Continuum Mechanics

Chapter 7
Linear Elasticity

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Infinitesimal Strains Framework

Displacements and *gradient of displacements* are infinitesimal

- No difference between *spatial* and *material configurations*
- No difference between *spatial* and *material coordinates*
- No difference between *spatial* and *material descriptions*
- No difference between *spatial* and *material differential operators*
- No difference between *spatial* and *material time derivatives*
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- *Linear* functions of the *gradient of displacements*

Linear Elastic Model

H1. Isothermal Processes

We consider that the processes are *isothermal*, such that,

$$\dot{\theta} = 0$$

The *internal dissipation rate per unit of volume* for an *isothermal process,* given by the *Clausius-Planck inequality*, may be written as,

$$\begin{aligned}
\mathcal{D}_{int} &:= \rho_0 \theta \dot{\eta} - \rho_0 r + \operatorname{div} \mathbf{q} \\
&= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \\
&= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \left(\dot{\psi} + \cancel{\eta \dot{\theta}} \right) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \dot{\psi} \geq 0
\end{aligned}$$

Linear Elastic Model

H2. Free Energy

We consider a *free energy per unit of volume* as a function of the *strain tensor*, such that for an *isothermal linear elastic model* is a *quadratic* function of the strain tensor given by,

$$\rho_0 \psi = \rho_0 \psi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \geq 0$$

where $\mathbb{C}$ is a *positive definite symmetric fourth-order tensor,* denoted as *isothermal constant elastic constitutive tensor*.

Linear Elastic Model

As the *isothermal elastic constitutive tensor* $\mathbb{C}$ is a symmetric fourth-order tensor, the following symmetry conditions hold,

$$\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \varepsilon_{ab} \mathbb{C}_{abcd} \varepsilon_{cd} = \varepsilon_{ab} \mathbb{C}_{cdab} \varepsilon_{cd} , \quad \mathbb{C}_{abcd} = \mathbb{C}_{cdab}$$

Furthermore, symmetry of the strain tensor yields the following *symmetry* conditions,

$$\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T : \mathbb{C} : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}^T , \quad \mathbb{C}_{abcd} = \mathbb{C}_{bacd} = \mathbb{C}_{abdc}$$

The symmtery conditions reduce from 81 to 21 the number of parameters needed to define the elastic constitutive tensor.

Linear Elastic Model

Internal Dissipation Inequality

Applying the chain rule, the *internal dissipation rate per unit of volume* for an *isothermal process,* given by the *Clausius-Planck inequality*, yields,

$$\mathcal{D}_{int} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \dot{\psi}$$

$$= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

Linear Elastic Model

Constitutive Equation and Reduced Dissipation

Following *Coleman's method*, the *internal dissipation rate per unit of volume* for an *isothermal process,* given by the *Clausius-Planck inequality*, must be satisfied for arbitrary thermodynamic processes, i.e. arbitrary strain rates,

$$\mathcal{D}_{int} := \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

yielding the following *isothermal linear elastic constitutive equation* and *zero internal reduced dissipation*, which characterizes a *reversible process*

$$\boxed{\boldsymbol{\sigma} = \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}, \quad \mathcal{D}_{int} = 0}$$

Linear Elastic Model

Constitutive Equation and Reduced Dissipation

The *constitutive equation and reduced internal dissipation* for an *isothermal linear elastic model* are given by,

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \boldsymbol{\varepsilon}, \quad \sigma_{ab} = \mathbb{C}_{abcd} \varepsilon_{cd}$$

$$\mathcal{D}_{int} = 0$$

Linear Elastic Model

Elastic Constitutive Tensor

The *isothermal elastic constitutive tensor* for a *linear elastic model* is given by,

$$\boxed{\mathbb{C} = \rho_0 \frac{\partial^2 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}, \quad \mathbb{C}_{abcd} = \rho_0 \frac{\partial^2 \psi(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ab} \otimes \partial \varepsilon_{cd}}}$$

Isotropic Linear Elastic Model

Isotropic Elastic Constitutive Tensor

A continuum medium is said to be *isotropic* if it has the same material properties in any direction.

For an *isotropic linear elastic material model*, the *physical property* of *isotropy* is translated into a *mathematical property* of *isotropy* of the *isothermal elastic constitutive tensor*, yielding,

$$\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}}, \quad \mathbb{C}_{abcd} = \lambda \delta_{ab} \delta_{cd} + \mu \left(\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} \right)$$

where $\lambda \geq 0, \mu \geq 0$, denoted as *isothermal Lamé parameters*, are the two material parameters characterizing an *isothermal isotropic linear elastic material model*.

Isotropy reduces from 21 to 2 the number of parameters needed to define the elastic constitutive tensor.

Isotropic Linear Elastic Model

Free Energy for an Isotropic Linear Elastic Model

The *free energy per unit of volume* for an *isothermal isotropic linear elastic material model* may be written as,

$$\rho_0 \psi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq 0$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Constitutive Equation

The *isothermal constitutive equation* for an *isotropic linear elastic material model* may be written as,

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$$

$$= \left(\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}} \right) : \boldsymbol{\varepsilon} = \lambda \left(\mathbf{1} : \boldsymbol{\varepsilon} \right) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}$$

$$= \lambda \left(\operatorname{tr} \boldsymbol{\varepsilon} \right) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}$$

$$\boxed{\boldsymbol{\sigma} = \lambda \left(\operatorname{tr} \boldsymbol{\varepsilon} \right) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}, \quad \sigma_{ab} = \lambda \varepsilon_{dd} \delta_{ab} + 2\mu \varepsilon_{ab}}$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Constitutive Equation

The *volumetric part* of the *constitutive equation* for an *isothermal isotropic linear elastic model* may be written as,

$$\operatorname{tr}\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{1} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1} : \mathbf{1} + 2\mu\boldsymbol{\varepsilon} : \mathbf{1} = (3\lambda + 2\mu)\operatorname{tr}\boldsymbol{\varepsilon}$$

Introducing the *mean stress*, *volumetric deformation* and *isothermal bulk modulus*, given by,

$$\sigma_m := \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}, \quad e := \operatorname{tr}\boldsymbol{\varepsilon}, \quad K := \lambda + \frac{2}{3}\mu$$

The *volumetric part* of the *constitutive equation* for an *isothermal isotropic linear elastic model* may be written as,

$$\boxed{\sigma_m = Ke}$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Constitutive Equation

The *deviatoric part* of the *constitutive equation* for an *isotropic linear elastic model* may be written as,

$$\operatorname{dev}\boldsymbol{\sigma} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\cancel{\operatorname{dev}\mathbf{1}} + 2\mu\operatorname{dev}\boldsymbol{\varepsilon} = 2\mu\operatorname{dev}\boldsymbol{\varepsilon}$$

Introducing the *shear modulus* given by,

$$G := \mu$$

The *deviatoric part* of the *constitutive equation* for an *isotropic linear elastic model* may be written as,

$$\boxed{\operatorname{dev}\boldsymbol{\sigma} = 2G\operatorname{dev}\boldsymbol{\varepsilon}}$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Inverse Constitutive Equation

The *inverse constitutive equation* for an *isothermal isotropic linear elastic model* may be written as,

$$\operatorname{tr}\boldsymbol{\varepsilon} = (3\lambda + 2\mu)^{-1}\operatorname{tr}\boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon} = -\frac{\lambda}{(3\lambda + 2\mu)\,2\mu}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Inverse Constitutive Equation

Let us introduce the *isothermal Young elastic modulus*, denoted as $E>0$, *isothermal Poisson's coefficient*, denoted as $0 \le \nu \le 1/2$, and *isothermal bulk modulus*, denoted as $K>0$, such that,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} > 0, \quad 0 \le \nu = \frac{\lambda}{2(\lambda+\mu)} \le \frac{1}{2}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \ge 0, \quad \mu = G = \frac{E}{2(1+\nu)} > 0$$

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} > 0$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Inverse Constitutive Equation

The *inverse constitutive equation* for an *isothermal isotropic linear elastic model* may be written as,

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E}\left(\operatorname{tr}\boldsymbol{\sigma}\right)\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}, \quad \varepsilon_{ab} = -\frac{\nu}{E}\sigma_{dd}\delta_{ab} + \frac{1+\nu}{E}\sigma_{ab}$$

Isotropic Linear Elastic Model

Isotropic Linear Elastic Inverse Constitutive Equation

Using *engineering notation*, the Cartesian components of the *inverse constitutive equation* for an *isothermal isotropic linear elastic model* may be written as,

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right), \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right), \quad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

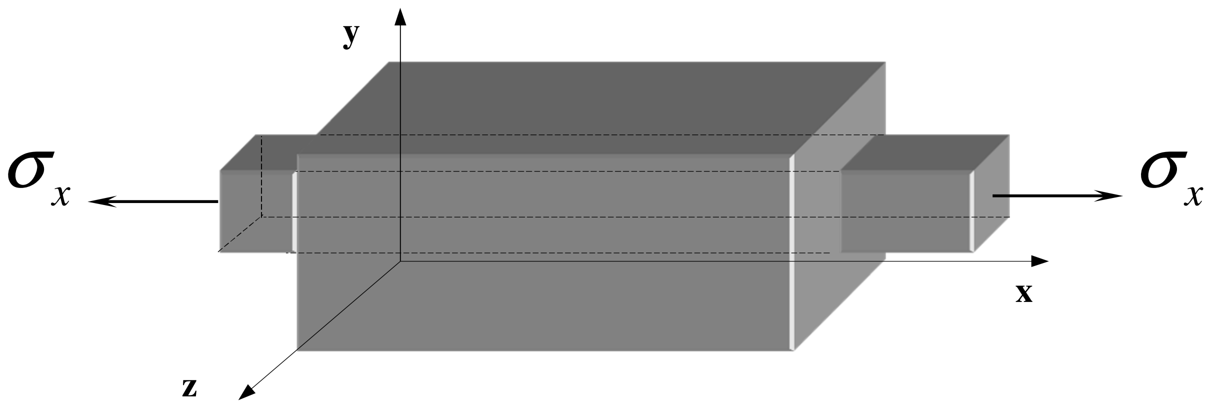
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right), \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

Isotropic Linear Elastic Model

Example 7.1

Let us consider an uniaxial traction test of an *isotropic linear elastic material model* such that,

$$\sigma_x > 0, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$



1) Obtain the components of the strain tensor
2) Consider as particular cases: (a) $\nu = 0$; (b) $\nu = 1/2$

Isotropic Linear Elastic Model

Example 7.1

Given a stress state such that,

$$\sigma_x > 0, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

the components of the strain tensor take the form,

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\cancel{\sigma_y} + \cancel{\sigma_z}\right)\right) = \frac{1}{E}\sigma_x > 0, \quad \gamma_{xy} = \frac{1}{G}\cancel{\tau_{xy}} = 0$$

$$\varepsilon_y = \frac{1}{E}\left(\cancel{\sigma_y} - \nu\left(\sigma_x + \cancel{\sigma_z}\right)\right) = -\frac{\nu}{E}\sigma_x \leq 0, \quad \gamma_{xz} = \frac{1}{G}\cancel{\tau_{xz}} = 0$$

$$\varepsilon_z = \frac{1}{E}\left(\cancel{\sigma_z} - \nu\left(\sigma_x + \cancel{\sigma_y}\right)\right) = -\frac{\nu}{E}\sigma_x \leq 0, \quad \gamma_{yz} = \frac{1}{G}\cancel{\tau_{yz}} = 0$$

Isotropic Linear Elastic Model

(a) Consider the case $\nu = 0$:

$$\varepsilon_x = \frac{1}{E}\sigma_x > 0, \quad \varepsilon_y = -\frac{\nu}{E}\sigma_x = 0, \quad \varepsilon_z = -\frac{\nu}{E}\sigma_x = 0$$

There is *no Poisson's effect*.

(b) Consider the case $\nu = 1/2$:

$$\varepsilon_x = \frac{1}{E}\sigma_x > 0, \; \varepsilon_y = -\frac{\nu}{E}\sigma_x = -\frac{1}{2E}\sigma_x \leq 0, \; \varepsilon_z = -\frac{\nu}{E}\sigma_x = -\frac{1}{2E}\sigma_x \leq 0$$

$$e = \operatorname{tr}\boldsymbol{\varepsilon} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E}\sigma_x - \frac{1}{2E}\sigma_x - \frac{1}{2E}\sigma_x = 0$$

The *volumetric strain* is *zero*, the *volume is preserved*, characte-rizing an *incompressible isotropic linear elastic material model*.

Isotropic Linear Elastic IBVP

Governing Equations

Let us consider the following *governing equations* in the space x time domain $\Omega\times\mathbb{I}=\Omega\times[0,T]$.

- Linear momentum balance · First Cauchy's motion equation

$$\operatorname{div}\boldsymbol{\sigma}+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2},\quad \sigma_{ab,b}+\rho_0 b_a=\rho_0\frac{\partial^2 u_a}{\partial t^2}\quad \text{in}\ \ \Omega\times\mathbb{I}$$

- Isothermal isotropic linear elastic constitutive equation

$$\boldsymbol{\sigma}=\lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon},\quad \sigma_{ab}=\lambda\,\varepsilon_{dd}\delta_{ab}+2\mu\varepsilon_{ab}\quad \text{in}\ \ \Omega\times\mathbb{I}$$

- Geometrical equations

$$\boldsymbol{\varepsilon}=\nabla^s\mathbf{u}=\frac{1}{2}\left(\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^T\right),\quad \varepsilon_{ab}=\frac{1}{2}\left(u_{a,b}+u_{b,a}\right)\quad \text{in}\ \ \Omega\times\mathbb{I}$$

Isotropic Linear Elastic IBVP

Boundary Conditions

Let us consider *prescribed displacements* and *prescribed tractions boundaries*, denoted as $\partial_u\Omega$ and $\partial_\sigma\Omega$, respectively, such that,

$$\partial_u\Omega \cup \partial_\sigma\Omega = \partial\Omega, \quad \partial_u\Omega \cap \partial_\sigma\Omega = \varnothing$$

with the following *bounday conditions* for the IBVP:

- Prescribed displacements boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}}, \quad u_a = \bar{u}_a \quad \text{on} \;\; \partial_u\Omega \times \mathbb{I}$$

- Prescribed tractions boundary conditions

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, \quad t_a = \sigma_{ab} n_b = \bar{t}_a \quad \text{on} \;\; \partial_\sigma\Omega \times \mathbb{I}$$

Isotropic Linear Elastic IBVP

Initial Conditions

Let us consider the following *initial conditions* for the IBVP,

- Initial displacements at time t=0

$$\mathbf{u}\big|_{t=0} = \mathbf{0}, \quad u_a\big|_{t=0} = 0 \quad \text{in} \;\; \Omega$$

- Initial velocities at time t=0

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{v}_0, \quad \left.\frac{\partial u_a}{\partial t}\right|_{t=0} = v_{0_a} \quad \text{in} \;\; \Omega$$

Isotropic Linear Elastic IBVP

Isotropic Linear Elastic IBVP

Find the *displacements, strains* and *stresses* in $\Omega \times \mathbb{I} = \Omega \times [0, T]$ such that the following equations are satisfied:

$$\left.\begin{aligned} &\operatorname{div}\boldsymbol{\sigma} + \rho_0 \mathbf{b} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \\ &\boldsymbol{\sigma} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \\ &\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2}\left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T\right) \end{aligned}\right\} \text{in } \ \Omega \times \mathbb{I}$$

$$\begin{aligned} &\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \ \partial_u \Omega \times \mathbb{I} \\ &\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \ \partial_\sigma \Omega \times \mathbb{I} \end{aligned}$$

$$\left.\begin{aligned} \mathbf{u}\big|_{t=0} &= \mathbf{0} \\ \frac{\partial \mathbf{u}}{\partial t}\bigg|_{t=0} &= \mathbf{v}_0 \end{aligned}\right\} \text{in } \ \Omega$$

Isotropic Linear Elastic IBVP

Unicity of the Solution of the Linear Elastic IBVP

The solution of the linear elastic IBVP is *unique* in *strains* and *stresses.*

The solution of the linear elastic IBVP is *unique* in *displacements* if the boundary conditions are such that *arbitrary rigid motions are not allowed*.

