be employed to solve for the two horizontal, regional, principal stresses in the rock.

INTRODUCTION

Well stimulation by hydraulic fracturing is a common practice today in the petroleum industry. However, this stimulation process is not a guaranteed success; hence, the deep interest shown by the petroleum companies in better understanding the mechanism that brings about rock fracturing, fracture extension and productivity increase. Geologists and mining people became interested in hydraulic fracturing from a different point of view: the method may possibly be employed to determine the magnitude and direction of the principal stresses of great depth.

Numerous articles in past years have dealt with the theory of hydraulic fracturing, but they all seem to underestimate the effect of stresses around the wellbore due to penetration of some of the injected fluid into the porous formation.[1-4] Excellent papers on stresses in porous materials due to fluid flow have been published[5,6] but no real attempt has been made to show the effect of these stresses in the form of a more complete criterion for vertical hydraulic fracturing initiation and extension. This paper is such an attempt.

ASSUMPTIONS

It is assumed that rock in the oil-bearing formation is elastic, porous, isotropic and homogeneous. The formation is under a nonhydrostatic state of regional stress with one of the principal regional stresses acting parallel to the vertical axis of the wellbore. This assumption is justified in areas where rock formations do not dip at steep angles and where th surface of the earth is relatively flat. This vertical principal regional stress equals the pressure of the overlying rock, i.e., $S_{33} = -\rho D$, where S_{33} is the total vertical principal stress (positive for tension), ρ is average density of the overlying material and D is the depth of the point where S_{33} is calculated. The wellbore wall in the formation is considered to be smooth and circular in cross-section The fluid flow through the porous elastic rock obeys Darcy's law. The whole medium is looked upon as an infinitely long cylinder with its axis along the axis of the wellbore. The radius of the cylinder is also very large. Over the range of depth at which the oil-bearing formation occurs, it will be assumed that any horizontal cross-section of the cylinder is subjected to the same stress distribution, and likewise that it will deform in the same manner.

STATE OF STRESS PRIOR TO DRILLING OF WELLBORE

At a depth D from the earth surface within the oil-bearing strata, it is assumed that the following are

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[1]References given at end of paper.

total principal regional stresses acting at any point.

1. S_{11}: minimum horizontal total principal stress (where tension is considered positive). It can be expressed as $S_{11} = \sigma_{11} - P_o$ where P_o is the formation pore fluid pressure and σ_{11} is the effective principal stress.

2. S_{22}: maximum horizontal total principal stress. It acts at right angles to the direction of S_{11}, and can be expressed as $S_{22} = \sigma_{22} - P_o$.

3. S_{33}: vertical total principal stress. Its value, as mentioned above, is assumed to be $S_{33} = \sigma_{33} - P_o = -\rho D$.

STATE OF STRESS AROUND THE WELLBORE

When the vertical wellbore (radius r_w) is drilled into the formation the above-mentioned state of stress in any horizontal cross-section is disturbed. The new stress distribution may be determined using the principle of superposition[2] as follows.

First, we have the stress field due to the two horizontal total principal stresses S_{11} and S_{22}. These can be regarded as stresses acting on the edges of a very large rectangular plate with a small circular hole (radius r_w) at its center. The resulting total stress distribution in terms of effective stresses, formation pressure (P_o) and polar coordinates is given by[7]

$$S_{rr}^{(1)} = \sigma_{rr}^{(1)} - P_o$$
$$= \left[\frac{\sigma_{11}+\sigma_{22}}{2}\left(1-\frac{r_w^2}{r^2}\right) + \frac{\sigma_{11}-\sigma_{22}}{2} \times \left(1+\frac{3r_w^4}{r^4}-\frac{4r_w^2}{r^2}\right)\cos 2\theta\right] - P_o$$

$$S_{\theta\theta}^{(1)} = \sigma_{\theta\theta}^{(1)} - P_o$$
$$= \left[\frac{\sigma_{11}+\sigma_{22}}{2}\left(1+\frac{r_w^2}{r^2}\right) - \frac{\sigma_{11}-\sigma_{22}}{2} \times \left(1+\frac{3r_w^4}{r^4}\right)\cos 2\theta\right] - P_o$$

$$S_{r\theta}^{(1)} = \frac{\sigma_{22}-\sigma_{11}}{2}\left(1-\frac{3r_w^4}{r^4}+\frac{2r_w^2}{r^2}\right)\sin 2\theta \qquad \ldots\ldots\ldots (1)$$

where r is horizontal radial distance from the center of the hole. θ is the angle measured counterclockwise from the radius in the direction of S_{11} (S_{11} direction corresponds to $\theta = 0$).

