Continuum Mechanics

Chapter 6
Balance Laws

C. Agelet de Saracibar
ETS Ingenieros de Caminos, Canales y Puertos, Universidad Politécnica de Cataluña (UPC), Barcelona, Spain
International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain

Contents

Chapter 6 · Balance Laws

Introduction

Introduction

The *fundamental laws* of the continuum mechanics are given by *four conservation/balance laws* plus a *restriction law*.

The four **conservation/balance laws** are:

- *Conservation of mass* · Mass continuity equation
- *Linear momentum balance* · Cauchy's first motion equation
- *Angular momentum balance* · Symmetry of Cauchy stress
- *First law of thermodynamics* · Energy balance equation

The **restriction law** is given by:

- *Second law of thermodynamics* · Clausius-Planck and heat conduction inequalities

Introduction

Introduction

The mathematical expressions arising from the *fundamental laws* will be given in:

- Global (or integral) form
  - Global (or integral) *spatial* form
  - Global (or integral) *material* form
- Local (or strong) form
  - Local (or strong) *spatial* form
  - Local (or strong) *material* form

Conservation of Mass

Conservation of Mass

We assume that during a motion there are neither *mass sources* (reservoirs that supply mass), nor *mass sinks* (reservoirs that absorb mass), so the **mass** of a continuum body is a *conserved quatity*.

$$m\left(\Omega_0\right) = m\left(\Omega\right) > 0$$

Then, the mass is independent of the motion and, hence, the *material time derivative* of the mass of a continuum medium (or a material volume) has to be zero,

$$\frac{d}{dt} m\left(\Omega_0\right) = \frac{d}{dt} m\left(\Omega\right) = 0$$

Conservation of Mass

Mass Density

The *mass* at the *material* (or *reference*) *configuration* may be characterized by a continuous positive scalar field, denoted as $\rho_0 = \rho_0(\mathbf{X}) > 0$, which is a *material property* called **material** (or **reference**) **mass density**, such that,

$$dm(\mathbf{X}) = \rho_0(\mathbf{X})\, dV > 0$$

The *mass* at the *spatial* (or *current*) *configuration* may be characterized by a continuous positive scalar field, denoted as $\rho = \rho(\mathbf{x}, t) > 0$, which is called **spatial** (or **current**) **mass density**, such that,

$$dm(\mathbf{x}, t) = \rho(\mathbf{x}, t)\, dv > 0$$

Note that, taking t=0 as reference time, $\rho(\mathbf{x}, 0) = \rho_0(\mathbf{X}) > 0.$

Conservation of Mass

Conservation of Mass: Global Material Form

The **mass** of a continuum medium (or a *material volume*) is a *conserved* quantity,

$$m(\Omega_0) = \int_{\Omega_0} \rho_0(\mathbf{X})\, dV = \int_{\Omega} \rho(\mathbf{x}, t)\, dv = m(\Omega) > 0$$

Using,

$$dv = J(\mathbf{X}, t)\, dV > 0$$

The *global material form* of the **conservation of mass** may be written as,

$$\boxed{\int_{\Omega_0} \rho_0(\mathbf{X})\, dV = \int_{\Omega_0} \rho(\boldsymbol{\varphi}(\mathbf{X}, t), t)\, J(\mathbf{X}, t)\, dV > 0}$$

Conservation of Mass

Conservation of Mass: Local Material Form

Let us consider the *global material form* of the **conservation of mass** given by,

$$\int_{\Omega_0} \rho_0\left(\mathbf{X}\right) dV = \int_{\Omega_0} \rho\left(\boldsymbol{\varphi}\left(\mathbf{X},t\right),t\right) J\left(\mathbf{X},t\right) dV > 0$$

Localizing the integral expression, the *local material form* of the **conservation of mass** reads,

$$\boxed{\rho_0\left(\mathbf{X}\right) = \rho\left(\boldsymbol{\varphi}\left(\mathbf{X},t\right),t\right) J\left(\mathbf{X},t\right) > 0}$$

Conservation of Mass

Conservation of Mass: Global Material Form

The *material time derivative* of the mass of a continuum medium (or a material volume) has to be *zero*,

$$\frac{d}{dt} m(\Omega_0) = \frac{d}{dt} \int_{\Omega_0} \rho_0(\mathbf{X})\, dV = \frac{d}{dt} \int_{\Omega} \rho(\mathbf{x},t)\, dv = \frac{d}{dt} m(\Omega) = 0$$

Using,

$$dv = J(\mathbf{X},t)\, dV > 0$$

The *global material form* of the **conservation of mass** may be written as,

$$\boxed{\frac{d}{dt} \int_{\Omega_0} \rho\left(\boldsymbol{\varphi}(\mathbf{X},t),t\right) J(\mathbf{X},t)\, dV = 0}$$

Conservation of Mass

Conservation of Mass: Local Material Form

Let us consider the *global material form* of the **conservation of mass** given by,

$$\frac{d}{dt}\int_{\Omega_0} \rho\left(\boldsymbol{\varphi}\left(\mathbf{X},t\right),t\right) J\left(\mathbf{X},t\right) dV = 0$$

$$\int_{\Omega_0} \frac{d}{dt}\left(\rho\left(\boldsymbol{\varphi}\left(\mathbf{X},t\right),t\right) J\left(\mathbf{X},t\right)\right) dV = 0$$

Localizing the integral expression, the *local material form* of the **conservation of mass** reads,

$$\boxed{\frac{d}{dt}\left(\rho\left(\boldsymbol{\varphi}\left(\mathbf{X},t\right),t\right) J\left(\mathbf{X},t\right)\right) = 0}$$

Conservation of Mass

Conservation of Mass: Global Spatial Form

The **mass** of a continuum medium (or a *material volume*) is a *conserved* quantity,

$$m(\Omega_0) = \int_{\Omega_0} \rho_0(\mathbf{X})\,dV = \int_{\Omega} \rho(\mathbf{x},t)\,dv = m(\Omega) > 0$$

Using,

$$dV = J^{-1}(\mathbf{x},t)\,dv > 0$$

The *global spatial form* of the **conservation of mass** may be written as,

$$\boxed{\int_{\Omega} \rho_0\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right) J^{-1}(\mathbf{x},t)\,dv = \int_{\Omega} \rho(\mathbf{x},t)\,dv > 0}$$

Conservation of Mass

Conservation of Mass: Local Spatial Form

Let us consider the *global spatial form* of the conservation of **mass** given by,

$$\int_\Omega \rho_0\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right) J^{-1}(\mathbf{x},t)\, dv = \int_\Omega \rho(\mathbf{x},t)\, dv > 0$$

Localizing the integral expression, the *local spatial form* of the **conservation of mass** reads,

$$\boxed{\rho_0\left(\boldsymbol{\varphi}^{-1}(\mathbf{x},t)\right) J^{-1}(\mathbf{x},t) = \rho(\mathbf{x},t) > 0}$$

Conservation of Mass

Conservation of Mass: Global Spatial Form

The *material time derivative* of the mass of a continuum medium (or a material volume) has to be *zero*,

$$\frac{d}{dt} m(\Omega_0) = \frac{d}{dt} \int_{\Omega_0} \rho_0(\mathbf{X})\, dV = \frac{d}{dt} \int_{\Omega} \rho(\mathbf{x},t)\, dv = \frac{d}{dt} m(\Omega) = 0$$

The *global spatial form* of the **conservation of mass** may be written as,

$$\boxed{\frac{d}{dt} \int_{\Omega} \rho(\mathbf{x},t)\, dv = 0}$$

Conservation of Mass

Conservation of Mass: Local Spatial Form

Let us consider the *global spatial form* of the conservation of **mass** given by,

$$\frac{d}{dt}\int_{\Omega}\rho(\mathbf{x},t)\,dv=0$$

$$\frac{d}{dt}\int_{\Omega}\rho dv=\frac{d}{dt}\int_{\Omega_0}\rho J dV=\int_{\Omega_0}\frac{d}{dt}(\rho J)\,dV=\int_{\Omega_0}\left(\dot{\rho}J+\rho\dot{J}\right)dV$$

$$=\int_{\Omega_0}\left(\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}\right)J dV=\int_{\Omega}\left(\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}\right)dv=0$$

Localizing the integral expression, the *local spatial form* of the **conservation of mass**, or **mass continuity equation**, reads,

$$\boxed{\dot{\rho}(\mathbf{x},t)+\rho(\mathbf{x},t)\,\mathrm{div}\,\mathbf{v}(\mathbf{x},t)=0}$$

Conservation of Mass

Conservation of Mass: Local Spatial Form

The *local spatial form* of the **conservation of mass**, or **mass continuity equation**, may be written as,

$$\dot{\rho}(\mathbf{x},t)+\rho(\mathbf{x},t)\operatorname{div}\mathbf{v}(\mathbf{x},t)=0$$

Using the following expressions,

$$\dot{\rho}=\frac{\partial\rho}{\partial t}+(\operatorname{grad}\rho)\cdot\mathbf{v},\quad \operatorname{div}(\rho\mathbf{v})=\rho\operatorname{div}\mathbf{v}+(\operatorname{grad}\rho)\cdot\mathbf{v}$$

The *local spatial form* of the **conservation of mass**, or **mass continuity equation**, may be alternatively written as,

$$\boxed{\frac{\partial\rho(\mathbf{x},t)}{\partial t}+\operatorname{div}\left(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\right)=0}$$

Conservation of Mass

Global and Local Spatial Forms

$$\frac{d}{dt}\int_{\Omega}\rho\,dv=\int_{\Omega}\left(\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}\right)\,dv=0$$

$$\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}=0,\quad \frac{\partial\rho}{\partial t}+\mathrm{div}\left(\rho\mathbf{v}\right)=0$$

Global and Local Material Forms

$$\frac{d}{dt}\int_{\Omega_0}\rho J dV=\int_{\Omega_0}\frac{d}{dt}\left(\rho J\right)dV=0$$

$$\frac{d}{dt}\left(\rho J\right)=0,\quad \rho J=\rho_0>0$$

Convective Flux

Convective Flux of an Arbitrary Property

Consider an *arbitrary property* $\mathcal{A}$ of a continuum medium and let us denote as $\psi(\mathbf{x},t)$ the *spatial description* of the amount of the *property per unit of mass*, $\rho(\mathbf{x},t)$ the *spatial density field* and $\mathbf{v}(\mathbf{x},t)$ the *spatial velocity field*.

The **convective flux** of the property $\mathcal{A}$ through a *fixed spatial surface* with unit normal $\mathbf{n}(\mathbf{x},t)$, i.e. the amount of the property crossing the spatial surface per unit of time due to the *convective flux*, is given by,

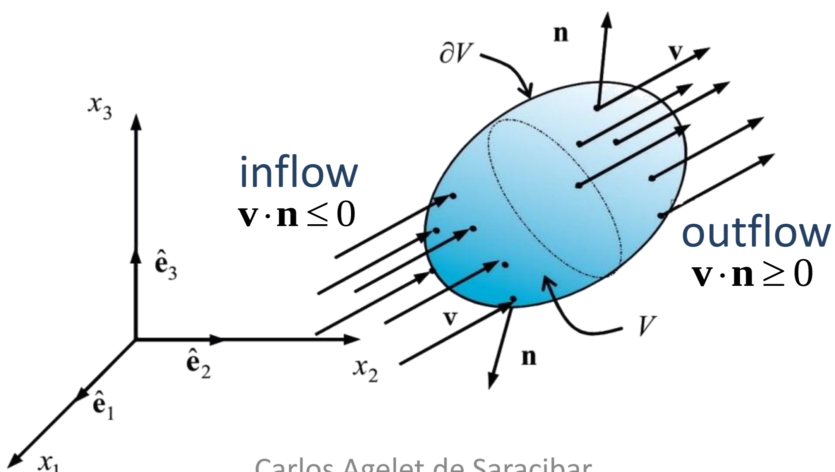
$$\boxed{\phi_{\mathcal{A}}(t) = \int_S \rho(\mathbf{x},t)\psi(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$

Convective Flux

Convective Flux of an Arbitrary Property

The **net outcoming convective flux** of the property $\mathcal{A}$ through a *fixed closed spatial surface* with unit outward normal $\mathbf{n}(\mathbf{x},t)$, i.e. the net amount of the property $\mathcal{A}$ leaving the spatial volume per unit of time due to the *convective flux*, i.e. outflow (+) plus inflow (-), is given by,

$$\boxed{\phi_{\mathcal{A}}(t) = \int_{\partial V} \rho(\mathbf{x},t)\psi(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$



Convective Flux

Mass Flux

Given a *spatial density field*, denoted as $\rho(\mathbf{x},t)$ and a *spatial velocity field*, denoted as $\mathbf{v}(\mathbf{x},t)$, the **mass flux** through a *fixed spatial surface* with unit normal $\mathbf{n}(\mathbf{x},t)$ is given by,

$$\boxed{\phi_M(t)=\int_s \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$

Note that the *mass flux* may be viewed as a particular case of the convective flux of an arbitrary property $\mathcal{A}$, setting $\psi=1$.