Isotropic Linear Elastic BVP

Quasistatic Linear Elastic BVP

A problem is said to be **quasistatic** if the acceleration term in the first Cauchy's motion equation is negligible.

A *quasistatic* linear elastic BVP *does not* involves any time derivative, hence *no time integration is involved* and *initial conditions are not needed* anymore.

Actions on the continuum body (forces, boundary conditions) may still be a function of *time* and, then, the *response* (displacements, strains, stresses) will be also a function of *time*. Here the time plays the role and may be viewed as a *loading parameter*.

Isotropic Linear Elastic IBVP

Quasistatic Isotropic Linear Elastic BVP

Find the *displacements, strains* and *stresses* in $\Omega \times \mathbb{I} = \Omega \times [0, T]$ such that the following equations are satisfied:

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0} \\ &\boldsymbol{\sigma} = \lambda (\operatorname{tr} \boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} \\ &\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} \end{aligned}\right\} \text{in} \quad \Omega \times \mathbb{I}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u \Omega \times \mathbb{I}$$

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \partial_\sigma \Omega \times \mathbb{I}$$

Solution of the Isotropic Linear Elastic IBVP

Isotropic Linear Elastic IBVP

Find the *displacements, strains* and *stresses* in $\Omega\times\mathbb{I}=\Omega\times[0,T]$ such that the following equations are satisfied:

$$\left.\begin{aligned}
&\operatorname{div}\boldsymbol{\sigma}+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\\
&\boldsymbol{\sigma}=\lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}\\
&\boldsymbol{\varepsilon}=\nabla^s\mathbf{u}=\frac{1}{2}\left(\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^T\right)
\end{aligned}\right\}\text{in }\ \Omega\times\mathbb{I}$$

$$\begin{aligned}
&\mathbf{u}=\bar{\mathbf{u}} \quad \text{on }\ \partial_u\Omega\times\mathbb{I}\\
&\mathbf{t}=\boldsymbol{\sigma}\mathbf{n}=\bar{\mathbf{t}} \quad \text{on }\ \partial_\sigma\Omega\times\mathbb{I}
\end{aligned}$$

$$\left.\begin{aligned}
&\mathbf{u}\big|_{t=0}=\mathbf{0}\\
&\frac{\partial\mathbf{u}}{\partial t}\bigg|_{t=0}=\mathbf{v}_0
\end{aligned}\right\}\text{in }\ \Omega$$

Solution of the Isotropic Linear Elastic IBVP

Method of Displacements · Navier's Equation

Stresses can be removed from the list of unknowns of the isothermal isotropic linear elastic IBVP using,

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} &= \operatorname{div}\left(\lambda\left(\operatorname{tr} \boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\right) \\ &= \lambda \operatorname{grad}\left(\operatorname{tr} \boldsymbol{\varepsilon}\right) + 2\mu \operatorname{div} \boldsymbol{\varepsilon} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\sigma}\mathbf{n} &= \left(\lambda\left(\operatorname{tr} \boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\right)\mathbf{n} \\ &= \lambda\left(\operatorname{tr} \boldsymbol{\varepsilon}\right)\mathbf{n} + 2\mu\boldsymbol{\varepsilon}\mathbf{n} \end{aligned}$$

Solution of the Isotropic Linear Elastic IBVP

Method of Displacements · Navier's Equation

Strains can can be removed from the list of unknowns of the isothermal isotropic linear elastic IBVP using,

$$\begin{aligned}\operatorname{div}\boldsymbol{\sigma} &= \lambda \operatorname{grad}(\operatorname{tr}\boldsymbol{\varepsilon}) + 2\mu \operatorname{div}\boldsymbol{\varepsilon} \\ &= (\lambda+\mu)\operatorname{grad}(\operatorname{div}\mathbf{u}) + \mu \operatorname{div}(\operatorname{grad}\mathbf{u})\end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}\mathbf{n} &= \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{n} + 2\mu\boldsymbol{\varepsilon}\mathbf{n} \\ &= \lambda(\operatorname{div}\mathbf{u})\mathbf{n} + \mu\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^{T}\right)\mathbf{n}\end{aligned}$$

Solution of the Isotropic Linear Elastic IBVP

Method of Displacements · Navier's Equation

The *first Cauchy's motion equation* written in terms of the displacements is denoted as **Navier's equation** and takes the form,

$$(\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u})+\mu\,\mathrm{div}(\mathrm{grad}\,\mathbf{u})+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\quad \text{in}\ \ \Omega\times\mathbb{I}$$

$$(\lambda+\mu)u_{b,ba}+\mu u_{a,bb}+\rho_0 b_a=\rho_0\frac{\partial^2 u_a}{\partial t^2}\quad \text{in}\ \ \Omega\times\mathbb{I}$$

Solution of the Isotropic Linear Elastic IBVP

Isotropic Linear Elastic IBVP

Find the *displacements* in $\Omega\times\mathbb{I}=\Omega\times[0,T]$ such that the following equations are satisfied:

$$(\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u})+\mu\,\mathrm{div}(\mathrm{grad}\,\mathbf{u})+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\quad \text{in}\ \ \Omega\times\mathbb{I}$$

$$\mathbf{u}=\bar{\mathbf{u}}\quad \text{on}\ \ \partial_u\Omega\times\mathbb{I}$$

$$\lambda(\mathrm{div}\,\mathbf{u})\,\mathbf{n}+\mu\left(\mathrm{grad}\,\mathbf{u}+(\mathrm{grad}\,\mathbf{u})^T\right)\mathbf{n}=\bar{\mathbf{t}}\quad \text{on}\ \ \partial_\sigma\Omega\times\mathbb{I}$$

$$\mathbf{u}\big|_{t=0}=\mathbf{0},\quad \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{t=0}=\mathbf{v}_0\quad \text{in}\ \ \Omega$$

Solution of the Isotropic Linear Elastic BVP

Quasistatic Isotropic Linear Elastic BVP

Find the *displacements* in $\Omega \times \mathbb{I} = \Omega \times [0, T]$ such that the following equations are satisfied:

$$(\lambda + \mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \mu\,\mathrm{div}(\mathrm{grad}\,\mathbf{u}) + \rho_0 \mathbf{b} = \mathbf{0} \quad \text{in} \quad \Omega \times \mathbb{I}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u \Omega \times \mathbb{I}$$

$$\lambda(\mathrm{div}\,\mathbf{u})\,\mathbf{n} + \mu\left(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^T\right)\mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \partial_\sigma \Omega \times \mathbb{I}$$

Solution of the Isotropic Linear Elastic IBVP

Isotropic Linear Elastic IBVP

While the *displacements* have been obtained, *strains* and *stresses* may be obtained as a *post-process* of the results,

- Geometrical equations

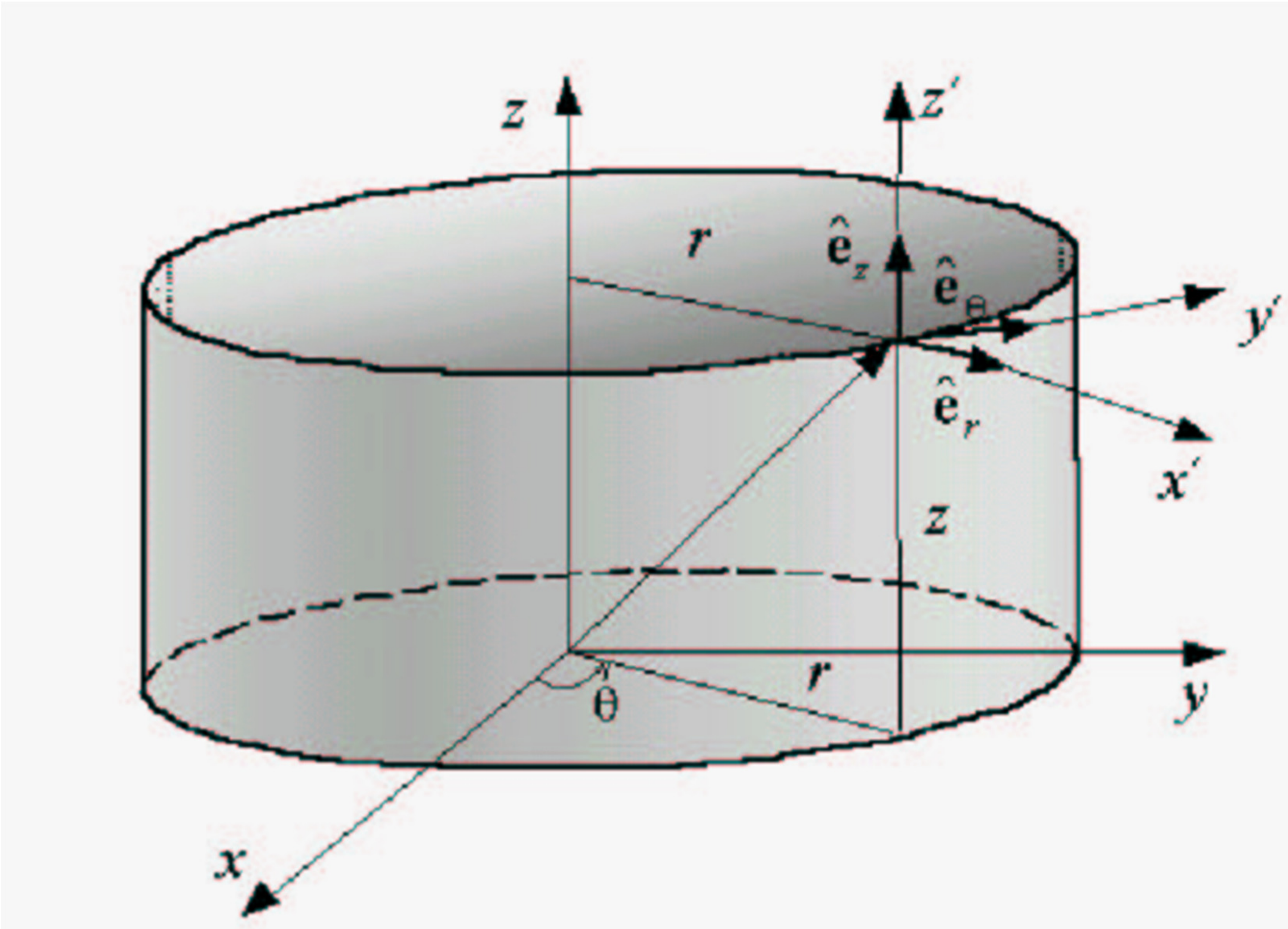
$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2}\left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T\right), \quad \varepsilon_{ab} = \frac{1}{2}\left(u_{a,b} + u_{b,a}\right) \quad \text{in} \ \ \Omega \times \mathbb{I}$$

- Isothermal isotropic linear elastic constitutive equations

$$\boldsymbol{\sigma} = \lambda \left(\mathrm{tr}\, \boldsymbol{\varepsilon}\right) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}, \quad \sigma_{ab} = \lambda \, \varepsilon_{dd} \delta_{ab} + 2\mu \varepsilon_{ab} \quad \text{in} \ \ \Omega \times \mathbb{I}$$

Cylindrical Coordinates

Cylindrical Coordinates



Cylindrical Coordinates

Cylindrical Coordinates

- Vector position

$$\mathbf{x} = \mathbf{x}(r, \theta, z) = r\cos\theta\,\hat{\mathbf{e}}_x + r\sin\theta\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$$

- Physical basis

$$\left.\begin{aligned}
\frac{\partial \mathbf{x}}{\partial r} &= \cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y \\
\frac{\partial \mathbf{x}}{\partial \theta} &= -r\sin\theta\,\hat{\mathbf{e}}_x + r\cos\theta\,\hat{\mathbf{e}}_y \\
\frac{\partial \mathbf{x}}{\partial z} &= \hat{\mathbf{e}}_z
\end{aligned}\right\}$$

Cylindrical Coordinates

Cylindrical Coordinates

- Euclidean norms of the physical basis vectors

$$\left\| \frac{\partial \mathbf{x}}{\partial r} \right\| = 1, \quad \left\| \frac{\partial \mathbf{x}}{\partial \theta} \right\| = r, \quad \left\| \frac{\partial \mathbf{x}}{\partial z} \right\| = 1,$$

- Local orthonormal basis vectors

$$\left.\begin{aligned} \hat{\mathbf{e}}_r(\theta) &= \cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\theta(\theta) &= -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z \end{aligned}\right\}$$

Cylindrical Coordinates

Navier's Equations

$$\left.\begin{aligned}
&(\lambda+2G)\frac{\partial e}{\partial r}-\frac{2G}{r}\frac{\partial \omega_z}{\partial \theta}+2G\frac{\partial \omega_\theta}{\partial z}+\rho_0 b_r=\rho_0\frac{\partial^2 u_r}{\partial t^2}\\
&(\lambda+2G)\frac{1}{r}\frac{\partial e}{\partial \theta}-2G\frac{\partial \omega_r}{\partial z}+2G\frac{\partial \omega_z}{\partial r}+\rho_0 b_\theta=\rho_0\frac{\partial^2 u_\theta}{\partial t^2}\\
&(\lambda+2G)\frac{\partial e}{\partial z}-\frac{2G}{r}\frac{\partial(\omega_\theta r)}{\partial r}+\frac{2G}{r}\frac{\partial \omega_r}{\partial \theta}+\rho_0 b_z=\rho_0\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\}$$

Cylindrical Coordinates

Navier's Equations

$$e = \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial\left(r u_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\left.\begin{aligned}
\omega_r &= -\Omega_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) \\
\omega_\theta &= -\Omega_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \\
\omega_z &= -\Omega_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)
\end{aligned}\right\}$$

Cylindrical Coordinates

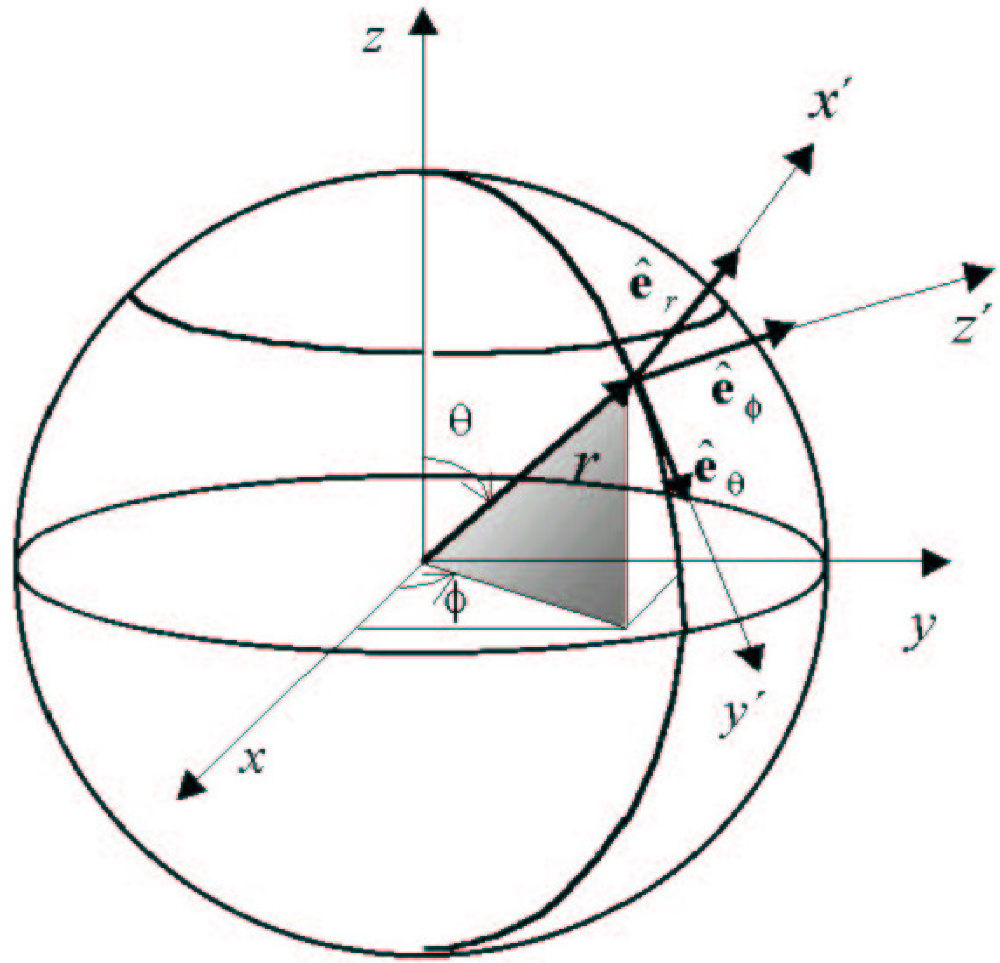
Components of the Strain Tensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

