
# Continuum Mechanics

## Chapter 5
## Stresses

C. Agelet de Saracibar
ETS Ingenieros de Caminos, Canales y Puertos, Universidad Politécnica de Cataluña (UPC), Barcelona, Spain
International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain

CIMNE

# Forces

## Chapter 5 · Stresses

1. Forces
2. Cauchy’s stress theorems
3. Stress tensors

# Forces

## Forces

We consider two types of forces that may act on a continuum body: **body** (or **mass** or **volume** or **internal**) **forces** and **surface forces.**

- **Body** (or **mass** or **volume** or **internal**) forces: Forces which act in the *volume* of a continuum medium. Typical examples of body forces are the gravity forces or the electromagnetic forces.
- **Surface forces**: Forces which act on the *surface* of a continuum medium due to the interaction with other bodies or the environtment. Typical examples of surface forces are the contact forces or applied loads.

# Body Forces

## Body Forces

**Body** (or **mass** or **volume** or **internal**) **forces** may be characterized by the **body forces per unit of mass** vector, denoted as $\mathbf{b}$.

*Spatial* and *material descriptions* of the **body forces per unit of mass** vector, denoted as $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{X},t)$, respectively, take the form,

$$\mathbf{b} = \mathbf{b}(\mathbf{x},t) = \mathbf{b}(\boldsymbol{\varphi}(\mathbf{X},t),t) = \mathbf{B}(\mathbf{X},t)$$

$$\mathbf{b} = \mathbf{B}(\mathbf{X},t) = \mathbf{B}(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t) = \mathbf{b}(\mathbf{x},t)$$

# Body Forces

## Differential Body Forces

The **differential body** (or **mass** or **volume** or **internal**) **force** acting in a differential of volume $dv$ in the *spatial configuration* takes the form,

$$d\mathbf{f}_v = \rho\left(\mathbf{x},t\right)\mathbf{b}\left(\mathbf{x},t\right)dv$$

The **differential body** (or **mass** or **volume** or **internal**) **force** acting in a differential of volume $dV$ in the *material configuration* takes the form,

$$d\mathbf{f}_v = \rho_0\left(\mathbf{X}\right)\mathbf{B}\left(\mathbf{X},t\right)dV$$

# Body Forces

## Total Body Forces

**The total body** (or **mass** or **volume** or **internal**) **forces** acting in a *spatial volume* $v$ of a continuum body, at a time *t*, may be written as,

$$\mathbf{F}_v = \int_v \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv$$

**The total body** (or **mass** or **volume** or **internal**) **forces** acting in a *material volume* $V$ of a continuum body, at a time *t*, may be written as,

$$\mathbf{F}_v = \int_V \rho_0(\mathbf{X})\,\mathbf{B}(\mathbf{X},t)\,dV$$

# Surface Forces

## Differential Surface Forces

The **differential surface force** acting on a differential of area $ds$ on the *spatial configuration* takes the form,

$$d\mathbf{f}_s = \mathbf{t}\left(\mathbf{x}, t\right) ds$$

where the **Cauchy** (or **true**) **traction** vector, denoted as $\mathbf{t}\left(\mathbf{x}, t\right)$, represents the *spatial description* of the surface force per unit of *spatial surface*.

The **differential surface force** acting on a differential of area $dS$ on the *material configuration* takes the form,

$$d\mathbf{f}_s = \mathbf{T}\left(\mathbf{X}, t\right) dS$$

where the **first Piola-Kirchhoff** (or **nominal**) **traction** vector, denoted as $\mathbf{T}\left(\mathbf{X}, t\right)$, represents the *material description* of the surface force per unit of *material surface*.

# Surface Forces

## Total Surface Forces

The **total surface forces** acting on a *spatial surface* $\partial v$ of a continuum body, at a time *t*, may be written in terms of the **Cauchy** (or **true**) **traction** vector as,

$$\mathbf{F}_{\partial v} = \int_{\partial v} \mathbf{t}(\mathbf{x}, t)\, ds$$

The **total surface forces** acting on a *material surface* $\partial V$ of a continuum body, at a time *t*, may be written in terms of the **first Piola-Kirchhoff** (or **nominal**) **traction** vector as,

