
# Continuum Mechanics

## Chapter 11
## Newtonian Fluids

C. Agelet de Saracibar
ETS Ingenieros de Caminos, Canales y Puertos, Universidad Politécnica de Cataluña (UPC), Barcelona, Spain
International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain

# Contents

## Chapter 11 · Newtonian Fluids

# Constitutive Equation

## Constitutive Equation for Newtonian Fluids

The *constitutive equation* for a *Newtonian fluid* may be written as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda(p,\theta)(\mathrm{tr}\,\mathbf{d})\mathbf{1} + 2\mu(p,\theta)\mathbf{d}$$

where $\lambda(p,\theta), \mu(p,\theta) \geq 0$ are two scalar-valued functions denoted as *dynamic viscosities.*

# Constitutive Equation

## Constitutive Equation for Newtonian Fluids

The *volumetric part* of the *constitutive equation* for a *Newtonian fluid* may be written as,

$$\begin{aligned}\operatorname{tr}\boldsymbol{\sigma} &= -p\operatorname{tr}\mathbf{1} + \lambda(p,\theta)(\operatorname{tr}\mathbf{d})\operatorname{tr}\mathbf{1} + 2\mu(p,\theta)\operatorname{tr}\mathbf{d} \\ &= -3p + \left(3\lambda(p,\theta) + 2\mu(p,\theta)\right)\operatorname{tr}\mathbf{d} \\ &= -3\overline{p}\end{aligned}$$

$$\begin{aligned}\overline{p} &= p - \left(\lambda(p,\theta) + \frac{2}{3}\mu(p,\theta)\right)\operatorname{tr}\mathbf{d} \\ &= p - K(p,\theta)\operatorname{tr}\mathbf{d}\end{aligned}$$

where $K(p,\theta) \geq 0$ is a non-negative scalar-valued function denoted as *dynamic bulk viscosity.*

# Constitutive Equation

## Thermodynamic Pressure vs Mean Pressure

The *thermodynamic pressure* and the *mean pressure* are related through the expression,

$$p = \overline{p} + K\left(p,\theta\right)\mathrm{tr}\,\mathbf{d} = \overline{p} + K\left(p,\theta\right)\mathrm{div}\,\mathbf{v}$$

The thermodynamic pressure will be equal to the mean pressure under any of the following conditions:

- Zero bulk viscosity (Stokes condition)
- Incompressible fluid
- Uniform velocity field
- Fluid at rest

# Governing Equations

## Governing Equations

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Balance of energy**

1 $$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \operatorname{div} \mathbf{q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho \theta \dot{\eta} - \rho r + \operatorname{div} \mathbf{q} \geq 0, \quad \mathcal{D}_{con} := -\mathbf{q} \cdot \operatorname{grad} \theta \geq 0$$

1 1

# Governing Equations

## Constitutive Equations

- **Thermo-mechanical constitutive equation for the stresses**

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda(p,\theta)(\operatorname{tr}\mathbf{d})\mathbf{1} + 2\mu(p,\theta)\mathbf{d}$$ 1

- **Thermo-mechanical constitutive equation for the entropy**

1 $$\eta = \eta(\mathbf{d}, p, \theta)$$

- **Thermal constitutive equation.** Fourier's law

3 $$\mathbf{q} = \mathbf{q}(\mathbf{v},\theta) = -\mathbf{k}(\mathbf{v},\theta)\operatorname{grad}\theta$$

- **Caloric state equation**

1 $$e = e(\rho,\theta)$$

- **Kinetic state equation**

1 $$\rho = \rho(p,\theta)$$

# Governing Equations

## Navier-Stokes Equation

The *Navier-Stokes equation* is obtained substituting the constitutive and geometric equations into the first Cauchy's motion equation, and assuming the viscosities are constants, yields,

$$\begin{aligned}
\operatorname{div}\boldsymbol{\sigma}+\rho\,\mathbf{b} &= \operatorname{div}\left(-p\mathbf{1}+\lambda\left(\operatorname{tr}\mathbf{d}\right)\mathbf{1}+2\mu\,\mathbf{d}\right)+\rho\,\mathbf{b} \\
&= -\operatorname{grad}p+\left(\lambda+\mu\right)\operatorname{grad}\left(\operatorname{div}\mathbf{v}\right)+\mu\operatorname{div}\left(\operatorname{grad}\mathbf{v}\right)+\rho\,\mathbf{b} \\
&= \rho\dot{\mathbf{v}} = \rho\frac{\partial\mathbf{v}}{\partial t}+\rho\left(\operatorname{grad}\mathbf{v}\right)\mathbf{v}
\end{aligned}$$

$$\boxed{-\operatorname{grad}p+\left(\lambda+\mu\right)\operatorname{grad}\left(\operatorname{div}\mathbf{v}\right)+\mu\operatorname{div}\left(\operatorname{grad}\mathbf{v}\right)+\rho\,\mathbf{b}=\rho\dot{\mathbf{v}}}$$

# Governing Equations

## Stress Power

The *stress power per unit of spatial volume* for a Newtonian fluid may be written as,

$$\boldsymbol{\sigma} : \mathbf{d} = -p\mathbf{1} : \mathbf{d} + \lambda \left( \operatorname{tr} \mathbf{d} \right) \mathbf{1} : \mathbf{d} + 2\mu \, \mathbf{d} : \mathbf{d}$$

$$= -p \operatorname{tr} \mathbf{d} + \lambda \left( \operatorname{tr} \mathbf{d} \right)^2 + 2\mu \, \mathbf{d} : \mathbf{d}$$

$$\boldsymbol{\sigma} : \mathbf{d} = -p \operatorname{tr} \mathbf{d} + \lambda \left( \operatorname{tr} \mathbf{d} \right)^2 + 2\mu \left( \frac{1}{3} \left( \operatorname{tr} \mathbf{d} \right)^2 + \operatorname{dev} \mathbf{d} : \operatorname{dev} \mathbf{d} \right)$$

$$= -p \operatorname{tr} \mathbf{d} + \left( \lambda + \frac{2}{3} \mu \right) \left( \operatorname{tr} \mathbf{d} \right)^2 + 2\mu \operatorname{dev} \mathbf{d} : \operatorname{dev} \mathbf{d}$$

$$= -p \operatorname{tr} \mathbf{d} + K \left( \operatorname{tr} \mathbf{d} \right)^2 + 2\mu \operatorname{dev} \mathbf{d} : \operatorname{dev} \mathbf{d}$$

# Governing Equations

## Mechanical Problem

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Mechanical constitutive equation** (thermal independent)

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda(\operatorname{tr}\mathbf{d})\mathbf{1} + 2\mu\,\mathbf{d}$$ 1

- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho(p)$$

# Governing Equations

## Mechanical Problem

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{div}\, \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 6

- **Mechanical constitutive equation**

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda \left( \mathrm{tr}\, \mathbf{d} \right) \mathbf{1} + 2\mu \mathbf{d}$$ 1

- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho\left( p \right)$$

# Governing Equations

## Mechanical Problem

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Navier-Stokes equation

$$-\operatorname{grad} p + (\lambda + \mu) \operatorname{grad}(\operatorname{div} \mathbf{v}) + \mu \operatorname{div}(\operatorname{grad} \mathbf{v}) + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$

3 1

- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho(p)$$

# Governing Equations

## Incompressible Mechanical Problem

- **Conservation of mass.** Mass continuity

1 $$\operatorname{div} \mathbf{v} = 0$$ 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho_0 \, \mathbf{b} = \rho_0 \, \dot{\mathbf{v}}$$ 6

- **Mechanical constitutive equation**

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu \, \mathbf{d}$$ 1

# Governing Equations

## Incompressible Mechanical Problem

- **Conservation of mass.** Mass continuity

1 $$\operatorname{div}\mathbf{v} = 0$$ 3

- **Balance of linear momentum.** Euler equation

3 $$-\operatorname{grad} p + \mu \operatorname{div}\left(\operatorname{grad}\mathbf{v}\right) + \rho_0\,\mathbf{b} = \rho_0\,\dot{\mathbf{v}}$$ 1

# Governing Equations

## Thermal Problem

- **Balance of energy**

$$\rho \dot{e} = -p \,\mathrm{tr}\,\mathbf{d} + K \left(\mathrm{tr}\,\mathbf{d}\right)^2 + 2\mu \,\mathrm{dev}\,\mathbf{d} : \mathrm{dev}\,\mathbf{d} + \rho r - \mathrm{div}\,\mathbf{q}$$

1     1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho \theta \dot{\eta} - \rho r + \mathrm{div}\,\mathbf{q} \geq 0, \quad \mathcal{D}_{con} := -\mathbf{q} \cdot \mathrm{grad}\,\theta \geq 0$$

1 1

- **Thermo-mechanical constitutive equation for the entropy**

1

$$\eta = \eta\left(\mathbf{d}, p, \theta\right)$$

- **Thermal constitutive equation.** Fourier's law

3

$$\mathbf{q} = \mathbf{q}\left(\mathbf{v}, \theta\right) = -\mathbf{k}\left(\mathbf{v}, \theta\right) \mathrm{grad}\,\theta$$

