Continuum Mechanics

Chapter 2
Kinematics: Motion

C. Agelet de Saracibar
ETS Ingenieros de Caminos, Canales y Puertos, Universidad Politécnica de Cataluña (UPC), Barcelona, Spain
International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain

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Continuum Body

Materials, such as solids, liquids, gases or intermediate states, are made of atoms, which may be grouped in molecules separated by empty space. Therefore, on a *microscopic scale*, materials are *not continuous*.

However, on a *macroscopic scale*, a length-scale much greater than that of inter-atomic distances, materials may be modeled as a *continuum body*, assuming that the matter is continuously distributed and fills the entire region of space it occupies, ignoring the discontinuities existing on a microscopic scale.

Continuum Mechanics

Continuum Mechanics

Continuum mechanics is a powerful and effective tool to succesfully describe **macroscopic systems** using a **continuum approach**. Such an approach leads to the **continuum theory**.

A **continuum body**, denoted by $\mathcal{B}$, is viewed as a *continuous medium*, having a continuous (or at least a piecewise continuous) distribution of matter in space and time. It may be imagined as being a composition of a (continuous) set of **particles** (or **material points**), represented by $\mathrm{P} \in \mathcal{B}$.

A **continuum body** is determined by **macroscopic quantities** which may be described by *continuous functions* with *continuous derivatives*.

Continuum Mechanics

Continuum Mechanics

Continuum mechanics includes the following key ingredients:

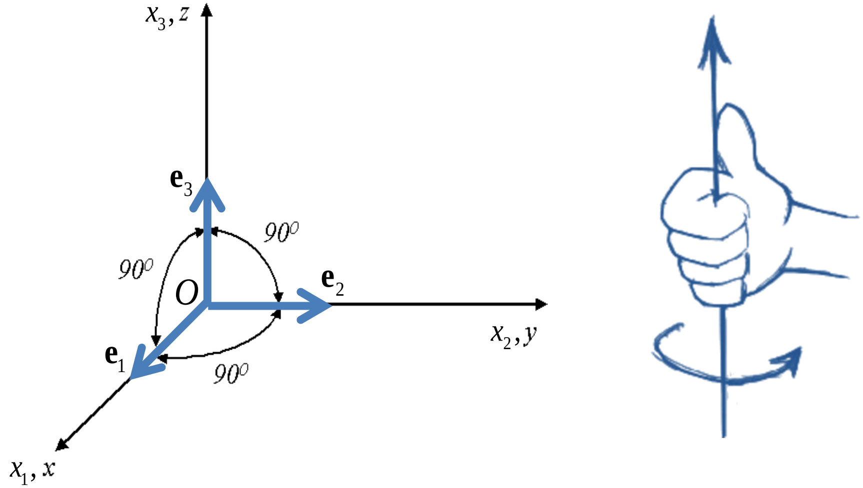
- **Kinematics**: Motion and deformations of a continuum body
- **Stresses**: Forces, stresses
- **Balance laws**: Fundamental laws of physics governing the motion of a continuum body
- **Constitutive equations**: Material characterization of a continuum body

Configurations

Configurations

Let us consider a **continuum body** $\mathcal{B}$ with **particle** $P \in \mathcal{B}$, which is embedded in the three-dimensional Euclidean space at a given instant of **time** *t*.

We introduce a **reference frame** of rectangular coordinate axes at a **fixed** origin O with **right-handed orthonormal basis vectors** $\mathbf{e}_a,\ a = 1, 2, 3$



Configurations

Configurations

As the **continuum body** $\mathcal{B}$ moves in space along the time it occupies a *continuous sequence of geometrical regions* denoted as **configurations** $\Omega_0,\ldots,\Omega_t$, which are determined uniquely at any instant of time *t*.

Any particle $\mathsf{P} \in \mathcal{B}$, at any time *t*, corresponds to a so-called geometrical **point** having a **position** in the configuration Ω_t.

Configurations

Reference Configuration

The geometrical region Ω_0 with the position of a typical point X corresponds to a fixed **reference time** and is denoted as **reference** (or **material** or **undeformed**) **configuration** of the body $\mathcal{B}$.

The point X corresponds to the position occupied by the particle $\mathsf{P} \in \mathcal{B}$ at the reference time. The particle P may be identified by the **position vector** (or **material** or **referential position**) $\mathbf{X}$ of the point X relative to the fixed origin O.

It is often convenient to call X as the **material point** $\mathbf{X}$ associated with the particle $\mathsf{P} \in \mathcal{B}$ at the fixed reference time.

Configurations

Initial Configuration

A geometrical region at **initial time** t=0 is referred to as the **initial configuration**.

We agree subsequently that the *initial configuration* coincides with the *reference configuration*, hence, we will assume that the **reference time** is t=0.

Configurations

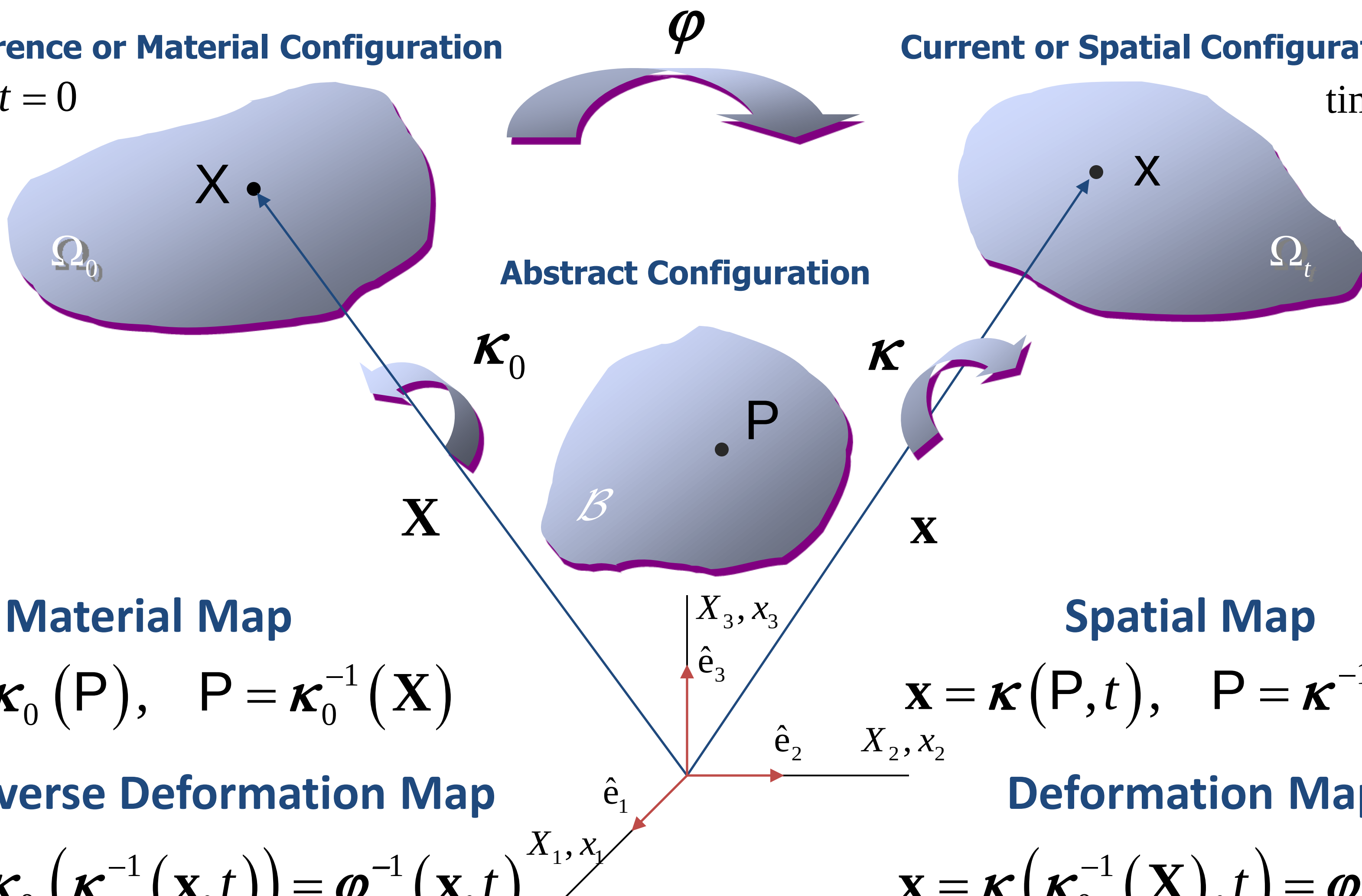
Current Configuration

The geometrical region Ω_t with the position of a typical point x corresponds to the **current time** *t>0* and is denoted as **current** (or **spatial** or **deformed**) **configuration** of the body $\mathcal{B}$.