When fluid is pumped into the borehole so as to increase the pressure on its vertical wall to P_w (Fig. 1), two additional stress fields are introduced, $S_{ij}^{(2)}$ and $S_{ij}^{(3)}(i,j = r,\theta)$. The increase in pressure at r_w from the original P_o to P_w, neglecting for the moment any fluid flow into the formation, introduces stress field $S_{ij}^{(2)}$.

$$S_{rr}^{(2)} = -\frac{r_w^2 r_e^2 p_w}{r^2(r_e^2-r_w^2)} + \frac{r_w^2 p_w}{r_e^2-r_w^2}$$

$$S_{\theta\theta}^{(2)} = \frac{r_w^2 r_e^2 p_w}{r^2(r_e^2-r_w^2)} + \frac{r_w^2 p_w}{r_e^2-r_w^2}$$

$$S_{r\theta}^{(2)} = 0 \qquad \ldots\ldots\ldots (2)$$

where $p_w = P_w - P_o$ and r_e is the external radius of the cylinder. If r_e is considered to be very large in comparison to r_w (say, $r_e > 10\,r_w$), Eqs. 2 can be reduced to

$$S_{rr}^{(2)} = -\frac{r_w^2}{r^2}p_w$$

$$S_{\theta\theta}^{(2)} = +\frac{r_w^2}{r^2}p_w$$

$$S_{r\theta}^{(2)} = 0 \qquad \ldots\ldots\ldots (3)$$

Simultaneously, the pressure difference between the wellbore fluid P_w and the pore fluid in the formation P_o will cause an outward radial flow, provided that the rock is permeable to the injected fluid. To simplify the flow problem, we assume that this fluid has properties similar to those of the formation pore fluid. This is not an unusual case in hydraulic fracturing. We assumed that the formation has a uniform permeability so that the fluid flow is axisymmetric. It has been shown that fluid flow through porous media gives rise to stresses and displacements in the material analogous to those caused by heat conduction through

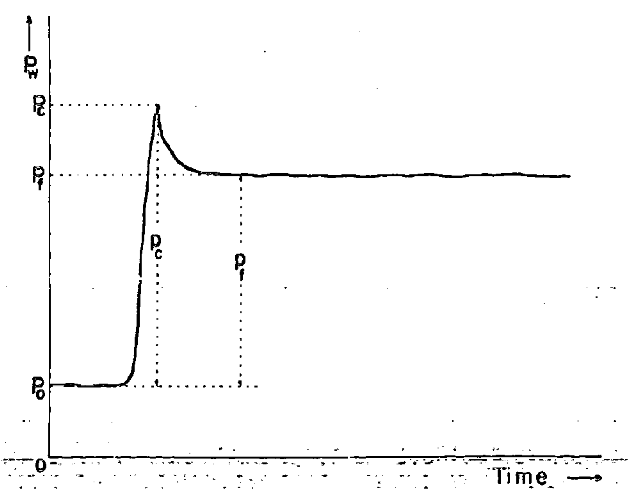

FIG. 1 — TYPICAL BOTTOM-HOLE PRESSURE P_w VS TIME RECORDING DURING HYDROFRACTURING OPERATION.

solids. Known results from the theory of thermoelasticity may be easily modified to solve problems in porous elastic materials.[6,8] In this case (using the solution for thermoelastic stress in a thick cylinder[7] one can express the third stress field $S_{ij}^{(3)}$ as

$$S_{rr}^{(3)} = \frac{\alpha(1-2\nu)}{r^2(1-\nu)}\left[\frac{r^2-r_w^2}{r_e^2-r_w^2}\int_{r_w}^{r_e} p(r)\,r\,dr - \int_{r_w}^{r} p(r)\,r\,dr\right]$$

$$S_{\theta\theta}^{(3)} = \frac{\alpha(1-2\nu)}{r^2(1-\nu)}\left[\frac{r^2+r_w^2}{r_e^2-r_w^2}\int_{r_w}^{r_e} p(r)\,r\,dr + \int_{r_w}^{r} p(r)\,r\,dr - p(r)\,r^2\right]$$

$$S_{r\theta}^{(3)} = 0 \quad \ldots\ldots\ldots\ldots \quad (4)$$

where ν is Poisson's ratio of the formation; $\alpha = (1 - C_r/C_b)$, C_r and C_b being the rock matrix compressibility and the rock bulk compressibility, respectively[8] (α can be obtained experimentally); and $p(r) = P(r) - P_o$ is the increase in pressure above the original P_o at a distance r from the center of the wellbore.

As pressure decreases with increase in r, if r_e is taken to be very large, Nowacki's results[10] can be used to reduce Eqs. 7 to

$$S_{rr}^{(3)} = -\frac{\alpha(1-2\nu)}{r^2(1-\nu)}\int_{r_w}^{r} p(r)\,r\,dr$$

$$S_{\theta\theta}^{(3)} = \frac{\alpha(1-2\nu)}{1-\nu}\left[\frac{1}{r^2}\int_{r_w}^{r} p(r)\,r\,dr - p(r)\right]$$

$$S_{r\theta}^{(3)} = 0 \quad \ldots\ldots\ldots\ldots \quad (5)$$

The complete distribution of stresses around the wellbore is thus obtained by superposing the three different stress fields

$$S_{ij} = S_{ij}^{(1)} + S_{ij}^{(2)} + S_{ij}^{(3)} \quad \ldots\ldots \quad (6)$$

To obtain the stress distribution (Eq. 6) in an explicit form, the distribution of $p(r)$ in the formation due to the fluid flow has to be known.