- **Caloric state equation**

1

$$e = e\left(\rho, \theta\right)$$

# Governing Equations

## Thermal Problem

- Balance of energy

$$\rho\,\dot{e} = -p\,\mathrm{tr}\,\mathbf{d} + K\left(\mathrm{tr}\,\mathbf{d}\right)^2 + 2\mu\,\mathrm{dev}\,\mathbf{d} : \mathrm{dev}\,\mathbf{d} + \rho\,r + \mathrm{div}\left(\mathbf{k}\left(\mathbf{v},\theta\right)\mathrm{grad}\,\theta\right)$$

1 1 1

- Clausius-Planck and heat conduction inequalities

$$\mathcal{D}_{int} := \rho\,\theta\,\dot{\eta} - \rho\,r - \mathrm{div}\left(\mathbf{k}\left(\mathbf{v},\theta\right)\mathrm{grad}\,\theta\right) \geq 0,$$

$$\mathcal{D}_{con} := \mathrm{grad}\,\theta \cdot \mathbf{k}\left(\mathbf{v},\theta\right)\mathrm{grad}\,\theta \geq 0$$

1

- Thermo-mechanical constitutive equation for the entropy

1

$$\eta = \eta\left(\mathbf{d}, p, \theta\right)$$

- Caloric state equation

1

$$e = e\left(\rho,\theta\right)$$

# Governing Equations

## Incompressible Thermal Problem

- **Balance of energy**

1 $$\rho_0\, \dot{e} = 2\mu\, \operatorname{dev} \mathbf{d} : \operatorname{dev} \mathbf{d} + \rho_0\, r + \operatorname{div}\left(\mathbf{k}\left(\mathbf{v}, \theta\right) \operatorname{grad} \theta\right)$$ 1 1

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho_0\, \theta \dot{\eta} - \rho_0\, r - \operatorname{div}\left(\mathbf{k}\left(\mathbf{v}, \theta\right) \operatorname{grad} \theta\right) \geq 0,$$

$$\mathcal{D}_{con} := \operatorname{grad} \theta \cdot \mathbf{k}\left(\mathbf{v}, \theta\right) \operatorname{grad} \theta \geq 0$$ 1

- **Thermo-mechanical constitutive equation for the entropy**

1 $$\eta = \eta\left(\mathbf{d}, p, \theta\right)$$

- **Caloric state equation**

1 $$e = e\left(\theta\right)$$

# Boundary Conditions

## Prescribed Velocity Boundary Conditions

- **Impenetrability condition**

The *impenetrability* boundary condition may be written as,

$$\left(\mathbf{v}\left(\mathbf{x},t\right)-\mathbf{v}^{*}\left(\mathbf{x},t\right)\right)\cdot\mathbf{n}\left(\mathbf{x},t\right)=0 \quad \forall\mathbf{x}\in\Gamma_{v}^{(1)},\forall t$$

where $\mathbf{v}^{*}\left(\mathbf{x},t\right)$ is velocity of the boundary.

If the boundary does not moves, then the *impenetrability* condition takes the form,

$$\mathbf{v}\left(\mathbf{x},t\right)\cdot\mathbf{n}\left(\mathbf{x},t\right)=0 \quad \forall\mathbf{x}\in\Gamma_{v}^{(1)},\forall t$$

The *impenetrability condition* is used for *ideal fluids*, while for *viscous fluids* a more restrictive boundary condition, denoted as *adherence condition*, is used.

# Boundary Conditions

## Prescribed Velocity Boundary Conditions

- **Adherence condition**

The *adherence* boundary condition may be written as,

$$\mathbf{v}(\mathbf{x},t)-\mathbf{v}^*(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x}\in \Gamma_v^{(1)}, \forall t$$

where $\mathbf{v}^*(\mathbf{x},t)$ is velocity of the boundary.

If the boundary does not moves, then the *adherence* condition takes the form,

$$\mathbf{v}(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x}\in \Gamma_v^{(1)}, \forall t$$

The *adherence condition* is used for *viscous fluids*, such as Newtonian fluids, while for *ideal fluids* the less restrictive *impenetrability condition*, is used.

# Boundary Conditions

## Prescribed Velocity Boundary Conditions

- **Prescribed velocity**

The *prescribed velocity* boundary condition may be written as,

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}^*(\mathbf{x},t) \quad \forall \mathbf{x} \in \Gamma_v^{(2)}, \forall t$$

where $\mathbf{v}^*(\mathbf{x},t)$ is the prescribed velocity.

# Boundary Conditions

## Prescribed Pressure Boundary Conditions

- **Prescribed pressure**

The *prescribed pressure* boundary condition may be written as,

$$p(\mathbf{x},t) = p^*(\mathbf{x},t) \quad \forall \mathbf{x} \in \Gamma_p^{(1)}, \forall t$$

where $p^*(\mathbf{x},t)$ is the prescribed thermodynamic pressure.

# Boundary Conditions

## Prescribed Pressure Boundary Conditions

- **Free surface**

At the *free surface* the pressure is prescribed to the *environmental pressure*, yielding,

$$p(\mathbf{x},t) = p_{atm}(\mathbf{x},t) \quad \forall \mathbf{x} \in \Gamma_p^{(2)}, \forall t$$

# Boundary Conditions

## Prescribed Traction Boundary Conditions

- **Prescribed traction**

The *prescribed traction boundary condition* may be written as,

$$\mathbf{t}(\mathbf{x},t)=\boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}(\mathbf{x},t)=\mathbf{t}^*(\mathbf{x},t) \quad \forall\mathbf{x}\in\Gamma_{\sigma}^{(1)},\forall t$$

# Boundary Conditions

## Prescribed Traction Boundary Conditions

- **Equilibrium at the interface**

At the *interface* between two immiscible fluid the *equilibrium boundary condition* may be written as,

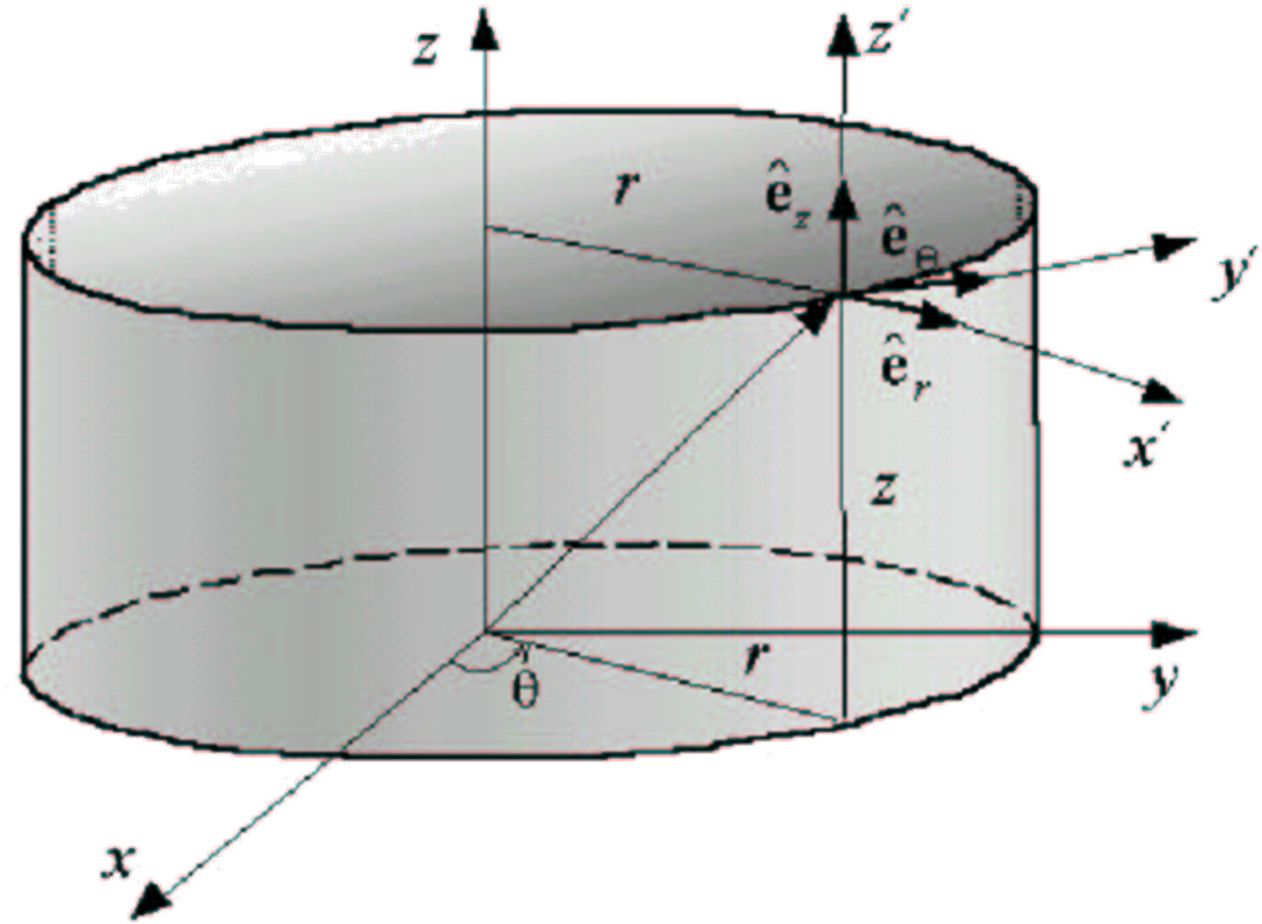
$$\mathbf{t}^{(1)}(\mathbf{x},t)+\mathbf{t}^{(2)}(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x}\in\Gamma_{\sigma}^{(1,2)}, \forall t$$

$$\boldsymbol{\sigma}^{(1)}(\mathbf{x},t)\mathbf{n}^{(1)}(\mathbf{x},t)+\boldsymbol{\sigma}^{(2)}(\mathbf{x},t)\mathbf{n}^{(2)}(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x}\in\Gamma_{\sigma}^{(1,2)}, \forall t$$

$$\left(\boldsymbol{\sigma}^{(1)}(\mathbf{x},t)-\boldsymbol{\sigma}^{(2)}(\mathbf{x},t)\right)\mathbf{n}(\mathbf{x},t)=\mathbf{0} \quad \forall \mathbf{x}\in\Gamma_{\sigma}^{(1,2)}, \forall t$$