Spherical Coordinates

Spherical Coordinates



Spherical Coordinates

Spherical Coordinates

- Vector position

$$\mathbf{x} = \mathbf{x}(r,\theta,\phi) = r\sin\theta\cos\phi\,\hat{\mathbf{e}}_x + r\sin\theta\sin\phi\,\hat{\mathbf{e}}_y + r\cos\theta\,\hat{\mathbf{e}}_z$$

- Physical basis

$$\left.\begin{aligned}
\frac{\partial \mathbf{x}}{\partial r} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y \\
\frac{\partial \mathbf{x}}{\partial \theta} &= r\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + r\cos\theta\sin\phi\,\hat{\mathbf{e}}_y - r\sin\theta\,\hat{\mathbf{e}}_z \\
\frac{\partial \mathbf{x}}{\partial \phi} &= -r\sin\theta\sin\phi\,\hat{\mathbf{e}}_x + r\sin\theta\cos\phi\,\hat{\mathbf{e}}_y
\end{aligned}\right\}$$

Spherical Coordinates

Spherical Coordinates

- Euclidean norms of the physical basis vectors

$$\left\|\frac{\partial \mathbf{x}}{\partial r}\right\| = \sin\theta, \quad \left\|\frac{\partial \mathbf{x}}{\partial r}\right\| = r, \quad \left\|\frac{\partial \mathbf{x}}{\partial z}\right\| = r\sin\theta,$$

- Local orthonormal basis vectors

$$\left.\begin{aligned} \hat{\mathbf{e}}_r(\phi) &= \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\theta(\theta,\phi) &= \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\phi(\phi) &= -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y \end{aligned}\right\}$$

Spherical Coordinates

Navier's Equations

$$\left.\begin{aligned}
(\lambda+2G)\frac{\partial e}{\partial r}-\frac{2G}{r\sin\theta}\frac{\partial\left(\omega_{\phi}\sin\theta\right)}{\partial\theta}+\frac{2G}{r\sin\theta}\frac{\partial\omega_{\theta}}{\partial\phi}+\rho_0 b_r&=\rho_0\frac{\partial^2 u_r}{\partial t^2}\\
(\lambda+2G)\frac{1}{r}\frac{\partial e}{\partial\theta}-\frac{2G}{r\sin\theta}\frac{\partial\omega_r}{\partial\phi}+\frac{2G}{r\sin\theta}\frac{\partial\left(r\omega_{\phi}\sin\theta\right)}{\partial r}+\rho_0 b_{\theta}&=\rho_0\frac{\partial^2 u_{\theta}}{\partial t^2}\\
(\lambda+2G)\frac{1}{r\sin\theta}\frac{\partial e}{\partial\phi}-\frac{2G}{r}\frac{\partial\left(\omega_{\theta}r\right)}{\partial r}+\frac{2G}{r}\frac{\partial\omega_r}{\partial\theta}+\rho_0 b_{\phi}&=\rho_0\frac{\partial^2 u_{\phi}}{\partial t^2}
\end{aligned}\right\}$$

Spherical Coordinates

Navier's Equations

$$e = \operatorname{div} \mathbf{u} = \frac{1}{r^2 \sin\theta} \left(\frac{\partial \left(r^2 u_r \sin\theta \right)}{\partial r} + \frac{\partial \left(r u_\theta \sin\theta \right)}{\partial \theta} + \frac{\partial \left(r u_\phi \right)}{\partial \phi} \right)$$

$$\left.\begin{aligned}
\omega_r &= -\Omega_{\theta\phi} = \frac{1}{2} \left(\frac{1}{r \sin\theta} \frac{\partial \left(u_\phi \sin\theta \right)}{\partial \theta} - \frac{1}{r \sin\theta} \frac{\partial u_\theta}{\partial \phi} \right) \\
\omega_\theta &= -\Omega_{\phi r} = \frac{1}{2} \left(\frac{1}{r \sin\theta} \frac{\partial u_r}{\partial \phi} - \frac{1}{r} \frac{\partial \left(r u_\phi \right)}{\partial r} \right) \\
\omega_\phi &= -\Omega_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial \left(r u_\theta \right)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right)
\end{aligned}\right\}$$

Spherical Coordinates

Components of the Strain Tensor

$$[\varepsilon]=\begin{bmatrix}\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi}\\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi}\\ \varepsilon_{r\phi} & \varepsilon_{\theta\phi} & \varepsilon_{\phi\phi}\end{bmatrix}$$

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}$$

$$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$$

$$\varepsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$$

$$\varepsilon_{r\phi}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{u}_r}{\partial\phi}+\frac{\partial \mathrm{u}_\phi}{\partial r}-\frac{\mathrm{u}_\phi}{r}\right)$$

$$\varepsilon_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial \mathrm{u}_\phi}{\partial\phi}+\frac{\mathrm{u}_\theta}{r}\cot\theta+\frac{\mathrm{u}_r}{r}$$

$$\varepsilon_{\theta\phi}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{u}_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial \mathrm{u}_\phi}{\partial\theta}-\frac{\mathrm{u}_\phi}{r}\cot\phi\right)$$

Assignments

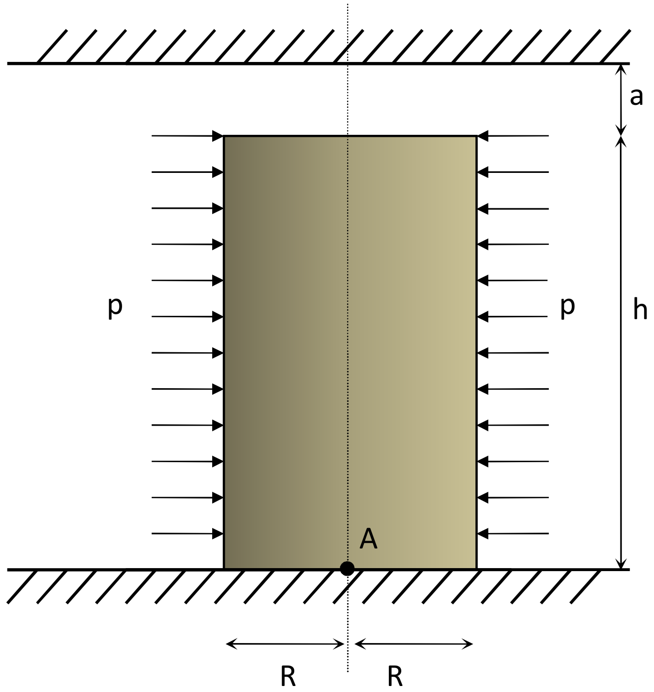
Assignment 7.1

A uniform compression pressure, denoted as p, is applied to the lateral surface of the cylinder of radius R of the figure. The material of the cylinder is assumed to be isotropic linear elastic with Lamé constants $\lambda = \mu$. There is an horizontal rigid surface on the top of the cylinder at a distance a<<R.

Body forces and frictional effects are neglected. Quasistatic conditions are assumed.

(1) Plot p vs δ, where δ is the radial displacement of the lateral surface of the cylinder.

(2) Plot p vs σ_z^A at point A.



Assignment 7.1

Assignment 7.1

Boundary conditions will depend on the value of the *applied pressure*. Let us consider the following problems,

- **Problem 1**: The cylinder is *not in contact* with the top horizontal surface. BC on the top surface of the cylinder are *zero tractions*. The range of values of the pressure is $0 \leq p \leq p^*$ where p^* is the pressure needed for the cylinder just to make contact (without contact pressure) with the top horizontal surface.
- **Problem 2**: The cylinder is *in contact* with the top horizontal surface. BC on the top surface of the cylinder are *zero incremental vertical displacements* (measure from the displacements at the end of Problem 1) and $p \geq p^*$.

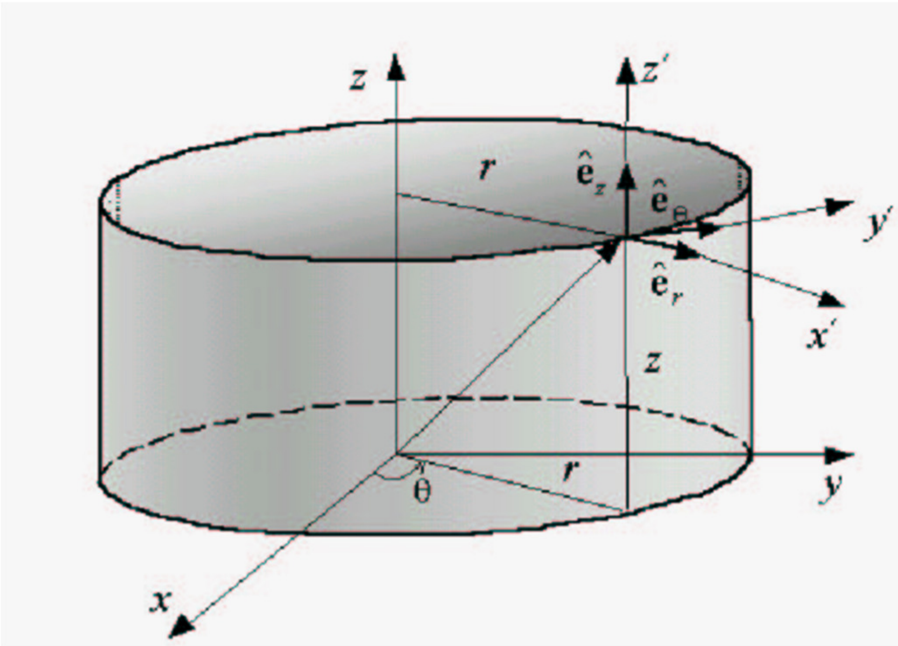
Assignment 7.1

Problem 1

We consider the following steps:

- *Step 1*. System of coordinates.

Taking into accoun the geometry of the problem, we select a *cylindrical system of coordinates* to solve the problem.



Assignment 7.1

- *Step 2*. Hypothesis on the displacements

Using cylindrical coordinates the displacement field takes the form,

$$[\mathbf{u}] = \left[u_r(r,\theta,z) \quad u_\theta(r,\theta,z) \quad u_z(r,\theta,z) \right]^T$$

Taking into account the *axial symmetry* of the geometry, loading and BC we consider a displacement field of the form,

$$[\mathbf{u}] = \left[u_r(r,z) \quad 0 \quad u_z(r,z) \right]^T$$

Furthermore, taking into account that the *pressure is uniform*, we introduce the additional hypothesis, yielding a displacement field of the form,

$$[\mathbf{u}] = \left[u_r(r) \quad 0 \quad u_z(z) \right]^T$$

Assignment 7.1

- *Step 3*. Navier's equations in cylindrical coordinates.

Taking into account that body forces are negligible,

$$(\lambda+2G)\frac{\partial e}{\partial r}-\frac{2G}{r}\frac{\partial \omega_z}{\partial \theta}+2G\frac{\partial \omega_\theta}{\partial z}+\cancel{\rho_0 b_r}=0$$

$$(\lambda+2G)\frac{1}{r}\frac{\partial e}{\partial \theta}-2G\frac{\partial \omega_r}{\partial z}+2G\frac{\partial \omega_z}{\partial r}+\cancel{\rho_0 b_\theta}=0$$

$$(\lambda+2G)\frac{\partial e}{\partial z}-\frac{2G}{r}\frac{\partial(\omega_\theta r)}{\partial r}+\frac{2G}{r}\frac{\partial \omega_r}{\partial \theta}+\cancel{\rho_0 b_z}=0$$

Assignment 7.1

- *Step 3*. Navier's equations in cylindrical coordinates.

Taking into account the hypotheses introduced on the displacement field,

$$e = \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z}$$

$$\omega_r = -\Omega_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\cancel{\frac{\partial u_z}{\partial \theta}} - \cancel{\frac{\partial u_\theta}{\partial z}}\right) = 0$$

$$\omega_\theta = -\Omega_{zr} = \frac{1}{2}\left(\cancel{\frac{\partial u_r}{\partial z}} - \cancel{\frac{\partial u_z}{\partial r}}\right) = 0$$

$$\omega_z = -\Omega_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\cancel{\frac{\partial(ru_\theta)}{\partial r}} - \frac{1}{r}\cancel{\frac{\partial u_r}{\partial \theta}}\right) = 0$$

Assignment 7.1

- *Step 3*. Navier's equations in cylindrical coordinates.

Then, the (non-trivial) Navier's equations take the form,

$$\frac{\partial e}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(ru_r(r)\right)}{\partial r} + \frac{\partial u_z(z)}{\partial z}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(ru_r(r)\right)}{\partial r}\right) = 0$$

$$\frac{\partial e}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\left(ru_r(r)\right)}{\partial r} + \frac{\partial u_z(z)}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial u_z(z)}{\partial z}\right) = 0$$

Integrating the Navier's equations yields,

$$u_r(r) = A_1 r + A_2\frac{1}{r}, \quad u_z(z) = A_3 z + A_4$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The BC on displacements for Problem 1 read,

$$u_r(r)\Big|_{r=0} = A_1 r + A_2 \frac{1}{r}\Big|_{r=0} = 0 \quad \Rightarrow \quad A_2 = 0$$

$$u_z(z)\Big|_{z=0} = A_3 z + A_4\Big|_{z=0} = 0 \quad \Rightarrow \quad A_4 = 0$$

Substituting into the equations of the components of the displacement yields,

$$u_r(r) = A_1 r, \quad u_z(z) = A_3 z$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The components of the strain tensor take the form,

$$[\varepsilon] = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

Taking into account that $\lambda = \mu$, the components of the stress tensor take the form,

$$[\sigma] = \lambda \begin{bmatrix} 4A_1 + A_3 & 0 & 0 \\ 0 & 4A_1 + A_3 & 0 \\ 0 & 0 & 2A_1 + 3A_3 \end{bmatrix}$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The BC on the traction vector for Problem 1, conveniently writting the non-trivial component in terms of the components of the stress tensor yields,

$$\sigma_r\big|_{r=R} = \lambda\left(4A_1 + A_3\right) = -p$$

$$\sigma_z\big|_{z=h} = \lambda\left(2A_1 + 3A_3\right) = 0$$

Solving the system of two equations, yields,

$$A_1 = -\frac{3p}{10\lambda}, \quad A_3 = \frac{p}{5\lambda}$$

Assignment 7.1

- *Step 5*. Solution of Problem 1

The solution in displacements, strains and stresses for Problem 1 takes the form,

$$[\mathbf{u}] = \frac{p}{10\lambda}\begin{bmatrix} -3r & 0 & 2z \end{bmatrix}^T$$

$$[\boldsymbol{\varepsilon}] = \frac{p}{10\lambda}\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assignment 7.1

- *Step 5*. Solution of Problem 1

The value of the pressure for which the cylinder makes contact with the top surface is given by,

$$u_z(z)\Big|_{z=h} = \frac{p^*}{5\lambda} h = a \quad \Rightarrow \quad p^* = \frac{5\lambda a}{h}$$

Then, the range of values of the pressure for the solution of the Problem 1 is given by,

$$0 \le p \le p^* = \frac{5\lambda a}{h}$$

Assignment 7.1

- *Step 6*. Curve p-δ.

