Continuum Mechanics

Chapter 4
Kinematics: Infinitesimal Strains

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Infinitesimal Strains

Chapter 4 · Infinitesimal Strains

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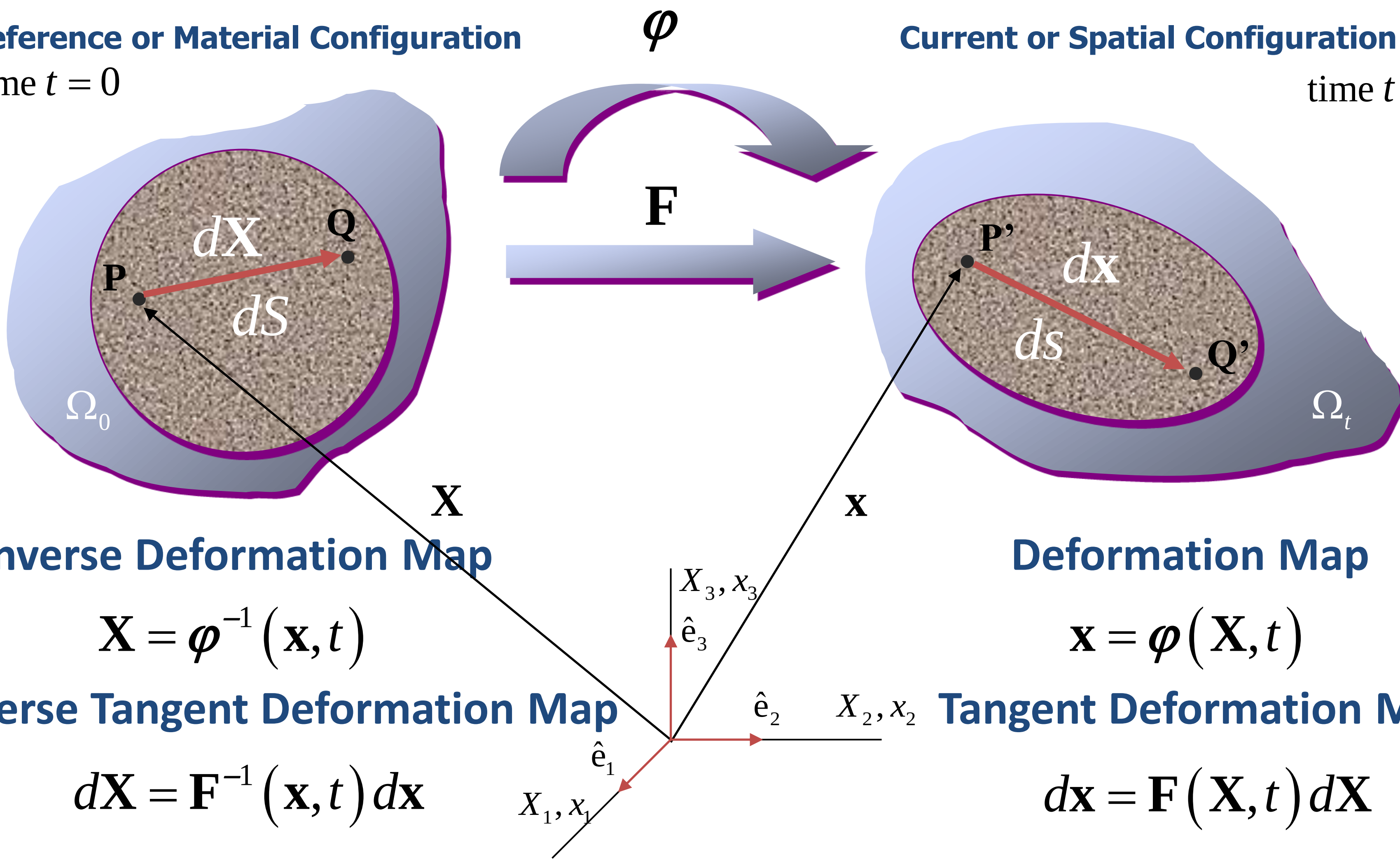
Infinitesimal Strains



Inverse Deformation Map

$$\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

Deformation Map

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

Inverse Tangent Deformation Map

$$d\mathbf{X} = \mathbf{F}^{-1}(\mathbf{x}, t)\, d\mathbf{x}$$

Tangent Deformation Map

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$$

Hypothesis

H1. Small Displacements

We assume that the **displacements** are **small**, such that,

- We do not make any difference between *spatial configuration* and *material configuration*,

$$\Omega_t \simeq \Omega_0$$

- We do not make any difference between *spatial points* and *material points*, or *spatial coordinates* and *material coordinates*,

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) \simeq \mathbf{X}, \quad [\mathbf{x}] \simeq [\mathbf{X}]$$

Hypothesis

H1. Small Displacements

We assume that the **displacements** are **small**, such that,

- We do not make any difference between *spatial description* and *material description*,

$$\gamma = \gamma(\mathbf{x},t) = \gamma\left(\boldsymbol{\varphi}(\mathbf{X},t),t\right) \simeq \gamma(\mathbf{X},t)$$

$$\gamma = \Gamma(\mathbf{X},t) = \Gamma\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t\right) \simeq \Gamma(\mathbf{x},t)$$

- We do not make any difference between *spatial differential operators* and *material differential operators*,

$$\nabla \simeq \bar{\nabla}, \quad \nabla^2 \simeq \bar{\nabla}^2, \quad \nabla \otimes \nabla \simeq \bar{\nabla} \otimes \bar{\nabla}$$

$$\mathrm{grad}[\circ] \simeq \mathrm{GRAD}[\circ], \quad \mathrm{div}[\circ] \simeq \mathrm{DIV}[\circ], \quad \mathrm{curl}[\circ] \simeq \mathrm{CURL}[\circ]$$

Hypothesis

H1. Small Displacements

We assume that the **displacements** are **small**, such that,

- We do not make any difference between *spatial time derivative* and *material time derivative,*

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma(\mathbf{x},t)}{\partial t} \simeq \frac{\partial \gamma(\mathbf{X},t)}{\partial t} \simeq \frac{d\gamma}{dt} = \dot{\gamma}$$

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{\partial \Gamma(\mathbf{X},t)}{\partial t} \simeq \frac{\partial \Gamma(\mathbf{x},t)}{\partial t} \simeq \frac{\partial \gamma}{\partial t}$$