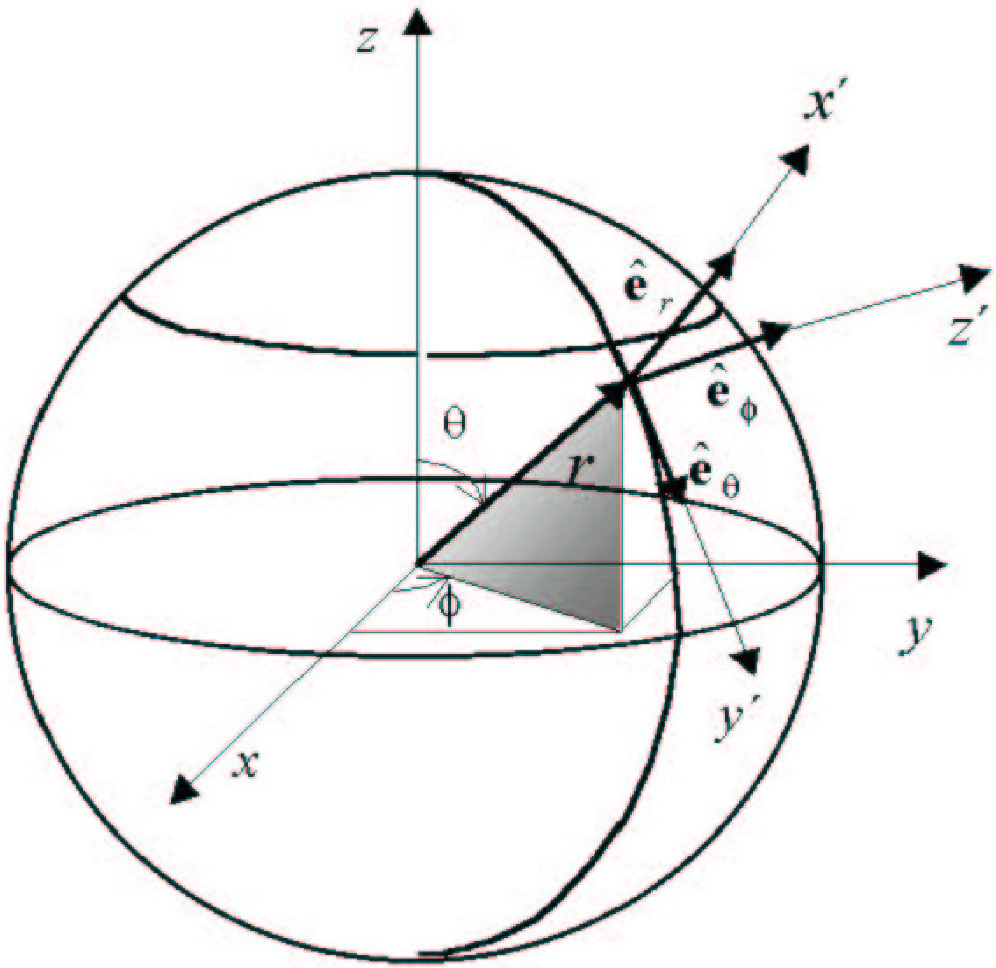
# Cylindrical Coordinates

## Cylindrical Coordinates



# Spherical Coordinates

## Spherical Coordinates



# Mass Continuity Equation

## Mass Continuity Equation

- Mass Continuity in Equation Cartesian Coordinates

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho v_x\right)+\frac{\partial}{\partial y}\left(\rho v_y\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)=0$$

- Mass Continuity Equation in Cylindrical Coordinates

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho v_\theta\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)=0$$

- Mass Continuity Equation in Spherical Coordinates

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho v_\theta \sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\rho v_\phi\right)=0$$

# Incompressible Navier-Stokes Equation

## Incompressible Navier-Stokes Equation in Cartesian Coordinates

$$-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho b_x=\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)$$

$$-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho b_y=\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)$$

$$-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho b_z=\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)$$

# Incompressible Navier-Stokes Equation

## Incompressible Navier-Stokes Equation in Cylindrical Coordinates

$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho b_r=$$

$$=\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho b_\theta=$$

$$=\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho b_z=\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

# Incompressible Navier-Stokes Equation

## Incompressible Navier-Stokes Equation in Spherical Coordinates

$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$+\rho b_r=\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$+\rho b_\theta=\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)$$

$$-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)$$

$$+\rho b_\phi=\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r v_\phi}{r}+\frac{v_\theta v_\phi\cot\theta}{r}\right)$$

# Solution Steps

## Solution Steps

- *Step* 1. Select appropriate *system of coordinates*
- *Step* 2. Introduce suitable *hypothesis* on the *velocity* and *pressure fields*
- *Step* 3. Solve mass continuity equation for an *incompressible fluid*
- *Step* 4. Solve the *Navier-Stokes equation* for an *incompressible Newtonian fluid* to get the velocity and pressure fields, and stresses using geometric and constitutive equations, in terms of some integration constants
- *Step* 5. Apply *boundary conditions* to determine the integration constants

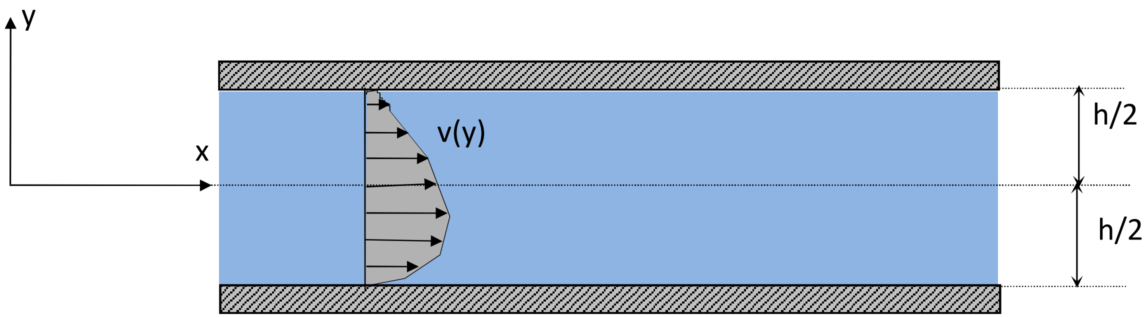
# Assignment 11.1

## Assignment 11.1

Let us consider the *stationary flow* of an *incompressible Newtonian fluid* with dynamic viscosity $\mu > 0$, flowing between two parallel horizontal plates as it is shown in the figure. *Body forces* are considered to be *negligible*. Obtain the velocity field assuming that the velocity and pressure fields are such that,

$$v_x = v(y), \quad v_y = 0, \quad v_z = 0$$

$$[\nabla p] = [-a \quad 0 \quad 0], \quad a = cte > 0$$



# Assignment 11.1

## Assignment 11.1

The *mass continuity equation* for an *incompressible fluid* may be written as,

$$\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v} = 0, \quad \dot{\rho} = 0 \quad \Rightarrow \quad \mathrm{div}\, \mathbf{v} = 0$$

The *Navier-Stokes equation* for an *incompressible Newtonian fluid* under *stationary* conditions and *negligible body forces* may be written as,

$$-\nabla p + \cancel{(\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\, \mathbf{v})} + \mu \nabla^2 \mathbf{v} + \cancel{\rho \mathbf{b}} = \rho \left( \cancel{\frac{\partial \mathbf{v}}{\partial t}} + (\mathrm{grad}\, \mathbf{v})\, \mathbf{v} \right)$$

$$-\nabla p + \mu \nabla^2 \mathbf{v} = \rho\, (\mathrm{grad}\, \mathbf{v})\, \mathbf{v}$$

# Assignment 11.1

The convective rate of the velocity is zero,

$$(\operatorname{grad}\mathbf{v})\,\mathbf{v} = \mathbf{0}$$

yielding,

$$-\nabla p + \mu \nabla^2 \mathbf{v} = 0 \quad \Rightarrow \quad a + \mu \frac{\partial^2 v(y)}{\partial y^2} = 0$$

Integrating yields,

$$v(y) = -\frac{a}{2\mu} y^2 + C_1 y + C_2$$

The *sticking boundary conditions* read,

$$v\left(\frac{h}{2}\right) = -\frac{ah^2}{8\mu} + C_1 \frac{h}{2} + C_2 = 0, \quad v\left(-\frac{h}{2}\right) = -\frac{ah^2}{8\mu} - C_1 \frac{h}{2} + C_2 = 0$$