INITIATION OF VERTICAL FRACTURE

Used here is the fracture initiation criterion propounded by M. K. Hubbert[2] (fracturing will first occur at a point on the boundary of the wellbore where the effective stress is equal to or greater than the tensile strength of the rock σ_t). This fracture will extend along a plane normal to the direction of the maximum principal stress S_{22}.

The total stresses at the wellbore boundary r_w are obtained by adding the stresses given in Eqs. 1, 2 and 4 for the case of finite r_e, or Eqs. 1, 3 and 5 for the case of $r_e \to \infty$ and replacing r by r_w and $p(r)$ by p_w. The latter case is particularly simple to handle as it does not require any knowledge of the distribution of $p(r)$,

$$S_{rr} = -P_o - p_w = -P_w$$

$$S_{\theta\theta} = \left[\sigma_{11} + \sigma_{22} - 2(\sigma_{11} - \sigma_{22})\cos 2\theta - P_o\right] + p_w - \alpha p_w \frac{1-2\nu}{1-\nu}$$

$$S_{r\theta} = 0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

From Eq. 7 we observed that the only effective stress that can become tensile at $r = r_w$ is the tangential one. The points where this stress will first become tensile are at $\theta = 0, \pi$, since for these values of θ we have $\cos 2\theta = 1$, and the tensile stress $[-(\sigma_{11} - \sigma_{22})\cos 2\theta]$ is the largest. Hence, for $\theta = 0, \pi$, we obtain from Eq. 7,

$$S_{\theta\theta} = \sigma_{\theta\theta} - P_w = 3\sigma_{22} - \sigma_{11} - P_o + p_w - \alpha p_w \frac{1-2\nu}{1-\nu} \quad (8)$$

and the tangential effective stress at r_w is

$$\sigma_{\theta\theta} = p_w\left(2 - \alpha\frac{1-2\nu}{1-\nu}\right) + 3\sigma_{22} - \sigma_{11} \quad \ldots\ldots\ldots\ldots \quad (9)$$

This stress can cause failure at the points $r = r_w$, $\theta = 0, \pi$, if $\sigma_{\theta\theta} \geq \sigma_t$, or equivalently, the minimum pressure difference required to induce fracture is given by

$$p_c = \frac{\sigma_t - 3\sigma_{22} + \sigma_{11}}{2 - \alpha\frac{1-2\nu}{1-\nu}} \quad \ldots\ldots \quad (10)$$

where p_c is p_w at the time of fracture initiation (Fig. 1). As $\phi \leq \alpha \leq 1$ (ϕ is the porosity of rock) and $0 \leq \nu \leq 0.5$ for rock, we have $0 \leq \alpha(1-2\nu)/(1-\nu) \leq 1$.

Eq. 10 is the criterion for fracture initiation in porous rock. Assuming that the rock parameters (σ_t, ν, a) and the effective stresses (σ_{11} and σ_{22}) are known, the breakdown pressure P_c may be predicted. When σ_{11} and σ_{22} are unknown, a pressure-vs-time chart taken during the fluid injection process may provide us with $p_c = P_c - P_o$, and so one equation in two unknown σ_{11} and σ_{22} is obtained.

In the case of nonpenetrating fluid, Eq. 8 becomes

$$S_{\theta\theta} = \sigma_{\theta\theta} - P_o = 3\sigma_{22} - \sigma_{11} - P_o + p_w \quad \ldots \ldots (11)$$

and the tangential effective stress is now

$$\sigma_{\theta\theta} = 3\sigma_{22} - \sigma_{11} + p_w \quad \ldots \ldots (12)$$

The reason for this discrepancy is that in the case of penetrating injection fluid, the pore pressure at the wellbore boundary is P_w (Fig. 3), while in the nonpenetrating case $S_{ij}^{(3)} = 0$ and the pore pressure is P_o everywhere (Fig. 2). Hence, for a nonpenetrating injection fluid, the criterion of fracture initiation is given by

$$p_c = \sigma_t - 3\sigma_{22} + \sigma_{11} \quad \ldots \ldots (13)$$

which is identical to the criterion obtained by Hubbert.[2]

EXTENSION OF VERTICAL FRACTURE

When the vertical fracture is initiated, it will extend as previously stated in a direction perpendicular to that of the maximum principal effective stress σ_{22}, provided enough fluid is pumped into it. We assume that this vertical fracture will initiate simultaneously at $\theta = 0$ and $\theta = \pi$ and will extend in both directions symmetrically (Fig. 4). It has been shown[11] that for a fracture longer than the diameter of the wellbore the circular hole has negligible influence on the stresses around the fracture; hence, Fig. 4 can be simplified to approximate the horizontal cross-section of the fracture to a very thin crack (Fig. 5).