Hypothesis

H2. Small Displacements Gradient

We assume that the **displacements gradient** are **small**, such that,

- The components of the displacement gradient satisfy,

$$\left|J_{aA}\right| = \left|\frac{\partial U_a}{\partial X_A}\right| \ll 1$$

- We consider a *linear theory*, i.e. any non-linear function of the displacements gradient is linearized,

$$\begin{aligned} f(\mathbf{J}) &= f(\mathbf{J})\big|_{\mathbf{J}=\mathbf{0}} + \partial_{\mathbf{J}} f(\mathbf{J})\big|_{\mathbf{J}=\mathbf{0}} \mathbf{J} + 0\left(\mathbf{J}^2\right) \\ &\simeq f(\mathbf{J})\big|_{\mathbf{J}=\mathbf{0}} + \partial_{\mathbf{J}} f(\mathbf{J})\big|_{\mathbf{J}=\mathbf{0}} \mathbf{J} \end{aligned}$$

Displacement Gradient Tensor

Displacements Gradient Tensor

As we do not make any difference between *material* and *spatial descriptions* and we do not make any difference between *material* and *spatial gradient*, then there is no difference between the *material* and *spatial displacement gradient*, yielding a single **displacement gradient tensor** such that,

$$\left.\begin{aligned} \mathbf{J} &:= \bar{\nabla} \otimes \mathbf{U}(\mathbf{X},t) = \mathrm{GRAD}\,\mathbf{U}(\mathbf{X},t) \\ \mathbf{j} &:= \nabla \otimes \mathbf{u}(\mathbf{x},t) = \mathrm{grad}\,\mathbf{u}(\mathbf{x},t) \end{aligned}\right\}$$

$$\Rightarrow \begin{cases} \mathbf{j} \simeq \mathbf{J} \\ \nabla \otimes \mathbf{u}(\mathbf{x},t) \simeq \bar{\nabla} \otimes \mathbf{U}(\mathbf{X},t) \\ \mathrm{grad}\,\mathbf{u}(\mathbf{x},t) \simeq \mathrm{GRAD}\,\mathbf{U}(\mathbf{X},t) \end{cases}$$

Infinitesimal Strain Tensor

Infinitesimal Strain Tensor

The *Green-Lagrange strain* tensor is a *non-linear function* of the *material displacement gradient* tensor,

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) = \frac{1}{2}\left(\left(\mathbf{1}+\mathbf{J}\right)^T\left(\mathbf{1}+\mathbf{J}\right) - \mathbf{1}\right) = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T\mathbf{J}\right)$$

The *linearization* of the Green-Lagrange strain tensor yields,

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T\mathbf{J}\right) \simeq \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right) := \boldsymbol{\varepsilon}$$

The **infinitesimal strain** tensor, denoted as $\boldsymbol{\varepsilon}$, may be defined as the *linearized Green-Lagrange strain* tensor, yielding,

$$\boxed{\boldsymbol{\varepsilon} := \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right), \quad \varepsilon_{ab} := \frac{1}{2}\left(u_{a,b} + u_{b,a}\right)}$$

Infinitesimal Strain Tensor

Infinitesimal Strain Tensor

The *Almansi strain* tensor is a *non-linear function* of the *spatial displacement gradient* tensor,

$$\mathbf{e} := \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\mathbf{1} - (\mathbf{1} - \mathbf{j})^{T}(\mathbf{1} - \mathbf{j})\right) = \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^{T} - \mathbf{j}^{T}\mathbf{j}\right)$$

The **infinitesimal strain** tensor, denoted as $\boldsymbol{\varepsilon}$, may be defined as the *linearized Almansi strain* tensor, and taking into account that we do not make any difference between material and spatial displacement gradient, yields,

$$\boxed{\boldsymbol{\varepsilon} := \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^{T}\right) \simeq \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^{T}\right), \quad \varepsilon_{ab} := \frac{1}{2}\left(u_{a,b} + u_{b,a}\right)}$$

Variation of Volume

Variation of Volume

The relation between the *differential of volume* at the *spatial* and *material configurations* is given by,

$$dv = J\,dV$$

where the *Jacobian* is a *non-linear* function of the components of the *displacement gradient* tensor given by,

$$J = \det \mathbf{F} = \det\left(\mathbf{1} + \mathbf{J}\right) = \det \begin{bmatrix} 1+u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & 1+u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & 1+u_{3,3} \end{bmatrix}$$

Variation of Volume

Variation of Volume

The linearization of the *Jacobian* yields,

$$J \simeq 1 + u_{1,1} + u_{2,2} + u_{3,3} = 1 + \operatorname{div} \mathbf{u} = 1 + \operatorname{tr} \boldsymbol{\varepsilon}$$

Then, within an infinitesimal strains framework, the relation between the spatial and material differential of volume takes the form,

$$\boxed{dv \simeq (1 + \operatorname{div} \mathbf{u})\, dV = (1 + \operatorname{tr} \boldsymbol{\varepsilon})\, dV}$$

Note that now, within the infinitesimal strains framework, the **incompressibility** condition reads,

$$\boxed{J = 1 \quad \Rightarrow \quad \operatorname{div} \mathbf{u} = \operatorname{tr} \boldsymbol{\varepsilon} = 0}$$

Polar Decomposition

Polar Decomposition

The **polar decomposition** of the **deformation gradient** tensor $\mathbf{F}$, reads,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \quad F_{aA} = R_{aB} U_{BA} = v_{ab} R_{bA}$$

where $\mathbf{U}$ is the **right** (or **material**) **stretch tensor**, $\mathbf{v}$ is the **left** (or **spatial**) **stretch tensor** and $\mathbf{R}$ is the **rotation tensor**, such that,

$$\mathbf{U} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2} = \mathbf{C}^{1/2}, \quad \mathbf{U} = \mathbf{U}^T, \quad \mathbf{x}\cdot\mathbf{U}\mathbf{x} > 0 \ \ \forall \mathbf{x} \neq 0, \quad \det\mathbf{U} = J$$

$$\mathbf{v} = \left(\mathbf{F}\mathbf{F}^T\right)^{1/2} = \mathbf{b}^{1/2}, \quad \mathbf{v} = \mathbf{v}^T, \quad \mathbf{x}\cdot\mathbf{v}\mathbf{x} > 0 \ \ \forall \mathbf{x} \neq 0, \quad \det\mathbf{v} = J$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{v}^{-1}\mathbf{F}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det\mathbf{R} = 1$$

Polar Decomposition

Linearized Right Stretch Tensor

The **right stretch** tensor is a *non-linear* function of the *material displacement gradient* tensor given by,

$$\mathbf{U} = \mathbf{C}^{1/2} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2} = \left[\left(\mathbf{1}+\mathbf{J}\right)^T\left(\mathbf{1}+\mathbf{J}\right)\right]^{1/2} = \left(\mathbf{1}+\mathbf{J}+\mathbf{J}^T+\mathbf{J}^T\mathbf{J}\right)^{1/2}$$

The *linearization* of the **right Cauchy-Green** tensor and the **right stretch** tensor yields,