The curve p-δ for Problem 1 is given by,

$$\delta := u_r(r)\big|_{r=R} = -\frac{3p}{10\lambda}R \quad \Rightarrow \quad p = -\frac{10\lambda}{3R}\delta$$

The stress σ_z^A for Problem 1 is zero for any value of p (within the range of values of p defining Problem 1).

Assignment 7.1

Problem 2

We have to use an *incremental formulation* and we may use the results obtained in *Steps 1-3 from Problem 1*, yielding an increment of displacements,

$$\Delta u_r\left(r\right)=B_1 r+B_2\frac{1}{r}, \quad \Delta u_z\left(z\right)=B_3 z+B_4$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The BC on displacements for Problem 2 (imposed on the reference or undeformed configuration) read,

$$\Delta u_r(r)\Big|_{r=0} = B_1 r + B_2 \frac{1}{r}\Big|_{r=0} = 0 \quad \Rightarrow \quad B_2 = 0$$

$$\Delta u_z(z)\Big|_{z=0} = B_3 z + B_4\Big|_{z=0} = 0 \quad \Rightarrow \quad B_4 = 0$$

$$\Delta u_z(z)\Big|_{z=h} = B_3 z\Big|_{z=h} = 0 \quad \Rightarrow \quad B_3 = 0$$

Substituting into the equations of the components of the displacement yields,

$$\Delta u_r(r) = B_1 r, \quad \Delta u_z(z) = 0$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The components of the incremental strain tensor take the form,

$$[\Delta\varepsilon] = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking into account that $\lambda = \mu$, the components of the incremental stress tensor take the form,

$$[\Delta\sigma] = \lambda \begin{bmatrix} 4B_1 & 0 & 0 \\ 0 & 4B_1 & 0 \\ 0 & 0 & 2B_1 \end{bmatrix}$$

Assignment 7.1

- *Step 4*. Boundary conditions.

The BC on the incremental traction vector for Problem 2, conveniently writting the non-trivial component in terms of the components of the incremental stress tensor yields,

$$\Delta\sigma_r\big|_{r=R} = \lambda\left(4B_1\right) = -\Delta p \quad \Rightarrow \quad B_1 = -\frac{\Delta p}{4\lambda}$$

Assignment 7.1

- *Step 5*. Solution of Problem 2

The incremental solution in displacements, strains and stresses for Problem 2 takes the form,

$$[\Delta \mathbf{u}] = -\frac{\Delta p}{4\lambda}\begin{bmatrix} r & 0 & 0 \end{bmatrix}^T$$

$$[\Delta \boldsymbol{\varepsilon}] = -\frac{\Delta p}{4\lambda}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\Delta \boldsymbol{\sigma}] = -\frac{\Delta p}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assignment 7.1

- *Step 6*. Curves Δp-Δδ and $\Delta\sigma_z^A$ - Δδ

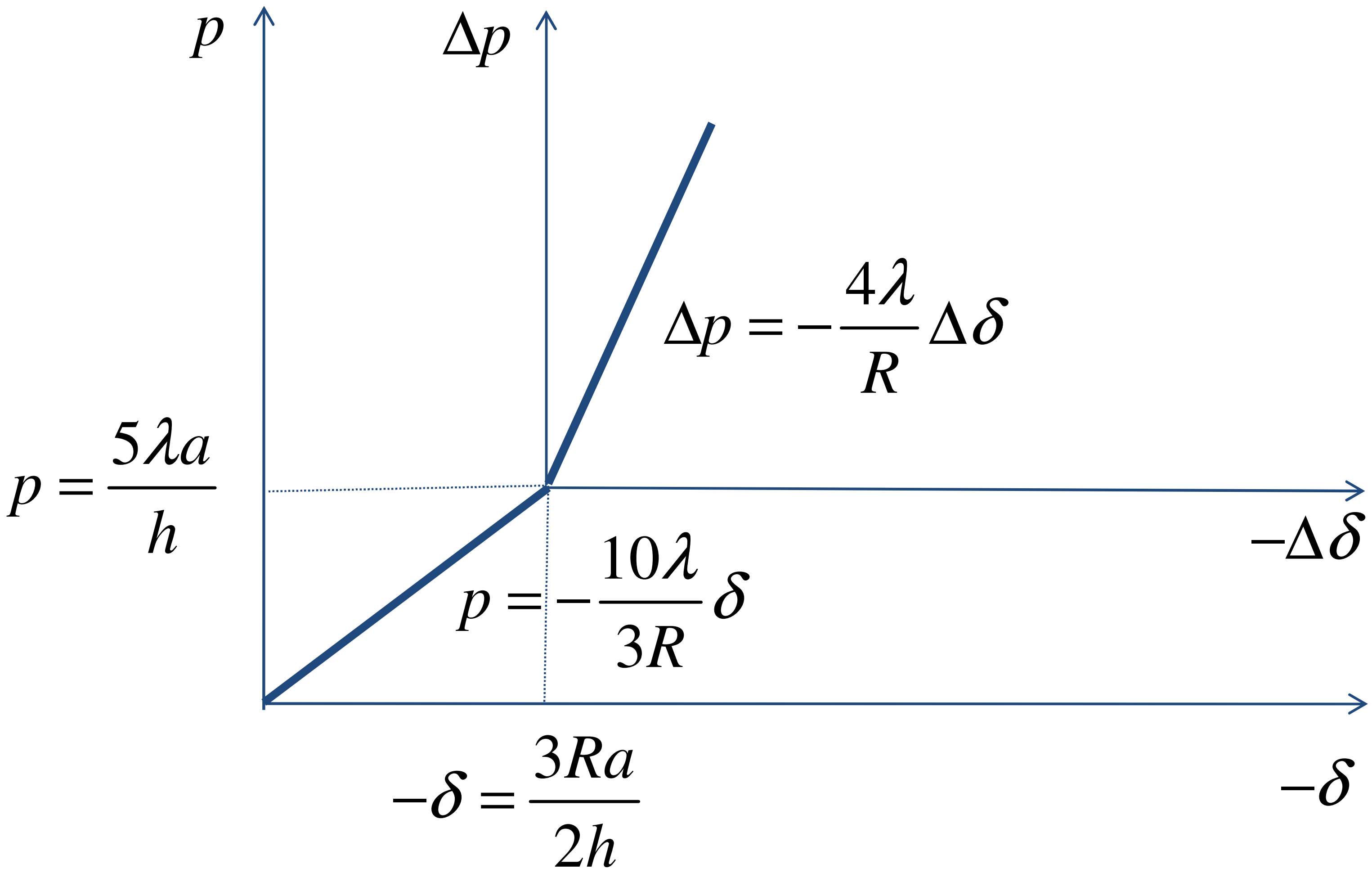
The curve Δp-Δδ for Problem 2 is given by,

$$\Delta\delta := \Delta u_r(r)\big|_{r=R} = -\frac{\Delta p}{4\lambda} R \quad \Rightarrow \quad \Delta p = -\frac{4\lambda}{R}\Delta\delta$$

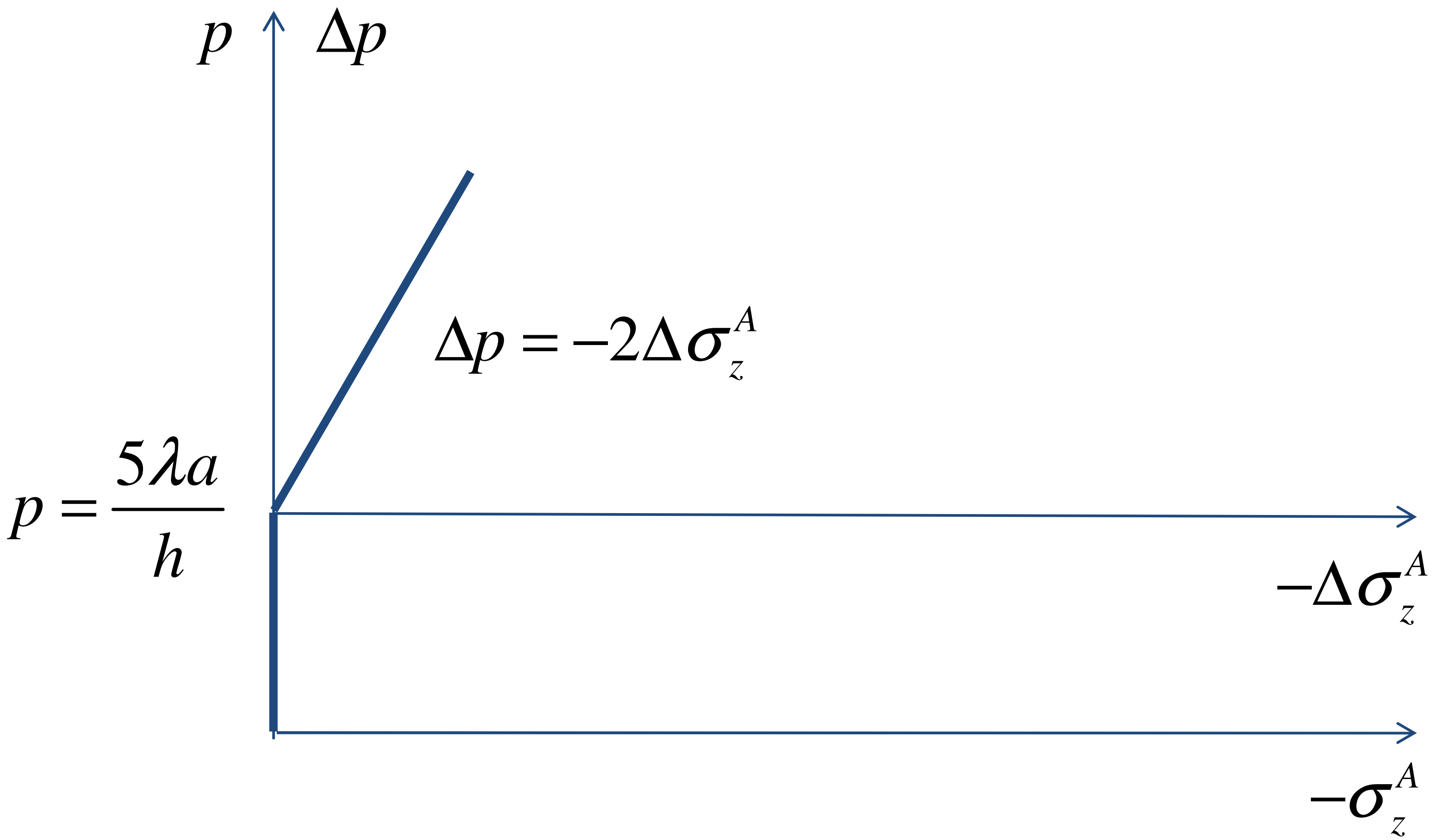
The curve $\Delta\sigma_z^A$ - Δδ for Problem 2 is given by,

$$\Delta\sigma_z^A := -\frac{\Delta p}{2} \quad \Rightarrow \quad \Delta p = -2\Delta\sigma_z^A$$

Assignment 7.1



Assignment 7.1

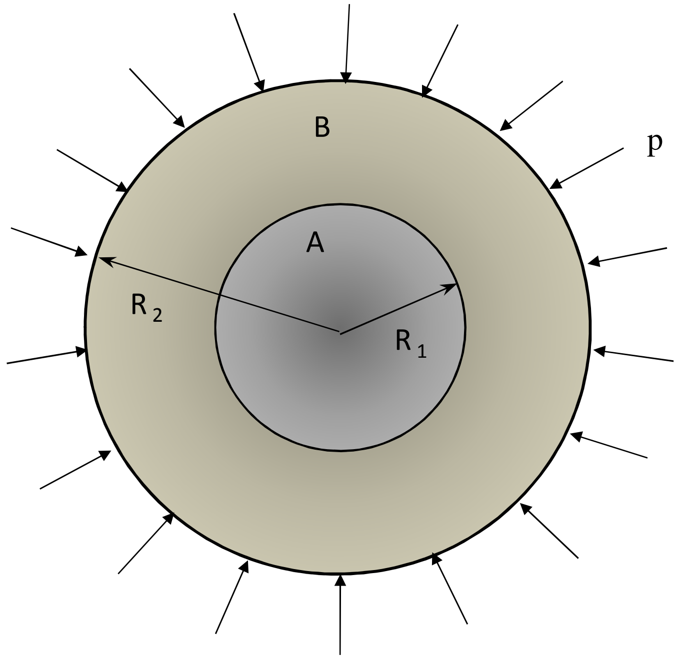


Assignment 7.2

Assignment 7.2 [Classwork]

Consider a sphere A with radius $R_1=R$ and a spherical crown B with external radius $R_2=2R$. The two spheres are of the same isotropic linear elastic material (Young modulus E and Poisson's coefficient =0). There is a small gap $a<<R$ between them. A uniform pressure p is applied on the external surface of the spherical crown B.

Body forces and frictional effects between the two spheres can be neglected. Quasi-static conditions can be assumed.



Assignment 7.2

Assignment 7.2 [Classwork]

1) Obtain the value of the pressure *p* for which the two bodies come into contact.
2) Plot the curve pressure-delta, where delta is defined as minus the radial displacement of the external surface of the spherical crown B.

Assignment 7.2

Step 1. Taking into account the geometry and BCs of the problem we will use a spherical coordinates system.