The *net outcoming mass flux* through a *closed* surface with unit *outward* normal $\mathbf{n}(\mathbf{x},t)$ is given by,

$$\boxed{\phi_M(t)=\int_{\partial v} \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$

Convective Flux

Volume Flux

Given a *spatial velocity field*, denoted as $\mathbf{v}(\mathbf{x},t)$, the **volume flux** through a *fixed spatial surface* with unit normal $\mathbf{n}(\mathbf{x},t)$ is given by,

$$\boxed{\phi_V(t) = \int_s \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$

Note that the *volume flux* may be viewed as a particular case of the convective flux of an arbitrary property $\mathcal{A}$, setting $\psi = \rho^{-1}$.

The *net outcoming volume flux* through a *closed* surface with unit *outward* normal $\mathbf{n}(\mathbf{x},t)$ is given by,

$$\boxed{\phi_V(t) = \int_{\partial v} \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,ds}$$

Reynolds Transport Theorem

Reynolds Lemma

Consider an *arbitrary property* $\mathcal{A}$ of a continuum medium and let us denote as $\psi(\mathbf{x},t)$ the *spatial description* of the amount of the *property per unit of mass*.

The *amount* of the property $\mathcal{A}$ may be written as,

$$\mathcal{A}(t)=\int_{\Omega}\rho\psi\,dv=\int_{\Omega_0}\rho_0\psi\,dV$$

The *material time derivative* of the property $\mathcal{A}$ may be written as,

$$\dot{\mathcal{A}}(t)=\frac{d}{dt}\int_{\Omega}\rho\psi\,dv=\frac{d}{dt}\int_{\Omega_0}\rho_0\psi\,dV$$

Reynolds Transport Theorem

Reynolds Lemma

The *material time derivative* of the property $\mathcal{A}$ may be written as,

$$\begin{aligned}\dot{\mathcal{A}}(t) &= \frac{d}{dt}\int_{\Omega} \rho\psi \, dv \\ &= \frac{d}{dt}\int_{\Omega_0} \rho\psi J dV = \int_{\Omega_0} \frac{d}{dt}(\rho\psi J) dV \\ &= \int_{\Omega_0} \frac{d}{dt}(\rho_0\psi) dV = \int_{\Omega_0} \rho_0\dot{\psi} dV \\ &= \int_{\Omega} \rho\dot{\psi} dv\end{aligned}$$

Reynolds Transport Theorem

Reynolds Lemma

The **Reynolds Lemma** for an *arbitrary property* $\mathcal{A}$ takes the form,

$$\boxed{\frac{d}{dt}\int_{\Omega}\rho\psi\,dv=\int_{\Omega}\rho\dot{\psi}\,dv}$$

Note that Reynolds lemma is obtained using mass conservation, taking into account that,

$$dm(\mathbf{x},t)=\rho(\mathbf{x},t)\,dv>0$$

Reynolds Transport Theorem

Reynolds Transport Theorem

The following key expression holds,

$$
\begin{aligned}
\rho\dot{\psi} &= \frac{d}{dt}(\rho\psi) - \dot{\rho}\psi \\
&= \frac{\partial}{\partial t}(\rho\psi) + \mathrm{grad}(\rho\psi)\cdot\mathbf{v} - \dot{\rho}\psi \\
&= \frac{\partial}{\partial t}(\rho\psi) + \mathrm{grad}(\rho\psi)\cdot\mathbf{v} + \rho\psi\,\mathrm{div}\,\mathbf{v} \\
&= \frac{\partial}{\partial t}(\rho\psi) + \mathrm{div}(\rho\psi\mathbf{v})
\end{aligned}
$$

Reynolds Transport Theorem

Reynolds Transport Theorem

The *material time derivative* of the property $\mathcal{A}$ may be written as,

$$\dot{\mathcal{A}}(t) = \frac{d}{dt}\int_{\Omega} \rho\psi\, dv = \int_{\Omega} \rho\dot{\psi}\, dv$$

$$= \int_{\Omega} \frac{\partial}{\partial t}(\rho\psi)\, dv + \int_{\Omega} \mathrm{div}(\rho\psi\mathbf{v})\, dv$$

$$= \int_{\Omega} \frac{\partial}{\partial t}(\rho\psi)\, dv + \int_{\partial\Omega} \rho\psi\mathbf{v}\cdot\mathbf{n}\, ds$$

Reynolds Transport Theorem

Reynolds Transport Theorem

The **Reynolds transport theorem** for an *arbitrary property* $\mathcal{A}$ may be written as,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\rho\psi\,dv &= \int_{\Omega}\rho\dot{\psi}\,dv \\ &= \int_{\Omega}\frac{\partial}{\partial t}(\rho\psi)\,dv + \int_{\Omega}\operatorname{div}(\rho\psi\mathbf{v})\,dv \\ &= \int_{\Omega}\frac{\partial}{\partial t}(\rho\psi)\,dv + \int_{\partial\Omega}\rho\psi\mathbf{v}\cdot\mathbf{n}\,ds\end{aligned}$$

Reynolds Transport Theorem

Reynolds Transport Theorem

Taking the *mass* as a particular case of an arbitrary property, the **Reynolds transport theorem** yields a *global spatial form* of the *conservation of mass* and take the form,

$$\frac{d}{dt}\int_{\Omega}\rho dv = 0$$

$$= \int_{\Omega}\frac{\partial}{\partial t}(\rho)\,dv + \int_{\Omega}\operatorname{div}(\rho\mathbf{v})\,dv$$

$$= \int_{\Omega}\frac{\partial}{\partial t}(\rho)\,dv + \int_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}ds$$

Reynolds Transport Theorem

Global Spatial Form

$$\frac{d}{dt}\int_{\Omega}\rho\psi\,dv=\int_{\Omega}\rho\dot{\psi}\,dv$$

$$=\int_{\Omega}\frac{\partial}{\partial t}\left(\rho\psi\right)dv+\int_{\Omega}\operatorname{div}\left(\rho\psi\mathbf{v}\right)dv$$

$$=\int_{\Omega}\frac{\partial}{\partial t}\left(\rho\psi\right)dv+\int_{\partial\Omega}\rho\psi\mathbf{v}\cdot\mathbf{n}\,ds$$

Local Spatial Form

$$\rho\dot{\psi}=\frac{\partial}{\partial t}\left(\rho\psi\right)+\operatorname{div}\left(\rho\psi\mathbf{v}\right)$$

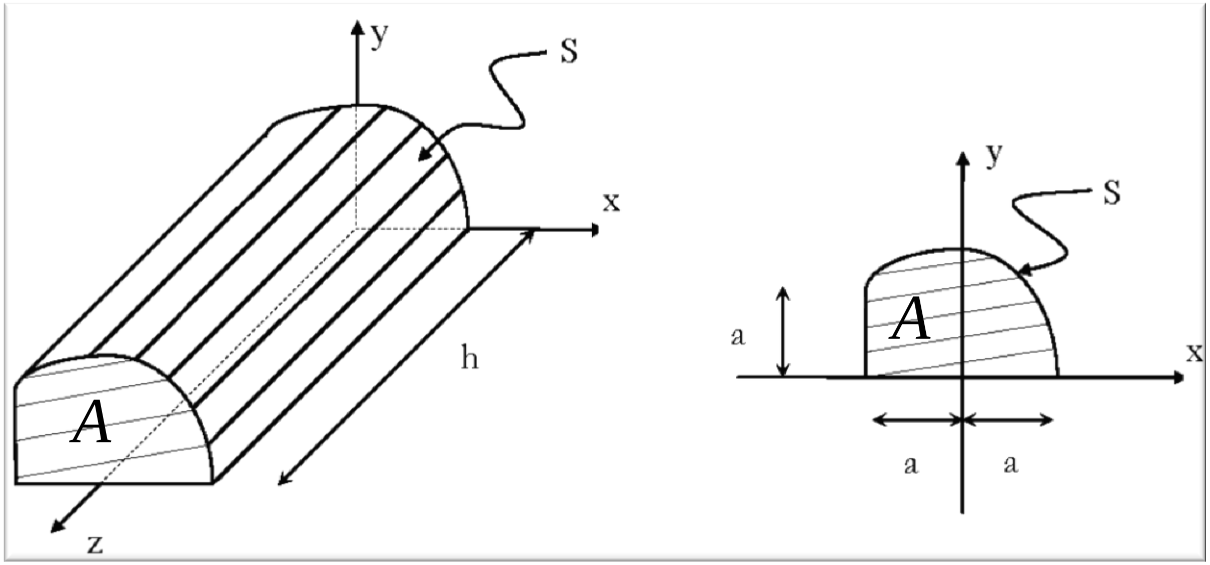
Assignment 6.1

Assignment 6.1

Consider the *spatial description* of a *velocity field* given by,

$$v_x = ye^{-t}, \quad v_y = y, \quad v_z = 0$$

The reference time is t=0. The mass density at the reference configuration is constant. Obtain the *spatial mass density* and the *mass flux* through the open cylindrical surface S of the figure.



Assignment 6.1

Assignment 6.1

Consider the *spatial description* of a *velocity field* given by,

$$v_x = ye^{-t}, \quad v_y = y, \quad v_z = 0$$

The reference time is t=0.

The mass density may be obtained from either the *local spatial* or *material forms* of the *mass continuity* equation.

Using the *local spatial form* of the mass continuity reads,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$

Where the spatial divergence of the velocity takes the value,

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = 1$$

Assignment 6.1

Integrating the scalar differential equation for the density yields,

$$\dot{\rho}+\rho=0 \quad \Rightarrow \quad \rho(t)=Ce^{-t}$$

The integration constant takes the value,

$$\left.\rho(t)\right|_{t=0}=C=\rho_0$$

The **spatial density** is given by,

$$\rho(t)=\rho_0 e^{-t}$$

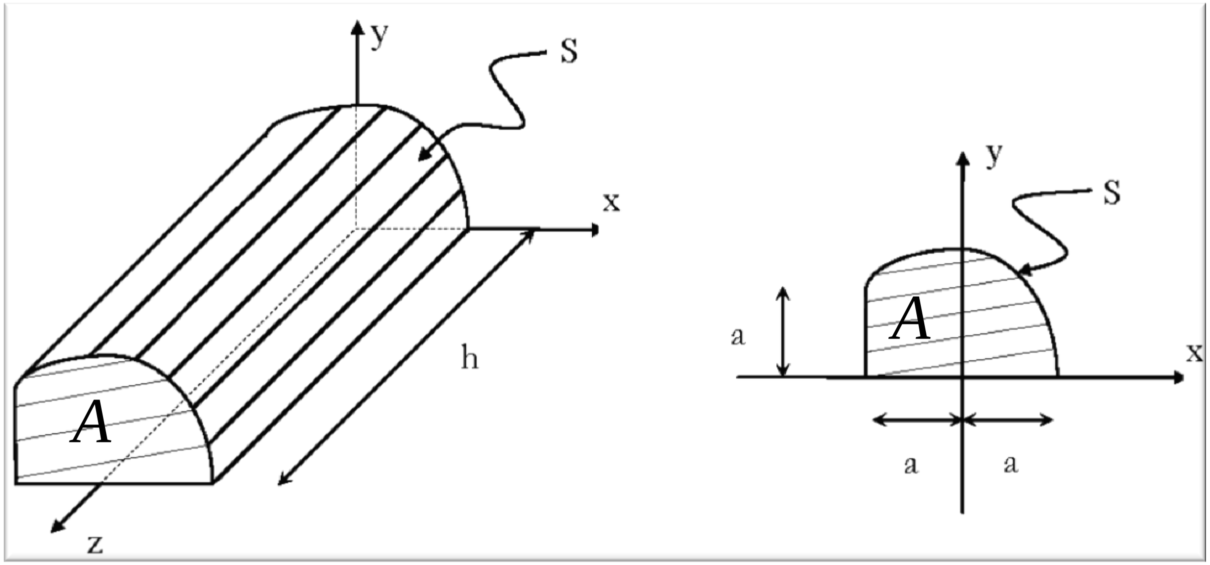
The *mass flux* through the open cylindrical surface S of the figure may be written as,

$$\phi_M=\int_S \rho\,\mathbf{v}\cdot\mathbf{n}\,dS$$

Assignment 6.1

As the equation of the surface is *unknown*, we cannot directly use this expression to compute the mass flux through S.

Let us consider a volume V with boundaries defined by: (S) the open cylindrical surface S; (S1) the Cartesian plane y=0; (S2) the plane x=-a; (S3) the plane z=0 and (S4) the plane z=h.