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T\mathbf{F} = \left(\mathbf{1}+\mathbf{J}\right)^T\left(\mathbf{1}+\mathbf{J}\right) = \mathbf{1}+\mathbf{J}+\mathbf{J}^T+\mathbf{J}^T\mathbf{J} \\ &\simeq \mathbf{1}+\mathbf{J}+\mathbf{J}^T = \mathbf{1}+2\boldsymbol{\varepsilon}\end{aligned}$$

$$\mathbf{U} = \mathbf{C}^{1/2} \simeq \left(\mathbf{1}+\mathbf{J}+\mathbf{J}^T\right)^{1/2} \simeq \mathbf{1}+\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right) = \mathbf{1}+\boldsymbol{\varepsilon}$$

Polar Decomposition

Linearized Inverse Right Stretch Tensor

The *linearization* of the **inverse right Cauchy-Green** tensor and the **right stretch** tensor yields,

$$\mathbf{C}^{-1} \simeq \left(\mathbf{1}+\mathbf{J}+\mathbf{J}^{T}\right)^{-1} \simeq \mathbf{1}-\left(\mathbf{J}+\mathbf{J}^{T}\right)=\mathbf{1}-2\boldsymbol{\varepsilon}$$

$$\mathbf{U}^{-1}=\mathbf{C}^{-1/2} \simeq \left(\mathbf{1}+\mathbf{J}+\mathbf{J}^{T}\right)^{-1/2} \simeq \mathbf{1}-\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^{T}\right)=\mathbf{1}-\boldsymbol{\varepsilon}$$

Polar Decomposition

Linearized Inverse Left Stretch Tensor

The inverse **left stretch** tensor is a *non-linear* function of the *spatial displacement gradient* tensor given by,

$$\mathbf{v}^{-1} = \mathbf{b}^{-1/2} = \left(\mathbf{F}^{-T}\mathbf{F}^{-1}\right)^{1/2} = \left[\left(\mathbf{1}-\mathbf{j}\right)^{T}\left(\mathbf{1}-\mathbf{j}\right)\right]^{1/2} = \left(\mathbf{1}-\mathbf{j}-\mathbf{j}^{T}+\mathbf{j}^{T}\mathbf{j}\right)^{1/2}$$

The *linearization* of the inverse **left Cauchy-Green** tensor and inverse **left stretch** tensor yields,

$$\mathbf{b}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} = \left(\mathbf{1}-\mathbf{j}\right)^{T}\left(\mathbf{1}-\mathbf{j}\right) = \mathbf{1}-\mathbf{j}-\mathbf{j}^{T}+\mathbf{j}^{T}\mathbf{j}$$

$$\cong \mathbf{1}-\mathbf{j}-\mathbf{j}^{T} \simeq \mathbf{1}-2\boldsymbol{\varepsilon}$$

$$\mathbf{v}^{-1} = \mathbf{b}^{-1/2} \cong \left(\mathbf{1}-\mathbf{j}-\mathbf{j}^{T}\right)^{1/2} \cong \mathbf{1}-\frac{1}{2}\left(\mathbf{j}+\mathbf{j}^{T}\right) \simeq \mathbf{1}-\boldsymbol{\varepsilon}$$

Polar Decomposition

Linearized Left Stretch Tensor

The *linearization* of the **left Cauchy-Green** tensor and **left stretch** tensor yields,

$$\mathbf{b} \simeq \left(\mathbf{1} - \mathbf{j} - \mathbf{j}^T\right)^{-1} \simeq \mathbf{1} + \left(\mathbf{j} + \mathbf{j}^T\right) \simeq \mathbf{1} + 2\boldsymbol{\varepsilon}$$

$$\mathbf{v} = \mathbf{b}^{1/2} \simeq \left(\mathbf{1} - \mathbf{j} - \mathbf{j}^T\right)^{-1/2} \simeq \mathbf{1} + \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^T\right) \simeq \mathbf{1} + \boldsymbol{\varepsilon}$$

Polar Decomposition

Linearized Rotation Tensor

The **rotation** tensor may be written as a *non-linear* function of the *material displacement gradient* tensor given by,

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{F}\mathbf{C}^{-1/2} = \mathbf{F}\left(\mathbf{F}^T\mathbf{F}\right)^{-1/2} = \left(\mathbf{1}+\mathbf{J}\right)\left(\left(\mathbf{1}+\mathbf{J}\right)^T\left(\mathbf{1}+\mathbf{J}\right)\right)^{-1/2}$$

The *linearization* of the inverse **rotation** tensor yields,

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \simeq \left(\mathbf{1}+\mathbf{J}\right)\left(\mathbf{1}-\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)\right) \simeq \mathbf{1}+\frac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right) := \mathbf{1}+\boldsymbol{\Omega}$$

where the *skew-symmetric* **infinitesimal rotation** tensor $\boldsymbol{\Omega}$ has been defined as,

$$\boxed{\boldsymbol{\Omega} := \frac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right)}$$

Polar Decomposition

Linearized Rotation Tensor

The *skew-symmetric* **infinitesimal rotation** tensor satisfies the following expressions,

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T , \quad \Omega_{ab} = -\Omega_{ab}^T = \Omega_{ba}$$

We may introduce an **axial** (or **dual**) **infinitesimal strain** vector such that,

$$\boldsymbol{\Omega}\, d\mathbf{X} = \boldsymbol{\theta} \times d\mathbf{X} \quad \forall d\mathbf{X}, \quad \Omega_{ab} dX_b = \varepsilon_{abc} \theta_b dX_c$$

$$\boldsymbol{\theta} = \frac{1}{2} \operatorname{curl} \mathbf{u} = \frac{1}{2} \nabla \times \mathbf{u}, \quad \theta_a = \frac{1}{2} \varepsilon_{abc} \frac{\partial u_c}{\partial X_b} = \frac{1}{2} \varepsilon_{abc} u_{c,b}$$

Polar Decomposition

Linearized Rotation Tensor

The *matrix of components* of the *skew-symmetric* **infinitesimal rotation** tensor and the *vector of components* of the **axial** (or **dual) infinitesimal strain** vector satisfy,

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$$

$$[\boldsymbol{\theta}] = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -\Omega_{23} \\ \Omega_{13} \\ -\Omega_{12} \end{bmatrix}$$

Polar Decomposition

Linearized Polar Decomposition

The *linearized,* right and left, **polar decomposition** takes the form,

$$\mathbf{F} = \mathbf{R}\mathbf{U} \simeq (\mathbf{1} + \boldsymbol{\Omega})(\mathbf{1} + \boldsymbol{\varepsilon}) \simeq \mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$$

$$\mathbf{F} = \mathbf{v}\mathbf{R} \simeq (\mathbf{1} + \boldsymbol{\varepsilon})(\mathbf{1} + \boldsymbol{\Omega}) \simeq \mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$$

yielding,

$$d\mathbf{x} = \mathbf{F}\,d\mathbf{X} \simeq (\mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega})\,d\mathbf{X}$$