Assuming that its pressure is sufficiently high, the injected fluid will penetrate into the fracture causing it to widen and extend. As the fluid penetrates there is a continuous loss in pressure with increase in length due to leakage into the formation. As the injection rate is increased, the

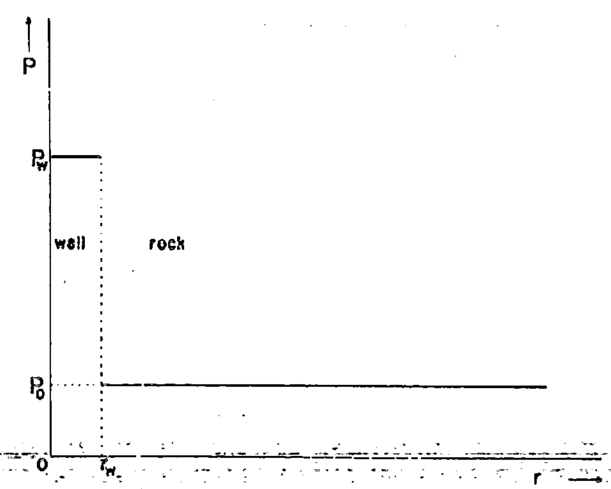

FIG. 2 — DISTRIBUTION OF PORE FLUID PRESSURE AROUND WELLBORE (RADIUS r_w) FOR NONPENETRATING FRACTURING FLUIDS.

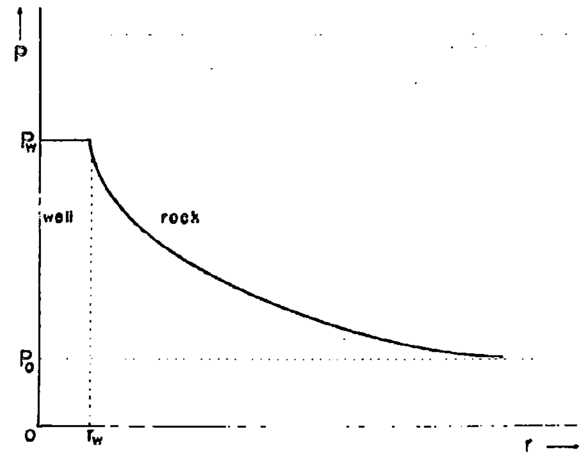

FIG. 3 — POSSIBLE DISTRIBUTION OF PORE FLUID PRESSURE AROUND WELLBORE (RADIUS r_w) FOR PENETRATING FRACTURING FLUID.

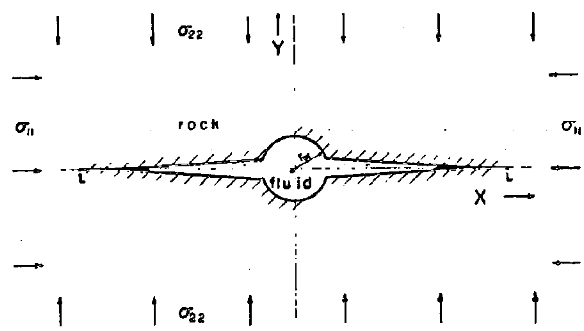

FIG. 4 — CROSS-SECTION OF VERTICALLY FRACTURED WELLBORE.

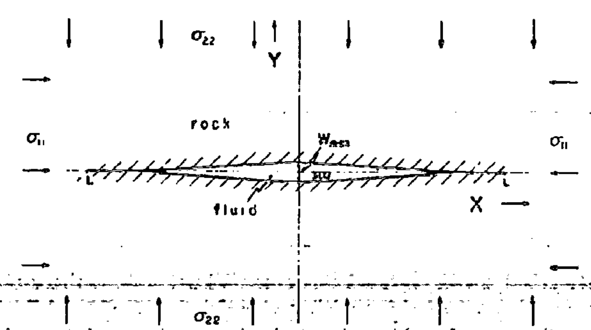

FIG. 5 — APPROXIMATED CROSS-SECTION OF VERTICAL FRACTURE (PLANE $z = x + iy$).

volume of the fracture will enlarge and both its width W and length L will extend accordingly. When the injection rate reaches a constant value, a state of equilibrium is finally obtained such that the rate of penetration into the formation is equal to the rate of injection and the flow in the porous rock is in a steady state ($\nabla^2 p = 0$, where ∇^2 is the Laplace operator). Hence, the volume of fluid in the fracture is also fixed, the pressure distribution along its length $p(x)$ (Fig. 5) is stationary and the dimensions of the fracture remain constant.

EFFECT OF REGIONAL STRESSES S_{11} AND S_{22} ON WELLBORE PRESSURE P_f FOR CONSTANT INJECTION RATE

To find the relation between wellbore pressure P_f (Fig. 1) and the stresses S_{11} and S_{22} (or the effective principal stresses σ_{11} and σ_{22}) for constant injection rate, we must first calculate the displacements around the fracture. These will be considered in two parts: (1) the displacement v_1 (in the y direction) at the fracture due to boundary loading and (2) the displacement v_2 (in the y direction) at the fracture due to fluid flow into the formation.

DISPLACEMENT v_1 AT THE FRACTURE CONTOUR DUE TO BOUNDARY LOADING

If u_1 and v_1 are denoted as displacements in the x and y directions, respectively, due to boundary loading, we can write[12]

$$2\mu(u_1 + iv_1) = k\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \text{ , } \quad (14)$$

where $\overline{\phi'(z)}$, $\overline{\psi(z)}$ are complex conjugates of the functions ϕ', ψ, which determine the Airy stress function in Muskelishvili's book[12] and $k = 3 - 4\nu$ (plane strain).