Step 2. Taking into account the spherical symmetry of the problem (geometry and BCs) we introduce the following hypothesis on the displacements (for any of the two spheres),

$$[\mathbf{u}] = \begin{bmatrix} u_r(r) & 0 & 0 \end{bmatrix}^T$$

Step 3. Using the assumed displacement field and taking into account that body forces are negligible, the (non-trivial) Navier equation for a quasistatic problem takes the form,

$$\frac{\partial e}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\left(\frac{\partial\left(r^2 u_r\right)}{\partial r}\right)\right) = 0$$

Assignment 7.2

Integrating the Navier equation yields,

$$u_r(r) = Ar + \frac{B}{r^2}$$

The non-zero spherical components of the stress tensor take the form,

$$\sigma_r(r) = E\left(A - \frac{2B}{r^3}\right), \quad \sigma_\theta(r) = \sigma_\phi(r) = E\left(A + \frac{B}{r^3}\right)$$

Assignment 7.2

Step 4. While the two bodies are not in contact, the displacements, strains and stresses in the body 1 are zero and the internal pressure on the body 2 is equal to zero. BCs for body 2, while the two bodies are not in contact, take the form,

$$\sigma_r^{(2)}(2R) = E\left(A - \frac{B}{4R^3}\right) = -p, \quad \sigma_r^{(2)}(R) = E\left(A - \frac{2B}{R^3}\right) = 0$$

Solving the system of equations yields,

$$A = -\frac{8}{7E}p, \quad B = -\frac{4}{7E}pR^3$$

Assignment 7.2

The radial displacement and normal radial stress for body 2 take the form,

$$u_r^{(2)}(r) = -\frac{8}{7E} p \left(r + \frac{R^3}{2r^2} \right), \quad \sigma_r^{(2)}(r) = -\frac{8}{7} p \left(1 - \frac{R^3}{r^3} \right)$$

The pressure needed for the two bodies come into contact is given by,

$$u_r^{(2)}(R) = -\frac{8}{7E} p \left(R + \frac{R}{2} \right) = -a \quad \Rightarrow \quad p = \frac{7Ea}{12R}$$

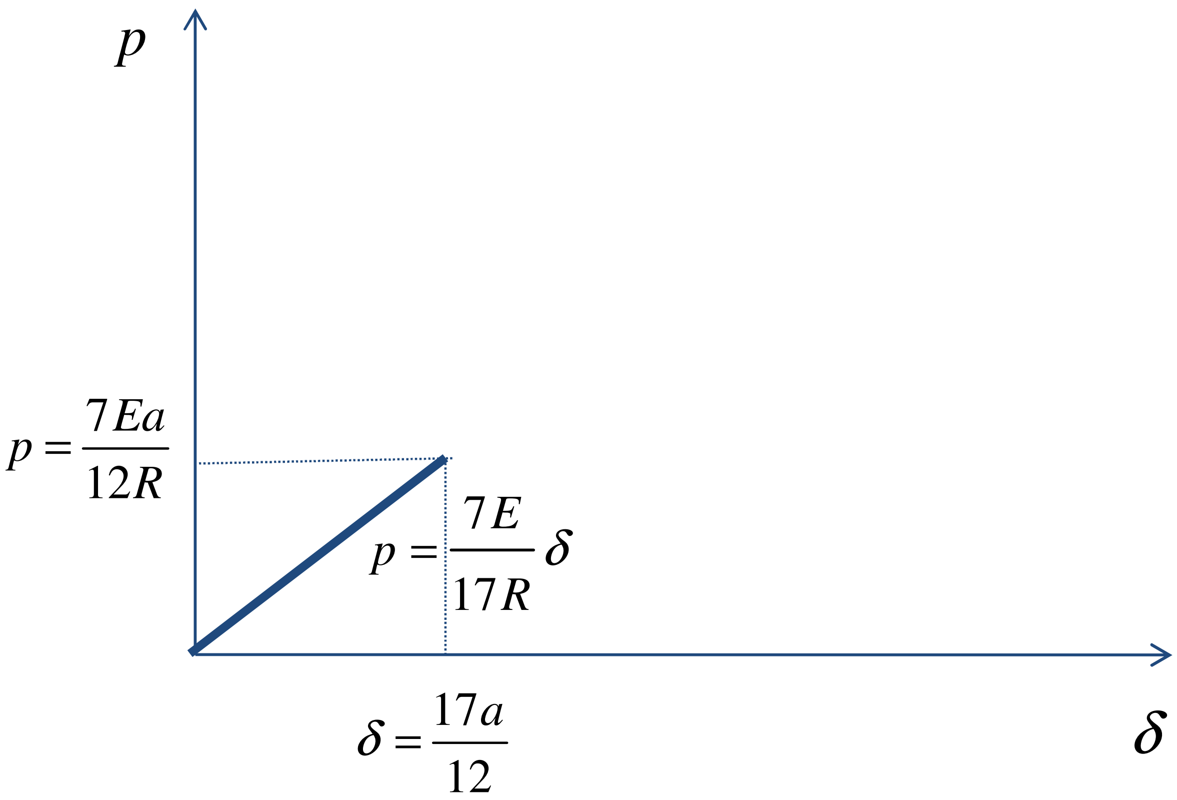
The solution obtained is valid for a range of values of the pressure given by,

$$0 \leq p \leq \frac{7Ea}{12R}$$

Assignment 7.2

The curve p-delta for this phase of the loading will be given by,

$$\delta := -u_r^{(2)}(2R) = \frac{8}{7E}p\left(2R + \frac{R}{8}\right) = \frac{17}{7E}pR \quad \Rightarrow \quad p = \frac{7E}{17R}\delta$$



Assignment 7.2

For higher values of the pressure, the two bodies will be in contact and therefore the boundary conditions will be different.

The incremental radial displacement and incremental normal radial stress fields for body 1 will be given by,

$$\Delta u_r^{(1)}(r) = A_1 r + \frac{B_1}{r^2}, \quad \Delta\sigma_r^{(1)}(r) = A_1 - \frac{2B_1}{r^3}$$

The incremental radial displacement and incremental normal radial stress fields for body 2 will be given by,

$$\Delta u_r^{(2)}(r) = A_2 r + \frac{B_2}{r^2}, \quad \Delta\sigma_r^{(2)}(r) = A_2 - \frac{2B_2}{r^3}$$

Assignment 7.2

Denoting as p* the uniform contact pressure between the two bodies, the boundary conditions for bodies 1 and take the form,

$$\Delta u_r^{(1)}\left(r=0\right)=A_1 r+\left.\frac{B_1}{r^2}\right|_{r=0}=0,\quad \Delta\sigma_r^{(1)}\left(R\right)=E\left(A_1-\frac{2B_1}{R^3}\right)=-p^*$$

$$\Delta\sigma_r^{(2)}\left(R\right)=E\left(A_2-\frac{2B_2}{R^3}\right)=-p^*,\quad \Delta\sigma_r^{(2)}\left(2R\right)=E\left(A_2-\frac{2B_2}{8R^3}\right)=-\Delta p$$

Solving the systems of equations, the constants are given by,

$$A_1=-\frac{p^*}{E},\quad B_1=0$$

$$A_2=\frac{1}{7E}\left(p^*-8\Delta p\right),\quad B_2=\frac{4R^3}{7E}\left(p^*-\Delta p\right)$$

Assignment 7.2

The contact pressure p* can be determined by imposing that the incremental radial displacement of the points of each one of the bodies at the contact surface has to be the same, yielding,

$$\Delta u_r^{(1)}(R) = \Delta u_r^{(2)}(R)$$

$$-\frac{1}{E}p^* = \frac{1}{7E}\left(p^* - 8\Delta p\right) + \frac{4}{7E}\left(p^* - \Delta p\right)$$

$$p^* = \Delta p$$

Substituting into the expressions of the constants, yields

$$A_1 = -\frac{\Delta p}{E}, \quad B_1 = 0$$

$$A_2 = -\frac{\Delta p}{E}, \quad B_2 = 0$$

Assignment 7.2

The incremental displacement and incremental stress fields for bodies 1 and 2 take the form,

$$\Delta u_r^{(1)}\left(r\right) = -\frac{\Delta p}{E} r, \quad \Delta \sigma_r^{(1)}\left(r\right) = -\Delta p$$

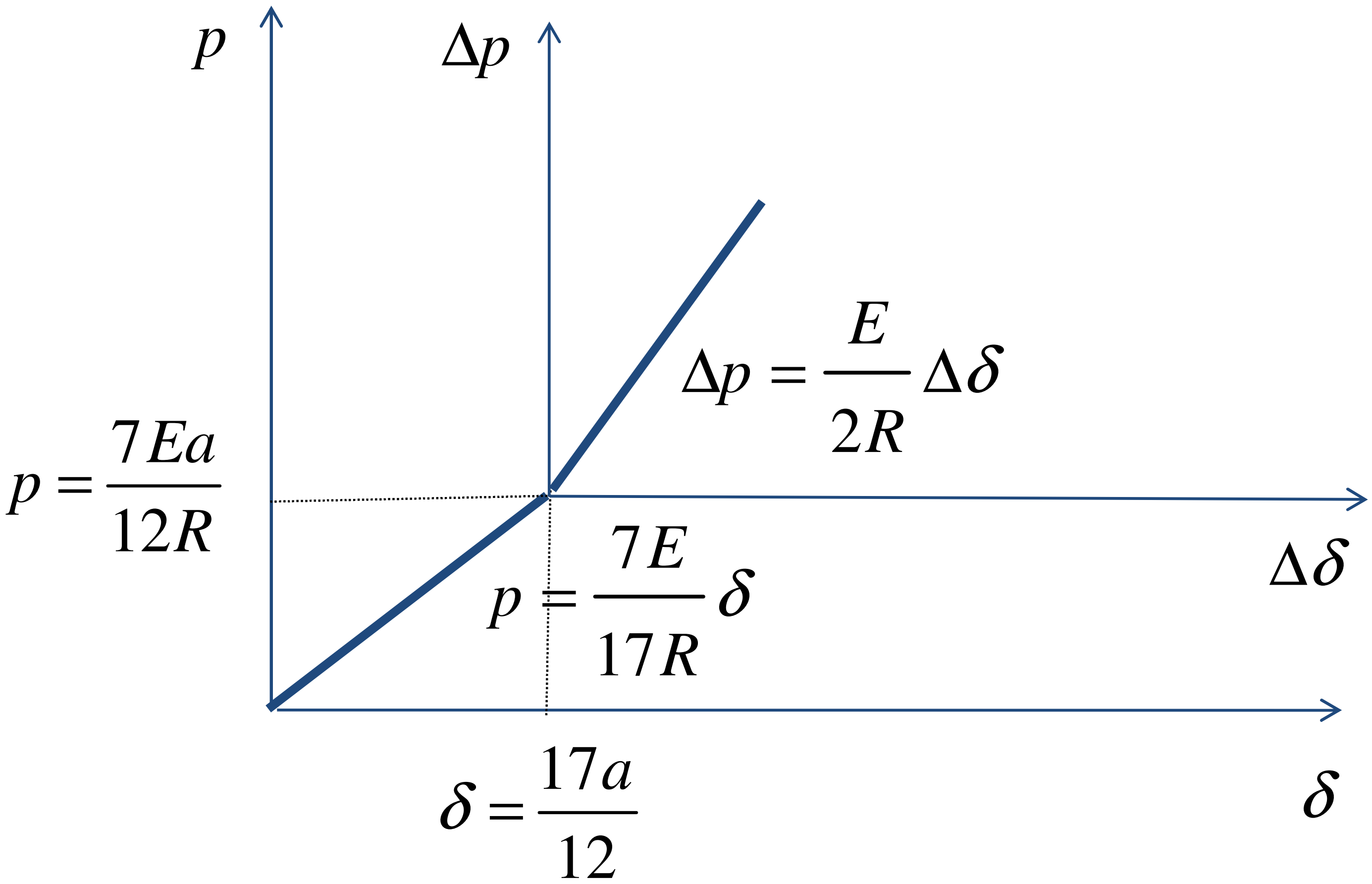
$$\Delta u_r^{(2)}\left(r\right) = -\frac{\Delta p}{E} r, \quad \Delta \sigma_r^{(2)}\left(r\right) = -\Delta p$$

Note that the two bodies have the *same incremental displacement* and *incremental stress fields*. This is because the two bodies have the *same material properties* and, since they are in contact, the two bodies deform as if they were a single sphere of radius 2R. In fact Problem 2 could have been solved considering the two bodies as if they were a single one.

Assignment 7.2

The incremental p-incremental delta curve is given by,

$$\Delta\delta := -\Delta u_r^{(2)}(2R) = \frac{2R}{E}\Delta p \quad \Rightarrow \quad \Delta p = \frac{E}{2R}\Delta\delta$$



Linear Thermoelastic Model (I)

Free Energy

We consider a *free energy per unit of volume* given by a quadratic function of the *strain tensor* and the *temperature*,

$$\rho_0\psi = \rho_0\psi(\varepsilon,\theta) = W(\varepsilon) + M(\varepsilon,\theta) + T(\theta)$$

where $W(\varepsilon)$ is the *isothermal elastic stored energy*, $M(\varepsilon,\theta)$ is the *coupled thermoelastic stored energy* and $T(\theta)$ is the *thermal stored energy*.

Linear Thermoelastic Model (I)

Elastic Stored Energy

The *isothermal elastic stored energy* for a *linear thermoelastic model* may be written as,

$$W(\boldsymbol{\varepsilon}) := \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \geq 0$$

where $\mathbb{C}$ is a *positive definite fourth-order tensor,* denoted as *isothermal constant elastic constitutive tensor*.

Symmetry of the strain tensor yields the following *symmetry* conditions on the *isothermal elastic constitutive tensor* $\mathbb{C}$,

$$\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T : \mathbb{C} : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}^T, \quad \mathbb{C}_{abcd} = \mathbb{C}_{bacd} = \mathbb{C}_{abdc}$$

Furthermore, the following symmetry conditions hold,

$$\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \varepsilon_{ab}\mathbb{C}_{abcd}\varepsilon_{cd} = \varepsilon_{ab}\mathbb{C}_{cdab}\varepsilon_{cd}, \quad \mathbb{C}_{abcd} = \mathbb{C}_{cdab}$$

Linear Thermoelastic Model (I)

Coupled Thermoelastic Stored Energy

The *coupled thermoelastic stored energy* for a *linear thermo-elastic model* may be written as,

$$M(\boldsymbol{\varepsilon},\theta) := -(\theta-\theta_0)\,\boldsymbol{\beta}:\boldsymbol{\varepsilon}$$

where $\boldsymbol{\beta}$ is a *positive semi-definite symmetric second-order tensor,* denoted as *stress-temperature constitutive tensor*, such that,

$$\boldsymbol{\beta} = \boldsymbol{\beta}^T, \quad \beta_{ab} = \beta_{ba}$$

and $\theta_0 > 0$ is a constant reference temperature.

Linear Thermoelastic Model (I)

Thermal Stored Energy

The *thermal stored energy* for a *linear thermo-elastic model* may be written as,

$$T(\theta) := -\frac{1}{2\theta_0}\rho_0 c_0 (\theta - \theta_0)^2$$

where $c_0 > 0$ is a positive scalar quantity denoted as *specific heat* at constant strain, i.e. heat supplied per unit of mass in order to achieve a unit temperature change while keeping the strain constant.