The *linearized,* either right or left, **polar decomposition** may be interpreted as the *sum* of an *infinitesimal deformation* (charaterized by the infinitesimal strain tensor) and an *infinitesimal rotation* (charaterized by the infinitesimal rotation tensor)

Linearized Tensors

Infinitesimal Strain and Rotation Tensors

$$\mathbf{E} \simeq \mathbf{e} \simeq \boldsymbol{\varepsilon} := \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right) \simeq \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^T\right), \quad \boldsymbol{\Omega} := \frac{1}{2}\left(\mathbf{J} - \mathbf{J}^T\right) \simeq \frac{1}{2}\left(\mathbf{j} - \mathbf{j}^T\right)$$

Linearized Cauchy-Green and Stretch Tensors

$$\mathbf{C} \simeq \mathbf{b} \simeq \mathbf{1} + 2\boldsymbol{\varepsilon}, \quad \mathbf{U} \simeq \mathbf{v} \simeq \mathbf{1} + \boldsymbol{\varepsilon}$$

$$\mathbf{C}^{-1} \simeq \mathbf{b}^{-1} \simeq \mathbf{1} - 2\boldsymbol{\varepsilon}, \quad \mathbf{U}^{-1} \simeq \mathbf{v}^{-1} \simeq \mathbf{1} - \boldsymbol{\varepsilon}$$

Linearized Rotation Tensor

$$\mathbf{R} \simeq \mathbf{1} + \boldsymbol{\Omega}$$

Stretches

Linearized Stretch

The stretch at a *material point* along a direction given by the *unit* vector $\mathbf{T}$ reads,

$$\lambda = \frac{ds}{dS} = \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}}$$

Using a *linear Taylor series expansion*, i.e.,

$$\lambda(x) = \sqrt{1 + 2x} \simeq \lambda(0) + \lambda'(0)\, x = 1 + x$$

The *linearized* **stretch** takes the form,

$$\lambda = \sqrt{1 + 2\mathbf{T} \cdot \mathbf{E}\mathbf{T}} \simeq 1 + \mathbf{T} \cdot \mathbf{E}\mathbf{T} \simeq 1 + \mathbf{T} \cdot \boldsymbol{\varepsilon}\mathbf{T}$$

Stretches

Linearized Stretch

The stretch at a *spatial point* along a direction given by the *unit* vector $\mathbf{t}$ reads,

$$\lambda = \frac{ds}{dS} = \frac{1}{\sqrt{1 - 2\,\mathbf{t} \cdot \mathbf{e}\,\mathbf{t}}}$$

Using a *linear Taylor series expansion*, i.e.,

$$\lambda(x) = \frac{1}{\sqrt{1 - 2x}} \simeq \lambda(0) + \lambda'(0)\,x = 1 + x$$

The *linearized* **stretch** takes the form,

$$\lambda = \frac{1}{\sqrt{1 - 2\,\mathbf{t} \cdot \mathbf{e}\,\mathbf{t}}} \simeq 1 + \mathbf{t} \cdot \mathbf{e}\,\mathbf{t} \simeq 1 + \mathbf{T} \cdot \boldsymbol{\varepsilon}\,\mathbf{T}$$

Stretches

Stretches along the Cartesian Axes

Taking *unit vectors* along the Cartesian axes,

$$[\mathbf{T}_x] = [1, \quad 0, \quad 0]^T, \quad [\mathbf{T}_y] = [0, \quad 1, \quad 0]^T, \quad [\mathbf{T}_z] = [0, \quad 0, \quad 1]^T$$

The **stretches** along the *Cartesian axes* take the form,

$$\lambda_x \simeq 1 + \mathbf{T}_x \cdot \boldsymbol{\varepsilon}\, \mathbf{T}_x = 1 + \varepsilon_{xx}, \quad \varepsilon_{xx} \simeq \lambda_x - 1 = \varepsilon_x$$
$$\lambda_y \simeq 1 + \mathbf{T}_y \cdot \boldsymbol{\varepsilon}\, \mathbf{T}_y = 1 + \varepsilon_{yy}, \quad \varepsilon_{yy} \simeq \lambda_y - 1 = \varepsilon_y$$
$$\lambda_z \simeq 1 + \mathbf{T}_z \cdot \boldsymbol{\varepsilon}\, \mathbf{T}_z = 1 + \varepsilon_{zz}, \quad \varepsilon_{zz} \simeq \lambda_z - 1 = \varepsilon_z$$

Variation of Angles

Linearized Variation of Angles

The **angle** formed by two material segments along unit vectors given by $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ on the material configuration, on the *spatial configuration* is given by,

$$\cos\theta = \frac{\mathbf{T}^{(1)}\cdot\left(\mathbf{1}+2\mathbf{E}\right)\mathbf{T}^{(2)}}{\sqrt{1+2\,\mathbf{T}^{(1)}\cdot\mathbf{E}\,\mathbf{T}^{(1)}}\sqrt{1+2\,\mathbf{T}^{(2)}\cdot\mathbf{E}\,\mathbf{T}^{(2)}}}$$

The *linearized* expression of the **angle** takes the form,

$$\cos\theta \simeq \mathbf{T}^{(1)}\cdot\left(\mathbf{1}+2\mathbf{E}\right)\mathbf{T}^{(2)} \simeq \cos\Theta + 2\,\mathbf{T}^{(1)}\cdot\boldsymbol{\varepsilon}\,\mathbf{T}^{(2)}$$

$$\cos\theta = \cos\left(\Theta+\Delta\theta\right) = \cos\Theta\cos\Delta\theta - \sin\Theta\sin\Delta\theta \simeq \cos\Theta - \Delta\theta\sin\Theta$$

$$\Delta\theta \simeq -\frac{2\,\mathbf{T}^{(1)}\cdot\boldsymbol{\varepsilon}\,\mathbf{T}^{(2)}}{\sin\Theta}$$

Variation of Angles

Linearized Variation of Angles

The **angle** formed by two spatial segments along unit vectors given by $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ on the spatial configuration, on the *material configuration* is given by,

$$\cos\Theta = \frac{\mathbf{t}^{(1)}\cdot(\mathbf{1}-2\mathbf{e})\mathbf{t}^{(2)}}{\sqrt{1-2\mathbf{t}^{(1)}\cdot\mathbf{e}\mathbf{t}^{(1)}}\sqrt{1-2\mathbf{t}^{(2)}\cdot\mathbf{e}\mathbf{t}^{(2)}}}$$