$$\mathbf{F}_{\partial V} = \int_{\partial V} \mathbf{T}(\mathbf{X}, t)\, dS$$

# Forces

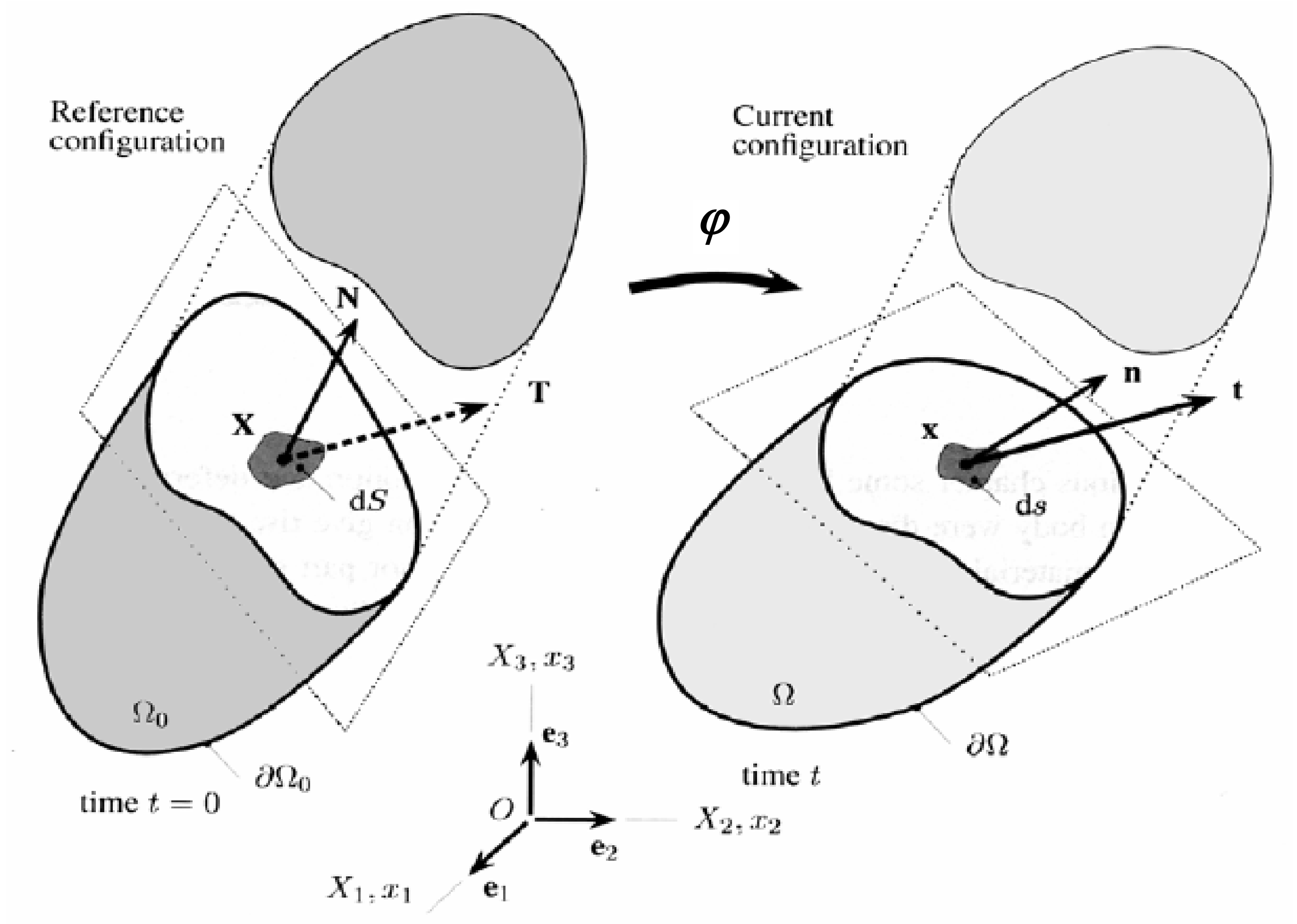
## Body Forces

$$\mathbf{F}_v = \int_v \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv = \int_V \rho_0(\mathbf{X})\,\mathbf{B}(\mathbf{X},t)\,dV$$

## Surface Forces

$$\mathbf{F}_{\partial v} = \int_{\partial v} \mathbf{t}(\mathbf{x},t)\,ds = \int_{\partial V} \mathbf{T}(\mathbf{X},t)\,dS$$