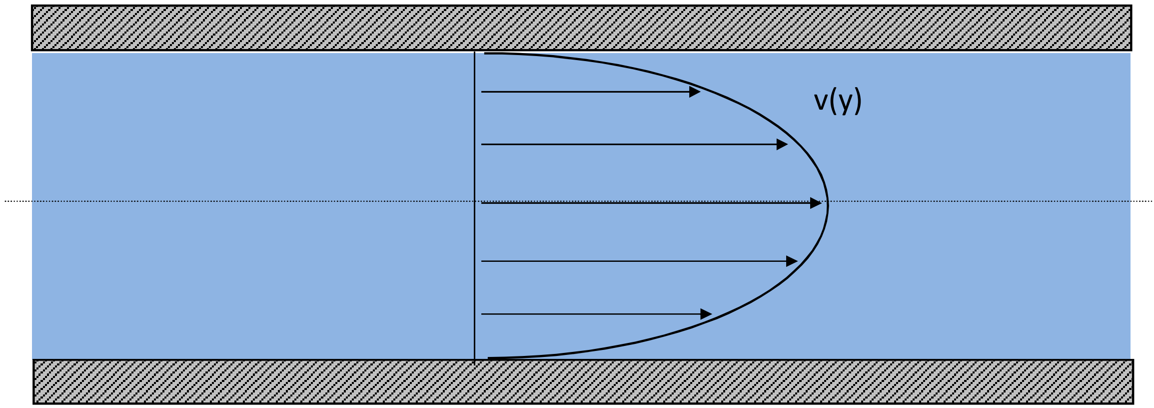
# Assignment 11.1

Solving the system of equations, the integration constants are given by,

$$C_1 = 0, \quad C_2 = \frac{ah^2}{8\mu}$$

And the velocity field (Poiseuille's flow) takes the value,

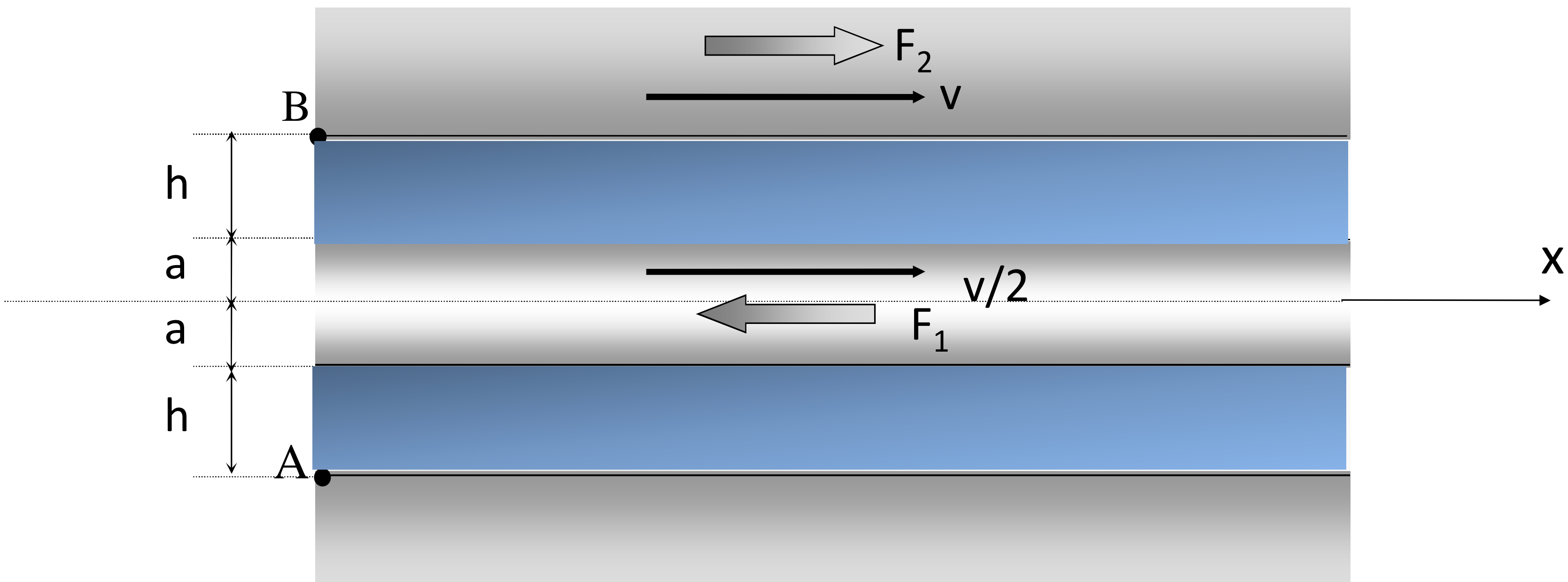
$$\boxed{v(y) = \frac{a}{2\mu}\left(\frac{h^2}{4} - y^2\right)}$$



# Assignment 11.2

## Assignment 11.2 [Classwork]

An horizontal plate of infinite length and thickness 2a is placed between two incompressible Newtonian fluids flowing in steady-state regime in the interior of two horizontal boundaries as it is shown on the figure.



# Assignment 11.2

## Assignment 11.2 [Classwork]

The distance of the plate surfaces to the boundaries is h. The upper boundary and the plate have prescribed velocities v* and v*/2, respectively. The pressure at the points A and B shown on the figure, are $p_A$ and $p_B$, respectively. It is assumed that the flow and its properties do not depend on x.

1) Compute the velocity, pressure and stress fields on each fluid
2) Compute the forces per unit of surface F1 and F2 acting on the plate and the upper boundary, respectively, needed to keep the described motion

# Assignment 11.2

## Assignment 11.2 [Classwork]

*Step* 1. We will solve the problem using *cartesian coordinates*.

*Step* 2. We introduce suitable *hypothesis* on the *pressure* and *velocity fields* for any of the two fluids. Let us a consider pressure and velocity fields of the form,

$$p = p(y), \quad \mathbf{v} = \begin{bmatrix} v_x(y) & 0 & 0 \end{bmatrix}^T$$

*Step 3*. The *mass continuity equation* for an *incompressible fluid,* taking into accoun the hypothesis on the velocity field, yields,

$$\operatorname{div} \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} = 0$$

# Assignment 11.2

*Step* 4. The *Navier Stokes* for an *incompressible Newtonian fluid* under *stationary flow*, taking into acount the hypothesis introdu-ced on the pressure and velocity fields, yields,

$$\mu \frac{\partial^2 v_x(y)}{\partial y^2} = 0$$

$$-\frac{\partial p(y)}{\partial y} - \rho g = 0$$

Integrating yields,

$$p(y) = -\rho g y + k_1, \quad v_x(y) = k_2 y + k_3$$

# Assignment 11.2

The *pressure* and *velocity fields* for fluid 1 take the form,

$$p^{(1)}(y) = -\rho_1 g y + k_1^{(1)}, \quad v_x^{(1)}(y) = k_2^{(1)} y + k_3^{(1)}$$

The *pressure* and *velocity fields* for fluid 2 take the form,

$$p^{(2)}(y) = -\rho_2 g y + k_1^{(2)}, \quad v_x^{(2)}(y) = k_2^{(2)} y + k_3^{(2)}$$

# Assignment 11.2

*Stresses* for fluid 1 take the form,

$$\sigma_x^{(1)}(y) = \sigma_y^{(1)}(y) = \sigma_z^{(1)}(y) = -p^{(1)}(y) = \rho_1 g y - k_1^{(1)}$$

$$\tau_{xy}^{(1)}(y) = \mu_1 \frac{\partial v_x^{(1)}(y)}{\partial y} = \mu_1 k_2^{(1)}$$

*Stresses* for fluid 2 take the form,

$$\sigma_x^{(2)}(y) = \sigma_y^{(2)}(y) = \sigma_z^{(2)}(y) = -p^{(2)}(y) = \rho_2 g y - k_1^{(2)}$$

$$\tau_{xy}^{(2)}(y) = \mu_2 \frac{\partial v_x^{(2)}(y)}{\partial y} = \mu_2 k_2^{(2)}$$

# Assignment 11.2

*Step* 5. The following *boundary conditions* are imposed to determine the integration constants for fluids 1 and 2,

$$v_x^{(1)}\left(-h-a\right)=-k_2^{(1)}\left(h+a\right)+k_3^{(1)}=0$$

$$v_x^{(1)}\left(-a\right)=-k_2^{(1)}a+k_3^{(1)}=v/2$$

$$v_x^{(2)}\left(a\right)=k_2^{(2)}a+k_3^{(2)}=v/2$$

$$v_x^{(2)}\left(h+a\right)=k_2^{(2)}\left(h+a\right)+k_3^{(2)}=v$$

$$p^{(1)}\left(-h-a\right)=\rho_1 g\left(h+a\right)+k_1^{(1)}=p_A$$

$$p^{(2)}\left(h+a\right)=-\rho_2 g\left(h+a\right)+k_1^{(2)}=p_B$$

# Assignment 11.2

The integration constants for fluids 1 and 2 are given by,

$$k_1^{(1)} = p_A - \rho_1 g\left(h+a\right), \quad k_2^{(1)} = \frac{v}{2h}, \quad k_3^{(1)} = \frac{v}{2}\left(1+\frac{a}{h}\right)$$

$$k_1^{(2)} = p_B + \rho_2 g\left(h+a\right), \quad k_2^{(2)} = \frac{v}{2h}, \quad k_3^{(2)} = \frac{v}{2}\left(1-\frac{a}{h}\right)$$