The *linearized* expression for the **angle** takes the form,

$$\cos\Theta \simeq \mathbf{t}^{(1)}\cdot(\mathbf{1}-2\mathbf{e})\mathbf{t}^{(2)} \simeq \cos\theta - 2\mathbf{t}^{(1)}\cdot\boldsymbol{\varepsilon}\mathbf{t}^{(2)}$$

$$\cos\theta = \cos(\Theta+\Delta\theta) = \cos\Theta\cos\Delta\theta - \sin\Theta\sin\Delta\theta \simeq \cos\Theta - \Delta\theta\sin\Theta$$

$$\Delta\theta \simeq -\frac{2\mathbf{t}^{(1)}\cdot\boldsymbol{\varepsilon}\mathbf{t}^{(2)}}{\sin\Theta} \simeq -\frac{2\mathbf{T}^{(1)}\cdot\boldsymbol{\varepsilon}\mathbf{T}^{(2)}}{\sin\Theta}$$

Variation of Angles

Variation of Cartesian Angles

Taking *unit vectors* along the Cartesian axes,

$$[\mathbf{T}_x] = [1, \quad 0, \quad 0]^T, \quad [\mathbf{T}_y] = [0, \quad 1, \quad 0]^T, \quad [\mathbf{T}_z] = [0, \quad 0, \quad 1]^T$$

The **change of angles** between segments oriented along the *Cartesian axes* takes the form,

$$\Delta\theta_{xy} \simeq -\frac{2\,\mathbf{T}_x \cdot \boldsymbol{\varepsilon}\,\mathbf{T}_y}{\sin\Theta_{xy}} = -2\varepsilon_{xy}, \qquad \varepsilon_{xy} \simeq -\frac{1}{2}\Delta\theta_{xy}$$

$$\Delta\theta_{xz} \simeq -\frac{2\,\mathbf{T}_x \cdot \boldsymbol{\varepsilon}\,\mathbf{T}_z}{\sin\Theta_{xz}} = -2\varepsilon_{xz}, \qquad \varepsilon_{xz} \simeq -\frac{1}{2}\Delta\theta_{xz}$$

$$\Delta\theta_{yz} \simeq -\frac{2\,\mathbf{T}_y \cdot \boldsymbol{\varepsilon}\,\mathbf{T}_z}{\sin\Theta_{yz}} = -2\varepsilon_{yz}, \qquad \varepsilon_{yz} \simeq -\frac{1}{2}\Delta\theta_{yz}$$

Matrix Notation

Tensorial and Engineering Matrix Notation

Using *tensorial notation*, the *matrix of components* of the *symmetric* **infinitesimal strain tensor** takes the form,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

Using *engineering notation*, the *matrix of components* of the *symmetric* **infinitesimal strain tensor** takes the form,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{z} \end{bmatrix}$$

Matrix Notation

Tensorial and Engineering Matrix Notation

Using *tensorial notation*, the associated *vector of components* of the *symmetric* **infinitesimal strain tensor** takes the form,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx}, & \varepsilon_{yy}, & \varepsilon_{zz}, & 2\varepsilon_{xy}, & 2\varepsilon_{xz}, & 2\varepsilon_{yz} \end{bmatrix}^T$$

$$= \begin{bmatrix} \varepsilon_{11}, & \varepsilon_{22}, & \varepsilon_{33}, & 2\varepsilon_{12}, & 2\varepsilon_{13}, & 2\varepsilon_{23} \end{bmatrix}^T$$

Using *engineering notation*, the *vector of components* of the *symmetric* **infinitesimal strain tensor** takes the form,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{x}, & \varepsilon_{y}, & \varepsilon_{z}, & \gamma_{xy}, & \gamma_{xz}, & \gamma_{yz} \end{bmatrix}^T$$

Material Time Derivatives

Material Time Derivative of the Infinitesimal Strain

Within the *infinitesimal strains framework*, the *deformation rate tensor*, the *material time derivative of the Green-Lagrange strain tensor* and the *material time derivative of the Almansi strain tensor* are approximated by the *time derivative of the infinitesimal strain tensor*, yielding

$$\mathbf{d} \simeq \dot{\mathbf{E}} \simeq \dot{\mathbf{e}} \simeq \dot{\boldsymbol{\varepsilon}}$$

Assignment 4.1

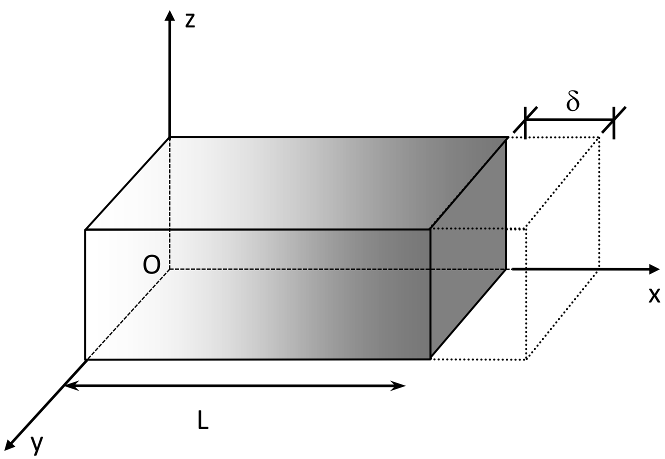
Assignment 4.1

The solid of the figure is subjected to a *uniform* strain state through a uniaxial traction/compression test, such that,

$$u_x(0,y,z)=0, \quad u_x(L,y,z)=\delta \quad \forall\ y,z$$

$$u_y(x,y,z)=u_z(x,y,z)=0 \quad \forall\ x,y,z$$

1) Obtain the *motion equations* and *displacement* field, indicating the range of values for δ/L.
2) Plot the *x-component* of the *Green-Lagrange*, *Almansi* and *infinitesimal strain* tensors vs the normalized displacement δ/L.