Assignment 6.1

Using the *conservation of mass* and making use of the *Reynolds transport theorem* yields,

$$\frac{d}{dt}\int_V \rho\, dV = \frac{\partial}{\partial t}\int_V \rho\, dV + \int_{\partial V} \rho\, \mathbf{v}\cdot\mathbf{n}\, dS$$

$$= \frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \rho\, \mathbf{v}\cdot\mathbf{n}\, dS + \sum_{i=1}^{4}\int_{S_i} \rho\, \mathbf{v}\cdot\mathbf{n}\, dS = 0$$

Then the *mass flux* through the open cylindrical surface S may be computed as,

$$\phi_M = \int_S \rho\, \mathbf{v}\cdot\mathbf{n}\, dS = -\frac{\partial}{\partial t}\int_V \rho\, dV - \sum_{i=1}^{4}\int_{S_i} \rho\, \mathbf{v}\cdot\mathbf{n}\, dS$$

Assignment 6.1

The *volume flux per unit of volume through S1* surface (y=0) is,

$$\mathbf{v}\cdot\mathbf{n}\big|_{y=0} = [\mathbf{v}]\big|_{y=0}\cdot\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T = -v_y\big|_{y=0} = 0$$

The *volume flux per unit of volume through S2* surface (x=-a) is,

$$\mathbf{v}\cdot\mathbf{n}\big|_{x=-a} = [\mathbf{v}]\big|_{x=-a}\cdot\begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T = -v_x\big|_{x=-a} = -ye^{-t}$$

The *volume flux per unit of volume through S3* surface (z=0) is,

$$\mathbf{v}\cdot\mathbf{n}\big|_{z=0} = [\mathbf{v}]\big|_{z=0}\cdot\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T = -v_z\big|_{z=0} = 0$$

The *volume flux per unit of volume through S4* surface (z=h) is,

$$\mathbf{v}\cdot\mathbf{n}\big|_{z=h} = [\mathbf{v}]\big|_{z=h}\cdot\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T = v_z\big|_{z=h} = 0$$

Assignment 6.1

The *spatial time derivative* of the mass takes the form,

$$\frac{\partial}{\partial t}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV = -\int_V \rho_0 e^{-t}\, dV = -\rho_0 e^{-t} A h$$

The *mass flux through S1* (y=0) takes the form,

$$\int_{S_1} \rho\, \mathbf{v}\cdot\mathbf{n}\, dS = -\int_{S_1} \rho\, v_y\Big|_{y=0} dS = 0$$

The *mass flux through S2* (x=-a) takes the form,

$$\int_{S_2} \rho\, \mathbf{v}\cdot\mathbf{n}\, dS = -\int_{S_2} \rho_0 e^{-t}\, v_x\Big|_{x=-a} dS = -\int_{S_2} \rho_0 y e^{-2t}\, dS = -\rho_0 e^{-2t} h \frac{a^2}{2}$$

Assignment 6.1

The *mass flux through S3* (z=0) takes the form,

$$\int_{S_3} \rho\,\mathbf{v}\cdot\mathbf{n}\,dS = -\int_{S_1} \rho_0 e^{-t}\, v_z\big|_{z=0}\, dS = 0$$

The *mass flux through S4* (z=h) takes the form,

$$\int_{S_4} \rho\,\mathbf{v}\cdot\mathbf{n}\,dS = \int_{S_4} \rho_0 e^{-t}\, v_z\big|_{z=h}\, dS = 0$$

Substituting , the **flux of mass** through the surface S is given by,

$$\phi_M = \int_S \rho\,\mathbf{v}\cdot\mathbf{n}\,dS = -\frac{\partial}{\partial t}\int_V \rho\,dV - \sum_{i=1}^{4}\int_{S_i} \rho\,\mathbf{v}\cdot\mathbf{n}\,dS$$

$$= \rho_0 e^{-t} Ah + \rho_0 e^{-2t} h\frac{a^2}{2} = \rho h\left(A + e^{-t}\frac{a^2}{2}\right) \quad \blacksquare$$

Linear Momentum Balance

Linear Momentum

The **linear momentum** of a material volume, denoted as $\mathbf{M}_L(t)$, is defined as a vector-valued function given by,

$$\begin{aligned}\mathbf{M}_L(t) &= \int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dv \\ &= \int_{\Omega_0} \rho_0(\mathbf{X})\,\mathbf{V}(\mathbf{X},t)\,dV\end{aligned}$$

Linear Momentum Balance

Linear Momentum

Using *Reynolds Lemma*, i.e. conservation of mass, the *material time derivative* of the **linear momentum** takes the form,

$$\dot{\mathbf{M}}_L(t) = \frac{d}{dt}\int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dv = \int_{\Omega} \rho(\mathbf{x},t)\,\dot{\mathbf{v}}(\mathbf{x},t)\,dv$$

$$= \frac{d}{dt}\int_{\Omega_0} \rho_0(\mathbf{X})\,\mathbf{V}(\mathbf{X},t)\,dV = \int_{\Omega_0} \rho_0(\mathbf{X})\,\dot{\mathbf{V}}(\mathbf{X},t)\,dV$$

Linear Momentum Balance

Resultant Force

The **resultant force** acting on a material volume, denoted as $\mathbf{F}(t)$, is defined as a vector-valued function given by,

$$\begin{aligned}\mathbf{F}(t) &= \int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv + \int_{\partial\Omega} \mathbf{t}(\mathbf{x},t)\,ds \\ &= \int_{\Omega_0} \rho_0(\mathbf{X})\,\mathbf{B}(\mathbf{X},t)\,dV + \int_{\partial\Omega_0} \mathbf{T}(\mathbf{X},t)\,dS\end{aligned}$$

Linear Momentum Balance

Linear Momentum Balance Law

The **linear momentum balance** law states that the *time-variation* of the *linear momentum* of a material volume is equal to the *resultant force* acting on that material volume.

$$\boxed{\dot{\mathbf{M}}_L(t) = \frac{d}{dt}\mathbf{M}_L(t) = \mathbf{F}(t)}$$

If the continuum body is in *equilibrium*, the *resultant force* is *zero* and the **linear momentum** is a *conserved* quantity,

$$\boxed{\dot{\mathbf{M}}_L(t) = \frac{d}{dt}\mathbf{M}_L(t) = \mathbf{0}}$$

Linear Momentum Balance

Linear Momentum Balance Law: Global Spatial Form

The *global material form* of the **linear momentum balance** law may be written as,

$$\frac{d}{dt}\int_{\Omega}\rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dv=\int_{\Omega}\rho(\mathbf{x},t)\,\dot{\mathbf{v}}(\mathbf{x},t)\,dv$$

$$=\int_{\Omega}\rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv+\int_{\partial\Omega}\mathbf{t}(\mathbf{x},t)\,ds$$

Linear Momentum Balance

Linear Momentum Balance Law: Global Spatial Form

The *surface forces* may be written as,

$$\int_{\partial\Omega} \mathbf{t}\, ds = \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\, ds = \int_{\Omega} \operatorname{div}\boldsymbol{\sigma}\, dv$$

Substituting into the global spatial form yields,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dv &= \int_{\Omega} \rho(\mathbf{x},t)\,\dot{\mathbf{v}}(\mathbf{x},t)\,dv \\ &= \int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv + \int_{\Omega} \operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t)\,dv \\ &= \int_{\Omega} \big(\rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t) + \operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t)\big)\,dv\end{aligned}$$

Linear Momentum Balance

Linear Momentum Balance Law: Local Spatial Form

Localizing, the *local spatial form* of the **linear momentum balance** law, known as **Cauchy's first equation of motion**, may be written as,

$$\boxed{\rho(\mathbf{x},t)\dot{\mathbf{v}}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t) + \operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t)}$$

If the *resultant force* is *zero*, the local spatial form of the **linear momentum balance** law may be written as,

$$\boxed{\mathbf{0} = \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t) + \operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t)}$$

Linear Momentum Balance

Linear Momentum Balance Law: Global Material Form

The *global material form* of the **linear momentum balance** law may be written as,

$$\frac{d}{dt}\int_{\Omega_0} \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)\,dV = \int_{\Omega_0} \rho_0(\mathbf{X})\dot{\mathbf{V}}(\mathbf{X},t)\,dV$$

$$= \int_{\Omega_0} \rho_0(\mathbf{X})\mathbf{B}(\mathbf{X},t)\,dV + \int_{\partial\Omega_0} \mathbf{T}(\mathbf{X},t)\,dS$$

Linear Momentum Balance

Linear Momentum Balance Law: Global Material Form

The *surface forces* may be written as,

$$\int_{\partial\Omega_0} \mathbf{T}\, dS = \int_{\partial\Omega_0} \mathbf{P}\mathbf{N}\, dS = \int_{\Omega_0} \mathrm{DIV}\,\mathbf{P}\, dV$$

Substituting into the global material form yields,

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_0} \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)\, dV &= \int_{\Omega_0} \rho_0(\mathbf{X})\dot{\mathbf{V}}(\mathbf{X},t)\, dV \\
&= \int_{\Omega_0} \rho_0(\mathbf{X})\mathbf{B}(\mathbf{X},t)\, dV + \int_{\Omega_0} \mathrm{DIV}\,\mathbf{P}(\mathbf{X},t)\, dV \\
&= \int_{\Omega_0} \big(\rho_0(\mathbf{X})\mathbf{B}(\mathbf{X},t) + \mathrm{DIV}\,\mathbf{P}(\mathbf{X},t)\big)\, dV
\end{aligned}$$

Linear Momentum Balance

Linear Momentum Balance Law: Local Material Form

Localizing, the *local material form* of the **linear momentum balance** law may be written as,

$$\boxed{\rho_0\left(\mathbf{X}\right)\dot{\mathbf{V}}\left(\mathbf{X},t\right)=\rho_0\left(\mathbf{X}\right)\mathbf{B}\left(\mathbf{X},t\right)+\mathrm{DIV}\,\mathbf{P}\left(\mathbf{X},t\right)}$$

If the *resultant force* is *zero*, the local spatial form of the **linear momentum balance** law may be written as,

$$\boxed{\mathbf{0}=\rho_0\left(\mathbf{X}\right)\mathbf{B}\left(\mathbf{X},t\right)+\mathrm{DIV}\,\mathbf{P}\left(\mathbf{X},t\right)}$$

Linear Momentum Balance

Global and Local Spatial Forms

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{v}\,dv=\int_{\Omega}\rho\dot{\mathbf{v}}\,dv=\int_{\Omega}\rho\mathbf{b}\,dv+\int_{\partial\Omega}\mathbf{t}\,ds$$

$$\rho\dot{\mathbf{v}}=\rho\mathbf{b}+\operatorname{div}\boldsymbol{\sigma}$$

Global and Local Material Forms

$$\frac{d}{dt}\int_{\Omega_0}\rho_0\mathbf{v}\,dV=\int_{\Omega_0}\rho_0\dot{\mathbf{v}}\,dV=\int_{\Omega_0}\rho_0\mathbf{b}\,dV+\int_{\partial\Omega_0}\mathbf{T}\,dS$$

$$\rho_0\dot{\mathbf{v}}=\rho_0\mathbf{b}+\mathrm{DIV}\,\mathbf{P}$$

Angular Momentum Balance

Angular Momentum

The **angular momentum** of a material volume about a *fixed spatial point* $\mathbf{x}_0$, denoted as $\mathbf{M}_A(t)$, is defined as a vector-valued function given by,

$$\mathbf{M}_A(t) = \int_{\Omega} \mathbf{r}(\mathbf{x},t) \times \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\,dv$$
$$= \int_{\Omega_0} \mathbf{R}(\mathbf{X},t) \times \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)\,dV$$

where the vector position $\mathbf{r} = \mathbf{r}(\mathbf{x},t) = \mathbf{R}(\mathbf{X},t)$ is defined as,

$$\mathbf{r} = \mathbf{r}(\mathbf{x},t) = \mathbf{x} - \mathbf{x}_0 = \boldsymbol{\varphi}(\mathbf{X},t) - \mathbf{x}_0 = \mathbf{R}(\mathbf{X},t)$$

Angular Momentum Balance

Angular Momentum

Using *Reynolds Lemma*, i.e. conservation of mass, the *material time derivative* of the **angular momentum** takes the form,

$$\begin{aligned}\dot{\mathbf{M}}_A(t) &= \frac{d}{dt}\int_{\Omega} \mathbf{r}(\mathbf{x},t)\times\rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dv \\ &= \int_{\Omega} \mathbf{r}(\mathbf{x},t)\times\rho(\mathbf{x},t)\,\dot{\mathbf{v}}(\mathbf{x},t)\,dv \\ &= \frac{d}{dt}\int_{\Omega_0} \mathbf{R}(\mathbf{X},t)\times\rho_0(\mathbf{X})\,\mathbf{V}(\mathbf{X},t)\,dV \\ &= \int_{\Omega_0} \mathbf{R}(\mathbf{X},t)\times\rho_0(\mathbf{X})\,\dot{\mathbf{V}}(\mathbf{X},t)\,dV\end{aligned}$$