# Assignment 11.2

The *pressure* and *velocity fields* for *fluid 1* take the form,

$$\boxed{\begin{aligned} p^{(1)}(y) &= p_A - \rho_1 g\left(y+h+a\right) \\ v_x^{(1)}(y) &= \frac{v}{2}\left(1+\frac{y+a}{h}\right) \end{aligned}}$$

The *pressure* and *velocity fields* for *fluid 2* take the form,

$$\boxed{\begin{aligned} p^{(2)}(y) &= p_B + \rho_2 g\left(h+a-y\right) \\ v_x^{(2)}(y) &= \frac{v}{2}\left(1+\frac{y-a}{h}\right) \end{aligned}}$$

# Assignment 11.2

The *stress field* for *fluid 1* takes the form,

$$\sigma_x^{(1)}(y) = \sigma_y^{(1)}(y) = \sigma_z^{(1)}(y) = -p^{(1)}(y) = -p_A + \rho_1 g (y + h + a)$$

$$\tau_{xy}^{(1)}(y) = \mu_1 \frac{\partial v_x^{(1)}(y)}{\partial y} = \mu_1 \frac{v}{2h}$$

The *stress field* for *fluid 2* takes the form,

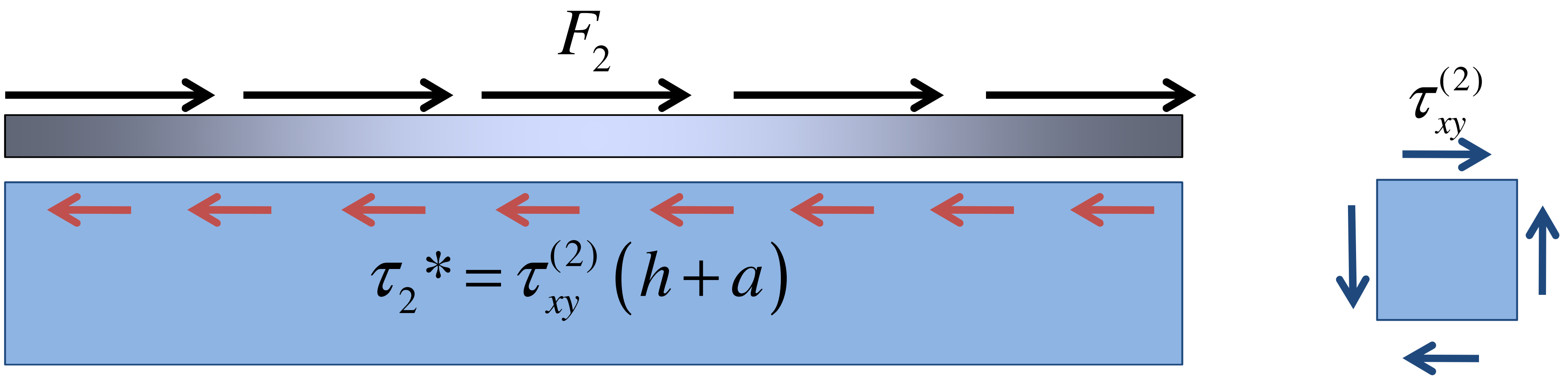
$$\sigma_x^{(2)}(y) = \sigma_y^{(2)}(y) = \sigma_z^{(2)}(y) = -p^{(2)}(y) = -p_B - \rho_2 g (h + a - y)$$

$$\tau_{xy}^{(2)}(y) = \mu_2 \frac{\partial v_x^{(2)}(y)}{\partial y} = \mu_2 \frac{v}{2h}$$

# Assignment 11.2

*Step 6. Equilibrium of horizontal forces* (per unit of surface)

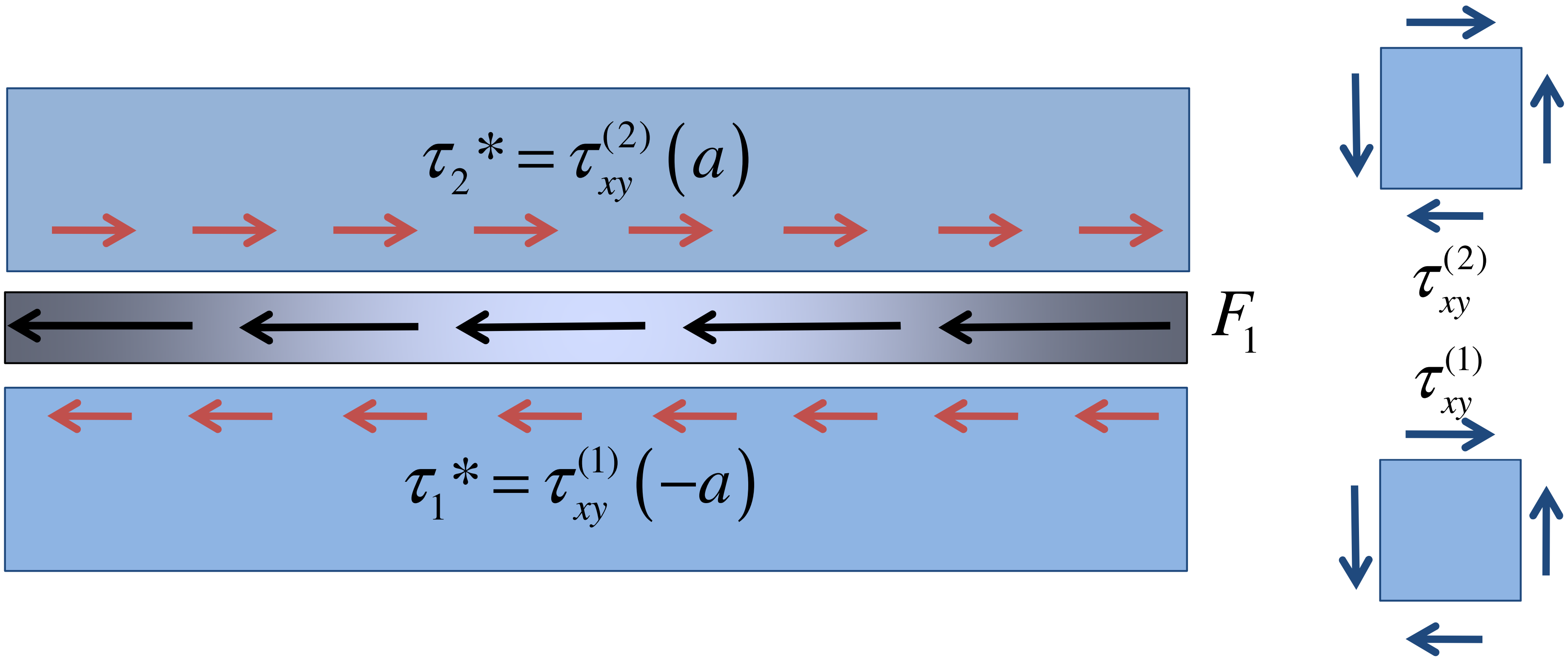
*Equilibrium of horizontal forces* (per unit of surface) at the upper plate yields,




$$\boxed{F_2 = \tau_2{}^* = \tau_{xy}^{(2)}(h+a) = \mu_2 \frac{v}{2h}}$$

# Assignment 11.2

*Equilibrium of horizontal forces* (per unit of surface) at the middle plate yields,