To solve Eq. 14 for our particular case, the appropriate boundary conditions must first be determined. As stated elsewhere in this paper, the original formation pressure in the rock is taken to be P_o. Hence, the differential pressure in the fracture is $p(x) = P(x) - P_o$. This pressure decreases from $p_f = P_f - P_o$ at the wellbore to $p_L = P_L - P_o$ at the tip of the crack as a continuous function. Although this function is not known exactly it can be approximated by a polynomial. As a first approximation, it is assumed[13]

$$p(x) = p_f - (p_f - p_L)\left(\frac{x}{L}\right)^2 \quad . . . . (15)$$

where, because of the symmetry of the function with respect to the y axis, only even powers of x are taken. The boundary conditions are as follows:

along the contour of the fracture ($y = 0$, $-L \leq x \leq L$)

$$S_{yy} = -P(x) = -p(x) - P_o$$

$$S_{xx} = S_{xy} = 0 \quad . . . . (16)$$

at infinity

$$S_{xx} = \sigma_{11} - P_o$$

$$S_{yy} = \sigma_{22} - P_o$$

$$S_{xy} = 0 \quad . . . . . . (17)$$

Following Muskelishvili's method,[12] the plane outside the fracture, of length $2L$ and negligible width, is mapped conformally onto the plane outside the unit circle. For this purpose dimensionless coordinates $x_D = x/L$, $y_D = y/L$ are introduced and $z_D = x_D + iy_D$ is mapped onto $\zeta = \xi + i\eta = Re^{i\gamma} = R\tau$ (where R and γ are polar coordinates in the new plane), using the mapping function (Fig. 6)

$$z_D = \omega(\zeta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \quad . . . . . (18)$$

In the new plane, the two Muskelishvili functions are defined as

$$\varphi(\zeta) = -\frac{1}{2\pi i}\int_\tau \frac{f}{\tau - \zeta}\, d\tau$$

$$\psi(\zeta) = -\frac{1}{2\pi i}\int_\tau \frac{\bar{f}}{\tau - \zeta}\, d\tau$$

$$-\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) \quad . . . . . (19)$$

In the stress field (ϕ_1, ψ_1) created by the given boundary conditions along the fracture (Eq. 16) and

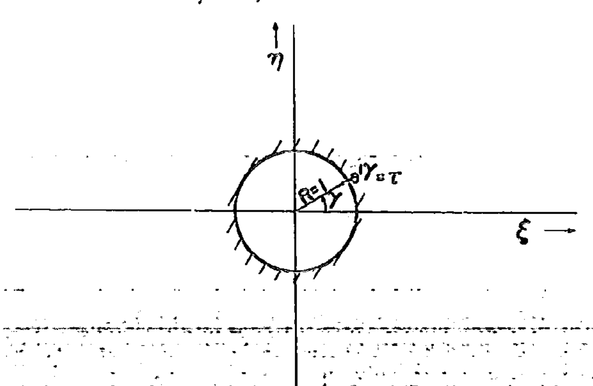

FIG. 6 — MAPPED PLANE $\zeta = \xi + i\eta$.

the condition of zero stresses at infinity, f is given by

$$f = \int(-p(x_D)\,dx_D$$

$$= -\left[p_f x_D - \frac{p_f - p_L}{3} x_D^3\right] \quad . . . . (20)$$

Hence,

$$2\pi i \varphi_1(\zeta) = \int_{e^0}^{e^{2\pi i}} \left[\frac{p_f}{2}\left(\tau + \frac{1}{\tau}\right) - \frac{1}{24}(p_f - p_L)\left(\tau + \frac{1}{\tau}\right)^3\right]\frac{d\tau}{\tau - \zeta}$$

$\therefore$

$$\varphi_1(\zeta) = -\frac{p_f}{2\zeta} + \frac{1}{24}(p_f - p_L)\left(\frac{3}{\zeta} + \frac{1}{\zeta^3}\right) \quad . (21)$$

In the stress field $(\phi_2 \cdot \psi_2)$ produced by the given boundary conditions at infinity with zero loads along the fracture, we have, following Muskelishvili,

$$\varphi_2(\zeta) = \frac{\sigma_{22}}{8}\left(\zeta - \frac{3}{\zeta}\right) + \frac{\sigma_{11}}{8}\left(\zeta + \frac{1}{\zeta}\right) \quad . . (22)$$

and

$$\varphi(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta) \quad . . . (23)$$

It is assumed that the crack is in equilibrium, and hence that the stresses at the tips are finite.[13] To ensure that this condition holds, it is required that $\phi'(\zeta)/\omega'(\zeta)$ is finite at $\zeta = \pm 1$.