Angular Momentum Balance

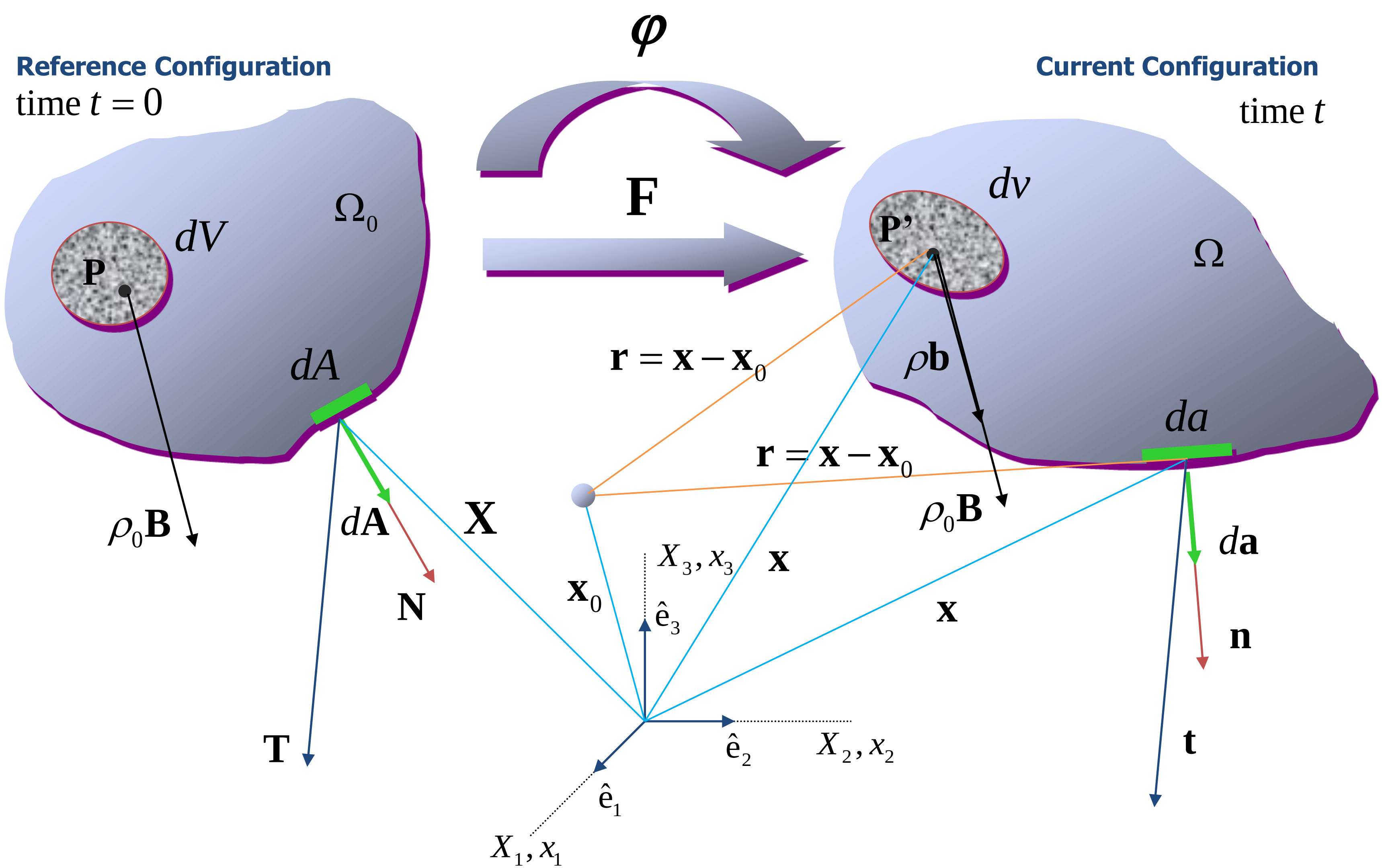
Resultant Moment

The **resultant moment** about a fixed spatial point $\mathbf{x}_0$, denoted as $\mathbf{M}(t)$, is defined as a vector-valued function given by,

$$\mathbf{M}(t) = \int_{\Omega} \mathbf{r}(\mathbf{x},t) \times \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv + \int_{\partial\Omega} \mathbf{r}(\mathbf{x},t) \times \mathbf{t}(\mathbf{x},t)\,ds$$

$$= \int_{\Omega_0} \mathbf{R}(\mathbf{X},t) \times \rho_0(\mathbf{X})\,\mathbf{B}(\mathbf{X},t)\,dV + \int_{\partial\Omega_0} \mathbf{R}(\mathbf{X},t) \times \mathbf{T}(\mathbf{X},t)\,dS$$

Angular Momentum Balance



Angular Momentum Balance

Angular Momentum Balance Law

The **angular momentum balance** law states that the *time-variation* of the *angular momentum* of a material volume about a fixed spatial point, is equal to the *resultant moment* about this fixed spatial point, acting on that material volume.

$$\boxed{\dot{\mathbf{M}}_A(t) = \frac{d}{dt}\mathbf{M}_A(t) = \mathbf{M}(t)}$$

If the *resultant moment* about the fixed spatial point is *zero*, then the **angular momentum** about this spatial point is a *conserved* quantity,

$$\boxed{\dot{\mathbf{M}}_A(t) = \frac{d}{dt}\mathbf{M}_A(t) = \mathbf{0}}$$

Angular Momentum Balance

Angular Momentum Balance: Global Spatial Form

The *global spatial form* of the **angular momentum balance** law may be written as,

$$\frac{d}{dt}\int_{\Omega} \mathbf{r}(\mathbf{x},t) \times \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,dV = \int_{\Omega} \mathbf{r}(\mathbf{x},t) \times \rho(\mathbf{x},t)\,\dot{\mathbf{v}}(\mathbf{x},t)\,dv$$

$$= \int_{\Omega} \mathbf{r}(\mathbf{x},t) \times \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,dv$$

$$+ \int_{\partial\Omega} \mathbf{r}(\mathbf{x},t) \times \mathbf{t}(\mathbf{x},t)\,ds$$

Angular Momentum Balance

Angular Momentum Balance: Global Spatial Form

The moment of the *surface forces* may be written as,

$$\int_{\partial\Omega} \mathbf{r}\times\mathbf{t}\,ds = \int_{\partial\Omega} \mathbf{r}\times\boldsymbol{\sigma}\mathbf{n}\,ds = \int_{\Omega} \mathbf{r}\times\operatorname{div}\boldsymbol{\sigma}\,dv + \int_{\Omega} \varepsilon_{abc}\sigma_{cb}\mathbf{e}_a\,dv$$

$$\begin{aligned}
\int_{\partial\Omega} \varepsilon_{abc} r_b t_c \mathbf{e}_a\,ds &= \int_{\partial\Omega} \varepsilon_{abc} r_b \sigma_{cd} n_d\,\mathbf{e}_a ds \\
&= \int_{\Omega} \varepsilon_{abc} \left(r_b\sigma_{cd}\right)_{,d} \mathbf{e}_a dv \\
&= \int_{\Omega} \varepsilon_{abc} r_b \sigma_{cd,d}\mathbf{e}_a dv + \int_{\Omega} \varepsilon_{abc} r_{b,d}\sigma_{cd}\mathbf{e}_a dv \\
&= \int_{\Omega} \varepsilon_{abc} r_b \left(\nabla\cdot\boldsymbol{\sigma}\right)_c \mathbf{e}_a dv + \int_{\Omega} \varepsilon_{abc}\sigma_{cb}\mathbf{e}_a\,dv
\end{aligned}$$

Angular Momentum Balance

Angular Momentum Balance: Global Spatial Form

Substituting into the global material form yields,

$$\frac{d}{dt}\int_{\Omega}\mathbf{r}(\mathbf{x},t)\times\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\,dV = \int_{\Omega}\mathbf{r}(\mathbf{x},t)\times\rho(\mathbf{x},t)\dot{\mathbf{v}}(\mathbf{x},t)\,dv$$

$$= \int_{\Omega}\mathbf{r}(\mathbf{x},t)\times\rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\,dv$$

$$+\int_{\Omega}\mathbf{r}(\mathbf{x},t)\times\operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t)\,dv$$

$$+\int_{\Omega}\varepsilon_{abc}\sigma_{cb}(\mathbf{x},t)\mathbf{e}_a\,dv$$

Angular Momentum Balance

Angular Momentum Balance: Local Spatial Form

Localizing, the *local spatial form* of the **angular momentum balance** law yields,

$$\mathbf{r} \times \rho\,\dot{\mathbf{v}} = \mathbf{r} \times \left(\rho\,\mathbf{b} + \operatorname{div}\boldsymbol{\sigma}\right) + \varepsilon_{abc}\sigma_{cb}\mathbf{e}_a$$

Using the *Cauchy's first motion equation* yields,

$$\varepsilon_{abc}\sigma_{cb}\mathbf{e}_a = \mathbf{0}$$

and the *local spatial form* of the **angular momentum balance** law may be written as,

$$\boxed{\boldsymbol{\sigma}\left(\mathbf{x},t\right) = \boldsymbol{\sigma}^T\left(\mathbf{x},t\right)}$$

Angular Momentum Balance

Angular Momentum Balance: Global Material Form

The *global material form* of the **angular momentum balance** law may be written as,

$$\frac{d}{dt}\int_{\Omega_0} \mathbf{R}(\mathbf{X},t)\times\rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)\,dV = \int_{\Omega_0} \mathbf{R}(\mathbf{X},t)\times\rho_0(\mathbf{X})\dot{\mathbf{V}}(\mathbf{X},t)\,dV$$

$$= \int_{\Omega_0} \mathbf{R}(\mathbf{X},t)\times\rho_0(\mathbf{X})\mathbf{B}(\mathbf{X},t)\,dV$$

$$+\int_{\partial\Omega_0} \mathbf{R}(\mathbf{X},t)\times\mathbf{T}(\mathbf{X},t)\,dS$$

Angular Momentum Balance

Angular Momentum Balance: Global Material Form

The moment of the *surface forces* may be written as,

$$\int_{\partial\Omega_0} \mathbf{r}\times\mathbf{T}dS = \int_{\partial\Omega_0} \mathbf{r}\times\mathbf{P}\mathbf{N}dS = \int_{\Omega_0} \mathbf{r}\times \mathrm{DIV}\,\mathbf{P}dV + \int_{\Omega_0} \varepsilon_{abc}\left(\mathbf{P}\mathbf{F}^T\right)_{cb}\mathbf{e}_a dV$$

$$\begin{aligned}
\int_{\partial\Omega_0} \varepsilon_{abc} r_b T_c \mathbf{e}_a dS &= \int_{\partial\Omega_0} \varepsilon_{abc} r_b P_{cD} N_D \mathbf{e}_a dS \\
&= \int_{\Omega_0} \varepsilon_{abc}\left(r_b P_{cD}\right)_{,D}\mathbf{e}_a dV \\
&= \int_{\Omega_0} \varepsilon_{abc} r_b P_{cD,D}\mathbf{e}_a dV + \int_{\Omega_0} \varepsilon_{abc} r_{b,D} P_{cD}\mathbf{e}_a dV \\
&= \int_{\Omega_0} \varepsilon_{abc} r_b \left(\bar{\nabla}\cdot\mathbf{P}\right)_c \mathbf{e}_a dV + \int_{\Omega_0} \varepsilon_{abc} F_{bD} P_{cD}\mathbf{e}_a dV \\
&= \int_{\Omega_0} \mathbf{r}\times \mathrm{DIV}\,\mathbf{P}dV + \int_{\Omega_0} \varepsilon_{abc}\left(\mathbf{P}\mathbf{F}^T\right)_{cb}\mathbf{e}_a dV
\end{aligned}$$

Angular Momentum Balance

Angular Momentum Balance: Global Material Form

Substituting into the global material form yields,

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_0} \mathbf{r}\times\rho_0\mathbf{v}\,dV &= \int_{\Omega_0} \mathbf{r}\times\rho_0\dot{\mathbf{v}}\,dV \\
&= \int_{\Omega_0} \mathbf{r}\times\rho_0\,\mathbf{b}\,dV + \int_{\Omega_0} \mathbf{r}\times\mathrm{DIV}\,\mathbf{P}\,dV \\
&\quad + \int_{\Omega_0} \varepsilon_{abc}\left(\mathbf{P}\mathbf{F}^T\right)_{cb}\mathbf{e}_a\,dV
\end{aligned}
$$

Angular Momentum Balance

Angular Momentum Balance: Local Material Form

Localizing, the *local material form* of the **angular momentum balance** law may be written as,

$$\mathbf{r} \times \rho_0 \dot{\mathbf{v}} = \mathbf{r} \times \rho_0 \, \mathbf{b} + \mathbf{r} \times \mathrm{DIV}\, \mathbf{P} + \varepsilon_{abc} \left(\mathbf{P}\mathbf{F}^T \right)_{cb} \mathbf{e}_a$$

Using the *Cauchy's first motion equation* yields,

$$\varepsilon_{abc} \left(\mathbf{P}\mathbf{F}^T \right)_{cb} \mathbf{e}_a = \mathbf{0}$$

and the *local spatial form* of the **angular momentum balance** law may be written as,

$$\boxed{\mathbf{P}(\mathbf{X},t)\mathbf{F}^T(\mathbf{X},t) = \mathbf{F}(\mathbf{X},t)\mathbf{P}^T(\mathbf{X},t)}$$

Angular Momentum Balance

Global and Local Spatial Forms

$$\frac{d}{dt}\int_{\Omega}\mathbf{r}\times\rho\,\mathbf{v}dV=\int_{\Omega}\mathbf{r}\times\rho\,\dot{\mathbf{v}}\,dv$$

$$=\int_{\Omega}\mathbf{r}\times\rho\,\mathbf{b}\,dv+\int_{\partial\Omega}\mathbf{r}\times\mathbf{t}\,ds$$