Linear Thermoelastic Model (I)

Internal Dissipation Inequality

Applying the chain rule, the *internal dissipation rate per unit of volume,* given by the *Clausius-Planck inequality*, yields,

$$\mathcal{D}_{int} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \left(\dot{\psi} + \eta \dot{\theta} \right)$$

$$= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \eta \dot{\theta} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon}, \theta)}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon}, \theta)}{\partial \theta} \dot{\theta}$$

$$= \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon}, \theta)}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \left(\rho_0 \eta + \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon}, \theta)}{\partial \theta} \right) \dot{\theta} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}, \dot{\theta}$$

Linear Thermoelastic Model (I)

Constitutive Equations and Reduced Dissipation

Following *Coleman's method*, the *internal dissipation rate per unit of volume,* must be satisfied for arbitrary thermodynamic processes, i.e. arbitrary strain rates and temperature rates,

$$\mathcal{D}_{int} := \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \left(\rho_0 \eta + \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \theta} \right) \dot{\theta} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}, \dot{\theta}$$

yielding the following *linear thermoelastic constitutive equations* and *zero internal reduced dissipation* which characterizes a *reversible* process,

$$\boxed{\boldsymbol{\sigma} = \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \boldsymbol{\varepsilon}}, \quad \rho_0 \eta = -\rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \theta}, \quad \mathcal{D}_{int} = 0}$$

Linear Thermoelastic Model (I)

Thermomechanical Constitutive Equations

The *stress and entropy constitutive equations* for a *linear thermoelastic constitutive model* may be written,

$$\boxed{\begin{aligned} \boldsymbol{\sigma} &= \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\beta}(\theta - \theta_0) \\ \rho_0 \eta &= -\rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon},\theta)}{\partial \theta} = \boldsymbol{\beta} : \boldsymbol{\varepsilon} + \rho_0 \frac{c_0}{\theta_0}(\theta - \theta_0) \end{aligned}}$$

Linear Thermoelastic Model (I)

Thermoelastic Constitutive Tensors

The *isothermal elastic constitutive tensor, stress-temperature tensor* and *specific heat* for a *linear thermoelastic model* are given by,

$$\mathbb{C} = \rho_0 \frac{\partial^2 \psi(\varepsilon,\theta)}{\partial \varepsilon \otimes \partial \varepsilon}, \quad \mathbb{C}_{abcd} = \rho_0 \frac{\partial^2 \psi(\varepsilon,\theta)}{\partial \varepsilon_{ab} \otimes \partial \varepsilon_{cd}}$$

$$\beta = -\rho_0 \frac{\partial^2 \psi(\varepsilon,\theta)}{\partial \varepsilon \otimes \partial \theta}, \quad \beta_{ab} = -\rho_0 \frac{\partial^2 \psi(\varepsilon,\theta)}{\partial \varepsilon_{ab} \otimes \partial \theta}$$

$$c_0 = -\theta_0 \frac{\partial^2 \psi(\varepsilon,\theta)}{\partial \theta^2}$$

Linear Thermoelastic Model (I)

Thermomechanical Constitutive Equations

Given the *stress and entropy constitutive equations* for a *linear thermoelastic constitutive model*,

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\beta}\left(\theta - \theta_0\right)$$

$$\rho_0 \eta = \boldsymbol{\beta} : \boldsymbol{\varepsilon} + \rho_0 \frac{c_0}{\theta_0}\left(\theta - \theta_0\right)$$

Using the *entropy constitutive equation*, the increment of temperature may be written as,

$$\theta - \theta_0 = \frac{\theta_0}{c_0}\eta - \frac{1}{\rho_0}\frac{\theta_0}{c_0}\boldsymbol{\beta} : \boldsymbol{\varepsilon}$$

Linear Thermoelastic Model (I)

Thermomechanical Constitutive Equations

Taking the *strain tensor* and *specific entropy* (instead of the temperature) as main variables, the *stress and entropy constitutive equations* for a *linear thermoelastic constitutive model* take the form,

$$\boldsymbol{\sigma} = \left(\mathbb{C} + \frac{\theta_0}{\rho_0 c_0} \boldsymbol{\beta} \otimes \boldsymbol{\beta} \right) : \boldsymbol{\varepsilon} - \boldsymbol{\beta} \frac{\theta_0}{c_0} \eta$$

$$:= \mathbb{C}^{\eta} : \boldsymbol{\varepsilon} - \boldsymbol{\beta} \frac{\theta_0}{c_0} \eta$$

$$\theta - \theta_0 = -\frac{\theta_0}{\rho_0 c_0} \boldsymbol{\beta} : \boldsymbol{\varepsilon} + \frac{\theta_0}{c_0} \eta$$

Linear Thermoelastic Model (I)

Thermomechanical Constitutive Equations

The *isentropic elastic constitutive tensor*, denoted as $\mathbb{C}^{\eta}$, and the *isothermal elastic constitutive tensor*, denoted as $\mathbb{C}$, are related through the following expression,

$$\mathbb{C}^{\eta} = \mathbb{C} + \frac{\theta_0}{\rho_0 c_0} \boldsymbol{\beta} \otimes \boldsymbol{\beta}$$

Linear Thermoelastic Model (II)

Internal Energy

We consider an *internal energy per unit of volume* given by a quadratic function of the *strain tensor* and the *specific entropy*,

$$\rho_0 e = \rho_0 e(\varepsilon, \eta) = W_\eta(\varepsilon) + N(\varepsilon, \eta) + Z(\eta)$$

where $W_\eta(\varepsilon)$ is the *isentropic elastic stored energy*, $N(\varepsilon, \eta)$ is the *coupled entropy-elastic stored energy* and $Z(\eta)$ is the *entropy stored energy*.

Linear Thermoelastic Model (II)

Elastic Stored Energy

The *isentropic elastic stored energy* for a *linear thermoelastic model* may be written as,

$$W_\eta(\boldsymbol{\varepsilon}) := \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^\eta : \boldsymbol{\varepsilon} \geq 0$$

where $\mathbb{C}^\eta$ is a *positive definite fourth-order tensor,* denoted as *isentropic constant elastic constitutive tensor*.

Symmetry of the strain tensor yields the following *symmetry* conditions on the *isentropic elastic constitutive tensor* $\mathbb{C}^\eta$,

$$\boldsymbol{\varepsilon} : \mathbb{C}^\eta : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T : \mathbb{C}^\eta : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \mathbb{C}^\eta : \boldsymbol{\varepsilon}^T, \quad \mathbb{C}^\eta_{abcd} = \mathbb{C}^\eta_{bacd} = \mathbb{C}^\eta_{abdc}$$

Furthermore, the following symmetry conditions hold,

$$\boldsymbol{\varepsilon} : \mathbb{C}^\eta : \boldsymbol{\varepsilon} = \varepsilon_{ab}\mathbb{C}^\eta_{abcd}\varepsilon_{cd} = \varepsilon_{ab}\mathbb{C}^\eta_{cdab}\varepsilon_{cd}, \quad \mathbb{C}^\eta_{abcd} = \mathbb{C}^\eta_{cdab}$$

Linear Thermoelastic Model (II)

Coupled Entropy-elastic Stored Energy

The *coupled entropy-elastic stored energy* for a *linear thermo-elastic model* may be written as,

$$N(\boldsymbol{\varepsilon},\eta) := -\frac{\theta_0}{c_0}(\boldsymbol{\beta} : \boldsymbol{\varepsilon})\eta$$

where $\boldsymbol{\beta}$ is a *positive semi-definite symmetric second-order tensor,* denoted as *stress-temperature constitutive tensor*, such that,

$$\boldsymbol{\beta} = \boldsymbol{\beta}^T, \quad \beta_{ab} = \beta_{ba}$$

$c_0 > 0$ is a positive scalar quantity denoted as *specific heat* at constant strain, and $\theta_0 > 0$ is the constant reference temperature.

Linear Thermoelastic Model (II)

Entropy Stored Energy

The *entropy stored energy* for a *linear thermoelastic model* may be written as,

$$Z(\eta) := \frac{1}{2}\frac{\rho_0\theta_0}{c_0}\eta^2 + \rho_0\theta_0\eta$$

where $c_0 > 0$ is a positive scalar quantity denoted as *specific heat* at constant strain, i.e. heat supplied per unit of mass in order to achieve a unit temperature change while keeping the strain constant.

Linear Thermoelastic Model (II)

Internal Dissipation Inequality

Applying the chain rule, the *internal dissipation rate per unit of volume,* given by the *Clausius-Planck inequality*, yields,

$$\mathcal{D}_{int} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right)$$

$$= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \rho_0 \theta \dot{\eta} - \rho_0 \frac{\partial e\left(\boldsymbol{\varepsilon}, \eta\right)}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \frac{\partial e\left(\boldsymbol{\varepsilon}, \eta\right)}{\partial \eta} \dot{\eta}$$

$$= \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial e\left(\boldsymbol{\varepsilon}, \eta\right)}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} + \left(\rho_0 \theta - \rho_0 \frac{\partial e\left(\boldsymbol{\varepsilon}, \eta\right)}{\partial \eta} \right) \dot{\eta} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}, \dot{\eta}$$

Linear Thermoelastic Model (II)

Constitutive Equations and Reduced Dissipation

Following *Coleman's method*, the *internal dissipation rate per unit of volume,* must be satisfied for arbitrary thermodynamic processes, i.e. arbitrary strain rates and entropy rates,

$$\mathcal{D}_{int} := \left(\boldsymbol{\sigma} - \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} + \left(\rho_0 \theta - \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \eta} \right) \dot{\eta} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}, \dot{\eta}$$

yielding the following *linear thermoelastic constitutive equations* and *zero internal reduced dissipation* which characterizes a *reversible* process,

$$\boxed{\boldsymbol{\sigma} = \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \boldsymbol{\varepsilon}}, \quad \rho_0 \theta = \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \eta}, \quad \mathcal{D}_{int} = 0}$$

Linear Thermoelastic Model (II)

Thermomechanical Constitutive Equations

The *stress and temperature constitutive equations* for a *linear thermoelastic constitutive model* may be written,

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \boldsymbol{\varepsilon}} = \mathbb{C}^{\eta} : \boldsymbol{\varepsilon} - \boldsymbol{\beta} \frac{\theta_0}{c_0} \eta$$

$$\rho_0 \theta = \rho_0 \frac{\partial e(\boldsymbol{\varepsilon}, \eta)}{\partial \eta} = -\frac{\theta_0}{c_0} \boldsymbol{\beta} : \boldsymbol{\varepsilon} + \frac{\rho_0 \theta_0}{c_0} \eta + \rho_0 \theta_0$$

Linear Thermoelastic Model (II)

Thermoelastic Constitutive Tensors

The *isentropic elastic constitutive tensor, stress-temperature tensor* and *inverse specific heat* for a *linear thermoelastic model* are given by,

$$\mathbb{C}^{\eta} = \rho_0 \frac{\partial^2 e(\varepsilon,\eta)}{\partial \varepsilon \otimes \partial \varepsilon}, \quad \mathbb{C}^{\eta}_{abcd} = \rho_0 \frac{\partial^2 e(\varepsilon,\eta)}{\partial \varepsilon_{ab} \otimes \partial \varepsilon_{cd}}$$

$$\beta = -\frac{c_0}{\theta_0} \rho_0 \frac{\partial^2 e(\varepsilon,\eta)}{\partial \varepsilon \otimes \partial \eta}, \quad \beta_{ab} = -\frac{c_0}{\theta_0} \rho_0 \frac{\partial^2 e(\varepsilon,\eta)}{\partial \varepsilon_{ab} \otimes \partial \eta}$$

$$\frac{1}{c_0} = \frac{1}{\theta_0} \frac{\partial^2 e(\varepsilon,\eta)}{\partial \eta^2}$$

Isotropic Linear Thermoelastic Model

Thermoelastic Constitutive Tensors

For an *isotropic linear thermoelastic material model*, the *physical property* of *isotropy* is translated into the *mathematical property* of *isotropy* of the *isothermal elastic and stress-temperature constitutive tensors*, yielding,

$$\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}}, \quad \mathbb{C}_{abcd} = \lambda \delta_{ab} \delta_{cd} + \mu \left(\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} \right)$$

$$\boldsymbol{\beta} = \beta \mathbf{1}, \quad \beta_{ab} = \beta \, \delta_{ab}$$

where $\lambda \geq 0, \mu \geq 0$, are the *isothermal Lamé parameters*, and $\beta \geq 0$ is the *stress-temperature coefficient*.

Isotropic Linear Thermoelastic Model

Thermoelastic Constitutive Tensors

The relationship between the *isothermal* and the *isentropic* *Lamé parameters* is given by,

$$\mathbb{C}_\eta := \mathbb{C} + \frac{\theta_0}{\rho_0 c_0} \boldsymbol{\beta} \otimes \boldsymbol{\beta}$$

$$= \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}} + \frac{\theta_0}{\rho_0 c_0} \beta^2 \mathbf{1} \otimes \mathbf{1} = \left(\lambda + \frac{\theta_0}{\rho_0 c_0} \beta^2 \right) \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}}$$

$$:= \lambda_\eta \mathbf{1} \otimes \mathbf{1} + 2\mu_\eta \hat{\mathbb{I}}$$

$$\lambda_\eta := \lambda + \frac{\theta_0}{\rho_0 c_0} \beta^2, \quad \mu_\eta := \mu$$

Isotropic Linear Thermoelastic Model

Free Energy

The *free energy per unit of volume* for an *isotropic linear thermo-elastic material model* may be written as,

$$\rho_0 \psi(\boldsymbol{\varepsilon},\theta) = W(\boldsymbol{\varepsilon}) + M(\boldsymbol{\varepsilon},\theta) + T(\theta)$$

$$= \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - (\theta - \theta_0)\boldsymbol{\beta} : \boldsymbol{\varepsilon} - \frac{1}{2\theta_0}\rho_0 c_0 (\theta - \theta_0)^2$$

$$= \frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \beta(\theta - \theta_0)\operatorname{tr}\boldsymbol{\varepsilon} - \frac{1}{2\theta_0}\rho_0 c_0 (\theta - \theta_0)^2$$

Isotropic Linear Thermoelastic Model

Thermoelastic Constitutive Equation

The *constitutive equation* for an *isotropic linear thermoelastic material model* may be written as,

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\beta}\left(\theta - \theta_0\right)$$

$$= \left(\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \hat{\mathbb{I}}\right) : \boldsymbol{\varepsilon} - \beta\left(\theta - \theta_0\right)\mathbf{1}$$

$$= \lambda\left(\mathbf{1} : \boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \beta\left(\theta - \theta_0\right)\mathbf{1}$$

$$= \lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \beta\left(\theta - \theta_0\right)\mathbf{1}$$

$$\boxed{\boldsymbol{\sigma} = \lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \beta\left(\theta - \theta_0\right)\mathbf{1}}$$

Isotropic Linear Thermoelastic Model

Thermoelastic Constitutive Equation

The *volumetric part* of the *constitutive equation* for an *isotropic linear thermoelastic model* may be written as,

$$\operatorname{tr}\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{1} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}:\mathbf{1} + 2\mu\boldsymbol{\varepsilon}:\mathbf{1} - \beta(\theta-\theta_0)\mathbf{1}:\mathbf{1}$$

$$= (3\lambda + 2\mu)\operatorname{tr}\boldsymbol{\varepsilon} - 3\beta(\theta-\theta_0)$$

$$\boldsymbol{\sigma}_m = \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = \left(\lambda + \frac{2}{3}\mu\right)e - \beta(\theta-\theta_0)$$

$$= Ke - \beta(\theta-\theta_0)$$

$$\boxed{\boldsymbol{\sigma}_m = Ke - \beta(\theta-\theta_0)}$$

Isotropic Linear Thermoelastic Model

Thermoelastic Constitutive Equation

The *deviatoric part* of the *constitutive equation* for an *isotropic linear thermoelastic model* may be written as,

$$\operatorname{dev}\boldsymbol{\sigma} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\cancel{\operatorname{dev}\mathbf{1}} + 2\mu\operatorname{dev}\boldsymbol{\varepsilon} - \beta(\theta-\theta_0)\cancel{\operatorname{dev}\mathbf{1}} = 2\mu\operatorname{dev}\boldsymbol{\varepsilon}$$

Introducing the *shear modulus* given by,

$$G := \mu$$

The *deviatoric part* of the *constitutive equation* for an *isotropic linear thermoelastic model* may be written as,

$$\boxed{\operatorname{dev}\boldsymbol{\sigma} = 2G\operatorname{dev}\boldsymbol{\varepsilon}}$$

Isotropic Linear Thermoelastic Model

Inverse Thermoelastic Constitutive Equation

The *inverse constitutive equation* for an *isotropic linear thermo-elastic model* may be written as,

$$\operatorname{tr}\boldsymbol{\varepsilon} = (3\lambda + 2\mu)^{-1}\operatorname{tr}\boldsymbol{\sigma} + (3\lambda + 2\mu)^{-1} 3\beta(\theta - \theta_0)$$

$$\boldsymbol{\varepsilon} = -\frac{\lambda}{(3\lambda + 2\mu)2\mu}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} + \frac{\beta}{3\lambda + 2\mu}(\theta - \theta_0)\mathbf{1}$$

Isotropic Linear Thermoelastic Model

Inverse Thermoelastic Constitutive Equation

Let us introduce the *isothermal Young elastic modulus*, denoted as $E>0$, *isothermal Poisson's coefficient*, denoted as $0 \le \nu \le 1/2$, *isothermal bulk modulus*, denoted as $K>0$ and *thermal expansion coefficient*, denoted as $\alpha \ge 0$, such that,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} > 0, \quad 0 \le \nu = \frac{\lambda}{2(\lambda+\mu)} \le \frac{1}{2}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \ge 0, \quad \mu = G = \frac{E}{2(1+\nu)} > 0$$

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} > 0, \quad \alpha = \frac{\beta}{3\lambda+2\mu} = \frac{\beta}{3K} = \frac{1-2\nu}{3E}\beta \ge 0$$

Isotropic Linear Thermoelastic Model

Inverse Thermoelastic Constitutive Equation

The *inverse constitutive equation* for an *isotropic linear thermo-elastic model* may be written as,

$$\boxed{\boldsymbol{\varepsilon} = -\frac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma} + \alpha(\theta - \theta_0)\mathbf{1}}$$

Isotropic Linear Thermoelastic Model

Inverse Thermoelastic Constitutive Equation

The *volumetric part* of the *inverse constitutive equation* for an *isotropic linear thermoelastic model* may be written as,

$$e = \operatorname{tr}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \mathbf{1} = -\frac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}:\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}:\mathbf{1} + \alpha(\theta - \theta_0)\mathbf{1}:\mathbf{1}$$

$$= -\frac{3\nu}{E}\operatorname{tr}\boldsymbol{\sigma} + \frac{1+\nu}{E}\operatorname{tr}\boldsymbol{\sigma} + 3\alpha(\theta - \theta_0) = \frac{1-2\nu}{E}\operatorname{tr}\boldsymbol{\sigma} + 3\alpha(\theta - \theta_0)$$

$$= \frac{1}{3K}\operatorname{tr}\boldsymbol{\sigma} + 3\alpha(\theta - \theta_0) = \frac{1}{K}\sigma_m + 3\alpha(\theta - \theta_0)$$

$$\boxed{e = \frac{1}{K}\sigma_m + 3\alpha(\theta - \theta_0)}$$

Isotropic Linear Thermoelastic IBVP

Governing Equations

Let us consider the following *governing equations* in the space x time domain $\Omega \times \mathbb{I} = \Omega \times [0, T]$.