The point x corresponds to the position occupied by the particle $\mathrm{P} \in \mathcal{B}$ at the current time *t>0*. The **position vector** (or **spatial** or **current position**) $\mathbf{x}$ serves as label for the associated point x relative to the fixed origin O.

It is often convenient to call x as the **spatial point** $\mathbf{x}$ associated with the particle $\mathrm{P} \in \mathcal{B}$ at the current time *t>0*.

Deformation Map



Material Map

$$\mathbf{X} = \boldsymbol{\kappa}_0(\mathsf{P}), \quad \mathsf{P} = \boldsymbol{\kappa}_0^{-1}(\mathbf{X})$$

Spatial Map

$$\mathbf{x} = \boldsymbol{\kappa}(\mathsf{P}, t), \quad \mathsf{P} = \boldsymbol{\kappa}^{-1}(\mathbf{x}, t)$$

Inverse Deformation Map

$$\mathbf{X} = \boldsymbol{\kappa}_0\left(\boldsymbol{\kappa}^{-1}(\mathbf{x}, t)\right) = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

Deformation Map

$$\mathbf{x} = \boldsymbol{\kappa}\left(\boldsymbol{\kappa}_0^{-1}(\mathbf{X}), t\right) = \boldsymbol{\varphi}(\mathbf{X}, t)$$

Deformation Map

Material Map

A particle P may be identified by the **position vector** (or **material** or **referential position**) of the point X relative to the fixed origin O, denoted as $\mathbf{X}$, through the *one-to-one* **material map**,

$$\mathbf{X} = \boldsymbol{\kappa}_0\left(\mathsf{P}\right), \quad \mathsf{P} = \boldsymbol{\kappa}_0^{-1}\left(\mathbf{X}\right)$$

Spatial Map

A particle P may be identified by the **position vector** (or **spatial** or **current position**) of the point x relative to the fixed origin O, denoted as $\mathbf{x}$, through the *one-to-one* **spatial map**,

$$\mathbf{x} = \boldsymbol{\kappa}\left(\mathsf{P}, t\right), \quad \mathsf{P} = \boldsymbol{\kappa}^{-1}\left(\mathbf{x}, t\right)$$

Deformation Map

Deformation Map

The composition of the spatial map and the inverse of the material map, yields the one-to-one **deformation map** defining the **equation of motion** given by,

$$\mathbf{x} = \boldsymbol{\kappa}\left(\boldsymbol{\kappa}_0^{-1}\left(\mathbf{X}\right), t\right) = \boldsymbol{\varphi}\left(\mathbf{X}, t\right)$$

Inverse Deformation Map

The composition of the material map and the inverse of the spatial map, yields the one-to-one **inverse deformation map** defining the **inverse of the equation of motion** given by,

$$\mathbf{X} = \boldsymbol{\kappa}_0\left(\boldsymbol{\kappa}^{-1}\left(\mathbf{x}, t\right)\right) = \boldsymbol{\varphi}^{-1}\left(\mathbf{x}, t\right)$$

Deformation Map

Material Coordinates

The vector position of a **material point,** denoted as $\mathbf{X}$, may be written as a linear combination of the orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, i.e., the *Cartesian basis*, such that,

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3 = X_A\mathbf{e}_A$$

where the components X_1, X_2, X_3 are denoted as **material coordinates**.

Using matrix notation, the **vector of material coordinates**, denoted as $[\mathbf{X}]$, takes the form,

$$[\mathbf{X}] = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T$$

Deformation Map

Spatial Coordinates

The vector position of a **spatial point,** denoted as $\mathbf{x}$, may be written as a linear combination of the orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, i.e., the *Cartesian basis*, such that,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_a\mathbf{e}_a$$

where the components x_1, x_2, x_3 are denoted as **spatial coordinates**.

Using matrix notation, the **vector of spatial coordinates**, denoted as $[\mathbf{x}]$, takes the form,

$$[\mathbf{x}] = [x_1 \quad x_2 \quad x_3]^T$$

Material Differential Operators

Material Nabla, Laplacian and Hessian

The **material nabla** vector differential operator, denoted as $\bar{\nabla}$, is defined as,

$$\bar{\nabla} = \sum_{A=1,3} \frac{\partial}{\partial X_A} \mathbf{e}_A = \frac{\partial}{\partial X_A} \mathbf{e}_A$$

The **material laplacian** scalar differential operator, denoted as $\bar{\Delta}$, is defined as,

$$\bar{\Delta} = \bar{\nabla} \cdot \bar{\nabla} = \frac{\partial^2}{\partial X_A^2}$$

The **material hessian** symmetric second-order tensor differential operator is defined as,

$$\bar{\nabla} \otimes \bar{\nabla} = \frac{\partial^2}{\partial X_A \partial X_B} \mathbf{e}_A \otimes \mathbf{e}_B$$

Material Differential Operators

Material Divergence, Curl and Gradient

The **material divergence** differential operator DIV (·) is defined as,

$$\mathrm{DIV}(\bullet) = \bar{\nabla} \cdot (\bullet) = \frac{\partial(\bullet)}{\partial X_A} \cdot \mathbf{e}_A$$

The **material curl** differential operator CURL (·) is defined as,

$$\mathrm{CURL}(\bullet) = \bar{\nabla} \times (\bullet) = \mathbf{e}_A \times \frac{\partial(\bullet)}{\partial X_A}$$

The **material gradient** differential operator GRAD (·) is defined as,

$$\mathrm{GRAD}(\bullet) = \bar{\nabla} \otimes (\bullet) = \frac{\partial(\bullet)}{\partial X_A} \otimes \mathbf{e}_A = \bar{\nabla}(\bullet) = \frac{\partial(\bullet)}{\partial X_A} \mathbf{e}_A$$

Spatial Differential Operators

Spatial Nabla, Laplacian and Hessian

The **spatial nabla** vector differential operator, denoted as ∇, is defined as,

$$\nabla = \sum_{a=1,3} \frac{\partial}{\partial x_a} \mathbf{e}_a = \frac{\partial}{\partial x_a} \mathbf{e}_a$$

The **spatial laplacian** scalar differential operator, denoted as Δ, is defined as,

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_a^2}$$

The **spatial hessian** symmetric second-order tensor differential operator is defined as,

$$\nabla \otimes \nabla = \frac{\partial^2}{\partial x_a \partial x_b} \mathbf{e}_a \otimes \mathbf{e}_b$$

Spatial Differential Operators

Spatial Divergence, Curl and Gradient

The **spatial divergence** differential operator div (·) is defined as,

$$\operatorname{div}(\bullet) = \nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_a} \cdot \mathbf{e}_a$$

The **spatial curl** differential operator curl (·) is defined as,

$$\operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_a \times \frac{\partial(\bullet)}{\partial x_a}$$