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^{T}$$

Global and Local Material Forms

$$\frac{d}{dt}\int_{\Omega_0}\mathbf{r}\times\rho_0\mathbf{v}dV=\int_{\Omega_0}\mathbf{r}\times\rho_0\dot{\mathbf{v}}dV$$

$$=\int_{\Omega_0}\mathbf{r}\times\rho_0\mathbf{b}\,dV+\int_{\partial\Omega_0}\mathbf{r}\times\mathbf{T}\,dS$$

$$\mathbf{P}\mathbf{F}^{T}=\mathbf{F}\mathbf{P}^{T}$$

Kinetic Energy

Kinetic Energy

The *global spatial form* of the **kinetic energy** of a continuum body, denoted as $\mathcal{K}(t)$, takes the form,

$$\mathcal{K}(t) = \frac{1}{2}\int_{\Omega} \rho(\mathbf{x},t)\left\|\mathbf{v}(\mathbf{x},t)\right\|^2 dv = \frac{1}{2}\int_{\Omega} \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,dv$$

The *global material form* of the **kinetic energy** of a continuum body, denoted as $\mathcal{K}(t)$, takes the form,

$$\mathcal{K}(t) = \frac{1}{2}\int_{\Omega_0} \rho_0(\mathbf{X})\left\|\mathbf{V}(\mathbf{X},t)\right\|^2 dV = \frac{1}{2}\int_{\Omega_0} \rho_0(\mathbf{X})\,\mathbf{V}(\mathbf{X},t)\cdot\mathbf{V}(\mathbf{X},t)\,dV$$

Internal Mechanical Power

Internal Mechanical Power

The **internal mechanical power** per unit of *spatial volume*, is the work done by the stresses per unit of time and unit of spatial volume, and may be written as,

$$\boldsymbol{\sigma}:\mathbf{d}=J^{-1}\boldsymbol{\tau}:\mathbf{d}$$

$$=J^{-1}\left(\mathbf{P}\mathbf{F}^{T}\right):\mathbf{d}=J^{-1}\mathbf{P}\mathbf{F}^{T}:\mathbf{l}=J^{-1}\mathbf{P}:\left(\mathbf{l}\mathbf{F}\right)=J^{-1}\mathbf{P}:\dot{\mathbf{F}}$$

$$=J^{-1}\left(\mathbf{F}\mathbf{S}\mathbf{F}^{T}\right):\mathbf{d}=J^{-1}\mathbf{S}:\left(\mathbf{F}^{T}\mathbf{d}\mathbf{F}\right)=J^{-1}\mathbf{S}:\dot{\mathbf{E}}$$

$$\sigma_{ab}d_{ab}=J^{-1}\tau_{ab}d_{ab}$$

$$=J^{-1}\left(P_{aA}F_{Ab}^{T}\right)d_{ab}=J^{-1}\left(P_{aA}F_{Ab}^{T}\right)l_{ab}=J^{-1}P_{aA}\left(l_{ab}F_{bA}\right)=J^{-1}P_{aA}\dot{F}_{aA}$$

$$=J^{-1}\left(F_{aA}S_{AB}F_{Bb}^{T}\right)d_{ab}=J^{-1}S_{AB}\left(F_{Aa}^{T}d_{ab}F_{bB}\right)=J^{-1}S_{AB}\dot{E}_{AB}$$

Internal Mechanical Power

Internal Mechanical Power

The **internal mechanical power** in the continuum body, denoted as $\mathcal{P}_{int}(t)$, i.e. work done per unit of time by the stresses, may be written as,

$$
\begin{aligned}
\mathcal{P}_{int}(t) &= \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}\, dv \\
&= \int_{\Omega} J^{-1}\boldsymbol{\tau} : \mathbf{d}\, dv = \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d}\, dV \\
&= \int_{\Omega} J^{-1}\mathbf{P} : \dot{\mathbf{F}}\, dv = \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}}\, dV \\
&= \int_{\Omega} J^{-1}\mathbf{S} : \dot{\mathbf{E}}\, dv = \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\, dV
\end{aligned}
$$

Internal Mechanical Power

Global Spatial Form

$$\mathcal{P}_{int}(t) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d} \, dv$$

Global Material Forms

$$\begin{aligned} \mathcal{P}_{int}(t) &= \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d} \, dV \\ &= \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}} \, dV \\ &= \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}} \, dV \end{aligned}$$

External Mechanical Power

External Mechanical Power

The **external mechanical power**, denoted as $\mathcal{P}_{ext}(t)$, is the work done per unit of time by the body forces and surface forces, and may be written as,

$$\begin{aligned}
\mathcal{P}_{ext}(t) &= \int_{\Omega} \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,dv + \int_{\partial\Omega} \mathbf{t}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,ds \\
&= \int_{\Omega_0} \rho_0(\mathbf{X})\mathbf{B}(\mathbf{X},t)\cdot\mathbf{V}(\mathbf{X},t)\,dV + \int_{\partial\Omega_0} \mathbf{T}(\mathbf{X},t)\cdot\mathbf{V}(\mathbf{X},t)\,dS
\end{aligned}$$

Mechanical Energy Balance

Mechanical Energy Balance

The *external mechanical power*, denoted as $\mathcal{P}_{ext}(t)$, i.e. work done per unit of time by the body forces and surface forces, may be written in *global spatial form* as,

$$\mathcal{P}_{ext}(t) = \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} \, ds$$

The *external mechanical power* of the *surface forces* may be written in *global spatial form* as,

$$\begin{aligned} \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} \, ds &= \int_{\partial\Omega} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} \, ds \\ &= \int_{\Omega} \operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma}) \, dv = \int_{\Omega} \mathbf{v} \cdot \operatorname{div} \boldsymbol{\sigma} \, dv + \int_{\Omega} \boldsymbol{\sigma} : \operatorname{grad} \mathbf{v} \, dv \end{aligned}$$

Mechanical Energy Balance

Then, using the *local spatial form* of the *linear momentum balance* equation, the *external mechanical power* may be written in *global spatial form* as,

$$\begin{aligned}
\mathcal{P}_{ext}(t) &= \int_{\Omega} \rho\,\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\Omega} \mathbf{t}\cdot\mathbf{v}\,ds \\
&= \int_{\Omega} \rho\,\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\Omega} \mathbf{v}\cdot\operatorname{div}\boldsymbol{\sigma}\,dv + \int_{\Omega} \boldsymbol{\sigma}:\operatorname{grad}\mathbf{v}\,dv \\
&= \int_{\Omega} \left(\rho\,\mathbf{b} + \operatorname{div}\boldsymbol{\sigma}\right)\cdot\mathbf{v}\,dv + \int_{\Omega} \boldsymbol{\sigma}:\operatorname{grad}\mathbf{v}\,dv \\
&= \int_{\Omega} \rho\,\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dv + \int_{\Omega} \boldsymbol{\sigma}:\mathbf{l}\,dv = \int_{\Omega} \frac{1}{2}\rho\,\frac{d}{dt}\left\|\mathbf{v}\right\|^{2} dv + \int_{\Omega} \boldsymbol{\sigma}:\mathbf{d}\,dv \\
&= \frac{d}{dt}\int_{\Omega} \frac{1}{2}\rho\left\|\mathbf{v}\right\|^{2} dv + \int_{\Omega} \boldsymbol{\sigma}:\mathbf{d}\,dv = \frac{d}{dt}\mathcal{K}(t) + \mathcal{P}_{int}(t)
\end{aligned}$$

Mechanical Energy Balance

Mechanical Energy Balance

The **mechanical energy balance** states that the *external mechanical power* supplied to the continuum body is spent in changing its *kinetic energy* and doing an *internal mechanical power*.

$$\boxed{\mathcal{P}_{ext}(t) = \frac{d}{dt}\mathcal{K}(t) + \mathcal{P}_{int}(t)}$$

Mechanical Energy Balance

Mechanical Energy Balance: Global Spatial Form

The *global spatial form* of the **mechanical energy balance** may be written as,

$$\mathcal{P}_{ext}(t) = \int_{\Omega} \rho \, \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} \, ds$$

$$= \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho \left\| \mathbf{v} \right\|^{2} dv + \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d} \, dv$$

Mechanical Energy Balance

Mechanical Energy Balance: Global Material Form

The *global material form* of the **mechanical energy balance** may be written as,

$$
\begin{aligned}
\mathcal{P}_{ext}(t) &= \int_{\Omega_0} \rho_0 \, \mathbf{b} \cdot \mathbf{v} \, dV + \int_{\partial\Omega_0} \mathbf{T} \cdot \mathbf{v} \, dS \\
&= \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 \|\mathbf{v}\|^2 \, dV + \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d} \, dV \\
&= \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 \|\mathbf{v}\|^2 \, dV + \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}} \, dV \\
&= \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 \|\mathbf{v}\|^2 \, dV + \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}} \, dV
\end{aligned}
$$

Mechanical Energy Balance

Quasistatic Problem

If the *material time derivative* of the *kinetic energy* is zero (or negligible), the problem is called **quasistatic** and the *mechanical energy balance* reads,

$$\boxed{\mathcal{P}_{ext}(t) = \mathcal{P}_{int}(t)}$$

Mechanical Energy Balance

Free Vibration Problem

If the *external mechanical power* is zero (or negligible), the problem is called **free vibration problem** and the *mechanical energy balance* reads,

$$\boxed{0 = \frac{d}{dt}\mathcal{K}(t) + \mathcal{P}_{int}(t)}$$

Mechanical Energy Balance

Conservative Mechanical System

If both the *external* and *internal mechanical power* derive from a *potential*, i.e. a *potential energy for the external loading* and a *strain energy*, such that,

$$\mathcal{P}_{ext}(t) = -\frac{d}{dt}\Pi_{ext}(t), \quad \mathcal{P}_{int}(t) = \frac{d}{dt}\Pi_{int}(t)$$

the problem is said to be **conservative** and the *mechanical energy balance* reads,

$$\boxed{\begin{aligned} -\frac{d}{dt}\Pi_{ext}(t) &= \frac{d}{dt}\mathcal{K}(t) + \frac{d}{dt}\Pi_{int}(t) \\ \frac{d}{dt}\big(\mathcal{K}(t) + \Pi_{int}(t) + \Pi_{ext}(t)\big) &= 0 \end{aligned}}$$

Mechanical Energy Balance

Conservative Mechanical System

The *total potential energy*, denoted as $\Pi(t)$, is defined as the sum of the *potential energy of the external loading*, denoted as $\Pi_{ext}(t)$ and the *strain energy*, denoted as $\Pi_{int}(t)$, yielding,

$$\boxed{\frac{d}{dt}\left(\mathcal{K}(t)+\Pi_{int}(t)+\Pi_{ext}(t)\right)=\frac{d}{dt}\left(\mathcal{K}(t)+\Pi(t)\right)=0}$$

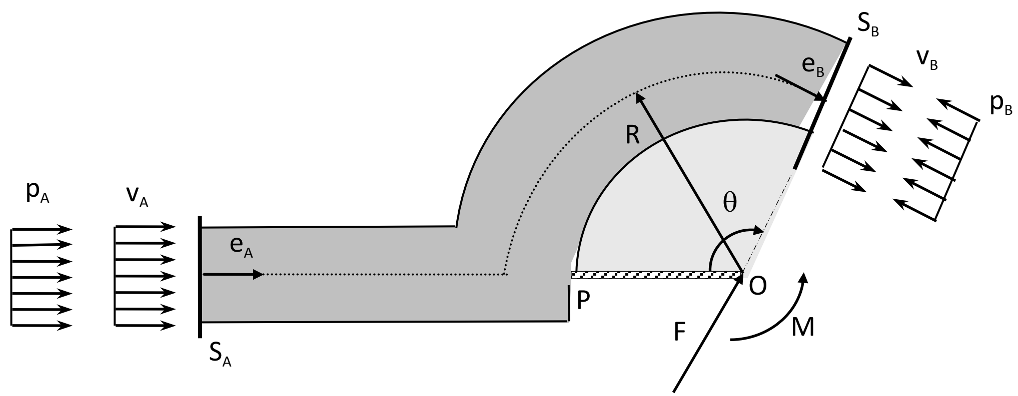
In a **conservative mechanical system** the sum of the *kinetic energy* and the *total potential energy* is a conserved quantity,

$$\boxed{\mathcal{K}(t)+\Pi(t)=\text{constant}}$$

Assignment 6.2

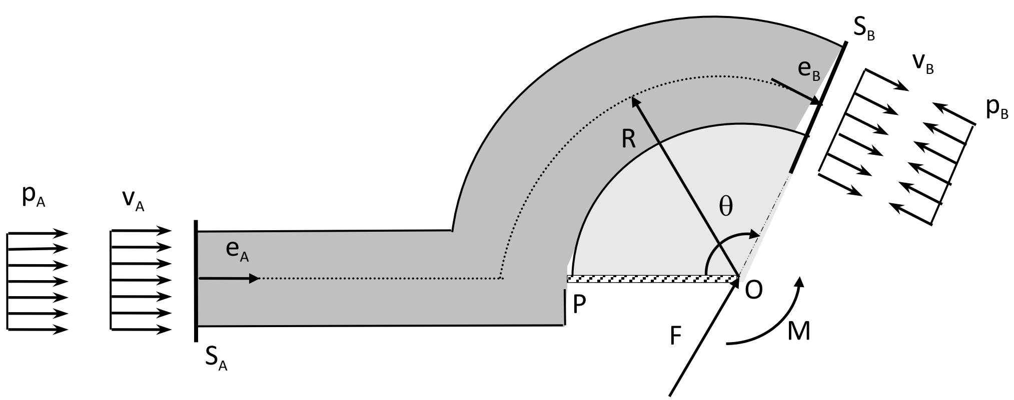
Assignment 6.2

A *volume flux* Q of an *incompressible* fluid flows in *stationary* conditions through the pipeline of the figure. *Velocity* and *pressure* distributions at the sections A and B are *uniform*. The pipeline is fixed through a rigid bar OP. The *weights* of the *pipeline*, *rigid bar* and *fluid* are *neglected*.