- Linear momentum balance · First Cauchy's motion equation

$$\operatorname{div} \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad \text{in} \quad \Omega \times \mathbb{I}$$

- Isotropic linear thermoelastic constitutive equation

$$\boldsymbol{\sigma} = \lambda (\operatorname{tr} \boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} - \beta (\theta - \theta_0) \mathbf{1} \quad \text{in} \quad \Omega \times \mathbb{I}$$

- Geometrical equations

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2} \left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T \right) \quad \text{in} \quad \Omega \times \mathbb{I}$$

Isotropic Linear Thermoelastic IBVP

Boundary Conditions

Let us consider *prescribed displacements* and *prescribed tractions boundaries*, denoted as $\partial_u \Omega$ and $\partial_\sigma \Omega$, respectively, such that,

$$\partial_u \Omega \cup \partial_\sigma \Omega = \partial \Omega, \quad \partial_u \Omega \cap \partial_\sigma \Omega = \emptyset$$

with the following *bounday conditions* for the IBVP:

- Prescribed displacements boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}}, \quad u_a = \bar{u}_a \quad \text{on} \quad \partial_u \Omega \times \mathbb{I}$$

- Prescribed tractions boundary conditions

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}}, \quad t_a = \sigma_{ab} n_b = \bar{t}_a \quad \text{on} \quad \partial_\sigma \Omega \times \mathbb{I}$$

Isotropic Linear Thermoelastic IBVP

Initial Conditions

Let us consider the following *initial conditions* for the IBVP,

- Initial displacements at time t=0

$$\mathbf{u}\big|_{t=0} = \mathbf{0}, \quad u_a\big|_{t=0} = 0 \quad \text{in} \ \ \Omega$$

- Initial velocities at time t=0

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{v}_0, \quad \left.\frac{\partial u_a}{\partial t}\right|_{t=0} = v_{0_a} \quad \text{in} \ \ \Omega$$

Isotropic Linear Thermoelastic IBVP

Isotropic Linear Thermoelastic IBVP

Find the *displacements, strains* and *stresses* in $\Omega\times\mathbb{I}=\Omega\times[0,T]$ such that the following equations are satisfied:

$$\left.\begin{aligned}
&\operatorname{div}\boldsymbol{\sigma}+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\\
&\boldsymbol{\sigma}=\lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}-\beta(\theta-\theta_0)\mathbf{1}\\
&\boldsymbol{\varepsilon}=\nabla^s\mathbf{u}=\frac{1}{2}\left(\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^T\right)
\end{aligned}\right\}\text{in }\ \Omega\times\mathbb{I}$$

$$\begin{aligned}
&\mathbf{u}=\bar{\mathbf{u}} \quad \text{on }\ \partial_u\Omega\times\mathbb{I}\\
&\mathbf{t}=\boldsymbol{\sigma}\mathbf{n}=\bar{\mathbf{t}} \quad \text{on }\ \partial_\sigma\Omega\times\mathbb{I}
\end{aligned}$$

$$\left.\begin{aligned}
&\mathbf{u}\big|_{t=0}=\mathbf{0}\\
&\frac{\partial\mathbf{u}}{\partial t}\bigg|_{t=0}=\mathbf{v}_0
\end{aligned}\right\}\text{in }\ \Omega$$

Solution of the Linear Thermoelastic IBVP

Method of Displacements · Navier's Equation

Stresses can be removed from the list of unknowns of the isotropic linear thermoelastic IBVP using, i.e. assuming that the material parameters are constant,

$$\begin{aligned}\operatorname{div}\boldsymbol{\sigma} &= \operatorname{div}\left(\lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \beta\left(\theta-\theta_0\right)\mathbf{1}\right) \\ &= \lambda\operatorname{grad}\left(\operatorname{tr}\boldsymbol{\varepsilon}\right) + 2\mu\operatorname{div}\boldsymbol{\varepsilon} - \beta\operatorname{grad}\left(\theta-\theta_0\right)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}\mathbf{n} &= \left(\lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \beta\left(\theta-\theta_0\right)\mathbf{1}\right)\mathbf{n} \\ &= \lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{n} + 2\mu\boldsymbol{\varepsilon}\mathbf{n} - \beta\left(\theta-\theta_0\right)\mathbf{n}\end{aligned}$$

Solution of the Linear Thermoelastic IBVP

Method of Displacements · Navier's Equation

Strains can can be removed from the list of unknowns of the isotropic linear thermoelastic IBVP using, i.e. assuming that the material parameters are constant,

$$\begin{aligned}\operatorname{div}\boldsymbol{\sigma} &= \lambda\,\operatorname{grad}\left(\operatorname{tr}\boldsymbol{\varepsilon}\right) + 2\mu\,\operatorname{div}\boldsymbol{\varepsilon} \\ &= \left(\lambda+\mu\right)\operatorname{grad}\left(\operatorname{div}\mathbf{u}\right) + \mu\,\operatorname{div}\left(\operatorname{grad}\mathbf{u}\right) - \beta\,\operatorname{grad}\left(\theta-\theta_0\right)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}\mathbf{n} &= \lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{n} + 2\mu\boldsymbol{\varepsilon}\mathbf{n} \\ &= \lambda\left(\operatorname{div}\mathbf{u}\right)\mathbf{n} + \mu\left(\operatorname{grad}\mathbf{u} + \left(\operatorname{grad}\mathbf{u}\right)^T\right)\mathbf{n} - \beta\left(\theta-\theta_0\right)\mathbf{n}\end{aligned}$$

Solution of the Linear Thermoelastic IBVP

Method of Displacements · Navier's Equation

The *first Cauchy's motion equation* written in terms of the displacements is denoted as **Navier's equation** and takes the form,

$$(\lambda+\mu)\operatorname{grad}(\operatorname{div}\mathbf{u})+\mu\operatorname{div}(\operatorname{grad}\mathbf{u})$$

$$-\beta\operatorname{grad}(\theta-\theta_0)+\rho_0\mathbf{b}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\quad\text{in}\quad\Omega\times\mathbb{I}$$

Introducing a *modified body force* per unit of mass vector defined as,

$$\hat{\mathbf{b}}:=\mathbf{b}-\frac{\beta}{\rho_0}\operatorname{grad}(\theta-\theta_0)$$

the *first Cauchy's motion equation* may be written as,

$$(\lambda+\mu)\operatorname{grad}(\operatorname{div}\mathbf{u})+\mu\operatorname{div}(\operatorname{grad}\mathbf{u})+\rho_0\hat{\mathbf{b}}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\quad\text{in}\quad\Omega\times\mathbb{I}$$

Solution of the Linear Thermoelastic IBVP

Method of Displacements · Prescribed Traction

The *prescribed traction* boundary condition takes the form,

$$\boldsymbol{\sigma}\mathbf{n} = \lambda(\operatorname{div}\mathbf{u})\,\mathbf{n} + \mu\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right)\mathbf{n} - \beta(\theta - \theta_0)\,\mathbf{n} = \bar{t}$$

Introducing a *modified prescribed traction* vector defined as,

$$\hat{\mathbf{t}} := \bar{\mathbf{t}} + \beta(\theta - \theta_0)\,\mathbf{n}$$

the (modified) *prescribed traction* boundary condition may be written as,

$$\boldsymbol{\sigma}\mathbf{n} = \lambda(\operatorname{div}\mathbf{u})\,\mathbf{n} + \mu\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right)\mathbf{n} = \hat{t}$$

Solution of the Linear Thermoelastic IBVP

Isotropic Linear Thermoelastic IBVP

Find the *displacements* in $\Omega \times \mathbb{I} = \Omega \times [0,T]$ such that the following equations are satisfied:

$$(\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \mu\,\mathrm{div}(\mathrm{grad}\,\mathbf{u}) + \rho_0 \hat{\mathbf{b}} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad \text{in} \quad \Omega \times \mathbb{I}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u \Omega \times \mathbb{I}$$

$$\lambda(\mathrm{div}\,\mathbf{u})\,\mathbf{n} + \mu\left(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^T\right)\mathbf{n} = \hat{\mathbf{t}} \quad \text{on} \quad \partial_\sigma \Omega \times \mathbb{I}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{0}, \quad \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega$$

Solution of the Linear Thermoelastic IBVP

Isotropic Linear Thermoelastic IBVP

While the *displacements* have been obtained, *strains* and *stresses* may be obtained as a *post-process* of the results,

- Geometrical equation

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2}\left(\nabla \otimes \mathbf{u} + \left(\nabla \otimes \mathbf{u} \right)^T \right) \quad \text{in} \;\; \Omega \times \mathbb{I}$$

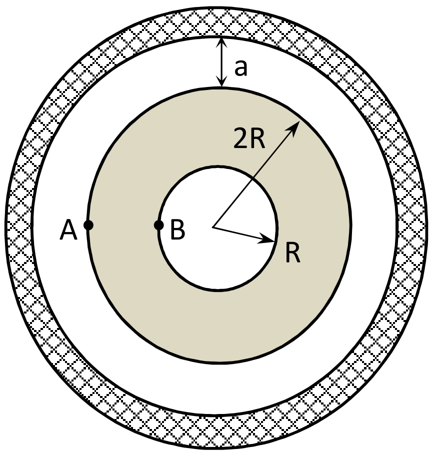
- Isotropic linear thermoelastic constitutive equation

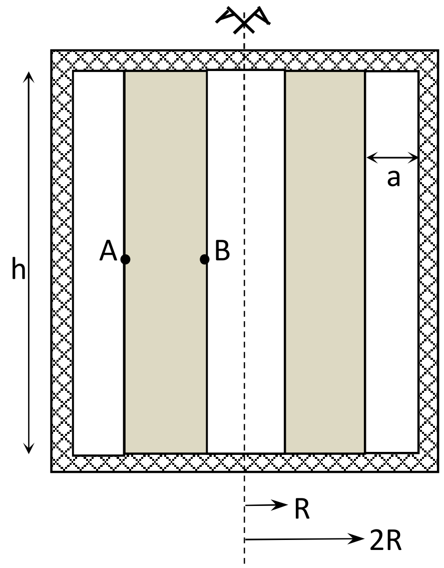
$$\boldsymbol{\sigma} = \lambda \left(\operatorname{tr} \boldsymbol{\varepsilon} \right) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} - \beta \left(\theta - \theta_0 \right) \mathbf{1} \quad \text{in} \;\; \Omega \times \mathbb{I}$$

Assignment 7.3

Assignment 7.3

A void cylinder of an isotropic linear thermoelastic material with height h, inner radius R and outer radius $2R$, is inside of a rigid void cylinder of the same height h and inner radius $2R+a$, such that there is a gap $a<<R$ between both of them, as it is shown in the figure. A uniform increment of temperature $\Delta\theta$ is applied to the inner cylinder.





Assignment 7.3

Assignment 7.3

The problem is considered as quasistatic. Body forces in the cylinder and frictional effects are negligible. The material properties of the inner cylinder are: Young modulus E, Poisson's coefficient $\nu = 0$ and thermal expansion coefficient α.

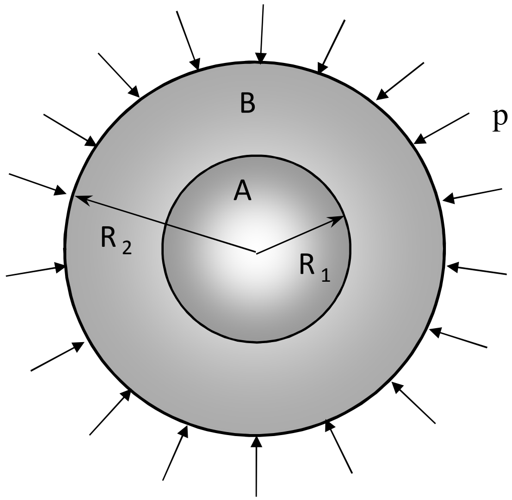
1) Compute the increment of temperature needed for the inner cylinder to come into contact with the rigid one.
2) Plot the δ vs $\Delta\theta$ curve, where δ is the radial displacement of the inner surface of the inner cylinder, and determine the increment of temperature needed for the inner radius of the cylinder to go back to its original position.
3) Plot the $\sigma_r \; vs \, \Delta\theta$, $\sigma_\theta \; vs \, \Delta\theta$ and $\sigma_z \; vs \, \Delta\theta$ curves at the inner and outer surfaces of the inner cylinder.

Assignment 7.4

Assignment 7.4

Let us consider a sphere of material 1 and radius $R_1 = 1$ placed inside a void sphere of material 2, inner radius $R_1 = 1$ and outer radius $R_2 = 2$. Both spheres are initially in contact without any pressure. An external uniform pressure p is applied to the external surface of the void sphere and a uniform increment of temperature $\Delta\theta$ is applied to both spheres.

Materials 1 and 2 are isotropic linear thermoelastic with the same Young modulus E, same Poisson's coefficient $\nu = 0$ and different thermal expansion coefficients, 2α for material 1 and α for material 2.



Assignment 7.4

Assignment 7.4

1) Determine all the possible values of p and $\Delta\theta$ (positives or negatives) such that the contact between the two spheres is maintained. Plot the result in a $p \; vs \; \Delta\theta$ axis.
2) Obtain the stresses in the each one of the two spheres for those values.

Plane Linear Elasticity

Plane Linear Elasticity

For some problems, i.e. geometry, loading and BCs, one of the *principal stress directions* is known "a priori", it does not play a role in the formulation, and the problem may be formulated on the plane defined by the other two principal directions.

Considering a local Cartesian axis, taking the z-axis as the one associated to the "a-priori" known principal direction, the *plane linear elastic* problem may be formulated on the x,y plane. Variables associated to the z-axis, are either known or they can be obtained as a post-process of the 2D results.