# The Traction Vector Picture



# Cauchy Stress Theorems

## First Cauchy Stress Theorem

The **Cauchy** (or **true**) **traction** vector at a spatial point $\mathbf{x}$, at a given time *t*, on a *spatial surface* with *unit outward normal* $\mathbf{n}$ at the spatial point $\mathbf{x}$, is only a function of the *spatial point*, the *time t* and the *unit outward normal* at the spatial point $\mathbf{x}$ at the time *t*,

$$\boxed{\mathbf{t} = \mathbf{t}\left(\mathbf{x}, t, \mathbf{n}\right), \quad t_a = t_a\left(\mathbf{x}, t, \mathbf{n}\right)}$$

# Cauchy Stress Theorems

## First Cauchy Stress Theorem

The **first Piola-Kirchhoff** (or **nominal**) **traction** vector at a material point $\mathbf{X}$, at a given time *t,* on a *material surface* with *unit outward normal* $\mathbf{N}$ at the material point $\mathbf{X}$, is only a function of the *material point*, the *time t* and the *unit outward normal* at the material point $\mathbf{X}$ at the time *t,*

$$\boxed{\mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N}), \quad T_a = T_a(\mathbf{X}, t, \mathbf{N})}$$

# Cauchy Stress Theorems

## Second Cauchy Stress Theorem

The **Cauchy** (or **true**) **traction** vector at a spatial point $\mathbf{x}$, at a given time *t*, on a *spatial surface* with *unit outward normal* $\mathbf{n}$ at the spatial point $\mathbf{x}$, is a *linear* function of the *unit outward normal* at the spatial point $\mathbf{x}$ at the time *t*, satisfying the so called **action-reaction principle**,

$$\boxed{\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}) = -\mathbf{t}(\mathbf{x}, t, -\mathbf{n}), \quad t_a = t_a(\mathbf{x}, t, \mathbf{n}) = -t_a(\mathbf{x}, t, -\mathbf{n})}$$