Assignment 6.2

1) Obtain the *velocities* at the sections A and B in terms of Q.
2) Obtain the *reaction force* F and *moment* M at the point O.
3) Obtain the values of the *angle* θ that maximize and minimize the reaction at the point O.
4) Obtain the *external power* needed to keep the volume flux Q if the fluid is an *incompressible ideal fluid* with a spherical stress state given byen by $\boldsymbol{\sigma} = -p\mathbf{1}$.



Assignment 6.2

The *local spatial form* of the *conservation of mass* or *mass continuity* equation, plus the *incompressibility* condition yields,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad \dot{\rho} = 0 \quad \Rightarrow \quad \operatorname{div} \mathbf{v} = 0$$

The *global spatial form* of the *conservation of mass* for an *incompressible* medium reads,

$$\begin{aligned}\int_V \operatorname{div} \mathbf{v} \, dV &= \int_{\partial V} \mathbf{v} \cdot \mathbf{n} \, dS \\ &= \int_{S_A} \mathbf{v} \cdot \mathbf{n} \, dS + \int_{S_B} \mathbf{v} \cdot \mathbf{n} \, dS \\ &= -v_A S_A + v_B S_B = 0\end{aligned}$$

Then the **velocities** at the sections A and B are given by,

$$v_A S_A = v_B S_B = Q \quad \Rightarrow \quad v_A = Q/S_A, \quad v_B = Q/S_B$$

Assignment 6.2

The *global spatial form* of the *linear momentum balance* for a *stationary* motion reads,

$$\mathbf{R}_{/f} = \frac{d}{dt}\int_V \rho \mathbf{v} dV$$

$$= \cancel{\frac{\partial}{\partial t}\int_V \rho \mathbf{v} dV} + \int_{\partial V} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS = \int_{\partial V} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS$$

$$= \int_{S_A} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS + \int_{S_B} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS$$

$$= -\rho v_A^2 S_A \mathbf{e}_A + \rho v_B^2 S_B \mathbf{e}_B = -\rho\frac{Q^2}{S_A}\mathbf{e}_A + \rho\frac{Q^2}{S_B}\mathbf{e}_B$$

Assignment 6.2

The *resultant force* acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible*, reads,

$$
\begin{aligned}
\mathbf{R}_{/f} &= \cancel{\int_V \rho \mathbf{b} dV} + \int_{\partial V} \mathbf{t} dS = \int_{\partial V} \mathbf{t} dS \\
&= \int_{wall} \mathbf{t} dS + \int_{S_A} \mathbf{t} dS + \int_{S_B} \mathbf{t} dS \\
&= \mathbf{R}_{wall/f} + \mathbf{R}_{p_A} + \mathbf{R}_{p_B} \\
&= \mathbf{R}_{wall/f} + p_A S_A \mathbf{e}_A - p_B S_B \mathbf{e}_B \\
&= -\rho \frac{Q^2}{S_A} \mathbf{e}_A + \rho \frac{Q^2}{S_B} \mathbf{e}_B
\end{aligned}
$$

Assignment 6.2

The *resultant force of the wall of the pipeline* acting *on the volume V of fluid*, reads,

$$\mathbf{R}_{wall/f} = \mathbf{R}_{/f} - p_A S_A \mathbf{e}_A + p_B S_B \mathbf{e}_B$$

$$= -\left(\rho \frac{Q^2}{S_A} + p_A S_A \right) \mathbf{e}_A + \left(\rho \frac{Q^2}{S_B} + p_B S_B \right) \mathbf{e}_B$$

Using the *action-reaction principle*, the *resultant force of the fluid* acting *on the wall of the pipeline*, reads,

$$\mathbf{R}_{f/wall} = -\mathbf{R}_{wall/f}$$

$$= \left(\rho \frac{Q^2}{S_A} + p_A S_A \right) \mathbf{e}_A - \left(\rho \frac{Q^2}{S_B} + p_B S_B \right) \mathbf{e}_B$$

Assignment 6.2

The *global spatial form* of the *angular momentum balance* about the point O, for a *stationary* motion, reads,

$$\mathbf{M}^O_{/f} = \frac{d}{dt}\int_V \mathbf{r} \times \rho \mathbf{v} dV$$

$$= \cancel{\frac{\partial}{\partial t}\int_V \mathbf{r} \times \rho \mathbf{v} dV} + \int_{\partial V} \mathbf{r} \times \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS = \int_{\partial V} \mathbf{r} \times \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS$$

$$= \cancel{\int_{S_A} \mathbf{r} \times \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS} + \int_{S_B} \mathbf{r} \times \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS$$

$$= -\rho v_B^2 S_B R\mathbf{e}_z = -\rho \frac{Q^2}{S_B} R\mathbf{e}_z$$

Assignment 6.2

The *resultant moment* about the point O acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible*, reads,

$$
\begin{aligned}
\mathbf{M}^{O}_{/f} &= \cancel{\int_V \mathbf{r}\times\rho\mathbf{b}\,dV} + \int_{\partial V}\mathbf{r}\times\mathbf{t}\,dS = \int_{\partial V}\mathbf{r}\times\mathbf{t}\,dS \\
&= \int_{wall}\mathbf{r}\times\mathbf{t}\,dS + \int_{S_A}\mathbf{r}\times\mathbf{t}\,dS + \int_{S_B}\mathbf{r}\times\mathbf{t}\,dS \\
&= \mathbf{M}^{O}_{wall/f} + \mathbf{M}^{O}_{p_A} + \mathbf{M}^{O}_{p_B} \\
&= \mathbf{M}^{O}_{wall/f} + p_B S_B R\mathbf{e}_z \\
&= -\rho\frac{Q^2}{S_B}R\mathbf{e}_z
\end{aligned}
$$

Assignment 6.2

The *resultant moment of the wall of the pipeline* acting *on the volume V of fluid* about the point O, reads,

$$\mathbf{M}^{O}_{wall/f} = \mathbf{M}^{O}_{/f} - p_B S_B R\mathbf{e}_z = -\left(\rho \frac{Q^2}{S_B} + p_B S_B \right) R\mathbf{e}_z$$

Using the *action-reaction principle*, the *resultant moment of the fluid* acting *on the wall of the pipeline* about the point O, reads,

$$\begin{aligned}\mathbf{M}^{O}_{f/wall} &= -\mathbf{M}^{O}_{wall/f} \\ &= \left(\rho \frac{Q^2}{S_B} + p_B S_B \right) R\mathbf{e}_z\end{aligned}$$

Assignment 6.2

The *equilibrium of forces and moments about the point O on the pipeline*, taking into account that the *weight of the pipeline and the rigid bar are negligible*, reads,

$$\mathbf{R}_{f/wall} + \cancel{\mathbf{W}} + \mathbf{F} = \mathbf{0}$$

$$\mathbf{M}^{O}_{f/wall} + \cancel{\mathbf{M}^{O}_{W}} + \mathbf{M} = \mathbf{0}$$

The **reaction force** F and **moment** M, at the point O, read,

$$\mathbf{F} = -\mathbf{R}_{f/wall} = -\left(\rho Q^2 / S_A + p_A S_A\right)\mathbf{e}_A + \left(\rho Q^2 / S_B + p_B S_B\right)\mathbf{e}_B$$

$$\mathbf{M} = -\mathbf{M}^{O}_{f/wall} = -\left(\rho \frac{Q^2}{S_B} + p_B S_B\right) R\mathbf{e}_z$$

Assignment 6.2

The *reactions force* F at the point O, reads,

$$\mathbf{F} = -\underbrace{\left(\rho Q^2/S_A + p_A S_A\right)}_{>0}\mathbf{e}_A + \underbrace{\left(\rho Q^2/S_B + p_B S_B\right)}_{>0}\mathbf{e}_B$$

The *norm* of the reaction force F will take a *maximum* for an **angle** such that,

$$\mathbf{e}_B = -\mathbf{e}_A \quad \Rightarrow \quad \theta = \frac{3\pi}{2}$$

The *norm* of the reaction force F will take a *minimum* for an **angle** such that,

$$\mathbf{e}_B = \mathbf{e}_A \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

Assignment 6.2

The **external mechanical power** needed to keep the volume flux Q, taking into account that the fluid is *incompressible*, *stationary* and the *stress state* is *spherical*, is given by,

$$\mathcal{P}_{ext} = \frac{d}{dt}\mathcal{K} + \mathcal{P}_{int} = \frac{d}{dt}\int_V \frac{1}{2}\rho\left\|\mathbf{v}\right\|^2 dV + \int_V \boldsymbol{\sigma} : \mathbf{d}\, dV$$

$$= \cancel{\frac{\partial}{\partial t}\int_V \frac{1}{2}\rho\left\|\mathbf{v}\right\|^2 dV} + \int_{\partial V} \frac{1}{2}\rho\left\|\mathbf{v}\right\|^2 \mathbf{v}\cdot\mathbf{n}\, dS + \cancel{\int_V -p\,\mathrm{tr}\,\mathbf{d}\, dV}$$

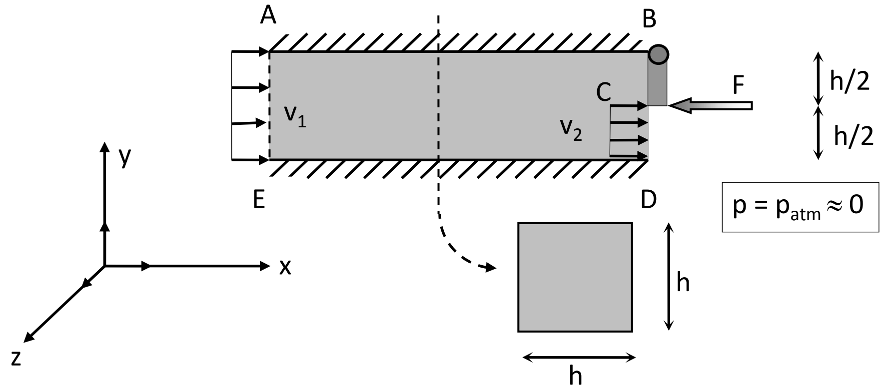
$$= \int_{S_A} \frac{1}{2}\rho\left\|\mathbf{v}\right\|^2 \mathbf{v}\cdot\mathbf{n}\, dS + \int_{S_B} \frac{1}{2}\rho\left\|\mathbf{v}\right\|^2 \mathbf{v}\cdot\mathbf{n}\, dS$$

$$= \frac{1}{2}\rho Q^3\left(\frac{1}{S_B^2} - \frac{1}{S_A^2}\right) \quad \blacksquare$$

Assignment 6.3

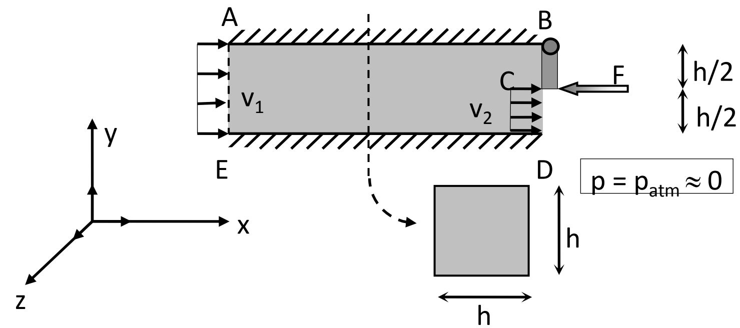
Assignment 6.3 [Classwork]

An *incompressible* fluid is flowing in *stationary* conditions through the pipeline of the figure. Velocities and pressures distributions are *uniform* at the sections AE and CD. Pressure on the walls is assumed to be *uniform*. There is a basculant barrier BC with a hinge on B. An horizontal force F, acting on the point C, is keeping the barrier in vertical position. Body forces in the fluid are *neglected*. The weight of the barrier is also *neglected*.



Assignment 6.3

1) Obtain the *velocity* v2 at the section CD in terms of the velocity v1 at the section AE.
2) Obtain the *resultant force and moment* acting *on the fluid* at the point B.
3) Obtain the *resultant force and moment* of the *fluid on the barrier* at the point B.
4) Obtain the *force* F and the *reaction* at the point B.
5) Obtain the *external mechanical power* needed, assuming the stress tensor in the fluid is spherical, given by $\boldsymbol{\sigma} = -p\mathbf{1}$.