Plane Linear Elasticity

Plane Linear Elasticity

Two cases of *plane linear elastic* problems may be considered,

- Plane stress problems
- Plane strain problems

For each one of them:

(i) We will introduce the hypothesis on the appropriate field,

(ii) We will obtain the consequences on the remaining fields,

(iii) We will show up some typical examples

Plane Stress Linear Elasticity

Hypothesis on the Stress Field

H1. The components of the stress tensor take the form,

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

H2. The non-zero components of the stress tensor take the form,

$$\sigma_x = \sigma_x(x, y), \quad \sigma_y = \sigma_y(x, y), \quad \tau_{xy} = \tau_{xy}(x, y)$$

Plane Stress Linear Elasticity

Strain Field

Using the inverse constitutive equations, the strains take the form,

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right), \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right), \quad \gamma_{xz} = 0$$

$$\varepsilon_z = -\frac{1}{E}\nu\left(\sigma_x + \sigma_y\right), \quad \gamma_{yz} = 0$$

The following relation holds,

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Plane Stress Linear Elasticity

Strain Field

Then, the strain field takes the form,

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

such that,

$$\varepsilon_x = \varepsilon_x(x, y), \quad \varepsilon_y = \varepsilon_y(x, y), \quad \gamma_{xy} = \gamma_{xy}(x, y)$$

and the following relation holds,

$$\varepsilon_z(x, y) = -\frac{\nu}{1-\nu}\left(\varepsilon_x(x, y) + \varepsilon_y(x, y)\right)$$

Plane Stress Linear Elasticity

Plane Stress Constitutive Equation

The constitutive equation for plane stress linear elasticity may be written in matrix form as,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Stress Linear Elasticity

Displacement Field

The in-plane components of the displacement field are obtained integrating the in-plane components of the strain field,

$$\varepsilon_x(x,y)=\frac{\partial u_x}{\partial x},\quad \varepsilon_y(x,y)=\frac{\partial u_y}{\partial y},\quad \gamma_{xy}(x,y)=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}$$

yielding,

$$u_x=u_x(x,y),\quad u_y=u_y(x,y)$$

Plane Stress Linear Elasticity

Displacement Field

The out-of-plane component of the displacements is obtained integrating the out-of-plane component of the strain field,

$$\varepsilon_z(x,y) = -\frac{\nu}{1-\nu}\left(\varepsilon_x(x,y)+\varepsilon_y(x,y)\right) = \frac{\partial u_z}{\partial z}$$

yielding,

$$u_z = u_z(x,y,z)$$

Plane Stress Linear Elasticity

Displacement Field

The *plane stress linear elastic* problem should be viewed as an *ideal* problem, which is not necessarily a particular case of the 3D linear elastic problem.

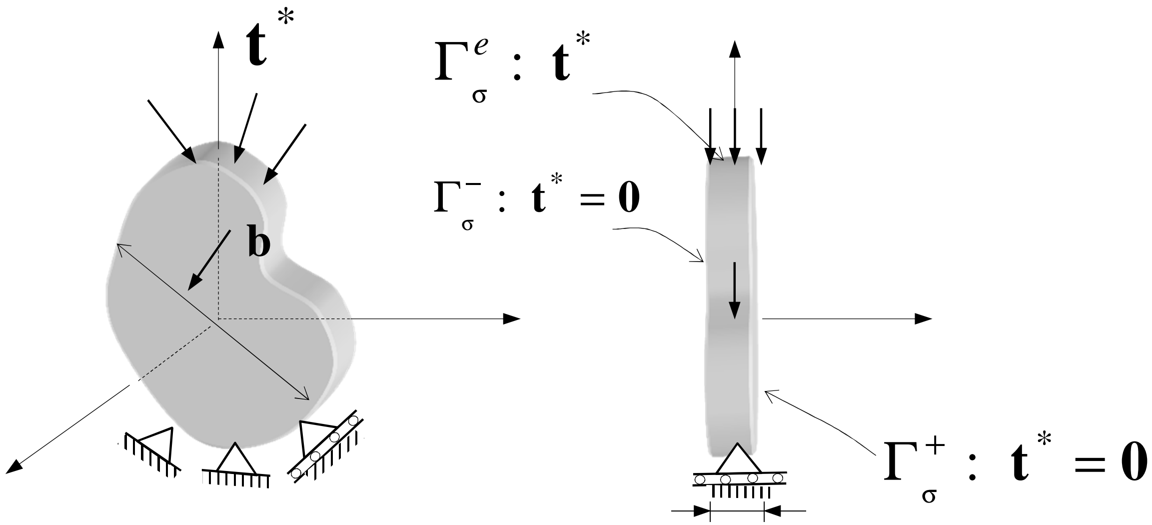
In particular, the following conditions are not necessarily satisfied,

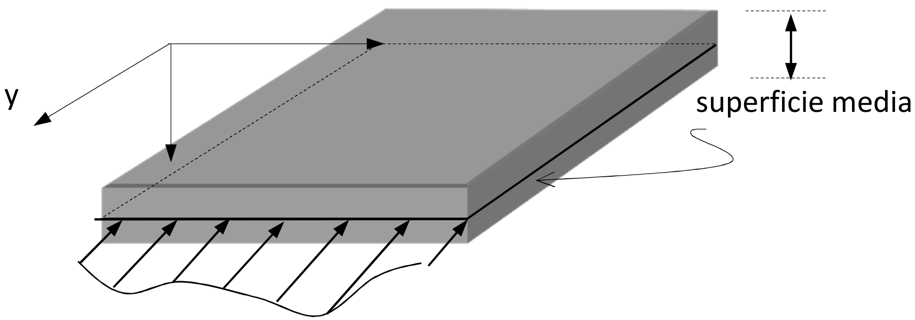
$$\gamma_{xz} = \cancel{\frac{\partial u_x(x,y)}{\partial z}} + \frac{\partial u_z(x,y,z)}{\partial x} = 0$$

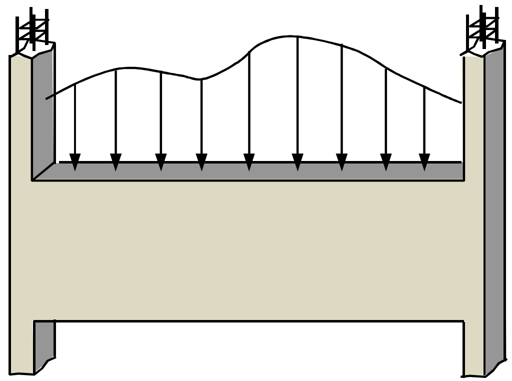
$$\gamma_{yz} = \cancel{\frac{\partial u_y(x,y)}{\partial z}} + \frac{\partial u_z(x,y,z)}{\partial y} = 0$$

Plane Stress Linear Elasticity

Examples







Plane Strain Linear Elasticity

Hypothesis on the Displacement Field

H1. The components of the displacement field take the form,

$$[\mathbf{u}] = \begin{bmatrix} u_x & u_y & 0 \end{bmatrix}^T$$

H2. The non-zero components of the displacements take the form,

$$u_x = u_x(x, y), \quad u_y = u_y(x, y)$$

Plane Strain Linear Elasticity

Strain Field

Using the geometric equations, strains take the form,

$$\varepsilon_x(x,y)=\frac{\partial u_x}{\partial x},\quad \varepsilon_y(x,y)=\frac{\partial u_y}{\partial y},\quad \gamma_{xy}(x,y)=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}$$

$$\varepsilon_z=\frac{\partial u_z}{\partial z}=0$$

$$\gamma_{xz}=\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}=0,\quad \gamma_{yz}=\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}=0$$

Plane Strain Linear Elasticity

Strain Field

Then the strain field takes the form,

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that,

$$\varepsilon_x = \varepsilon_x(x, y), \quad \varepsilon_y = \varepsilon_y(x, y), \quad \gamma_{xy} = \gamma_{xy}(x, y)$$

Plane Strain Linear Elasticity

Tensor Field

Using the constitutive equations for linear elasticity, the stresses take the form,

$$\sigma_x = \lambda\left(\varepsilon_x + \varepsilon_y\right) + 2\mu\varepsilon_x = \left(\lambda + 2G\right)\varepsilon_x + \lambda\varepsilon_y \qquad \tau_{xy} = G\,\gamma_{xy}$$

$$\sigma_y = \lambda\left(\varepsilon_x + \varepsilon_y\right) + 2\mu\varepsilon_y = \left(\lambda + 2G\right)\varepsilon_y + \lambda\varepsilon_x \qquad \tau_{xz} = G\,\gamma_{xz} = 0$$

$$\sigma_z = \lambda\left(\varepsilon_x + \varepsilon_y\right) \qquad \tau_{yz} = G\,\gamma_{yz} = 0$$

The following relation holds,

$$\sigma_z = \frac{\lambda}{2(\lambda + \mu)}\left(\sigma_x + \sigma_y\right) = \nu\left(\sigma_x + \sigma_y\right)$$

Plane Strain Linear Elasticity

Tensor Field

Then, the stress field takes the form,

$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{xy} & \sigma_y & 0\\ 0 & 0 & \sigma_z\end{bmatrix}$$

such that,

$$\sigma_x=\sigma_x(x,y),\quad \sigma_y=\sigma_y(x,y),\quad \tau_{xy}=\tau_{xy}(x,y)$$

$$\sigma_z=\sigma_z(x,y)=\frac{\lambda}{2(\lambda+\mu)}(\sigma_x+\sigma_y)=\nu(\sigma_x+\sigma_y)$$

Plane Strain Linear Elasticity

Plane Strain Constitutive Equation

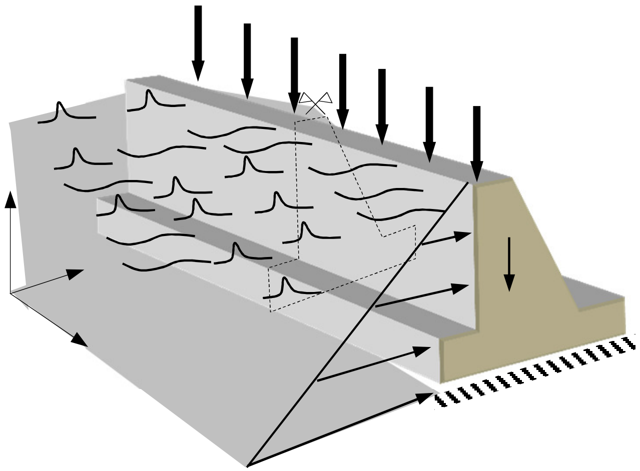
The constitutive equation for *plane strain linear elasticity* may be written in matrix form as,

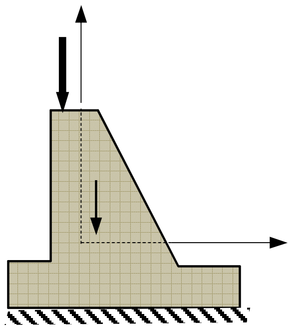
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

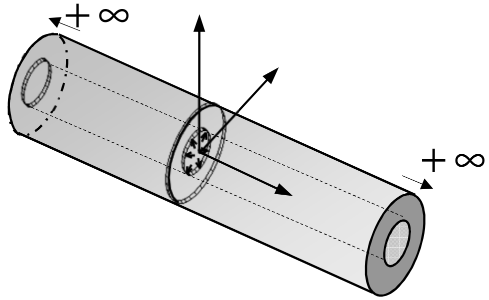
$$\sigma_z = \sigma_z(x,y) = \frac{\lambda}{2(\lambda+\mu)}\left(\sigma_x+\sigma_y\right) = \nu\left(\sigma_x+\sigma_y\right)$$

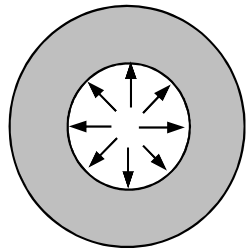
Plane Strain Linear Elasticity

Examples









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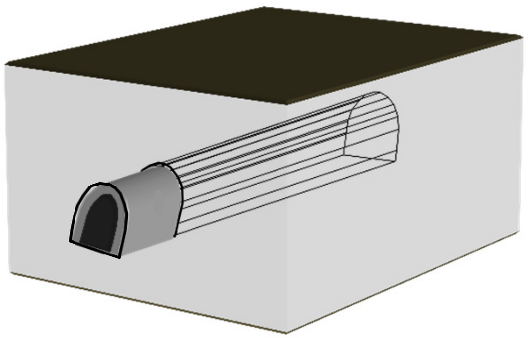

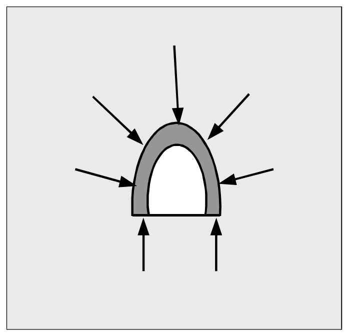

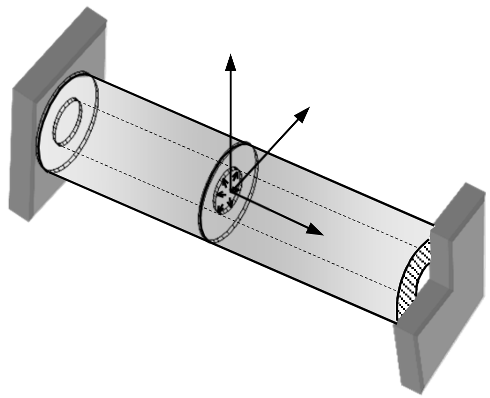

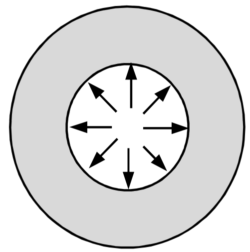

Sección
transversal

Sección transversal

Plane Linear Elastic IVBP

First Cauchy's Motion Equation

The *first Cauchy's motion equation* for a plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, may be written as,

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho_0 b_x = \rho_0 \frac{\partial^2 u_x}{\partial t^2} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho_0 b_y = \rho_0 \frac{\partial^2 u_y}{\partial t^2} \end{aligned}\right\} \quad \text{in } \Omega \times \mathbb{I}$$

Plane Linear Elastic IVBP

Plane Linear Elastic Constitutive Equation

The *constitutive equations* for a plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, may be written as,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{\bar{E}}{1-\bar{\nu}^2} \begin{bmatrix} 1 & \bar{\nu} & 0 \\ \bar{\nu} & 1 & 0 \\ 0 & 0 & (1-\bar{\nu})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{Plane stress:} \begin{cases} \bar{E} = E \\ \bar{\nu} = \nu \end{cases} \qquad \text{Plane strain:} \begin{cases} \bar{E} = \dfrac{E}{1-\nu^2} \\ \bar{\nu} = \dfrac{\nu}{(1-\nu)} \end{cases}$$

Plane Linear Elastic IVBP

Kinematic or Geometric Equations

The *kinematic* or *geometric equations* for a plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, may be written as,

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_y = \frac{\partial u_y}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Plane Linear Elastic IVBP

Boundary Conditions

Boundary conditions for a plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, may be written as,

1. Prescribed displacements

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_x^* \\ u_y^* \end{bmatrix} \quad \text{on } \partial_u \Omega \times \mathbb{I}$$

2. Prescribed tractions

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} t_x^*(x, y, t) \\ t_y^*(x, y, t) \end{bmatrix} \quad \text{on } \partial_\sigma \Omega \times \mathbb{I}$$

Plane Linear Elastic IVBP

Initial Conditions

Initial conditions for a plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, may be written as,

1. Initial displacements

$$\mathbf{u} = \mathbf{0} \quad \text{in} \;\; \Omega$$

2. Initial velocities

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v}_0 \quad \text{in} \;\; \Omega$$

Plane Linear Elastic IVBP

Post-process of In-plane Results

While the solution for the plane linear elastic IBVP, either plane stress or plane strain linear elastic IBVP, has been obtained the following post-process of results yields,

1. For *plane stress* linear elastic IBVP (assuming $u_z\big|_{z=0} = 0$)

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right), \quad u_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) z$$

2. For *plane strain* linear elastic IBVP

$$\sigma_z = \frac{\lambda}{2(\lambda+\mu)}\left(\sigma_x + \sigma_y\right) = \nu\left(\sigma_x + \sigma_y\right)$$