Assignment 4.1

Assignment 4.1

The solid of the figure is subjected to a *uniform* strain state through a uniaxial traction/compression test, such that,

$$u_x(0,y,z)=0, \quad u_x(L,y,z)=\delta \quad \forall\ y,z$$

$$u_y(x,y,z)=u_z(x,y,z)=0 \quad \forall\ x,y,z$$

As the strain state is *uniform* the deformation gradient and the displacement gradient are *uniform* and the displacement field is *linear*,

$$\mathbf{F}(\mathbf{X},t)=\mathbf{F}(t)=\mathbf{1}+\mathbf{J}(t) \qquad \forall\, t, \forall\, \mathbf{X}$$

$$d\mathbf{U}=\mathbf{J}(t)\,d\mathbf{X} \qquad \Rightarrow \qquad \mathbf{U}=\mathbf{J}(t)\,\mathbf{X}+\mathbf{C}(t)$$

Assignment 4.1

The **displacement** vector field is *linear* and its components are given by,

$$
\begin{aligned}
U_X &= J_{11}X + J_{12}Y + J_{13}Z + C_1 \\
U_Y &= J_{21}X + J_{22}Y + J_{23}Z + C_2 \\
U_Z &= J_{31}X + J_{32}Y + J_{33}Z + C_3
\end{aligned}
$$

Imposing the *boundary conditions*, the integration constants read,

$$
\begin{aligned}
&J_{11} = \delta/L, \quad J_{12} = J_{13} = C_1 = 0 \\
&J_{21} = J_{22} = J_{23} = C_2 = 0 \\
&J_{31} = J_{32} = J_{33} = C_3 = 0
\end{aligned}
$$

Assignment 4.1

The components of the **displacement** vector field are given by,

$$[\mathbf{u}] = \begin{bmatrix} \dfrac{\delta}{L} X & 0 & 0 \end{bmatrix}^T$$

The components of the **motion equation** are given by,

$$[\mathbf{x}] = \begin{bmatrix} \left(1 + \dfrac{\delta}{L}\right) X & Y & Z \end{bmatrix}^T$$

Assignment 4.1

The components of the **deformation gradient** are given by,

$$[\mathbf{F}] = \begin{bmatrix} 1+\delta/L & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The *Jacobian* has to be positive at any time, yielding the following range of admissible values,

$$\det \mathbf{F} = 1 + \frac{\delta}{L} > 0 \quad \Rightarrow \quad \delta > -L \Rightarrow \quad -1 < \frac{\delta}{L} < \infty$$

Assignment 4.1

The components of the **Green-Lagrange**, **Almansi** and **infinitesimal** strain tensors are given by,

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\delta}{L}\left(1+\dfrac{\delta}{2L}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{e}] = \frac{1}{\left(1+\dfrac{\delta}{L}\right)^2} \begin{bmatrix} \dfrac{\delta}{L}\left(1+\dfrac{\delta}{2L}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} \delta/L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assignment 4.1

The x-components of the **Green-Lagrange**, **Almansi** and **infinite-simal** strain tensors satisfy,

$$\lim_{\delta/L \to -1} E_{xx} = \lim_{\delta/L \to -1} \frac{\delta}{L}\left(1+\frac{\delta}{2L}\right) = -\frac{1}{2}$$

$$\lim_{\delta/L \to -1} e_{xx} = \lim_{\delta/L \to -1} \frac{\frac{\delta}{L}\left(1+\frac{\delta}{2L}\right)}{\left(1+\frac{\delta}{L}\right)^2} = -\infty$$

$$\lim_{\delta/L \to -1} \varepsilon_{xx} = \lim_{\delta/L \to -1} \frac{\delta}{L} = -1$$

Assignment 4.1

The x-components of the **Green-Lagrange**, **Almansi** and **infinite-simal** strain tensors satisfy,

$$\lim_{\delta/L\to\infty} E_{xx} = \lim_{\delta/L\to\infty} \frac{\delta}{L}\left(1+\frac{\delta}{2L}\right) = \infty$$

$$\lim_{\delta/L\to\infty} e_{xx} = \lim_{\delta/L\to\infty} \frac{\frac{\delta}{L}\left(1+\frac{\delta}{2L}\right)}{\left(1+\frac{\delta}{L}\right)^2} = \frac{1}{2}$$

$$\lim_{\delta/L\to\infty} \varepsilon_{xx} = \lim_{\delta/L\to\infty} \frac{\delta}{L} = \infty$$

Assignment 4.1

The x-components of the **Green-Lagrange**, **Almansi** and **infinite-simal** strain tensors satisfy,

$$\lim_{\delta/L \to -1} \frac{dE_{xx}}{d\left(\delta/L\right)} = \lim_{\delta/L \to -1} \left(1+\frac{\delta}{L}\right) = 0$$

$$\lim_{\delta/L \to -1} \frac{de_{xx}}{d\left(\delta/L\right)} = \lim_{\delta/L \to -1} \frac{1}{\left(1+\dfrac{\delta}{L}\right)^3} = \infty$$

$$\lim_{\delta/L \to -1} \frac{d\varepsilon_{xx}}{d\left(\delta/L\right)} = 1$$

Assignment 4.1

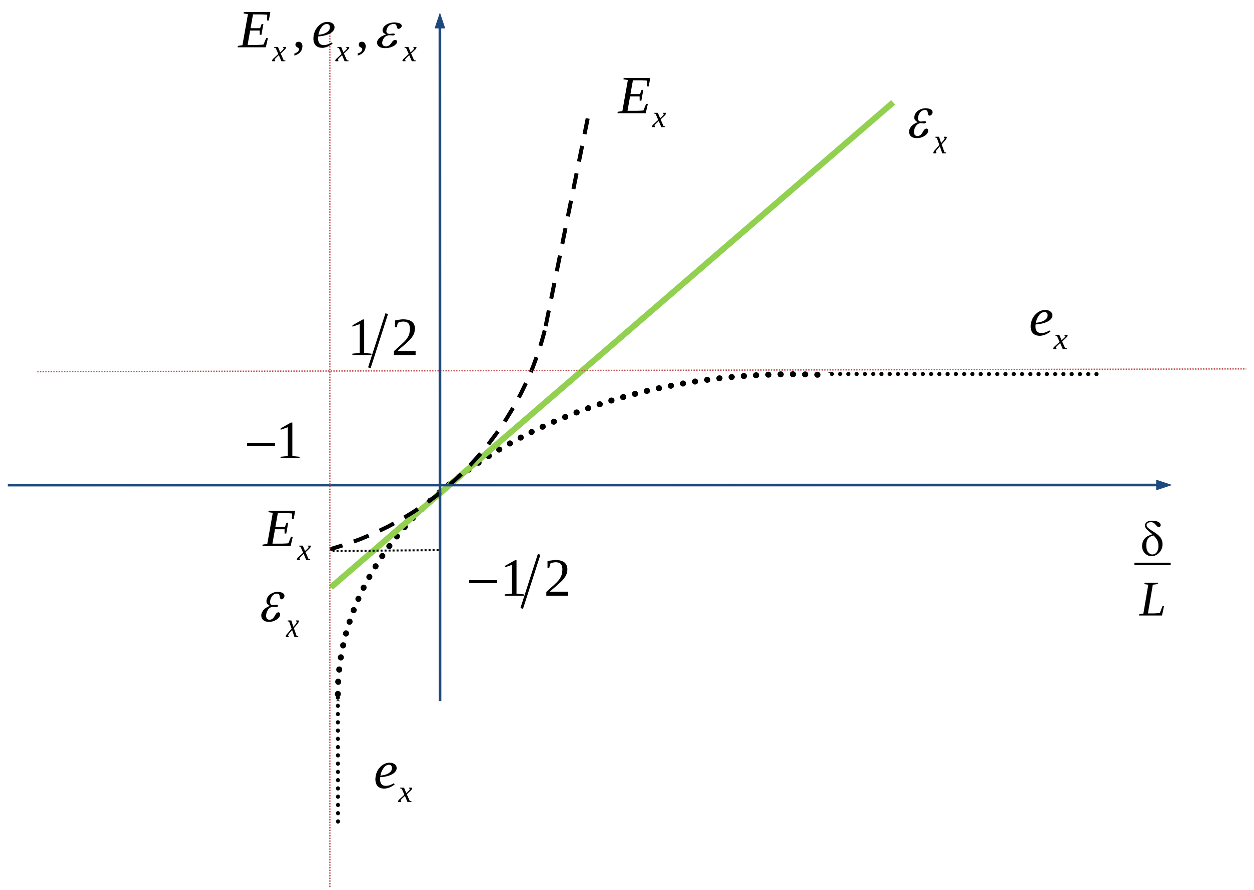
The x-components of the **Green-Lagrange**, **Almansi** and **infinite-simal** strain tensors satisfy,