The **spatial gradient** differential operator grad (·) is defined as,

$$\operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_a} \otimes \mathbf{e}_a = \nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_a} \mathbf{e}_a$$

Deformation Map

Deformation Map

The **deformation map** $\mathbf{x} = \boldsymbol{\varphi}\left(\mathbf{X}, t\right)$ has to satisfy the following conditions,

1. **Continuous** with continuous derivatives up to the required continuity degree
2. **Consistency** condition, i.e. taking *t*=0 as reference time,

$$\mathbf{X} = \boldsymbol{\varphi}\left(\mathbf{X}, 0\right)$$

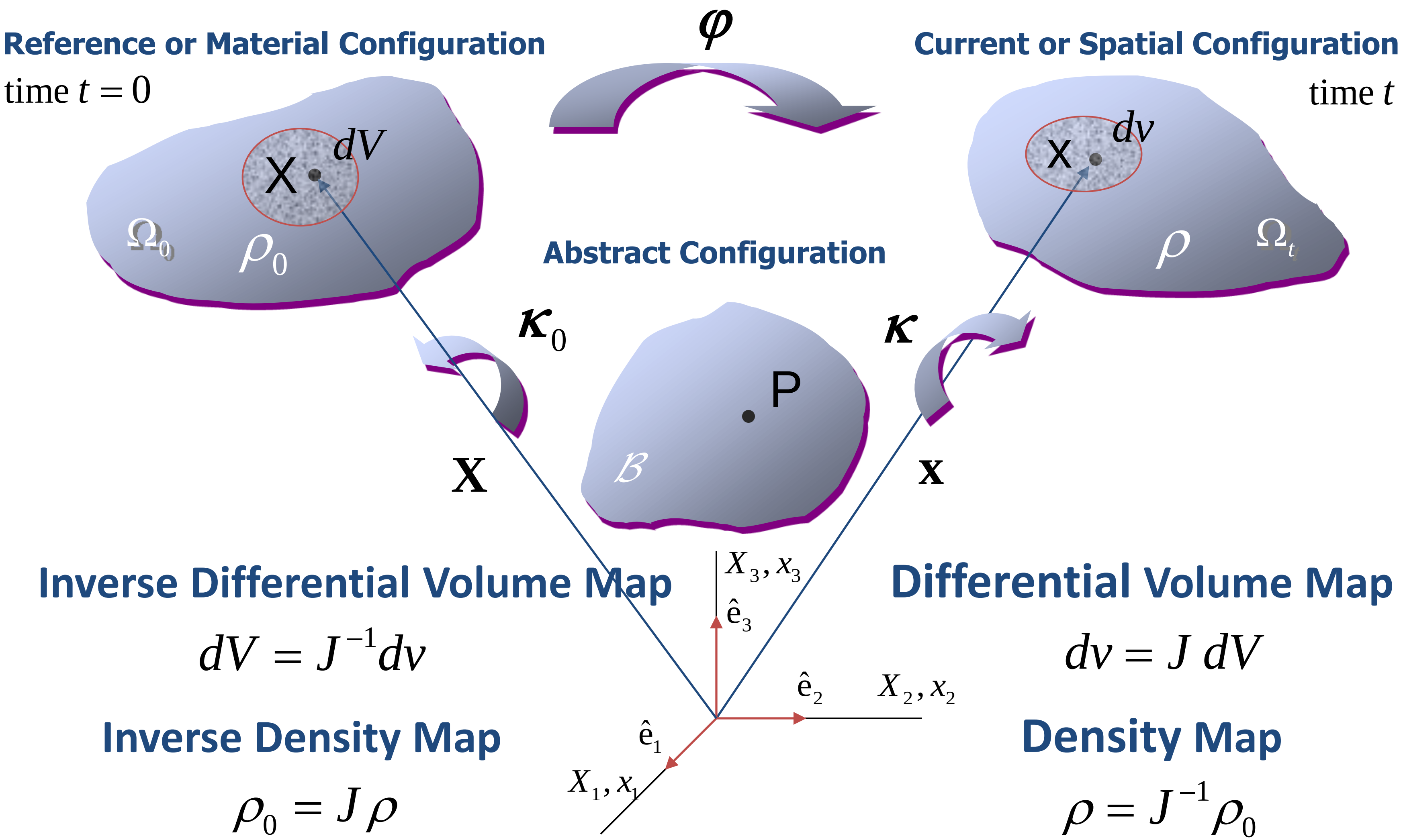
3. **One-to-one map**, i.e. there exists the inverse of the deforma-tion map,

$$\mathbf{X} = \boldsymbol{\varphi}^{-1}\left(\mathbf{x}, t\right)$$

4. **Positive Jacobian**, i.e. positive differential of volume,

$$J := \det\left[\mathrm{GRAD}\,\boldsymbol{\varphi}\left(\mathbf{X}, t\right)\right] > 0$$

Deformation Map



Inverse Differential Volume Map

$$dV = J^{-1} dv$$

Differential Volume Map

$$dv = J\, dV$$

Inverse Density Map

$$\rho_0 = J \rho$$

Density Map

$$\rho = J^{-1} \rho_0$$

Deformation Map

Jacobian

The **jacobian** of the deformation map is a positive real value and takes the form,

$$J := \det\left[\mathrm{GRAD}\,\boldsymbol{\varphi}\left(\mathbf{X},t\right)\right] > 0$$

and the following relation holds,

$$dv = J\,dV$$

Note that at the reference time for t=0,

$$dv = J\,dV = dV \quad \Rightarrow \quad J = 1$$

Material and Spatial Descriptions

Material Description

Using a **material description**, any arbitrary property γ (of any tensorial order) involved in the description of a continuum body, is mathematically described as a function of the **material points** (or **material vector positions**) $\mathbf{X}$ and the time *t*, i.e.,

$$\gamma = \Gamma\left(\mathbf{X}, t\right)$$

Spatial Description

Using a **spatial description**, any arbitrary property γ (of any tensorial order) involved in the description of a continuum body, is mathematically described as a function of the **spatial points** (or **spatial vector positions**) $\mathbf{x}$ and the time *t*, i.e.,

$$\gamma = \gamma\left(\mathbf{x}, t\right)$$

Material and Spatial Descriptions

Material Description

The **material description** of an arbitrary property γ (of any tensorial order) provides the time-evolution of the property for a given **particle** or **material point** $\mathbf{X}$ and is typically used in *solid mechanics*.

$$\gamma = \Gamma(\mathbf{X}, t)$$

Spatial Description

The **spatial description** of an arbitrary property γ (of any tensorial order) provides the time-evolution of the property at a fixed **spatial point** $\mathbf{x}$ and is typically used in *fluid mechanics*.

$$\gamma = \gamma(\mathbf{x}, t)$$

Material and Spatial Descriptions

Material and Spatial Descriptions

Giving the **material description** of an arbitrary property $\gamma = \Gamma(\mathbf{X}, t)$ and the inverse of the motion equation $\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$, the **spatial description** of the property reads,

$$\gamma = \Gamma(\mathbf{X}, t) = \Gamma\left(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t\right) = \gamma(\mathbf{x}, t)$$

Giving the **spatial description** of an arbitrary property $\gamma = \gamma(\mathbf{x}, t)$ and the motion equation $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$, the **material description** of the property reads,

$$\gamma = \gamma(\mathbf{x}, t) = \gamma\left(\boldsymbol{\varphi}(\mathbf{X}, t), t\right) = \Gamma(\mathbf{X}, t)$$

Material and Spatial Time Derivatives

Material Time Derivative

Giving the *material description* of an arbitrary property, $\gamma = \Gamma(\mathbf{X}, t)$ the **material time derivative** of the property is given by,