$$\frac{\varphi'(\zeta)}{\omega'(\zeta)} = \frac{1}{1 - \frac{1}{\zeta^2}}\left[\frac{p_f}{2\zeta^2} - \frac{p_f - p_L}{24}\left(\frac{3}{\zeta^2} + \frac{3}{\zeta^4}\right) + \frac{\sigma_{22}}{8}\left(1 + \frac{3}{\zeta^2}\right) + \frac{\sigma_{11}}{8}\left(1 - \frac{1}{\zeta^2}\right)\right]$$

$$. . . . . (24)$$

Since $(1 - 1/\zeta^2) = 0$ at $\zeta^2 = 1$, the only way in which $\phi'(\zeta)/\omega'(\zeta)$ can be finite is for

$$\frac{p_f}{2} - \frac{6}{24}(p_f - p_L) + \frac{4}{8}\sigma_{22} = 0$$

or

$$\sigma_{22} = -\frac{1}{2}(p_f + p_L) \quad . . . . (25)$$

In our case, $f(\tau) = \overline{f(\tau)}$, so $\psi(\zeta)$ given in Eq. 19 is actually equal to

$$\psi(\zeta) = \varphi(\zeta) - \frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) \quad . . (26)$$

Hence, Eq. 14 becomes (on the fracture contour)

$$2\mu(u_1 + iv_1) = L\left[k\varphi(\tau) - \overline{\varphi(\tau)}\right] \quad . . (27)$$

Using Eq. 25, we have

$$\varphi(\zeta) = \frac{\sigma_{11} - p_f}{8}\left(\zeta + \frac{1}{\zeta}\right) + \frac{\sigma_{22} + p_f}{4}\left[\frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right) + \frac{1}{3\zeta^3}\right] \quad . . . . (28)$$

Hence,

$$2\mu(u_1 + iv_1) = \frac{L}{4}(k-1)(\sigma_{11} - p_f)\cos\gamma + \frac{L}{12}(k-1)(\sigma_{22} + p_f)\cos 3\gamma + \frac{iL}{4}(k+1)(\sigma_{22} + p_f)\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right) \quad . . . . . . (29)$$

The contribution to the width of the crack W_1 due to the boundary loading is thus given by

$$W_1 = 2v_1 = \frac{\sigma_{22} + p_f}{\mu}L(1-\nu)\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right) \quad . . . . . . (30)$$

or

$$W_1 = 2L\frac{1-\nu^2}{E}(\sigma_{22} + p_f)\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right)$$

$$. . . . . (31)$$

Clearly, the well fluid pressure p_f is independent of the minimum regional effective principal stress σ_{11}.

DISPLACEMENT v_2 AT FRACTURE CONTOUR DUE TO FLUID FLOW INTO THE FORMATION

The pressure distribution has been approximated (above the original P_o) along the crack $-1 \leq x_D = x/L \leq 1$ as

$$p(x) = p_f - (p_f - p_L)\,x_D^2 \quad . . . . (15)$$

At a large distance r_e from the crack, pressure p_e can be expected to be

$$p_e = P_e - P_o = 0 \quad \ldots\ldots (32)$$

To find the pressure distribution in the original plane, the transformation given by Eq. 18 is performed. The boundary conditions for this new plane are

at $R = 1$ $\quad p(\gamma) = \frac{p_f + p_L}{2} - \frac{p_f - p_L}{2}\cos 2\gamma$

at $R = R_e$ $\quad p(\gamma) = 0 \quad , \ldots\ldots (33)$

where R_e is the transformation of r_e.

Using the method of Muskelishvili, the pressure distribution is found to be

$$p(R,\gamma) = -\frac{p_f + p_L}{2\ln R_e}\ln R + \frac{p_f + p_L}{2} + \frac{p_f - p_L}{2(R_e^4 - 1)}R^2\cos 2\gamma - \frac{R_e^4(p_f - p_L)}{2(R_e^4 - 1)R^2}\cos 2\gamma \quad \ldots (34)$$

where ln is the natural logarithm.

Since the case of steady-state pressure is being considered for which $\nabla^2 p = 0$, p is harmonic and can be considered as the real part of a complex variable function $F(\zeta)$. $F(\zeta)$ is found to be

$$F(\zeta) = -\frac{p_f + p_L}{2\ln R_e}\ln\zeta + \frac{p_f + p_L}{2} + \frac{p_f - p_L}{2(R_e^4 - 1)}\zeta^2 - \frac{R_e^4(p_f - p_L)}{2(R_e^4 - 1)\zeta^2}$$

$$\ldots\ldots (35)$$

If $F_1(z_D)$ is the complex function in the z_D plane whose real part is pressure $p(x,y)$, then $F(\zeta)$ is the representation of $F_1(z_D)$ where z_D has been replaced by the mapping variable ζ in the following way.

$$F_1(z_D) = F_1(\omega(\zeta)) = F(\zeta) \quad \ldots (36)$$

Again assuming a complete analogy between stresses and displacements due to heat conduction and those due to fluid flow through porous media, it can be shown[12,14] that $F_1(z_D)$ causes displacements u_2 and v_2 given by

$$u_2 + i v_2 = \frac{\alpha L}{2(\lambda+\mu)}\int F_1(z_D)\,dz_D \quad \ldots (37)$$