$$\lim_{\delta/L\to\infty} \frac{dE_{xx}}{d\left(\delta/L\right)} = \lim_{\delta/L\to\infty} \left(1+\frac{\delta}{L}\right) = \infty$$

$$\lim_{\delta/L\to\infty} \frac{de_{xx}}{d\left(\delta/L\right)} = \lim_{\delta/L\to\infty} \frac{1}{\left(1+\frac{\delta}{L}\right)^3} = 0$$

$$\lim_{\delta/L\to\infty} \frac{d\varepsilon_{xx}}{d\left(\delta/L\right)} = 1$$

Assignment 4.1

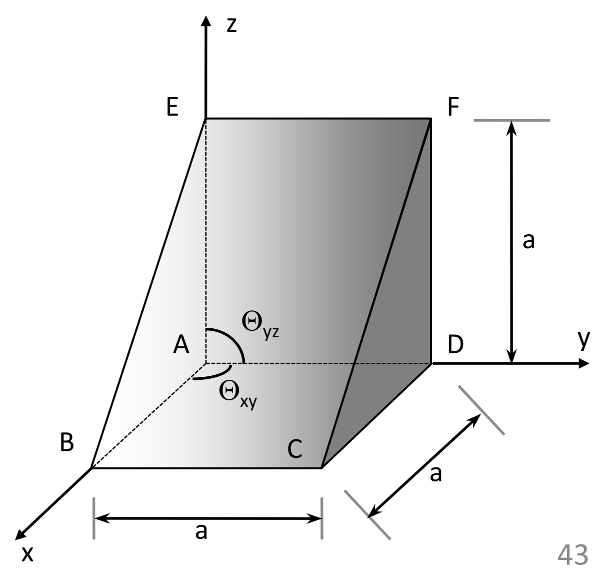


Assignment 4.2

Assignment 4.2 [Classwork]

The solid of the figure is subjected to a *uniform* strain state such that: (1) Point A does not moves, point B remains on the x-axis and point E remains on z-axis; (2) The volume remains constant; (3) The angle Θ_{xy} remains constant; (4) The angle Θ_{yz} increases in r radians; (5) The length of the segment AF becomes 1+p times the initial one; (6) The length of the segment AC becomes 1-q times the initial one.

Obtain the *infinitesimal strain tensor*, the *infinitesimal rotation tensor* and the *displacements* in terms of p, q and r.



Assignment 4.2

Assignment 4.2 [Classwork]

As the strain state is *uniform* the deformation gradient and the displacement gradient are *uniform* and the displacement field is *linear*,

$$\mathbf{F}(\mathbf{X},t)=\mathbf{F}(t)=\mathbf{1}+\mathbf{J}(t) \qquad \forall\, t, \forall\, \mathbf{X}$$

$$d\mathbf{U}=\mathbf{J}(t)\,d\mathbf{X} \qquad \Rightarrow \qquad \mathbf{U}=\mathbf{J}(t)\,\mathbf{X}+\mathbf{C}(t)$$

The components of the linear displacement field take the form,

$$\begin{aligned} U_X &= J_{11}X + J_{12}Y + J_{13}Z + C_1 \\ U_Y &= J_{21}X + J_{22}Y + J_{23}Z + C_2 \\ U_Z &= J_{31}X + J_{32}Y + J_{33}Z + C_3 \end{aligned}$$

Assignment 4.2

Condition 1. Point A does not moves, point B remains on the x-axis and point E remains on the z-axis.

$$\mathbf{U}_A = \mathbf{J}(t)\,\mathbf{X}_A + \mathbf{C}(t) = \mathbf{0} \quad \Rightarrow \quad \mathbf{C}(t) = \mathbf{0}$$

$$\mathbf{U}_B = \mathbf{J}(t)\,\mathbf{X}_B \quad \Rightarrow \quad J_{21}(t) = J_{31}(t) = 0$$

$$\mathbf{U}_E = \mathbf{J}(t)\,\mathbf{X}_E \quad \Rightarrow \quad J_{13}(t) = J_{23}(t) = 0$$

$$\left[\mathbf{J}(t)\right] = \begin{bmatrix} J_{11} & J_{12} & 0 \\ 0 & J_{22} & 0 \\ 0 & J_{32} & J_{33} \end{bmatrix}, \qquad \left[\boldsymbol{\varepsilon}(t)\right] = \begin{bmatrix} J_{11} & J_{12}/2 & 0 \\ J_{12}/2 & J_{22} & J_{32}/2 \\ 0 & J_{32}/2 & J_{33} \end{bmatrix}$$

Assignment 4.2

Condition 2. The volume remains constant.

As the strain is *uniform*, if the volume remains constant, the medium is *incompressible* yielding,

$$\mathrm{tr}\left[\boldsymbol{\varepsilon}(t)\right] = J_{11}(t) + J_{22}(t) + J_{33}(t) = 0$$

Condition 3. The angle Θ_{xy} remains constant.

$$\Delta\Theta_{xy} = 0 \quad \Rightarrow \quad \varepsilon_{xy} = J_{12}/2 = 0 \quad \Rightarrow \quad J_{12} = 0$$

Condition 4. The angle Θ_{yz} increases in r radians.

$$\Delta\Theta_{yz} = r \quad \Rightarrow \quad \varepsilon_{yz} = J_{32}/2 = -r/2 \quad \Rightarrow \quad J_{32} = -r$$

Assignment 4.2

Then the *gradient of displacements* and the *infinitesimal strain tensor* take the form,

$$\left[\mathbf{J}(t)\right] = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & -r & -J_{11} - J_{22} \end{bmatrix}$$

$$\left[\boldsymbol{\varepsilon}(t)\right] = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & -r/2 \\ 0 & -r/2 & -J_{11} - J_{22} \end{bmatrix}$$

Assignment 4.2

Condition 5. The length of the segment AF becomes 1+p times the initial one.