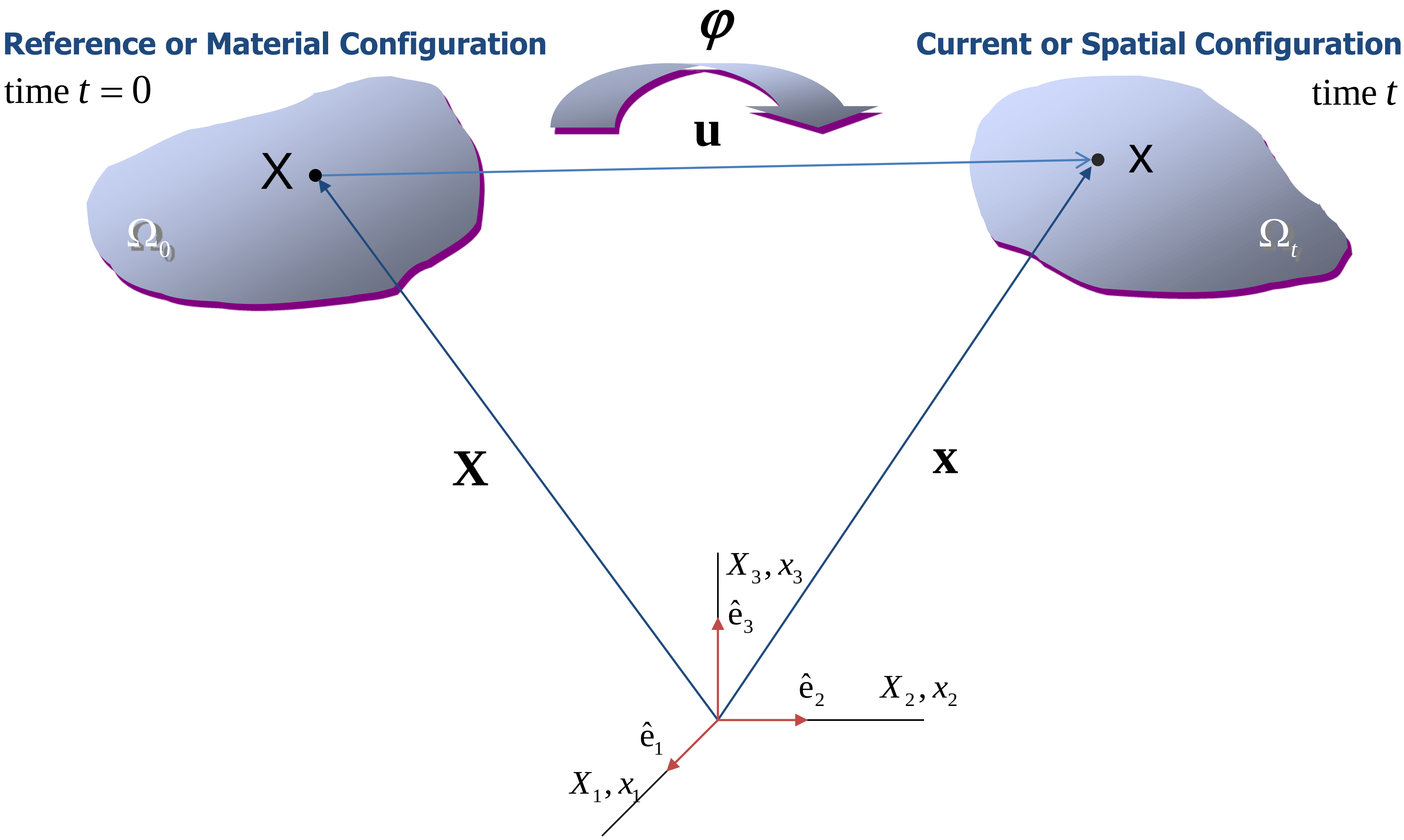
$$\dot{\gamma} = \frac{d\gamma}{dt} = \left.\frac{\partial \Gamma(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}} = \frac{\partial \Gamma(\mathbf{X}, t)}{\partial t}$$

Spatial Time Derivative

Giving the *spatial description* of an arbitrary property, $\gamma = \gamma(\mathbf{x}, t)$ the **spatial** (or **local**) **time derivative** of the property is given by,

$$\frac{\partial \gamma}{\partial t} = \left.\frac{\partial \gamma(\mathbf{x}, t)}{\partial t}\right|_{\mathbf{x}} = \frac{\partial \gamma(\mathbf{x}, t)}{\partial t}$$

Displacement Vector Field



Displacement Vector Field

Displacement Vector Field

The **displacement** vector field, denoted as $\mathbf{u}$, is defined as,

$$\mathbf{u} = \mathbf{x} - \mathbf{X}$$

The *material description* of the **displacement** vector field takes the form,

$$\mathbf{u} = \mathbf{U}(\mathbf{X}, t) = \boldsymbol{\varphi}(\mathbf{X}, t) - \mathbf{X}$$

The *spatial description* of the **displacement** vector field takes the form,

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$$

Velocity Vector Field

Velocity Vector Field

The **velocity** vector field, denoted as $\mathbf{v}$, is defined as,

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$

The *material description* of the **velocity** vector field takes the form,

$$\mathbf{v} = \mathbf{V}(\mathbf{X},t) = \frac{\partial \varphi(\mathbf{X},t)}{\partial t}$$

The *spatial description* of the **velocity** vector field takes the form,

$$\mathbf{v} = \mathbf{V}\left(\varphi^{-1}(\mathbf{x},t),t\right) = \mathbf{v}(\mathbf{x},t)$$

Acceleration Vector Field

Acceleration Vector Field

The **acceleration** vector field, denoted as $\mathbf{a}$, is defined as,

$$\mathbf{a} = \frac{d}{dt}\left(\frac{d\mathbf{x}}{dt}\right) = \frac{d\mathbf{v}}{dt}$$

The *material description* of the **acceleration** vector field takes the form,

$$\mathbf{a} = \mathbf{A}\left(\mathbf{X},t\right) = \frac{\partial^2 \varphi\left(\mathbf{X},t\right)}{\partial t^2} = \frac{\partial \mathbf{V}\left(\mathbf{X},t\right)}{\partial t}$$

The *spatial description* of the **acceleration** vector field takes the form,

$$\mathbf{a} = \mathbf{A}\left(\varphi^{-1}\left(\mathbf{x},t\right),t\right) = \mathbf{a}\left(\mathbf{x},t\right)$$

Material Time Derivative

Material Time Derivative

Giving the **spatial description** of an arbitrary property, $\gamma = \gamma(\mathbf{x}, t)$, the **material time derivative** of the property can be written as,

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma(\mathbf{x}(t), t)}{\partial t} = \frac{\partial \gamma(\mathbf{x}, t)}{\partial t} + \frac{\partial \gamma(\mathbf{x}, t)}{\partial x_a} \cdot \frac{dx_a(t)}{dt}$$

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma(\mathbf{x}(t), t)}{\partial t} = \frac{\partial \gamma(\mathbf{x}, t)}{\partial t} + \underbrace{\left(\operatorname{grad} \gamma(\mathbf{x}, t)\right) \cdot \mathbf{v}(\mathbf{x}, t)}_{\text{Convective time derivative}}$$

The **material time derivative** of an arbitrary propert given in spatial description may be written as the sum of its *spatial* (or *local*) *time derivative* and its *convective time derivative*.