In terms of the ζ variable, we have

$$u_2 + i v_2 = \frac{\alpha L}{2(\lambda+\mu)}\int F(\zeta)\omega'(\zeta)\,d\zeta \quad \ldots (38)$$

Using Eq. 30 and the relation

$$\omega'(\zeta) = \frac{1}{2}\left(1 - \frac{1}{\zeta^2}\right) \quad \ldots\ldots (39)$$

we obtain

$$u_2 + i v_2 = \frac{\alpha L}{2(\lambda+\mu)}\Big[-A\left(\zeta + \frac{1}{\zeta}\right)\ln\zeta - B\frac{R_e^4}{3\zeta^3} + (-A + A\ln R_e + BR_e^4)\frac{1}{\zeta} + (A + A\ln R_e - B)\zeta + \frac{1}{3}B\zeta^3 \quad \ldots (40)$$

where $A = (p_f + p_L)/(4\ln R_e)$; $B = (p_f - p_L)/4(R_e^4 - 1)$.

This expression is not single valued because the first term includes the factor $(\zeta + 1/\zeta)\ln\zeta$, and additional displacements have to be superposed to eliminate this term. However, in this particular case we are interested in the displacement v_2 at the boundary of the crack when r_e (or its equivalent R_e) is very large. For these conditions ($R = 1$, $R_e \to \infty$), Eq. 40 to a first approximation yields

$$v_2 = \frac{-\alpha L}{2(\lambda+\mu)}\left[\frac{p_f - p_L}{4}\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right)\right]$$

$$\ldots\ldots (41)$$

Using the relation in Eq. 25 and expressing λ and μ in terms of E and ν, the contribution to the width of crack W_2 is

$$W_2 = -\frac{\alpha L}{E}(1+\nu)(1-2\nu)(\sigma_{22} + p_f)\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right) \quad \ldots (42)$$

TOTAL FRACTURE WIDTH IN TERMS OF σ_{22} AND p_f

From Eqs. 31 and 42 we obtain for the total width of the fracture W

$$W = W_1 + W_2 = \frac{1+\nu}{E}L(\sigma_{22} + p_f)[2(1-\nu) - \alpha(1-2\nu)]\left(\sin\gamma - \frac{1}{3}\sin 3\gamma\right) \quad \ldots (43)$$

The width of the fracture at the wellbore is given by $W_{\pi/2}$ which is also the maximum width

(W_{max}).

$$W_{max} = W_{\frac{\pi}{2}} = \frac{4(1+\nu)}{3E} L\,(\sigma_{22} + p_f)\Big[2(1 - \nu) - \alpha(1-2\nu)\Big] \quad \ldots\ldots\ldots (44)$$

Eq. 44 is useful for a variety of applications, provided at least two of the three variables are known. It gives the maximum width if the length of the crack and the smallest regional pressure ($-\sigma_{22}$) can be estimated. On the other hand, if by some logging method the maximum width can be measured, Eq. 44 will give the length of the fracture, provided σ_{22} is known.

Similarly, if the ratio W_{max}/L can be approximated for an applied p_f in a certain field, the term σ_{22} can be calculated. In this case Eq. 4 can be rewritten in the form

$$\sigma_{22} = \frac{3EW_{max}}{4(1+\nu)L}\Big[2(1-\nu) - \alpha(1-2\nu)\Big]^{-1} - p_f \quad . (45)$$

From Eq. 10 we obtain σ_{11},

$$\sigma_{11} = p_c\Big[2 - \alpha\frac{1-2\nu}{1-\nu}\Big] - \sigma_t + 3\sigma_{22} \quad . . (46)$$

and the two principal regional effective stresses are determined.

NUMERICAL EXAMPLES

The ratio W_{max}/L can be calculated if there exists an estimate of σ_{22}. For instance, $\sigma_{22} = -1{,}400$ psi (estimated), $E = 5 \times 10^6$ psi (measured), $\nu = 0.2$ (measured), $\alpha = 0.8$ (measured) and p_f = 1,800 psi (recorded). Using Eq. 46 we obtain

$$\frac{W_{max}}{L} = \frac{4}{3}\;\frac{1.2}{5 \times 10^6}(-1{,}400 + 1{,}800)$$

$$\times (1.60 - 0.48) \cong 1.4 \times 10^{-4}.$$

For example, the width at the wellbore of a 2,000-in. long vertical fracture is approximately 0.28 in. Fig. 7 shows how the ratio W_{max}/L can be obtained if the maximum horizontal principal stress is known.

If the ratio W_{max}/L is known, the two horizontal principal stresses can be calculated. For example, $W_{max}/L = 0.5 \times 10^{-4}$, $E = 10 \times 10^6$ psi, $\nu = 0.25$, $\alpha = 0.9$, $p_f = 2{,}000$ psi, $p_c = 2{,}200$ psi and σ_t = 100 psi. Using Eq. 45,

$$\sigma_{22} = \frac{3 \times (10\times10^6) \times (0.5 \times 10^{-4})}{4 \times (1 + 0.25)}$$

$$\times [2(1 - 0.25) - 0.9\,(1-0.5)]^{-1} - 2{,}000$$

$$\cong -1{,}700 \text{ psi}.$$

Inserting this result in Eq. 46, σ_{11} = 2,200 $[2 - 0.9(0.5/0.75)] - 100 - (3 \times 1{,}700) \cong -2{,}120$ psi, and thus the two principal horizontal effective stresses are obtained.