Assignment 6.3

The *local spatial form* of the *conservation of mass* or *mass continuity* equation, plus the incompressibility condition yields,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad \dot{\rho} = 0 \quad \Rightarrow \quad \operatorname{div} \mathbf{v} = 0$$

The *global spatial form* of the *conservation of mass* for an *incompressible* medium reads,

$$\int_V \operatorname{div} \mathbf{v} \, dV = \int_{\partial V} \mathbf{v} \cdot \mathbf{n} \, dS$$

$$= \cancel{\int_{wall+barrier} \mathbf{v} \cdot \mathbf{n} \, dS} + \int_{S_1} \mathbf{v} \cdot \mathbf{n} \, dS + \int_{S_2} \mathbf{v} \cdot \mathbf{n} \, dS$$

$$= -v_1 S_1 + v_2 S_2 = -v_1 h^2 + v_2 \frac{h^2}{2} = 0$$

$$v_2 = 2 v_1$$

Assignment 6.3

The *global spatial form* of the *linear momentum balance* for a *stationary* motion reads,

$$\begin{aligned}
\mathbf{R}_{/f} &= \frac{d}{dt}\int_V \rho \mathbf{v} dV \\
&= \cancel{\frac{\partial}{\partial t}\int_V \rho \mathbf{v} dV} + \int_{\partial V} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS = \int_{\partial V} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS \\
&= \cancel{\int_{wal+barrier} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS} + \int_{S_1} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS + \int_{S_2} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS \\
&= -\rho v_1^2 S_1 \mathbf{e}_x + \rho v_2^2 S_2 \mathbf{e}_x \\
&= \rho v_1^2 h^2 \mathbf{e}_x
\end{aligned}$$

Assignment 6.3

The *resultant force* acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible, the pressure on the walls of the pipeline is uniform* and the *pressure at the section CD is the atmospheric, which is negligible*, reads,

$$
\begin{aligned}
\mathbf{R}_{/f} &= \cancel{\int_V \rho \mathbf{b} dV} + \int_{\partial V} \mathbf{t} dS = \int_{\partial V} \mathbf{t} dS \\
&= \int_{wall} \mathbf{t} dS + \int_{barrier} \mathbf{t} dS + \int_{S_1} \mathbf{t} dS + \int_{S_2} \mathbf{t} dS \\
&= \mathbf{R}_{wall/f} + \mathbf{R}_{barrier/f} + \mathbf{R}_{p_1} + \mathbf{R}_{p_2} \\
&= \cancel{\mathbf{R}_{wall/f}} + \mathbf{R}_{barrier/f} + p_1 S_1 \mathbf{e}_x - \cancel{p_{atm} S_2} \mathbf{e}_x \\
&= \rho v_1^2 h^2 \mathbf{e}_x
\end{aligned}
$$

Assignment 6.3

The *resultant force of the barrier* acting *on the fluid*, reads,

$$\begin{aligned}\mathbf{R}_{barrier/f} &= \mathbf{R}_{/f} - p_1 S_1 \mathbf{e}_x \\ &= \left(\rho v_1^2 - p_1\right) h^2 \mathbf{e}_x\end{aligned}$$

Using the *action-reaction principle*, the *resultant force of the fluid* acting *on the barrier*, reads,

$$\begin{aligned}\mathbf{R}_{f/barrier} &= -\mathbf{R}_{barrier/f} \\ &= \left(p_1 - \rho v_1^2\right) h^2 \mathbf{e}_x\end{aligned}$$

Assignment 6.3

The *global spatial form* of the *angular momentum balance* about the point B, for a *stationary* motion, reads,

$$\begin{aligned}
\mathbf{M}^{B}_{/f} &= \frac{d}{dt}\int_V \mathbf{r}\times\rho\mathbf{v}dV \\
&= \cancel{\frac{\partial}{\partial t}\int_V \mathbf{r}\times\rho\mathbf{v}dV} + \int_{\partial V} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS \\
&= \cancel{\int_{wall+barrier} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS} + \int_{S_1+S_2} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS \\
&= -\rho v_1^2 S_1 \frac{h}{2}\mathbf{e}_z + \rho v_2^2 S_2 \frac{3h}{4}\mathbf{e}_z \\
&= \rho v_1^2 h^3 \mathbf{e}_z
\end{aligned}$$

Assignment 6.3

The *resultant moment* about the point B acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible,* the *pressure at the walls of the pipeline is uniform* and the *atmospheric pressure can be neglected*, reads,

$$\mathbf{M}^{B}_{/f} = \cancel{\int_{V} \mathbf{r} \times \rho \mathbf{b} dV} + \int_{\partial V} \mathbf{r} \times \mathbf{t} dS = \int_{\partial V} \mathbf{r} \times \mathbf{t} dS$$

$$= \int_{wall} \mathbf{r} \times \mathbf{t} dS + \int_{barrier} \mathbf{r} \times \mathbf{t} dS + \int_{S_1} \mathbf{r} \times \mathbf{t} dS + \int_{S_2} \mathbf{r} \times \mathbf{t} dS$$

$$= \mathbf{M}^{B}_{wall/f} + \mathbf{M}^{B}_{barrier/f} + \mathbf{M}^{B}_{p_1} + \mathbf{M}^{B}_{p_2}$$

$$= \cancel{\mathbf{M}^{B}_{wall/f}} + \mathbf{M}^{B}_{barrier/f} + p_1 S_1 \frac{h}{2} \mathbf{e}_z + \cancel{p_{atm} S_2 \frac{3h}{2} \mathbf{e}_z}$$

$$= \rho v_1^2 h^3 \mathbf{e}_z$$

Assignment 6.3

The *resultant moment of the barrier* acting *on the volume V of fluid,* about the point B, reads,

$$\mathbf{M}^{B}_{barrier/f} = \mathbf{M}^{B}_{/f} - p_1 S_1 \frac{h}{2} \mathbf{e}_z = \left(\rho v_1^2 - \frac{1}{2} p_1 \right) h^3 \mathbf{e}_z$$

Using the *action-reaction principle*, the *resultant moment of the fluid* acting *on the barrier,* about the point B, reads,

$$\begin{aligned} \mathbf{M}^{B}_{f/barrier} &= -\mathbf{M}^{B}_{barrier/f} \\ &= \left(\frac{1}{2} p_1 - \rho v_1^2 \right) h^3 \mathbf{e}_z \end{aligned}$$

Assignment 6.3

The *equilibrium of forces and moments about the point B on the barrier*, taking into account that the *weight of the barrier is negligible*, reads,

$$\mathbf{R}_{f/barrier} + \cancel{\mathbf{W}} + \mathbf{R}' + \mathbf{F} = \mathbf{0}$$

$$\mathbf{M}^{B}_{f/barrier} + \cancel{\mathbf{M}^{B}_{W}} + \cancel{\mathbf{M}'} + \mathbf{r} \times \mathbf{F} = \mathbf{0}$$

where $\mathbf{R}'$ is the reaction force at B, $\mathbf{M}' = \mathbf{0}$ is the reaction moment at B (zero because there is a hinge) and the force $\mathbf{F}$ may be written as,

$$\mathbf{F} = -F\mathbf{e}_x$$

$$\mathbf{r} \times \mathbf{F} = -\frac{h}{2}\mathbf{e}_y \times (-F)\mathbf{e}_x = -\frac{h}{2}F\mathbf{e}_z$$

Assignment 6.3

The *equilibrium of moments about the point B on the barrier* yields,

$$\mathbf{M}^{B}_{f/barrier} + \mathbf{r} \times \mathbf{F} = \left(\frac{1}{2} p_1 - \rho v_1^2 \right) h^3 \mathbf{e}_z - \frac{h}{2} F \mathbf{e}_z = \mathbf{0}$$

Then, the **force** F takes the value,

$$\boxed{\begin{aligned} F &= \left(p_1 - 2\rho v_1^2 \right) h^2 \\ \mathbf{F} &= -F\mathbf{e}_x = -\left(p_1 - 2\rho v_1^2 \right) h^2 \mathbf{e}_x \end{aligned}}$$

Assignment 6.3

The *equilibrium of forces on the barrier* yields,

$$\mathbf{R}_{f/barrier} + \mathbf{R}' + \mathbf{F} = \mathbf{0}$$

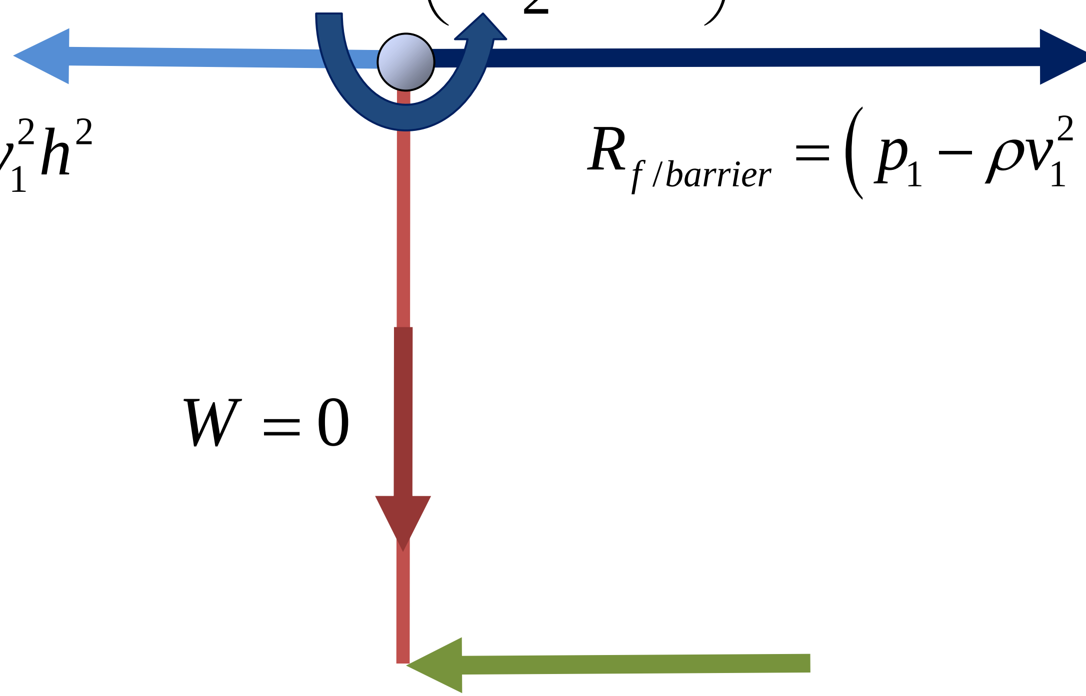
$$\left(p_1 - \rho v_1^2 \right) h^2 \mathbf{e}_x + \mathbf{R}' - \left(p_1 - 2\rho v_1^2 \right) h^2 \mathbf{e}_x = \mathbf{0}$$

Then, the reaction R' takes the value,

$$\boxed{\mathbf{R}' = -\rho v_1^2 h^2 \mathbf{e}_x}$$

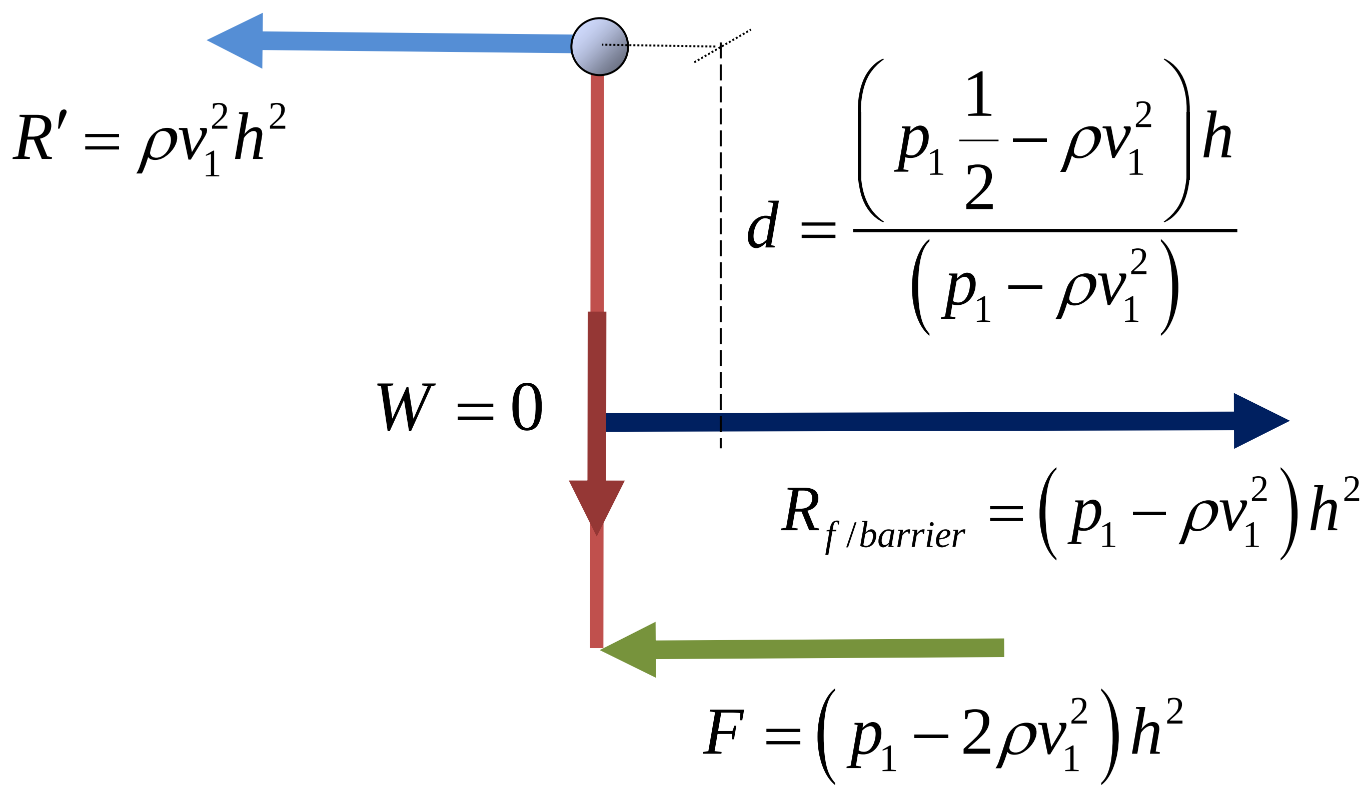
Assignment 6.3

Equilibrium at the barrier



Assignment 6.3

Equilibrium at the barrier



Assignment 6.3

The **external mechanical power** needed to keep the flux, taking into account that the fluid is *incompressible*, *stationary* and the *stress state* is *spherical*, $\boldsymbol{\sigma} = -p\mathbf{1}$, is given by,