Material Time Derivative

Convective Time Derivative

The **convective time derivative** of an arbitrary property given in spatial description, $\gamma = \gamma(\mathbf{x}, t)$, may be defined as the difference between its *material time derivative* and its *spatial* (or *local*) *time derivative*, yielding,

$$\left(\operatorname{grad}\gamma(\mathbf{x},t)\right)\cdot\mathbf{v}(\mathbf{x},t) = \dot{\gamma} - \frac{\partial\gamma(\mathbf{x},t)}{\partial t} = \frac{d\gamma}{dt} - \frac{\partial\gamma(\mathbf{x},t)}{\partial t}$$

Material Time Derivative

Acceleration

The **acceleration** vector field may be also written as the *material time derivative* of the *spatial description of the velocity vector field*, yielding,

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d\mathbf{v}(\mathbf{x},t)}{dt}=\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t}+\left(\operatorname{grad}\mathbf{v}(\mathbf{x},t)\right)\mathbf{v}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)$$

$$a_a=\frac{dv_a}{dt}=\frac{dv_a(\mathbf{x},t)}{dt}=\frac{\partial v_a(\mathbf{x},t)}{\partial t}+\left(\operatorname{grad}v_a(\mathbf{x},t)\right)\mathbf{v}(\mathbf{x},t)$$

$$=\frac{\partial v_a(\mathbf{x},t)}{\partial t}+v_{a,b}(\mathbf{x},t)\,v_b(\mathbf{x},t)=a_a(\mathbf{x},t)$$

Kinematics of Deformation

Displacement Vector Field

$$\mathbf{U}(\mathbf{X},t)=\boldsymbol{\varphi}(\mathbf{X},t)-\mathbf{X},\quad \mathbf{u}(\mathbf{x},t)=\mathbf{x}-\boldsymbol{\varphi}^{-1}(\mathbf{x},t)$$

Velocity Vector Field

$$\mathbf{V}(\mathbf{X},t)=\frac{\partial\varphi(\mathbf{X},t)}{\partial t},\quad \mathbf{v}(\mathbf{x},t)=\mathbf{V}\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t\right)$$

Acceleration Vector Field

$$\mathbf{A}(\mathbf{X},t)=\frac{\partial\mathbf{V}(\mathbf{X},t)}{\partial t},\quad \mathbf{a}(\mathbf{x},t)=\mathbf{A}\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t\right)$$

$$=\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t}+\left(\operatorname{grad}\mathbf{v}(\mathbf{x},t)\right)\mathbf{v}(\mathbf{x},t)$$

Assignment 2.1

Assignment 2.1

The Cartesian components of the *spatial description* of the *velocity* field is,

$$v_x(\mathbf{x},t) = x - z, \quad v_y(\mathbf{x},t) = z\left(e^t + e^{-t}\right), \quad v_z(\mathbf{x},t) = 0$$

Compute the *acceleration* at the fixed *spatial point* with Cartesian coordinates (1,1,1) at time *t*=2.

Assignment 2.1

Assignment 2.1

The Cartesian components of the *spatial description* of the *velocity* field is,

$$v_x(\mathbf{x},t) = x - z, \quad v_y(\mathbf{x},t) = z\left(e^{t} + e^{-t}\right), \quad v_z(\mathbf{x},t) = 0$$

Compute the *acceleration* at the fixed *spatial point* with Cartesian coordinates (1,1,1) at time *t*=2.

The *spatial description* of the **acceleration** vector field can be directly computed from the spatial description of the velocity vector field,

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \left(\operatorname{grad} \mathbf{v}(\mathbf{x},t)\right)\mathbf{v}(\mathbf{x},t)$$

Assignment 2.1

The Cartesian components of the **acceleration** vector field in *spatial description* are given by,

$$a_x(\mathbf{x},t) = \frac{\partial v_x(\mathbf{x},t)}{\partial t} + \left(\operatorname{grad} v_x(\mathbf{x},t)\right)\cdot \mathbf{v}(\mathbf{x},t)$$

$$a_y(\mathbf{x},t) = \frac{\partial v_y(\mathbf{x},t)}{\partial t} + \left(\operatorname{grad} v_y(\mathbf{x},t)\right)\cdot \mathbf{v}(\mathbf{x},t)$$

$$a_z(\mathbf{x},t) = \frac{\partial v_z(\mathbf{x},t)}{\partial t} + \left(\operatorname{grad} v_z(\mathbf{x},t)\right)\cdot \mathbf{v}(\mathbf{x},t)$$

Assignment 2.1

The Cartesian components of the **spatial** (or **local**) **time derivative** of the **velocity** in *spatial description* are given by,

$$\frac{\partial v_x\left(\mathbf{x},t\right)}{\partial t}=0$$

$$\frac{\partial v_y\left(\mathbf{x},t\right)}{\partial t}=z\left(e^t-e^{-t}\right)$$

$$\frac{\partial v_z\left(\mathbf{x},t\right)}{\partial t}=0$$

Assignment 2.1

The Cartesian components of the *spatial gradient* of each one of the components of the *velocity* vector field in spatial *description* take the form,

$$\left[\operatorname{grad} v_x(\mathbf{x},t)\right] = \begin{bmatrix}1 & 0 & -1\end{bmatrix}^T$$

$$\left[\operatorname{grad} v_y(\mathbf{x},t)\right] = \begin{bmatrix}0 & 0 & e^t + e^{-t}\end{bmatrix}^T$$

$$\left[\operatorname{grad} v_z(\mathbf{x},t)\right] = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$$

yielding,

$$\left(\operatorname{grad} v_x(\mathbf{x},t)\right)\cdot\mathbf{v}(\mathbf{x},t) = x - z$$

$$\left(\operatorname{grad} v_y(\mathbf{x},t)\right)\cdot\mathbf{v}(\mathbf{x},t) = 0$$

$$\left(\operatorname{grad} v_z(\mathbf{x},t)\right)\cdot\mathbf{v}(\mathbf{x},t) = 0$$

Assignment 2.1

The Cartesian components of the **acceleration** vector field in *spatial description* are given by,

$$a_x(\mathbf{x},t) = x - z$$

$$a_y(\mathbf{x},t) = z\left(e^t - e^{-t}\right)$$

$$a_z(\mathbf{x},t) = 0$$

The Cartesian components of the **acceleration** vector field at the *spatial point* with Cartesian coordinates (1,1,1), at time *t*=2, are given by,

$$\boxed{\begin{aligned} a_x(\mathbf{x},t) &= 0 \\ a_y(\mathbf{x},t) &= e^2 - e^{-2} \\ a_z(\mathbf{x},t) &= 0 \end{aligned}}$$

Assignment 2.2

Assignment 2.2 [Classwork]

The Cartesian components of the *canonical form* of a *motion equation*, i.e. deformation map, are given by,

$$x(\mathbf{X},t) = Xe^{t}, \quad y(\mathbf{X},t) = Ye^{t}, \quad z(\mathbf{X},t) = Z + Xt$$

1) Compute the *acceleration* vector field at the fixed *spatial point* with Cartesian coordinates (1,1,1).
2) Compute the *acceleration* vector field at the fixed *material point* with Cartesian coordinates (1,1,1).
3) Compute the *rate of change* of the *velocity* vector field per unit of time at the fixed *spatial point* with Cartesian coordinates (1,1,1).