CONCLUSIONS

An analysis was made of the influence of fluid penetration on the pressures required to initiate and extend vertical hydraulic fractures in porous permeable formations. It is believed that the results improve existing theories of hydraulic fracturing. They may also be of value in attempts to interpret fracturing data in order to determine the regional state of stress at depth.

The results do not constitute a complete solution to the problem inasmuch as a prior knowledge of one of the fracture dimensions is necessary to accurately determine the other. However, it is felt that a very good estimate of the volume of the fracture as a function of the injection rate may be obtained from the theory of hydrodynamics in which case both L and W_{max} could be calculated using Eq. 44. Eq. 44 was obtained by considering the two-fold effect of the fracturing fluid of the penetrating type on the width of the crack. One effect is the gradual decrease in the fluid pressure along the fracture, and the other is the displacements in the rock due to flow of the penetrating fluid into the formation. Impression packers could be used to determine the general direction of the vertical fracture; by varying the injection rate, the length or the width of the crack could be controlled.

It is shown in Eq. 45 that for the same regional stress field (σ_{11}, σ_{22} – constant), i.e., in many cases for the same oil field, the relationship between the ratio W_{max}/L and the wellbore pressure p_f is linear.

To those whose main interest is determining stresses underground, Eqs. 45 and 46 will provide a better approximation of the regional stresses in porous formations when the crack dimensions are known. However, these equations show that,

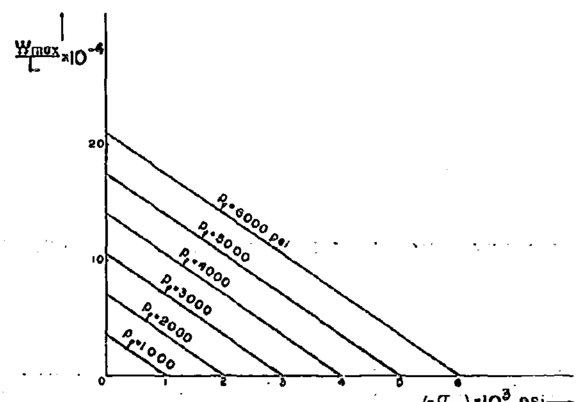

FIG. 7 — RELATION BETWEEN DIMENSIONLESS MAXIMUM WIDTH OF FRACTURE W_{max}/L, MAXIMUM PRINCIPAL HORIZONTAL EFFECTIVE STRESS σ_{22} AND BOTTOM-HOLE PRESSURE p_f.

contrary to what was previously believed,[3,4] the porous elastic parameters of the rock (E, ν, α) must be known to obtain realistic results.

NOMENCLATURE

C_b = rock bulk compressibility
C_r = rock matrix compressibility
D = depth
E = Young's modulus of rock
F = complex function whose real part is equal to p (when $\nabla^2 p = 0$)
i = $\sqrt{-1}$
L = half length of crack
ln = natural logarithm
P = pressure
P_o = original formation pore pressure (datum pressure)
P_w = pressure at wellbore
P_c = fracture initiation pressure at wellbore
P_f = final fracture extension pressure at wellbore
p = $P - P_o$
p_w = $P_w - P_o$
p_c = $P_c - P_o$
p_f = $P_f - P_o$
R = radial distance in the transformed plane
R_e = external radius in the transformed plane
r = radial distance
r_w = wellbore radius
r_e = external radius of the horizontal plane
S_{ij} = total stress tensor
u = displacement at the crack in the x direction
v = displacement at the crack in the y direction
x = Cartesion coordinate in the original plane
x_D = dimensionless Cartesion coordinate
y = Cartesion coordinate in the original plane
y_D = dimensionless Cartesion coordinate
W = width of fracture
W_{max} = width of fracture at wellbore
z = original plane in complex notation ($z = x + iy$)
z_D = dimensionless original plane
α = porous elastic constant of the material, equal to $(1 - C_r/C_b)$
γ = polar coordinate in the transformed plane
∇^2 = Laplace operator
ζ = transformed plane
η = Cartesion coordinate in the transformed plane
θ = polar coordinate in the original plane
k = elastic parameter of the rock, equal to $(3 - 4\nu)$ for plane strain
λ = Lamé constant of rock
μ = Lamé constant of rock (shear modulus)
ν = Poisson's ratio of rock
ξ = Cartesion coordinate in the transformed plane
ρ = average density of overlying rock
σ_{ij} = effective stress tensor
σ_t = tensile strength of rock
τ = position of a point on the unit circle in the transformed plane, equal to $e^{i\gamma}$
Φ = rock porosity
ϕ, ψ = Muskelishvili's stress functions
$\bar{\phi}, \bar{\psi}$ = complex conjugates of ϕ and ψ
$\phi'(z)$ = derivative of ϕ with respect to z
ω = mapping function from the ζ plane to the original z_D plane

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