# Cauchy Stress Theorems

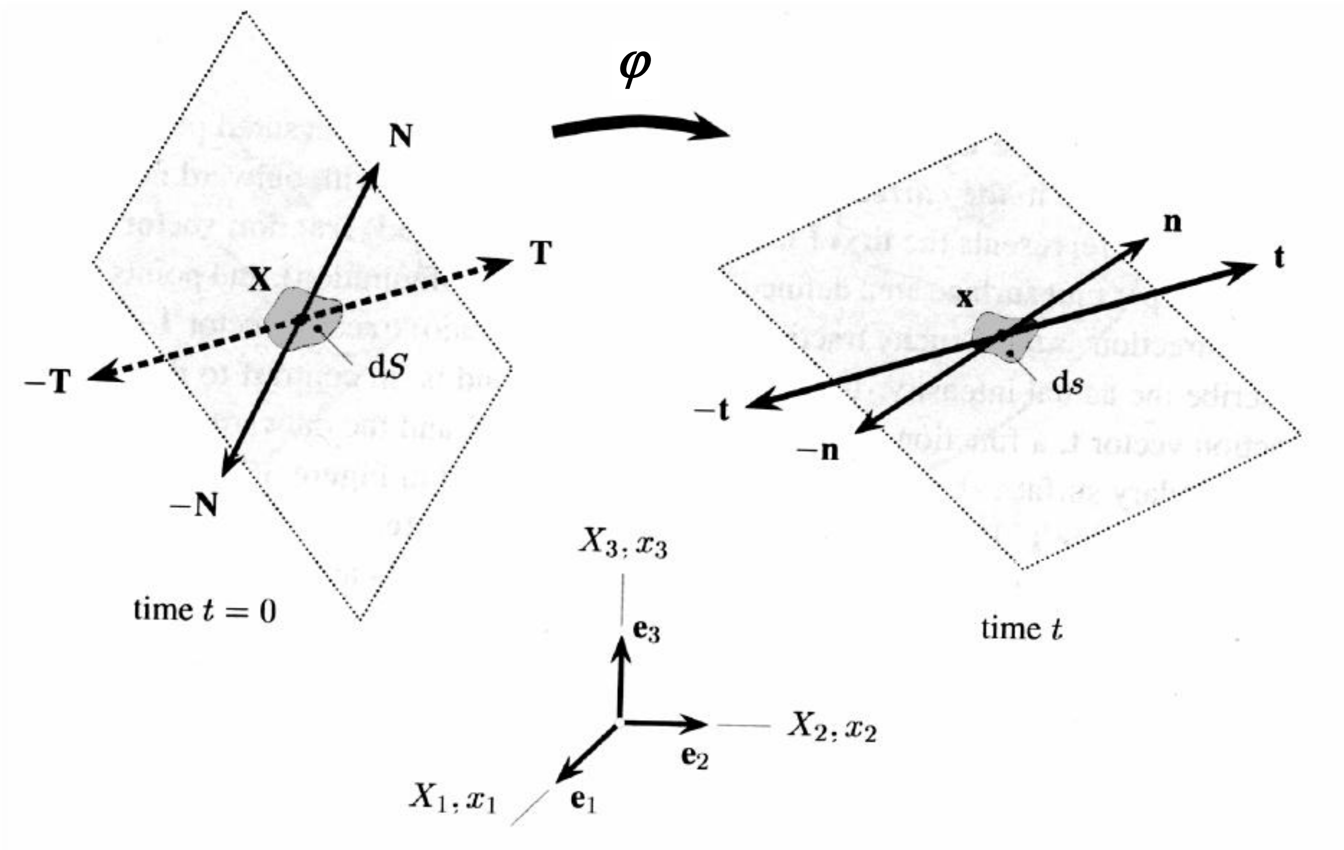
## Second Cauchy Stress Theorem

The **first Piola-Kirchhoff** (or **nominal**) **traction** vector at a material point $\mathbf{X}$, at a given time *t*, on a *material surface* with *unit outward normal* $\mathbf{N}$ at the material point $\mathbf{X}$, is a *linear* function of the *unit outward normal* at the material point $\mathbf{X}$ at the time *t*, satisfying the so called **action-reaction principle**,

$$\boxed{\mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N}) = -\mathbf{T}(\mathbf{X}, t, -\mathbf{N}), \quad T_a = T_a(\mathbf{X}, t, \mathbf{N}) = -T_a(\mathbf{X}, t, -\mathbf{N})}$$

# Cauchy Stress Theorems

## Second Cauchy Stress Theorem



# Stress Tensors

## Cauchy Stress Tensor

The **Cauchy** (or **true**) **stress** tensor, denoted as $\boldsymbol{\sigma}$, is a *symmetric* spatial second-order tensor, such that,

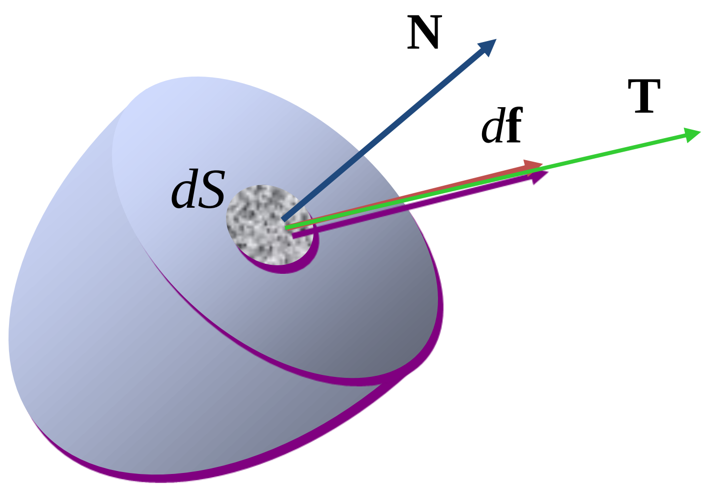
$$\boxed{\mathbf{t}(\mathbf{x},t,\mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}, \quad t_a(\mathbf{x},t,\mathbf{n}) = \sigma_{ab}(\mathbf{x},t)n_b}$$

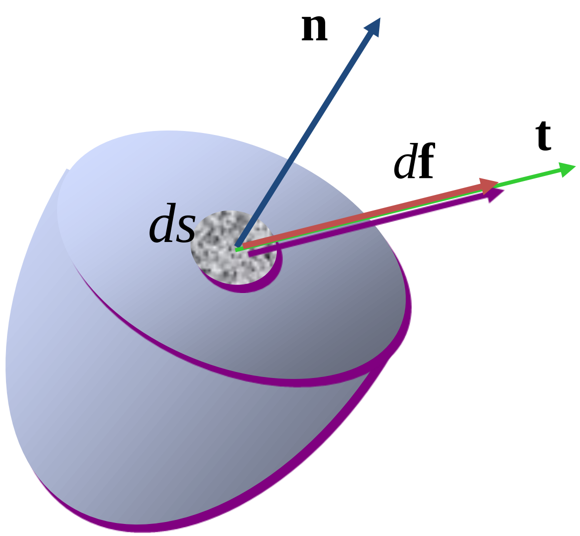
## First Piola-Kirchhoff Stress Tensor

The **first Piola-Kirchhoff** (or **nominal**) **stress** tensor, denoted as $\mathbf{P}$, is a *non-symmetric* two-point second-order tensor, such that,