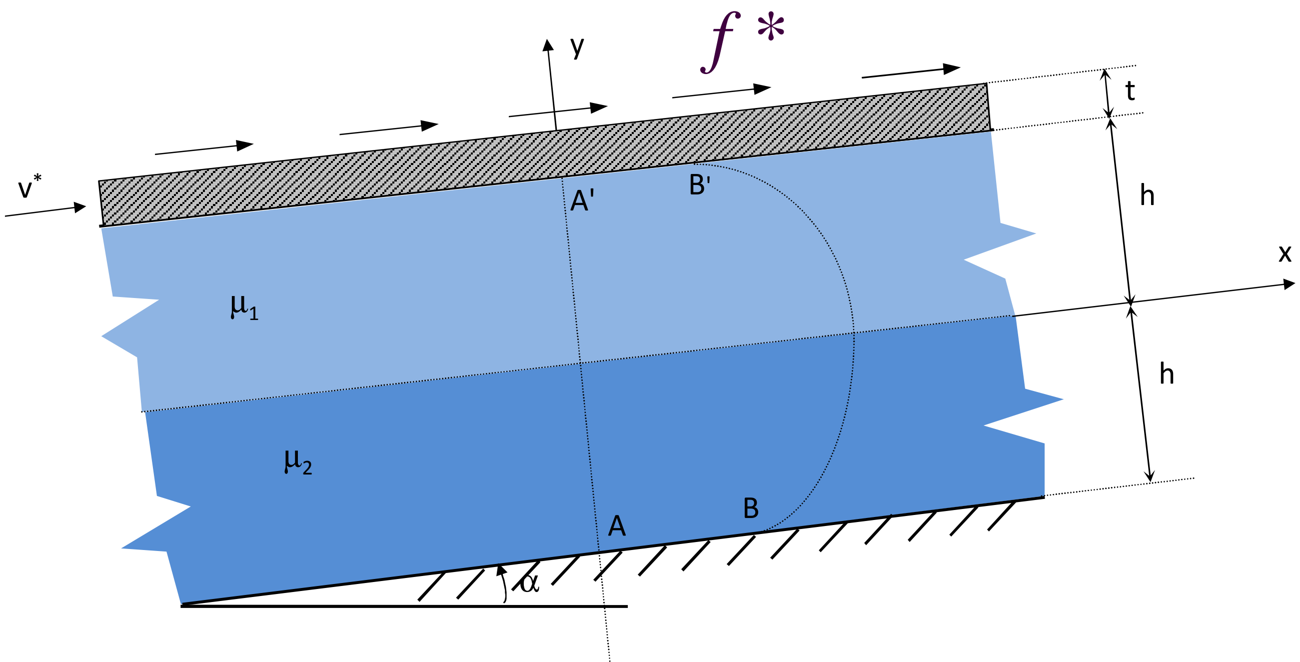



$$\boxed{F_1 = \tau_2{}^* - \tau_1{}^* = \tau_{xy}^{(2)}(a) - \tau_{xy}^{(1)}(-a) = (\mu_2 - \mu_1)\frac{v}{2h}}$$

# Assignment 11.3

## Assignment 11.3 [Homework]

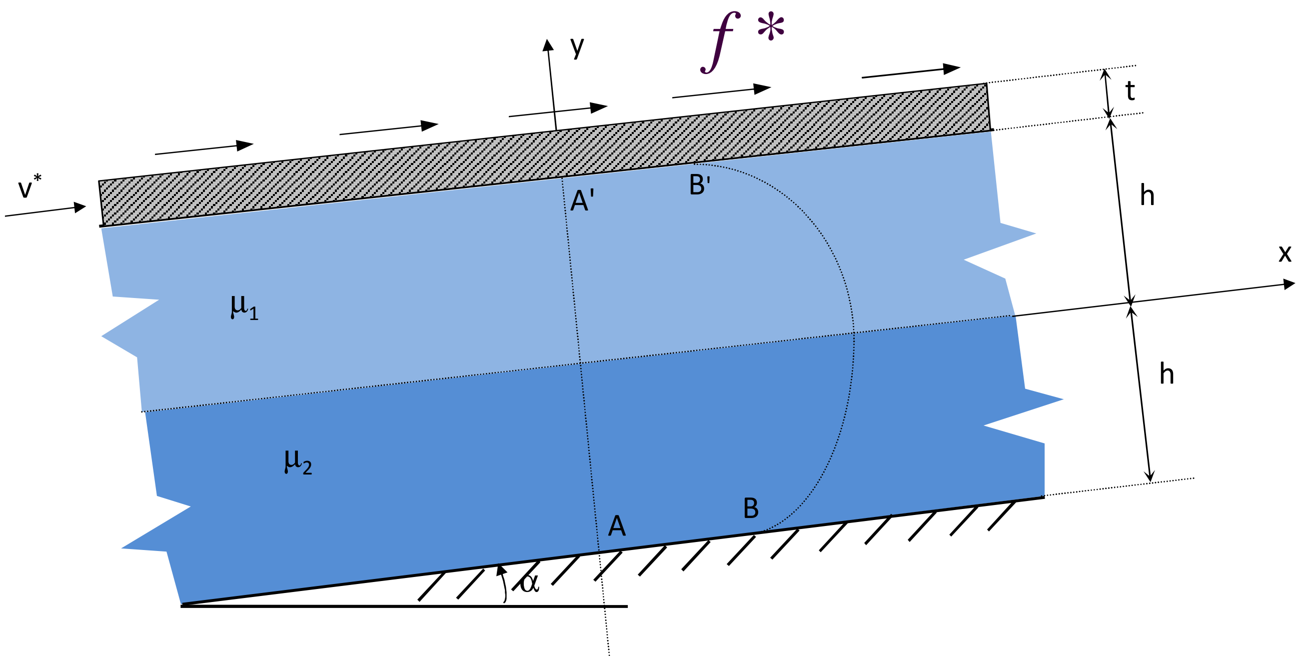
Consider stationary flows of the two immiscid incompressible Newtonian fluids with dynamic viscosities $\mu_1, \mu_2$, as shown in the figure. Body forces in the fluids and environmental pressure are neglected. The plate has density $\rho^*$ and thickness $t$. On the top of the plate a tangential force $f^*$ per unit of surface is applied.



# Assignment 11.3

## Assignment 11.3 [Homework]

1) Compute the velocity, pressure and stress fields in each one of the fluids
2) Compute the volume flux Q accross the circular section BB’
3) Compute the velocity v* of the plate



# Assignment 11.3

## Assignment 11.3 [Homework]

*Step 1*. We will solve the problem in 2D using local x,y cartesian coordinates as shown on the figure.

*Step 2*. We introduce suitable hypothesis on the pressure and velocity fields for any of the two fluids. Let us a consider pressure and velocity fields of the form,

$$p = p(x, y), \quad \mathbf{v} = \begin{bmatrix} v_x(x, y) & 0 & 0 \end{bmatrix}^T$$

*Step 3*. The *mass continuity equation* for an *incompressible fluid,* taking into accoun the hypothesis on the velocity field, yields,

$$\mathrm{div}\,\mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow \quad v_x = v_x(y)$$

# Assignment 11.3

*Step* 4. The *Navier Stokes* for an *incompressible Newtonian fluid* under *stationary flow*, taking into acount that body forces are negligible and the hypothesis introduced for the pressure and velocity fields, yields,

$$-\frac{\partial p(x,y)}{\partial x}+\mu\frac{\partial^2 v_x(y)}{\partial y^2}=0$$

$$-\frac{\partial p(x,y)}{\partial y}=0$$

From the second equation we get,

$$p=p(x)$$

# Assignment 11.3

Then, from the first equation we get,

$$\frac{\partial p(x)}{\partial x} = \mu \frac{\partial^2 v_x(y)}{\partial y^2} = k = cte$$

Integrating yields,

$$\frac{\partial p(x)}{\partial x} = k \quad \Rightarrow \quad p(x) = kx + k_1$$

$$\mu \frac{\partial^2 v_x(y)}{\partial y^2} = k \quad \Rightarrow \quad v_x(y) = \frac{k}{2\mu} y^2 + k_2 y + k_3$$

# Assignment 11.3

The *pressure* and *velocity fields* for fluid 1 take the form,

$$p^{(1)}(x) = k^{(1)}x + k_1^{(1)}$$

$$v_x^{(1)}(y) = \frac{k^{(1)}}{2\mu_1} y^2 + k_2^{(1)} y + k_3^{(1)}$$

The *pressure* and *velocity fields* for fluid 2 take the form,

$$p^{(2)}(x) = k^{(2)}x + k_1^{(2)}$$

$$v_x^{(2)}(y) = \frac{k^{(2)}}{2\mu_2} y^2 + k_2^{(2)} y + k_3^{(2)}$$

# Assignment 11.3

*Stresses* for fluid 1 take the form,

$$\sigma_x^{(1)}(x) = \sigma_y^{(1)}(x) = \sigma_z^{(1)}(x) = -p^{(1)}(x) = -k^{(1)}x - k_1^{(1)}$$

$$\tau_{xy}^{(1)}(y) = \mu_1 \frac{\partial v_x^{(1)}(y)}{\partial y} = k^{(1)}y + \mu_1 k_2^{(1)}$$

*Stresses* for fluid 2 take the form,

$$\sigma_x^{(2)}(x) = \sigma_y^{(2)}(x) = \sigma_z^{(2)}(x) = -p^{(2)}(x) = -k^{(2)}x - k_1^{(2)}$$

$$\tau_{xy}^{(2)}(y) = \mu_2 \frac{\partial v_x^{(2)}(y)}{\partial y^2} = k^{(2)}y + \mu_2 k_2^{(2)}$$

# Assignment 11.3

*Step* 5. The following *boundary conditions* are imposed to determine the integration constants for fluids 1 and 2,

$$v_x^{(1)}(h) = \frac{k^{(1)}}{2\mu_1} h^2 + k_2^{(1)} h + k_3^{(1)} = v^*$$

$$v_x^{(2)}(-h) = \frac{k^{(2)}}{2\mu_2} h^2 - k_2^{(2)} h + k_3^{(2)} = 0$$

$$v_x^{(1)}(0) = v_x^{(2)}(0) \quad \Rightarrow \quad k_3^{(1)} = k_3^{(2)}$$

$$p^{(1)}(x) = k^{(1)} x + k_1^{(1)} = \rho^* g t \cos\alpha \quad \forall x$$

$$p^{(1)}(x) = p^{(2)}(x) \ \text{ en } y = 0, \forall x \quad \Rightarrow \quad k^{(1)} x + k_1^{(1)} = k^{(2)} x + k_1^{(2)}$$

$$\tau_{xy}^{(1)}(0) = \tau_{xy}^{(2)}(0) \quad \Rightarrow \quad \mu_1 k_2^{(1)} = \mu_2 k_2^{(2)}$$