Assignment 2.2

Assignment 2.2 [Classwork]

The Cartesian components of the *canonical form* of a *motion equation*, i.e. deformation map, are given by,

$$x(\mathbf{X},t) = Xe^{t}, \quad y(\mathbf{X},t) = Ye^{t}, \quad z(\mathbf{X},t) = Z + Xt$$

The Cartesian components of the **velocity vector field** in *material description* are given by,

$$V_x(\mathbf{X},t) = Xe^{t}, \quad V_y(\mathbf{X},t) = Ye^{t}, \quad V_z(\mathbf{X},t) = X$$

and, using the (inverse) motion equations, in *spatial description* are given by,

$$v_x(\mathbf{x},t) = x, \quad v_y(\mathbf{x},t) = y, \quad v_z(\mathbf{x},t) = xe^{-t}$$

Assignment 2.2

The Cartesian components of the **acceleration vector field** in *material description* are given by,

$$A_x(\mathbf{X},t) = Xe^t, \quad A_y(\mathbf{X},t) = Ye^t, \quad A_z(\mathbf{X},t) = 0$$

and, using the (inverse) motion equations, in *spatial description* are given by,

$$a_x(\mathbf{x},t) = x, \quad a_y(\mathbf{x},t) = y, \quad a_z(\mathbf{x},t) = 0$$

The Cartesian components of the **spatial** (or **local**) **time derivative** of the **velocity** vector field in *spatial description* are given by,

$$\frac{\partial v_x(\mathbf{x},t)}{\partial t} = 0, \quad \frac{\partial v_y(\mathbf{x},t)}{\partial t} = 0, \quad \frac{\partial v_z(\mathbf{x},t)}{\partial t} = -xe^{-t}$$

Assignment 2.2

The Cartesian components of the **acceleration** vector field at the fixed *spatial point* x* with Cartesian coordinates (1,1,1) are,

$$\boxed{a_x\left(\mathbf{x}^*,t\right)=1,\quad a_y\left(\mathbf{x}^*,t\right)=1,\quad a_z\left(\mathbf{x}^*,t\right)=0}$$

The Cartesian components of the **acceleration** vector field at the fixed *material point* X* with Cartesian coordinates (1,1,1) are,

$$\boxed{A_x\left(\mathbf{X}^*,t\right)=e^t,\quad A_y\left(\mathbf{X}^*,t\right)=e^t,\quad A_z\left(\mathbf{X}^*,t\right)=0}$$

The Cartesian components of the **rate of change** of the **velocity** vector field per unit of time at the fixed *spatial point* x* with Cartesian coordinates (1,1,1) are,

$$\boxed{\frac{\partial v_x\left(\mathbf{x}^*,t\right)}{\partial t}=0,\quad \frac{\partial v_y\left(\mathbf{x}^*,t\right)}{\partial t}=0,\quad \frac{\partial v_z\left(\mathbf{x}^*,t\right)}{\partial t}=-e^{-t}}$$

Stationary Field

Stationary Field

An arbitrary property given in spatial description as $\gamma = \gamma(\mathbf{x}, t)$ is said to be **stationary** *if and only if* the following condition is satisfied,

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma(\mathbf{x}, t)}{\partial t} = 0 \quad \Leftrightarrow \quad \gamma = \gamma(\mathbf{x})$$

If an arbitrary property is stationary, its material time derivative does not needs to be stationary and, in general, will be different than zero,

$$\dot{\gamma} = \frac{d\gamma}{dt} = \cancel{\frac{\partial \gamma(\mathbf{x})}{\partial t}} + \left(\operatorname{grad} \gamma(\mathbf{x})\right) \cdot \mathbf{v}(\mathbf{x}, t) \neq 0$$

Stationary Velocity Vector Field

Stationary Velocity Vector Field

The velocity vector field is said to be **stationary** *if and only if* the following condition is satisfied,

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{v}(\mathbf{x})$$

If the velocity vector field is stationary, the *acceleration* vector field has to be also *stationary*, but, in general, different than zero. Note that the opposite is not true.

$$\mathbf{a}(\mathbf{x}) = \cancel{\frac{\partial \mathbf{v}(\mathbf{x})}{\partial t}} + \left(\operatorname{grad} \mathbf{v}(\mathbf{x})\right)\mathbf{v}(\mathbf{x}) = \left(\operatorname{grad} \mathbf{v}(\mathbf{x})\right)\mathbf{v}(\mathbf{x}) \neq \mathbf{0}$$

Uniform Velocity Vector Field

Uniform Velocity Vector Field

A velocity vector field is said to be **uniform** *if and only if* the following condition is satisfied,

$$\mathbf{v} = \mathbf{v}(\mathbf{x},t) = \mathbf{v}(t) \quad \forall \mathbf{x} \in \Omega_t$$

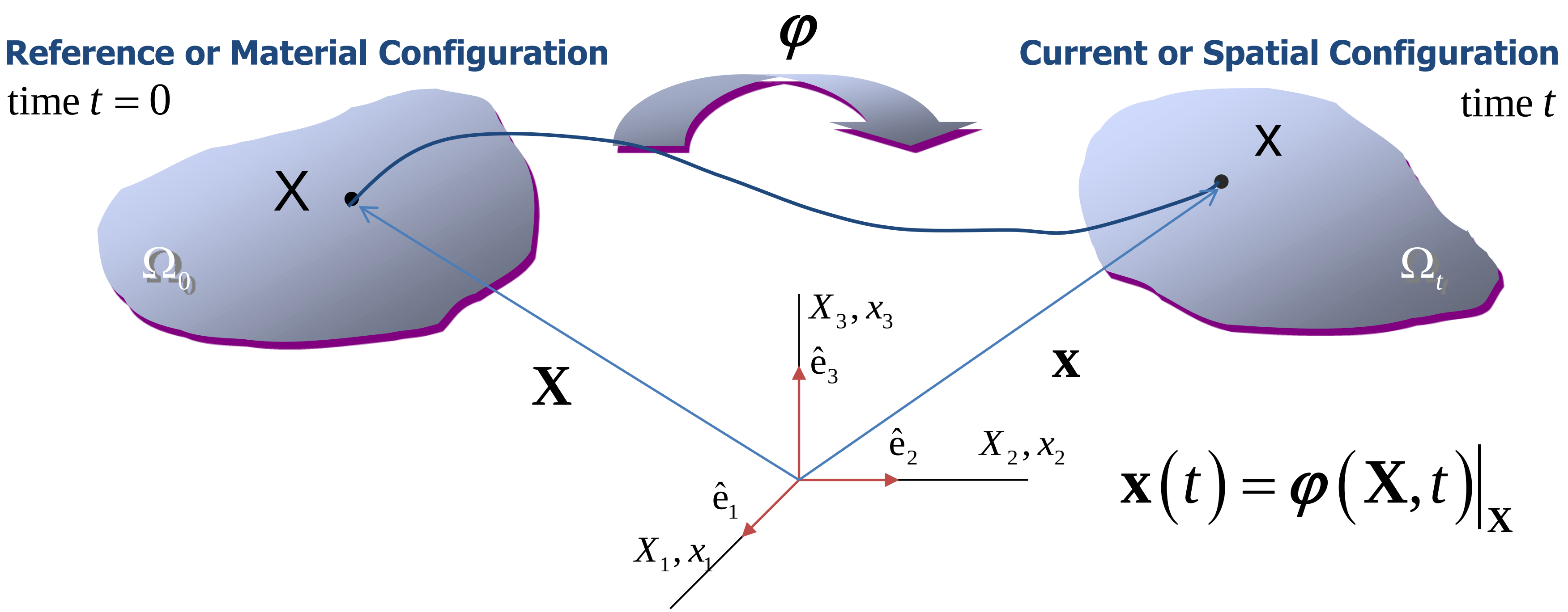
If the velocity vector field is uniform, the *acceleration* vector field has to be also *uniform*, but, in general, different than zero. Note that the opposite is not true.

$$\mathbf{a}(t) = \frac{\partial \mathbf{v}(t)}{\partial t} + \cancel{\left(\operatorname{grad} \mathbf{v}(t)\right)\mathbf{v}(t)} = \frac{\partial \mathbf{v}(t)}{\partial t} \neq \mathbf{0}$$

Trajectories

Trajectories

The **motion equation** provides the sequence of spatial positions occupied for any particle at any time, defining a *time-parame-trized family of curves* denoted as **trajectories** (or **path lines**).