$$\boxed{\mathbf{T}(\mathbf{X},t,\mathbf{N}) = \mathbf{P}(\mathbf{X},t)\mathbf{N}, \quad T_a(\mathbf{X},t,\mathbf{N}) = P_{aA}(\mathbf{X},t)N_A}$$

# Stress Tensors




$$d\mathbf{f} = \mathbf{T}dS = \mathbf{t}\, ds$$

$$\mathbf{T} = d\mathbf{f}/dS$$

$$\mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N}) = \mathbf{P}(\mathbf{X}, t)\mathbf{N}$$

$$\mathbf{t} = d\mathbf{f}/ds$$

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}$$

# Piola Transformation

## Piola Transformation

The *Cauchy* (or *true*) *traction* vector and the *first Piola-Kirchhoff* (or *nominal*) *traction* vector are related through the expression,

$$d\mathbf{f} = \mathbf{t}\, ds = \mathbf{T}\, dS$$

Introducing the *Cauchy* (or *true*) *stress* tensor and the *first Piola-Kirchhoff* (or *nominal*) *stress* tensor, yields,

$$d\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}\, ds = \mathbf{P}\mathbf{N}\, dS$$

and using *Nanson*'s formula, given by,

$$\mathbf{n}\, ds = J\, \mathbf{F}^{-T}\mathbf{N}\, dS$$

yields the so called **Piola transformation**, given by,

$$\boxed{\mathbf{P} = J\boldsymbol{\sigma}\,\mathbf{F}^{-T}, \quad P_{aA} = J\sigma_{ab}F_{bA}^{-T}}$$

# Piola Transformation

## Piola Identity

Using the diverge theorem, the following useful identity holds,

$$\int_{\partial\Omega} \mathbf{n}\,ds = \int_{\partial\Omega} \mathbf{1}\mathbf{n}\,ds = \int_{\Omega} \operatorname{div}\mathbf{1}\,dv = \mathbf{0}$$

Using Nanson's formula and the divergence theore yields,

$$\int_{\partial\Omega} \mathbf{n}\,ds = \int_{\partial\Omega_0} J\mathbf{F}^{-T}\mathbf{N}\,dS = \int_{\Omega_0} \mathrm{DIV}\left(J\mathbf{F}^{-T}\right)dV = \mathbf{0}$$

And in local form, we obtain the so called **Piola identity** given by,

$$\boxed{\mathrm{DIV}\left(J\mathbf{F}^{-T}\right) = \mathbf{0}, \quad \left(JF_{aA}^{-T}\right)_{,A} = 0}$$

# Piola Transformation

## Piola Transformation

Using the **Piola transformation** and the **identity** given by,

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \mathrm{DIV}\left(J\mathbf{F}^{-T}\right) = \mathbf{0},$$

the following expression holds,

$$\mathrm{DIV}\,\mathbf{P} = \mathrm{DIV}\left(J\boldsymbol{\sigma}\mathbf{F}^{-T}\right) = J\mathbf{F}^{-T}\,\mathrm{DIV}\,\boldsymbol{\sigma} + \cancel{\boldsymbol{\sigma}\cdot\mathrm{DIV}\left(J\mathbf{F}^{-T}\right)} = J\,\mathrm{div}\,\boldsymbol{\sigma}$$

yielding the useful expression,

$$\boxed{\mathrm{DIV}\,\mathbf{P} = J\,\mathrm{div}\,\boldsymbol{\sigma}, \quad P_{aA,A} = J\sigma_{ab,b}}$$

# Piola Transformation

## Symmetry Restriction

The *Piola transformation* yields to the following relations between the *Cauchy* (or *true*) *stress* tensor and the *first Piola-Kirchhoff* (or *nominal*) *stress* tensor,