# Assignment 11.3

The velocity, pressure and stress fields are given by,

$$v_x^{(1)}(y) = \frac{v^*}{1+\mu_1/\mu_2}\left(\frac{y}{h}+\frac{\mu_1}{\mu_2}\right), \quad v_x^{(2)}(y) = \frac{\mu_1}{\mu_2}\frac{v^*}{1+\mu_1/\mu_2}\left(\frac{y}{h}+1\right)$$

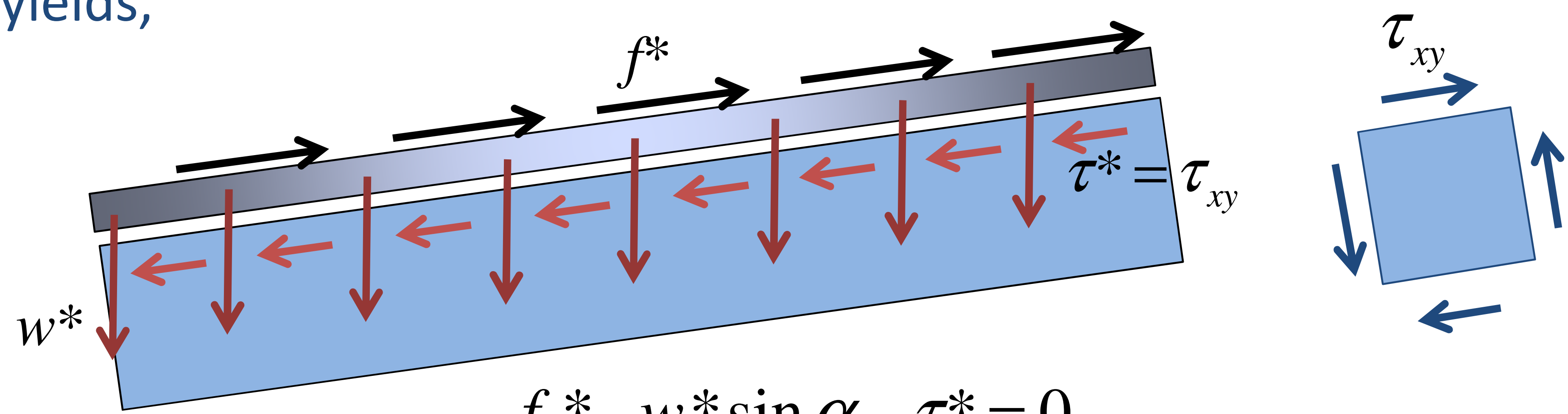
$$p^{(1)} = p^{(2)} = \rho^* g t \cos\alpha \quad \forall x$$

$$\tau_{xy}^{(1)} = \tau_{xy}^{(2)} = \frac{\mu_1}{h}\frac{v^*}{1+\mu_1/\mu_2} \quad \forall x$$

# Assignment 11.3

*Step 6. Equilibrium of tangential forces* (per unit of length)

*Equilibrium of tangential forces* (per unit of length) on the plate yields,




$$f^* - w^* \sin\alpha - \tau^* = 0$$

$$f^* = \rho^* g t \sin\alpha + \tau^* = \rho^* g t \sin\alpha + \frac{\mu_1}{h} \frac{v^*}{1 + \mu_1/\mu_2}$$

$$v^* = \frac{h}{\mu_1}\left(1 + \mu_1/\mu_2\right)\left(f^* - \rho^* g t \sin\alpha\right)$$

# Assignment 11.3

*Step* 7. Volume flux accross circular surface BB'.

As the fluid is incompressible, the volume flux through BB' is the same than the one accross any other section, i.e. a plane AA',

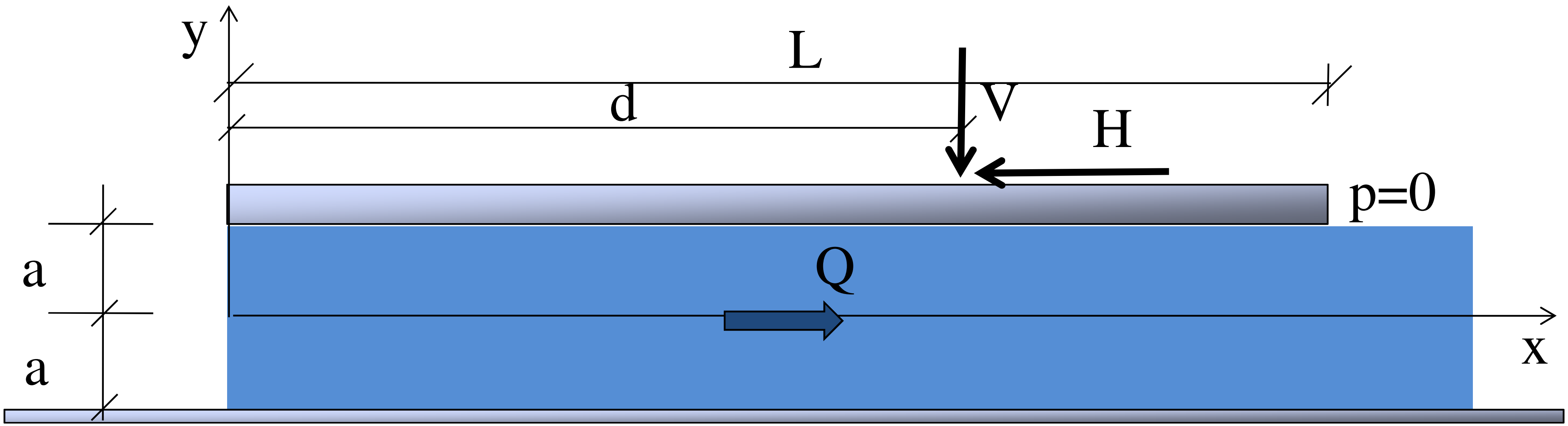
$$Q = \int_{-h}^{h} \mathbf{v} \cdot \mathbf{n} dS = \int_{-h}^{h} v_x(y)\, dy = \int_{0}^{h} v_x^{(1)}(y)\, dy + \int_{-h}^{0} v_x^{(2)}(y)\, dy$$

$$Q = v^* h \left( \frac{1}{2} + \frac{\mu_1}{\mu_1 + \mu_2} \right) \quad \blacksquare$$

# Assignment 11.4

## Assignment 11.4 [Classwork]

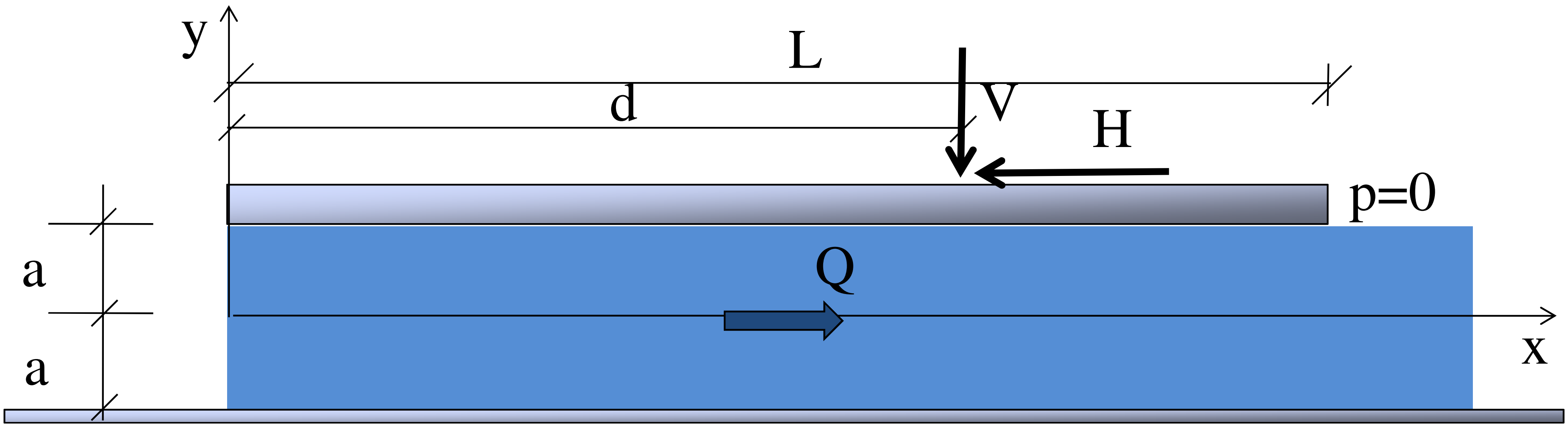
Consider the stationary flow of an incompressible Newtonian fluid with dynamic viscosity $\mu$, under an horizontal plate of length L as shown in the figure. Inertial forces, environmental pressure and body forces in the plate are neglected. The pressure at the point (x=l, y=a) is zero. The volume flux per unit of width is Q.



# Assignment 11.4

The plate remains fixed and horizontal due to the application of forces per unit of width H and V, at a distance d of the left edge of the plate. The flow is parallel to the plane x-y.