$$\mathbf{x}(t) = \varphi(\mathbf{X}, t)\big|_{\mathbf{X}}$$

Differential Equation of the Trajectories

Differential Equation of the Trajectories

A **trajectory** can be described in **differential form** by means of the *spatial velocity* vector field, through the following set of differential equations,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}\left(\mathbf{x}(t), t\right)$$

Integrating the set of differential equations yields,

$$\mathbf{x} = \boldsymbol{\phi}\left(\mathbf{C}, t\right)$$

where $\mathbf{C}$ is a *vector of integration constants* with Cartesian components given by,

$$[\mathbf{C}] = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}^T$$

Differential Equation of the Trajectories

Imposing the **consistency condition**, taking *t*=0 as reference time, yields,

$$\mathbf{X} = \boldsymbol{\phi}(\mathbf{C}, 0)$$

Then, the vector of integration constants can be expressed in terms of the vector of material points, yielding,

$$\mathbf{C} = \boldsymbol{\phi}^{-1}(\mathbf{X}, 0)$$

Then, the **canonical form** of the **motion equation** reads,

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{C}, t) = \boldsymbol{\phi}\left(\boldsymbol{\phi}^{-1}(\mathbf{X}, 0), t\right) = \boldsymbol{\varphi}(\mathbf{X}, t)$$

Assignment 2.3

Assignment 2.3

The Cartesian components of the *spatial description* of the *velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad v_z(\mathbf{x},t) = z$$

Compute the *canonical form* of the trajectories.

Assignment 2.3

Assignment 2.3

The Cartesian components of the *spatial description* of the *velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad v_z(\mathbf{x},t) = z$$

The Cartesian components of the *differential equation of motion* read,

$$\frac{dx}{dt} = v_x(\mathbf{x},t) = y, \quad \frac{dy}{dt} = v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad \frac{dz}{dt} = v_z(\mathbf{x},t) = z$$

Assignment 2.3

Integrating the differential equations yields,

$$\frac{dy}{dt} = \frac{y}{1+t} \quad \Rightarrow \quad \frac{dy}{y} = \frac{dt}{1+t} \quad \Rightarrow \quad \log \frac{y}{C_2} = \log(1+t)$$

$$\Rightarrow \quad y = C_2 (1+t)$$

$$\frac{dx}{dt} = y = C_2 (1+t) \quad \Rightarrow \quad x = C_1 + C_2 t + \frac{1}{2} C_2 t^2$$

$$\frac{dz}{dt} = z \quad \Rightarrow \quad \frac{dz}{z} = dt \quad \Rightarrow \quad \log \frac{z}{C_3} = t \quad \Rightarrow \quad z = C_3 e^t$$

Assignment 2.3

Integrating, the **motion equations** read,

$$x = C_1 + C_2 t + \frac{1}{2} C_2 t^2, \quad y = C_2 \left(1 + t\right), \quad z = C_3 e^t$$

Imposing the *consistency condition*, taking *t*=0 as reference time, yields,

$$C_1 = X, \quad C_2 = Y, \quad C_3 = Z$$

The Cartesian components of the **canonical form** of the **equation of motion** reads,

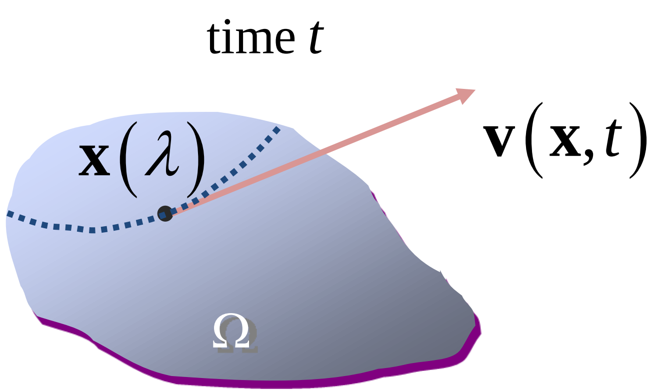
$$\boxed{x = X + Yt + \frac{1}{2} Yt^2, \quad y = Y\left(1 + t\right), \quad z = Ze^t}$$

Streamlines

Streamlines

The **streamlines** are a time-dependent family of spatial curves which at any time *t* are the *envelope of the spatial velocity vector field*, i.e. the velocity vector field is tangent to the streamlines at any spatial point, at any time *t*.

Current or Spatial Configuration



Differential Equation of the Streamlines

Differential Equation of the Streamlines

The **differential equation** of the **streamlines** may be obtained imposing the condition that the spatial vector velocity field $\mathbf{v}\left(\mathbf{x}(\lambda),t\right)$ is *tangent* to the streamlines $\mathbf{x}(\lambda)$. The parameter of the stramlines, denoted as λ, is chosen such that the velocity is *equal* to the tangent to the streamlines, yielding,

$$\frac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}\left(\mathbf{x}(\lambda),t\right)$$

Integrating the differential equations, collecting the integration constants in vector form, yields the equation of the **streamlines**,

$$\mathbf{x} = \psi\left(\mathbf{C},\lambda,t\right)$$

Streamlines for a Stationary Motion

Streamlines for a Stationary Motion

If the velocity vector field is *stationary* the **streamlines** are stationary and coincide with the **trajectories**.

If the velocity vector field is *stationary*, the **trajectories** and **streamlines** have the same differential equations, yielding,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t)) \quad \leftrightarrow \quad \frac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}(\mathbf{x}(\lambda))$$

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{C}, t) \quad \leftrightarrow \quad \mathbf{x} = \boldsymbol{\phi}(\mathbf{C}, \lambda)$$

Assignment 2.4

Assignment 2.4

The Cartesian components of the *spatial description* of the *velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad v_z(\mathbf{x},t) = z$$

Compute the equation of the *streamlines*.

Assignment 2.4

Assignment 2.4

The Cartesian components of the *spatial description* of the *velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad v_z(\mathbf{x},t) = z$$

The Cartesian components of the *differential equation of the streamlines* take the form,

$$\frac{dx}{d\lambda} = v_x(\mathbf{x},t) = y, \quad \frac{dy}{d\lambda} = v_y(\mathbf{x},t) = \frac{y}{1+t}, \quad \frac{dz}{d\lambda} = v_z(\mathbf{x},t) = z$$

Assignment 2.4

Integrating the differential equations yields,

$$\frac{dy}{d\lambda} = \frac{y}{1+t} \quad \Rightarrow \quad \frac{dy}{y} = \frac{d\lambda}{1+t} \quad \Rightarrow \quad \log\frac{y}{C_2} = \frac{\lambda}{1+t} \quad \Rightarrow \quad y = C_2 e^{\frac{\lambda}{1+t}}$$

$$\frac{dx}{d\lambda} = y = C_2 e^{\frac{\lambda}{1+t}} \quad \Rightarrow \quad x = C_1 + C_2\left(1+t\right)e^{\frac{\lambda}{1+t}}$$

$$\frac{dz}{d\lambda} = z \quad \Rightarrow \quad \frac{dz}{z} = d\lambda \quad \Rightarrow \quad \log\frac{z}{C_3} = \lambda \quad \Rightarrow \quad z = C_3 e^{\lambda}$$

The Cartesian components of the **streamlines** read,

$$\boxed{x = C_1 + C_2\left(1+t\right)e^{\frac{\lambda}{1+t}}, \quad y = C_2 e^{\frac{\lambda}{1+t}}, \quad z = C_3 e^{\lambda}}$$

Assignment 2.5

Assignment 2.5 [Classwork]

The Cartesian components of the *streamlines* are given by,

$$x = C_1 e^{\lambda t}, \quad y = C_2 e^{\lambda t}, \quad z = C_3 e^{2\lambda t}$$

where the parameter λ is such that the velocity is *equal* to the tangent to the streamlines. Compute the *canonical form* of the trajectories, taking *t*=0 as reference time.