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{T}$$

The *symmetry* of the *Cauchy stress* tensor, i.e.,

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^{T}, \quad \sigma_{ab} = \sigma_{ab}^{T} = \sigma_{ba}}$$

yields the following *symmetry restriction* on the *first Piola-Kirchhoff stress* tensor,

$$\boxed{\mathbf{P}\mathbf{F}^{T} = \mathbf{F}\mathbf{P}^{T}, \quad P_{aA}F_{Ab}^{T} = P_{aA}F_{bA} = F_{aA}P_{Ab}^{T} = F_{aA}P_{bA}}$$

# Stress Tensors

## Cauchy Stress Tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \quad \sigma_{ab} = \sigma_{ab}^T = \sigma_{ba}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T, \quad \sigma_{ab} = J^{-1}P_{aA}F_{Ab}^T = J^{-1}P_{aA}F_{bA}$$

$$\operatorname{div}\boldsymbol{\sigma} = J^{-1}\,\mathrm{DIV}\,\mathbf{P}, \quad \sigma_{ab,b} = J^{-1}P_{aA,A}$$

## First Piola-Kirchhoff Stress Tensor

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \quad P_{aA}F_{Ab}^T = P_{aA}F_{bA} = F_{aA}P_{Ab}^T = F_{aA}P_{bA}$$

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad P_{aA} = J\sigma_{ab}F_{bA}^{-T} = J\sigma_{ab}F_{Ab}^{-1}$$

$$\mathrm{DIV}\,\mathbf{P} = J\operatorname{div}\boldsymbol{\sigma}, \quad P_{aA,A} = J\sigma_{ab,b}$$

# Stress Tensor Components

## Cauchy Stress Tensor Components

The *Cauchy* (or *true*) *traction* vector at a *spatial point*, along the *Cartesian planes*, i.e. on planes with *unit vectors* along the Cartesian axes at the spatial configuration, read,

$$\mathbf{t}_1 = \boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3$$

$$\mathbf{t}_2 = \boldsymbol{\sigma}\mathbf{e}_2 = \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3$$

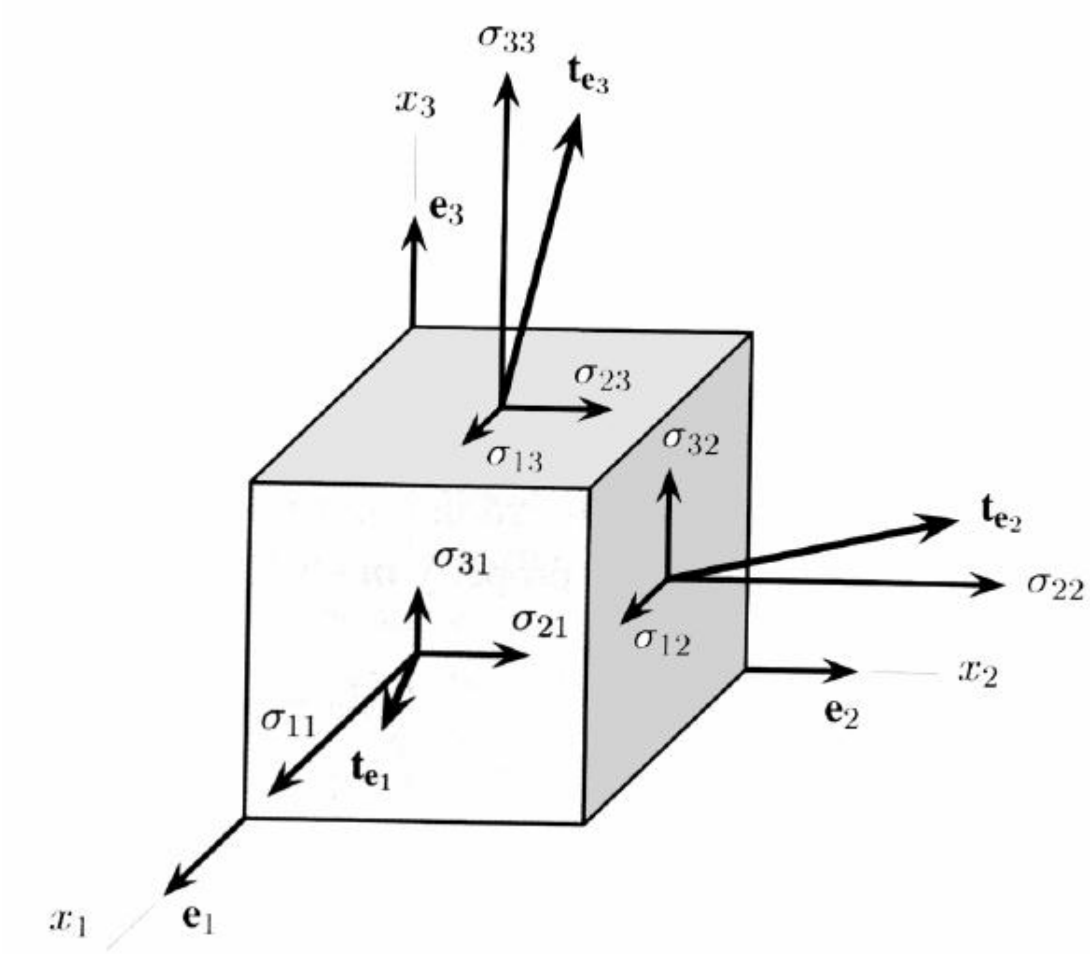
$$\mathbf{t}_3 = \boldsymbol{\sigma}\mathbf{e}_3 = \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3$$