As the strain is *uniform*, the stretch will be also *uniform* yielding,

$$\lambda_{AF}(t) = 1 + \mathbf{T}_{AF} \cdot \boldsymbol{\varepsilon}(t)\mathbf{T}_{AF} = 1 + p$$

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} J_{11} & 0 & 0 \\ J_{12}/2 & J_{22} & -r/2 \\ 0 & -r/2 & -J_{11} - J_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -\frac{1}{2} r + J_{11} = p$$

$$J_{11} = -r - 2p$$

Assignment 4.2

Then the *gradient of displacements* and the *infinitesimal strain tensor* take the form,

$$\left[\mathbf{J}(t)\right] = \begin{bmatrix} -r-2p & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & -r & r+2p-J_{22} \end{bmatrix}$$

$$\left[\boldsymbol{\varepsilon}(t)\right] = \begin{bmatrix} -r-2p & 0 & 0 \\ 0 & J_{22} & -r/2 \\ 0 & -r/2 & r+2p-J_{22} \end{bmatrix}$$

Assignment 4.2

Condition 6. The length of the segment AC becomes 1-q times the initial one.

As the strain is *uniform*, the stretch will be also *uniform* yielding,

$$\lambda_{AC}(t) = 1 + \mathbf{T}_{AC} \cdot \boldsymbol{\varepsilon}(t)\mathbf{T}_{AC} = 1 - q$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -r-2p & 0 & 0 \\ 0 & J_{22} & -r/2 \\ 0 & -r/2 & r+2p-J_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -q$$

$$\frac{1}{2}\left(-r-2p+J_{22}\right) = -q \quad \Rightarrow \quad J_{22} = r+2p-2q$$

Assignment 4.2

Then the *gradient of displacements* and the *infinitesimal strain tensor* take the form,

$$[\mathbf{J}(t)] = \begin{bmatrix} -r-2p & 0 & 0 \\ 0 & r+2p-2q & 0 \\ 0 & -r & 2q \end{bmatrix}$$

$$[\boldsymbol{\varepsilon}(t)] = \begin{bmatrix} -r-2p & 0 & 0 \\ 0 & r+2p-2q & -r/2 \\ 0 & -r/2 & 2q \end{bmatrix}$$

Assignment 4.2

The *infinitesimal rotation tensor* takes the form,

$$[\boldsymbol{\Omega}(t)] = \frac{1}{2}\left[\mathbf{J}(t) - \mathbf{J}^T(t)\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r/2 \\ 0 & -r/2 & 0 \end{bmatrix}$$

The *displacement vector field* takes the form,

$$[\mathbf{U}(t)] = \begin{bmatrix} -r-2p & 0 & 0 \\ 0 & r+2p-2q & 0 \\ 0 & -r & 2q \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Compatibility Equations

Compatibility Equations

Given an *arbitrary displacement* vector field $\mathbf{u}$ (with the required continuity degree), it is always possible to get the associated symmetrical **infinitesimal strain** tensor defined as,

$$\boldsymbol{\varepsilon} := \frac{1}{2}\left(\nabla \otimes \mathbf{u} + \left(\nabla \otimes \mathbf{u}\right)^T\right), \quad \varepsilon_{ab} = \frac{1}{2}\left(u_{a,b} + u_{b,a}\right)$$

The opposite statement is *not* true.

Compatibility Equations

Compatibility Equations

Given an *arbitrary symmetrical infinitesimal strain tensor* $\boldsymbol{\varepsilon}$, i.e. given six *arbitrary* functions as the matrix components of a symmetric second-order tensor, it is *not* always possible to get the associated **displacement** vector field such that,

$$\boldsymbol{\varepsilon} := \frac{1}{2}\left(\nabla \otimes \mathbf{u} + \left(\nabla \otimes \mathbf{u}\right)^T\right), \quad \varepsilon_{ab} = \frac{1}{2}\left(u_{a,b} + u_{b,a}\right)$$

This gives a system of 6 PDE with 3 unknowns and, hence, the system *may not be integrable* and *may not have a solution*.

In order to get an *integrable system* of PDE and guarantee the *existence* of the *displacement* vector field, the infinitesimal strain tensor has to satisfy the **compatibility equations**.

Compatibility Equations

Compatibility Equations

The **compatibility equations** for the infinitesimal strain tensor may be written as,

$$\mathbf{S} := \nabla \times \left(\nabla \times \boldsymbol{\varepsilon}^T \right)^T = \nabla \times \left(\nabla \times \boldsymbol{\varepsilon} \right)^T = \mathbf{0}$$

$$\mathbf{S} := \mathbf{e}_f \times \left(\varepsilon_{cde} \varepsilon_{bd,c} \mathbf{e}_b \otimes \mathbf{e}_e \right)_{,f} = \varepsilon_{fba} \varepsilon_{cde} \varepsilon_{bd,cf} \mathbf{e}_a \otimes \mathbf{e}_e$$

$$S_{xx} \stackrel{def}{=} \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} - 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = 0 \qquad S_{xy} \stackrel{def}{=} -\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial}{\partial z} \left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right) = 0$$

$$S_{yy} \stackrel{def}{=} \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} - 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0 \qquad S_{xz} \stackrel{def}{=} -\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} + \frac{\partial}{\partial y} \left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{xz}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) = 0$$

$$S_{zz} \stackrel{def}{=} \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \qquad S_{yz} \stackrel{def}{=} -\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial}{\partial x} \left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{xz}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) = 0$$

Compatibility Equations

Compatibility Equations

The **compatibility tensor** S satisfies the following equation,

$$\operatorname{div}\mathbf{S} = \nabla\cdot\mathbf{S} := \nabla\cdot\left(\nabla\times\left(\nabla\times\boldsymbol{\varepsilon}\right)^{T}\right) = \mathbf{0}$$

$$\begin{cases} \dfrac{\partial S_{xx}}{\partial x} + \dfrac{\partial S_{xy}}{\partial y} + \dfrac{\partial S_{xz}}{\partial z} = 0 \\ \dfrac{\partial S_{xy}}{\partial x} + \dfrac{\partial S_{yy}}{\partial y} + \dfrac{\partial S_{yz}}{\partial z} = 0 \\ \dfrac{\partial S_{xz}}{\partial x} + \dfrac{\partial S_{yz}}{\partial y} + \dfrac{\partial S_{zz}}{\partial z} = 0 \end{cases}$$