1) Compute the velocity and pressure fields
2) Compute the horizontal component H and vertical component V of the point load needed to keep the plate horizontal and fixed



# Assignment 11.4

## Assignment 11.4 [Classwork]

*Step 1*. We will solve the problem in 2D using Cartesian coordinates as shown in the figure.

*Step 2*. We introduce suitable hypothesis on the pressure and velocity fields for any of the two fluids. Let us a consider pressure and velocity fields of the form,

$$p = p(x, y), \quad \mathbf{v} = \begin{bmatrix} v_x(x, y) & 0 & 0 \end{bmatrix}^T$$

*Step 3*. The *mass continuity equation* for an *incompressible fluid,* taking into accoun the hypothesis on the velocity field, yields,

$$\operatorname{div} \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow \quad v_x = v_x(y)$$

# Assignment 11.4

*Step* 4. The *Navier Stokes* for an *incompressible Newtonian fluid* under *stationary flow*, taking into acount the hypothesis introdu-ced on the pressure and velocity fields, yields,

$$-\frac{\partial p(x,y)}{\partial x}+\mu\frac{\partial^2 v_x(y)}{\partial y^2}=0$$

$$-\frac{\partial p(x,y)}{\partial y}-\rho g=0$$

From the second equation we get,

$$p=-\rho g y+\varphi(x)$$

# Assignment 11.4

Then, from the first equation we get,

$$\frac{\partial \varphi(x)}{\partial x} = \mu \frac{\partial^2 v_x(y)}{\partial y^2} = C_1 = cte$$

Integrating yields,

$$\frac{\partial \varphi(x)}{\partial x} = C_1 \quad \Rightarrow \quad \varphi(x) = C_1 x + C_2$$

$$\mu \frac{\partial^2 v_x(y)}{\partial y^2} = C_1 \quad \Rightarrow \quad v_x(y) = \frac{C_1}{2\mu} y^2 + C_3 y + C_4$$

$$p = -\rho g y + \varphi(x) = -\rho g y + C_1 x + C_2$$

# Assignment 11.4

*Step* 5. The following *boundary conditions* are imposed to determine the integration constants,

$$v_x(a) = \frac{1}{2\mu} C_1 a^2 + C_3 a + C_4 = 0$$

$$v_x(-a) = \frac{1}{2\mu} C_1 a^2 - C_3 a + C_4 = 0$$

$$p(l,a) = -\rho g a + C_1 l + C_2 = 0$$

$$Q = \int_{-a}^{a} v_x(y)\, dy = \int_{-a}^{a} \left( \frac{1}{2\mu} C_1 y^2 + C_3 y + C_4 \right) dy$$

# Assignment 11.4

The integration constants are given by,

$$C_1 = -\frac{3\mu}{2a^3}Q, \quad C_2 = \rho g a + \frac{3\mu}{2a^3}Ql, \quad C_3 = 0, \quad C_4 = \frac{3}{4a}Q$$

The *velocity* and *pressure* fields take the form,

$$\boxed{\begin{aligned} v_x(y) &= \frac{3Q}{4a^3}\left(a^2 - y^2\right) \\ p(x,y) &= \rho g\left(a - y\right) - \frac{3\mu}{2a^3}Q\left(x - l\right) \end{aligned}}$$

# Assignment 11.4

*Step* 6. The *Cauchy stress tensor* is given by,

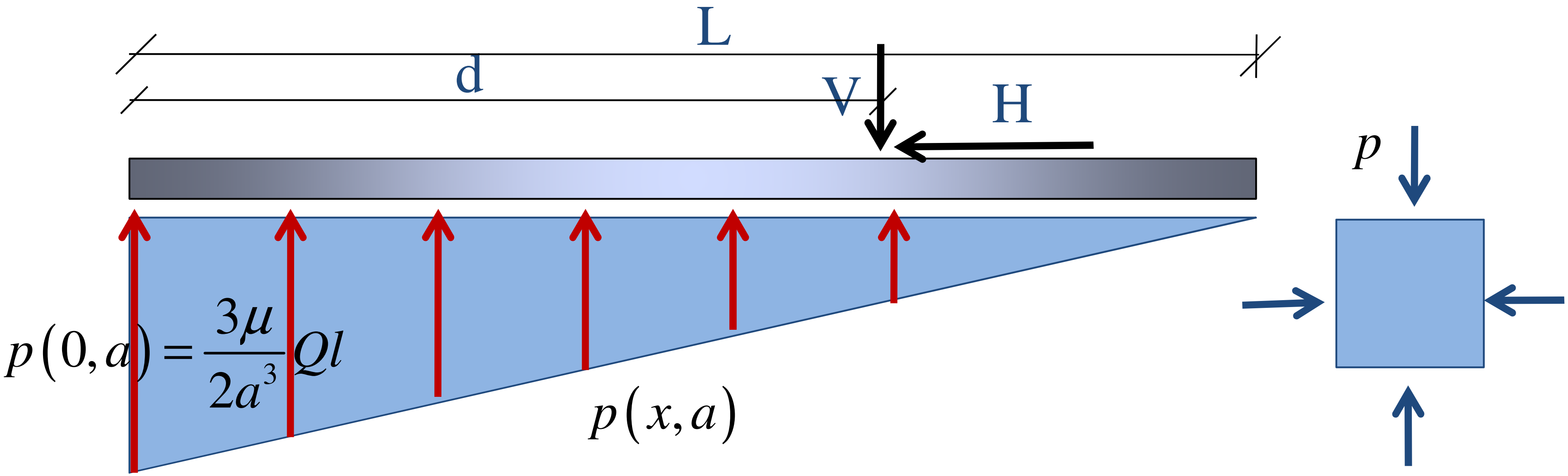
$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda\left(\cancel{\operatorname{tr}\mathbf{d}}\right)\mathbf{1} + 2\mu\mathbf{d} = -p\mathbf{1} + 2\mu\mathbf{d}$$

$$[\mathbf{d}] = \begin{bmatrix} 0 & d_{xy} & 0 \\ d_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad d_{xy} = \frac{1}{2}\frac{\partial v_x(y)}{\partial y} = -\frac{3Q}{4a^3}y$$

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}, \quad \sigma_x = \sigma_y = \sigma_z = -p(x,y), \quad \tau_{xy} = -\frac{3Q}{2a^3}\mu y$$

# Assignment 11.4

*Step 7*. Using the *action-reaction principle*, the *equilibrium of vertical forces and moments* on the plate yields,

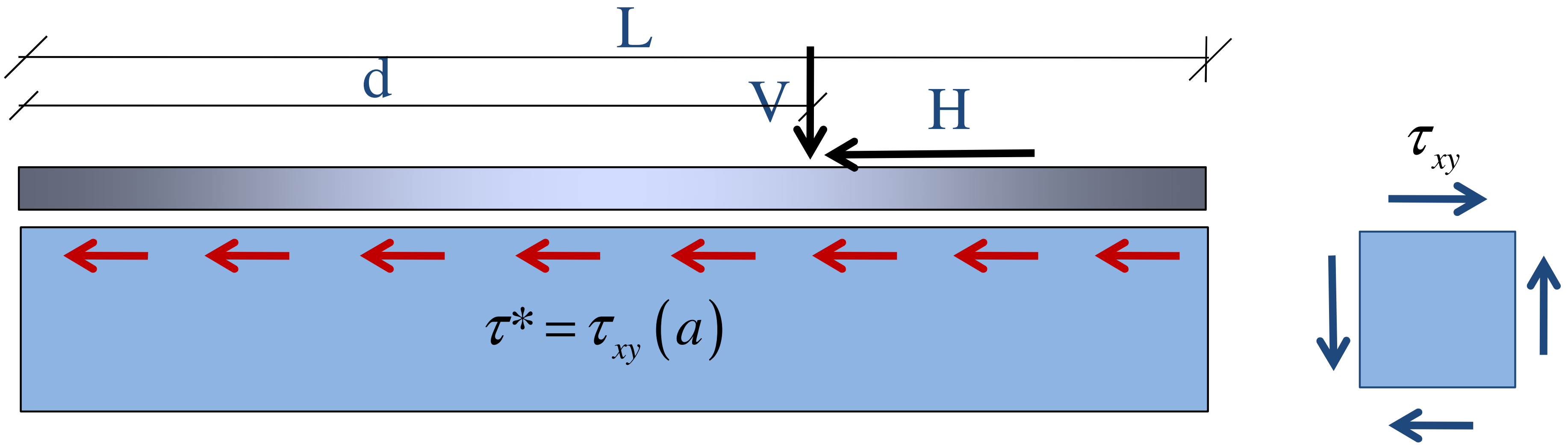



$$\left.\begin{aligned} V-\tfrac{1}{2}\,p(0,a)\,l=0 \\ Vd-\tfrac{1}{2}\,p(0,a)\,l\,\tfrac{1}{3}\,l=0 \end{aligned}\right\}$$

$$\boxed{V=\frac{1}{2}\,p(0,a)\,l=\frac{3\mu}{4a^3}Ql^2,\quad d=\frac{1}{3}l}$$

# Assignment 11.4

*Step 7*. Using the *action-reaction principle*, the *equilibrium of horizontal forces* on the plate yields,




$$H + \tau_{xy}(a)\,l = 0$$

$$\boxed{H = -\tau_{xy}(a)\,l = \frac{3\mu}{2a^2}Ql}$$