Note that the *ab*-component of the *Cauchy stress* tensor may be computed as,

$$\sigma_{ab} = \mathbf{e}_a \cdot \mathbf{t}_b = \mathbf{e}_a \cdot \boldsymbol{\sigma}\mathbf{e}_b$$

# Stress Tensor Components

## Cauchy Stress Tensor Components



# Stress Tensor Components

## Cauchy Stress Tensor Components

Using *index notation*, the *matrix* of *Cartesian components* of the *symmetric Cauchy* (or *true*) *stress* tensor takes the form,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

Being a *symmetric* second-order tensor, we may collect the six components into a vector of components, such that,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{13} & \sigma_{23} \end{bmatrix}^T$$

# Stress Tensor Components

## Cauchy Stress Tensor Components

Using *engineering notation*, the *matrix* of *Cartesian components* of the *symmetric Cauchy* (or *true*) *stress* tensor takes the form,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zz} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Being a *symmetric* second-order tensor, we may collect the six components into a vector of components, such that,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{bmatrix}^T$$

# Other Stress Tensors

## Kirchhoff Stress Tensor

The **Kirchhoff stress** tensor, denoted as , is a *symmetric spatial* second-order tensor and it may be defined in terms of the *Cauchy stress* tensor as,

$$\boxed{\boldsymbol{\tau} = J\boldsymbol{\sigma}, \quad \tau_{ab} = J\sigma_{ab}}$$

## Second Piola-Kirchhoff Stress Tensor

The **second Piola-Kirchhoff stress** tensor, denoted as , is a *symmetric material* second-order tensor and it may be defined in terms of the *Kirchhoff stress* tensor as,

$$\boxed{\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}, \quad S_{AB} = F_{Aa}^{-1}\tau_{ab}F_{bB}^{-T} = F_{Aa}^{-1}\tau_{ab}F_{Bb}^{-1}}$$

# Push-forward / Pull-back operations

## Push-forward / Pull-back Operations

The *Kirchhoff stress* tensor may be viewed as the **push-forward** of the *second Piola-Kirchhoff stress* tensor, satisfying,

$$\boxed{\boldsymbol{\tau} = \varphi_{\#}\left(\mathbf{S}\right) = \mathbf{F}\mathbf{S}\mathbf{F}^{T}}$$

The *second Piola-Kirchhoff stress* tensor may be viewed as the **pull-back** of the *Kirchhoff stress* tensor, satisfying,

$$\boxed{\mathbf{S} = \boldsymbol{\varphi}_{\#}^{-1}\left(\boldsymbol{\tau}\right) = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}}$$

# Stress Tensors

## Cauchy Stress Tensor

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{\tau} = J^{-1}\mathbf{P}\mathbf{F}^{T} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{T}$$

## Kirchhoff Stress Tensor

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^{T} = \mathbf{F}\mathbf{S}\mathbf{F}^{T}$$

## First Piola-Kirchhoff Stress Tensor

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} = \boldsymbol{\tau}\mathbf{F}^{-T} = \mathbf{F}\mathbf{S}$$

## Second Piola-Kirchhoff Stress Tensor

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{P}$$