Assignment 2.5

Assignment 2.5 [Classwork]

The Cartesian components of the streamlines are given by,

$$x = C_1 e^{\lambda t}, \quad y = C_2 e^{\lambda t}, \quad z = C_3 e^{2\lambda t}$$

As a first step we will compute the *spatial description* of the **velocity**. Using the differential equation of the *streamlines* yields,

$$\frac{dx}{d\lambda} = C_1 t e^{\lambda t} = v_x, \quad \frac{dy}{d\lambda} = C_2 t e^{\lambda t} = v_y, \quad \frac{dz}{d\lambda} = 2C_3 t e^{2\lambda t} = v_z$$

Using the equations of the streamlines, the components of the *spatial description* of the **velocity** read,

$$v_x = xt, \quad v_y = yt, \quad v_z = 2zt$$

Assignment 2.5

The Cartesian components of the *differential motion equation* read,

$$\frac{dx}{dt} = v_x(\mathbf{x},t) = xt, \quad \frac{dy}{dt} = v_y(\mathbf{x},t) = yt, \quad \frac{dz}{dt} = v_z(\mathbf{x},t) = 2zt$$

Integrating the differential equations yields,

$$\frac{dx}{dt} = xt \quad \Rightarrow \quad \frac{dx}{x} = tdt \quad \Rightarrow \quad \log\frac{x}{C_1} = \frac{1}{2}t^2 \quad \Rightarrow \quad x = C_1 e^{t^2/2}$$

$$\frac{dy}{dt} = yt \quad \Rightarrow \quad \frac{dy}{y} = tdt \quad \Rightarrow \quad \log\frac{y}{C_2} = \frac{1}{2}t^2 \quad \Rightarrow \quad y = C_2 e^{t^2/2}$$

$$\frac{dz}{dt} = 2zt \quad \Rightarrow \quad \frac{dz}{z} = 2tdt \quad \Rightarrow \quad \log\frac{z}{C_3} = t^2 \quad \Rightarrow \quad z = C_3 e^{t^2}$$

Assignment 2.5

The Cartesian components of the **motion equation** read,

$$x = C_1 e^{t^2/2}, \quad y = C_2 e^{t^2/2}, \quad z = C_3 e^{t^2}$$

Imposing the *consistency condition* at reference time *t*=0 yields,

$$C_1 = X, \quad C_2 = Y, \quad C_3 = Z$$

The Cartesian components of the *canonical form* of the **motion equation** reads,

$$\boxed{x = X e^{t^2/2}, \quad y = Y e^{t^2/2}, \quad z = Z e^{t^2}}$$

Assignment 2.6

Assignment 2.6 [Classwork]

The Cartesian components of the *spatial description* of a *stationary velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = y, \quad v_z(\mathbf{x},t) = z$$

Compute the *canonical form* of the *trajectories* and the equation of the *streamlines*.

Assignment 2.6

Assignment 2.6 [Classwork]

The Cartesian components of the *spatial description* of a *stationary velocity* vector field are,

$$v_x(\mathbf{x},t) = y, \quad v_y(\mathbf{x},t) = y, \quad v_z(\mathbf{x},t) = z$$

As the velocity vector field is *stationary* the trajectories and streamlines are the same curves.

The Cartesian components of the *differential equations* of the *trajectories* take the form,

$$\frac{dx}{dt} = v_x(\mathbf{x},t) = y, \quad \frac{dy}{dt} = v_y(\mathbf{x},t) = y, \quad \frac{dz}{dt} = v_z(\mathbf{x},t) = z$$

Assignment 2.6

Integrating the differential equations yields,

$$\frac{dy}{dt} = y \quad \Rightarrow \quad \frac{dy}{y} = dt \quad \Rightarrow \quad \log \frac{y}{C_2} = t \quad \Rightarrow \quad y = C_2 e^t$$

$$\frac{dx}{dt} = y = C_2 e^t \quad \Rightarrow \quad x = C_1 + C_2 e^t$$

$$\frac{dz}{dt} = z \quad \Rightarrow \quad \frac{dz}{z} = dt \quad \Rightarrow \quad \log \frac{z}{C_3} = t \quad \Rightarrow \quad z = C_3 e^t$$

and the Cartesian components of the **trajectories** take the form,

$$x = C_1 + C_2 e^t, \quad y = C_2 e^t, \quad z = C_3 e^t$$

Assignment 2.6

Imposing the *consistency condition*, taking *t*=0 as reference time, yields,

$$C_1 + C_2 = X\,, \quad C_2 = Y\,, \quad C_3 = Z$$

$$C_1 = X - Y\,, \quad C_2 = Y\,, \quad C_3 = Z$$

and the Cartesian components of the *canonical form* of the **trajectories** take the form,

$$\boxed{x = X + Y\left(e^t - 1\right), \quad y = Ye^t\,, \quad z = Ze^t}$$

Assignment 2.6

As the velocity vector field is *stationary*, the **streamlines** do not need to be integrated and can be obtained directly substituting the time t by λ in the expression of the trajectories (written in terms of the integration constants, i.e. before having used the consistency condition), yielding,

$$\boxed{x = C_1 + C_2 e^{\lambda}, \quad y = C_2 e^{\lambda}, \quad z = C_3 e^{\lambda}}$$

Material Surface

Material Surface

A **material surface** is defined by the different positions occupied in the space by the particles that at the reference time were on a given surface.

The **material description** of a **material surface** may be written as,

$$S = \left\{ \mathbf{X} \mid F\left(\mathbf{X}\right) = 0 \right\}$$

where the time-independency of the material function guaran-tees that the particles satisfying this equation are always the same ones, for any time t.

Note that,

$$F\left(\mathbf{X}, t\right) = F\left(\mathbf{X}\right) \quad \Leftrightarrow \quad \frac{\partial F\left(\mathbf{X}, t\right)}{\partial t} = 0$$

Material Surface

Material Surface

The spatial description of the function may be obtained using the inverse motion equation yielding,

$$F\left(\mathbf{X}\right) = F\left(\boldsymbol{\varphi}^{-1}\left(\mathbf{x},t\right)\right) = f\left(\mathbf{x},t\right)$$

Additionally,

$$\frac{\partial F\left(\mathbf{X}\right)}{\partial t} = \frac{df\left(\mathbf{x},t\right)}{dt} = 0$$

The **spatial description** of a **material surface** may be written as,

$$S = \left\{ \mathbf{x} \,|\, f\left(\mathbf{x},t\right) = 0 \quad \text{and} \quad \frac{df\left(\mathbf{x},t\right)}{dt} = 0 \right\}$$

Spatial Surface

Spatial Surface

A **spatial surface** is defined by the same fixed spatial points at any time t. Then, at different times t, different particles will be on a spatial surface.

The **spatial description** of a **spatial surface** may be written as,

$$S = \left\{ \mathbf{x} \,|\, f\left(\mathbf{x}\right) = 0 \right\}$$

where the time-independency of the spatial function guarantees that the spatial points satisfying this equation are always the same ones, for any time t.

Material Volume

Material Volume

A **material volume** is a volume defined by a closed material surface.

A **material volume**, written in **material description**, takes the form,

$$V = \left\{ \mathbf{X} \mid F\left(\mathbf{X}\right) \leq 0 \right\}$$

and, in **spatial description**, takes the form,

$$V = \left\{ \mathbf{x} \mid f\left(\mathbf{x}, t\right) \leq 0 \quad \text{and} \quad \frac{df\left(\mathbf{x}, t\right)}{dt} = 0 \right\}$$

Spatial Volume

Spatial Volume

A **spatial volume** is a volume defined by a closed spatial surface.

The **spatial description** of a **spatial volume** may be written as,

$$V = \left\{ \mathbf{x} \mid f\left(\mathbf{x}\right) \leq 0 \right\}$$