$$\mathcal{P}_{ext} = \frac{d}{dt}\mathcal{K} + \mathcal{P}_{int} = \frac{d}{dt}\int_V \frac{1}{2}\rho\|\mathbf{v}\|^2 \, dV + \int_V \boldsymbol{\sigma} : \mathbf{d} \, dV$$

$$= \cancel{\frac{\partial}{\partial t}\int_V \frac{1}{2}\rho\|\mathbf{v}\|^2 \, dV} + \int_{\partial V} \frac{1}{2}\rho\|\mathbf{v}\|^2 \, \mathbf{v}\cdot\mathbf{n} \, dS + \cancel{\int_V -p \operatorname{div}\mathbf{v} \, dV}$$

$$= \int_{S_1} \frac{1}{2}\rho\|\mathbf{v}\|^2 \, \mathbf{v}\cdot\mathbf{n} \, dS + \int_{S_2} \frac{1}{2}\rho\|\mathbf{v}\|^2 \, \mathbf{v}\cdot\mathbf{n} \, dS$$

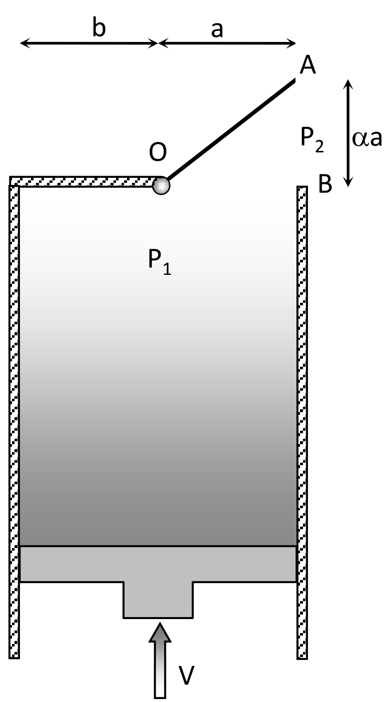
$$= -\frac{1}{2}\rho v_1^3 h^2 + \frac{1}{2}\rho v_2^3 \frac{h^2}{2} = \frac{3}{2}\rho v_1^3 h^2 \qquad \blacksquare$$

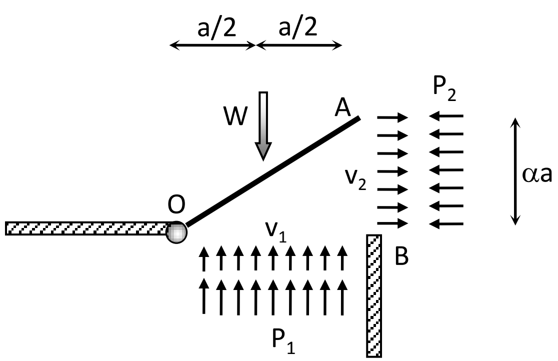
Assignment 6.4

Assignment 6.4 [Homework]

The figure shows the longitudinal section of a pump with a valve OA of weight W per unit of width (normal to the plane of the figure). There is a hinge on O. The velocity of the pump is V.

The fluid is incompressible and the motion stationary. Uniform pressure distributions on the sections OB and AB are p1 and p2=0, respectively. Velocity distributions at the sections OB and AB are uniform. Body forces in the fluid are negligible.





Detail

Assignment 6.4

Assignment 6.4 [Homework]

1) Obtain the uniform velocities v1 and v2 at the sections OB and AB, respectively, in terms of the velocity v of the pumping tool
2) Obtain the resultant of the forces per unit of width given by the fluid on the valve OA
3) Obtain the moment per unit of width at the point O of the forces given by the fluid on the valve OA
4) Obtain the weight W of the valve OA per unit of width. Environmental pressure p2 is neglected.

Assignment 6.4

The *local spatial form* of the *conservation of mass* or *mass continuity* equation, plus the incompressibility condition yields,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad \dot{\rho} = 0 \quad \Rightarrow \quad \operatorname{div} \mathbf{v} = 0$$

The *global spatial form* of the *conservation of mass* for an *incompressible* medium reads,

$$\int_V \operatorname{div} \mathbf{v} \, dV = \int_{\partial V} \mathbf{v} \cdot \mathbf{n} \, dS$$

$$= \cancel{\int_{walls} \mathbf{v} \cdot \mathbf{n} \, dS} + \int_{pump} \mathbf{v} \cdot \mathbf{n} \, dS + \int_{OB} \mathbf{v} \cdot \mathbf{n} \, dS$$

$$= -v(a+b) + v_1 a = 0$$

$$\boxed{v_1 = \frac{a+b}{a} v}$$

Assignment 6.4

The *global spatial form* of the *conservation of mass* for an *incompressible* medium reads,

$$\int_V \operatorname{div}\mathbf{v}\,dV = \int_{\partial V} \mathbf{v}\cdot\mathbf{n}\,dS$$

$$= \cancel{\int_{valve} \mathbf{v}\cdot\mathbf{n}\,dS} + \int_{OB} \mathbf{v}\cdot\mathbf{n}\,dS + \int_{AB} \mathbf{v}\cdot\mathbf{n}\,dS$$

$$= -v_1 a + v_2 \alpha a = 0$$

$$\boxed{v_2 = \frac{1}{\alpha} v_1 = \frac{1}{\alpha}\frac{a+b}{a} v}$$

Assignment 6.4

The *global spatial form* of the *linear momentum balance* for a *stationary* motion reads,

$$
\begin{aligned}
\mathbf{R}_{/f} &= \frac{d}{dt}\int_V \rho\mathbf{v}dV \\
&= \cancel{\frac{\partial}{\partial t}\int_V \rho\mathbf{v}dV} + \int_{\partial V} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS = \int_{\partial V} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS \\
&= \cancel{\int_{valve} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS} + \int_{OB} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS + \int_{AB} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS \\
&= \rho a\left(-v_1^2\mathbf{e}_y + v_2^2\alpha\mathbf{e}_x\right) \\
&= \rho v^2\frac{(a+b)^2}{a}\left(\frac{1}{\alpha}\mathbf{e}_x - \mathbf{e}_y\right)
\end{aligned}
$$

Assignment 6.4

The *resultant force* acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible* and the *pressure at the section AB is negligible*, reads,

$$\begin{aligned}
\mathbf{R}_{/f} &= \cancel{\int_V \rho \mathbf{b}\, dV} + \int_{\partial V} \mathbf{t}\, dS = \int_{\partial V} \mathbf{t}\, dS \\
&= \int_{valve} \mathbf{t}\, dS + \int_{OB} \mathbf{t}\, dS + \int_{AB} \mathbf{t}\, dS \\
&= \mathbf{R}_{valve/f} + \mathbf{R}_{p_1} + \mathbf{R}_{p_2} \\
&= \mathbf{R}_{valve/f} + p_1 a \mathbf{e}_y - \cancel{p_{atm} \alpha a \mathbf{e}_x} \\
&= \rho v^2 \frac{(a+b)^2}{a} \left(\frac{1}{\alpha} \mathbf{e}_x - \mathbf{e}_y \right)
\end{aligned}$$

Assignment 6.4

The *resultant force of the valve OA* acting *on the fluid*, reads,

$$\mathbf{R}_{valve/f} = \mathbf{R}_{/f} - p_1 a \mathbf{e}_y$$

$$= \rho v^2 \frac{(a+b)^2}{a} \left(\frac{1}{\alpha} \mathbf{e}_x - \mathbf{e}_y \right) - p_1 a \mathbf{e}_y$$

Using the *action-reaction principle*, the *resultant force of the fluid* acting *on the valve OA*, reads,

$$\boxed{\begin{aligned} \mathbf{R}_{f/valve} &= -\mathbf{R}_{valve/f} \\ &= -\rho v^2 \frac{(a+b)^2}{a} \left(\frac{1}{\alpha} \mathbf{e}_x - \mathbf{e}_y \right) + p_1 a \mathbf{e}_y \end{aligned}}$$

Assignment 6.4

The *global spatial form* of the *angular momentum balance* about the point O, for a *stationary* motion, reads,

$$
\begin{aligned}
\mathbf{M}^{O}_{/f} &= \frac{d}{dt}\int_{V} \mathbf{r}\times\rho\mathbf{v}dV \\
&= \cancel{\frac{\partial}{\partial t}\int_{V} \mathbf{r}\times\rho\mathbf{v}dV} + \int_{\partial V} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS \\
&= \int_{OB} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS + \int_{AB} \mathbf{r}\times\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right)dS \\
&= \int_{OB} -\rho v_1^2\mathbf{r}\times\mathbf{e}_y dS + \int_{AB} \rho v_2^2\mathbf{r}\times\mathbf{e}_x dS \\
&= -\rho v_1^2 a\frac{a}{2}\mathbf{e}_z - \rho v_2^2\alpha a\frac{\alpha a}{2}\mathbf{e}_z = -\rho v^2\left(a+b\right)^2\mathbf{e}_z
\end{aligned}
$$

Assignment 6.4

The *resultant moment* about the point O acting on the volume V of fluid, taking into account that the *weight of the fluid is negligible* and the *atmospheric pressure can be neglected*, reads,

$$\mathbf{M}^{O}_{/f} = \cancel{\int_V \mathbf{r}\times\rho\mathbf{b}\,dV} + \int_{\partial V}\mathbf{r}\times\mathbf{t}\,dS$$

$$= \int_{valve/f}\mathbf{r}\times\mathbf{t}\,dS + \int_{OB}\mathbf{r}\times\mathbf{t}\,dS + \int_{AB}\mathbf{r}\times\mathbf{t}\,dS$$

$$= \mathbf{M}^{O}_{valve/f} + \mathbf{M}^{O}_{p_1} + \mathbf{M}^{O}_{p_2}$$

$$= \mathbf{M}^{O}_{valve/f} + p_1 a\frac{a}{2}\mathbf{e}_z + \cancel{p_{atm}\alpha a\frac{\alpha a}{2}}\mathbf{e}_z$$

$$= -\rho v^2\left(a+b\right)^2\mathbf{e}_z$$

Assignment 6.4

The *resultant moment of the valve* acting *on the volume V of fluid* about the point O, reads,

$$\mathbf{M}^{O}_{valve/f} = \mathbf{M}^{O}_{/f} - p_1 a \frac{a}{2} \mathbf{e}_z = -\rho v^2 \left(a+b\right)^2 \mathbf{e}_z - \frac{1}{2} p_1 a^2 \mathbf{e}_z$$

Using the *action-reaction principle*, the *resultant moment of the fluid* acting *on the valve* about the point O, reads,

$$\begin{aligned} \mathbf{M}^{O}_{f/valve} &= -\mathbf{M}^{O}_{valve/f} \\ &= \rho v^2 \left(a+b\right)^2 \mathbf{e}_z + \frac{1}{2} p_1 a^2 \mathbf{e}_z \end{aligned}$$

Assignment 6.4

The *equilibrium of forces and moments about the point O on the valve*, reads

$$\mathbf{R}_{f/valve} + \mathbf{W} + \mathbf{R}' = \mathbf{0}$$

$$\mathbf{M}^{O}_{f/valve} + \mathbf{M}^{O}_{W} + \cancel{\mathbf{M}'} = \mathbf{0}$$

where $\mathbf{R}'$ is the reaction force at O and $\mathbf{M}' = \mathbf{0}$ is the reaction moment at O (zero because there is a hinge).

The equilibrium of moments yields,

$$\left(\rho v^2 \left(a + b \right)^2 + p_1 \frac{a^2}{2} \right) \mathbf{e}_z - W \frac{a}{2} \mathbf{e}_z = \mathbf{0}$$

Assignment 6.4

The *equilibrium of moments about the point O on the valve* yields,

$$\mathbf{M}^{O}_{f/valve} + \mathbf{M}^{O}_{W} = \left(\rho v^2 (a+b)^2 + p_1 \frac{a^2}{2} \right) \mathbf{e}_z - W \frac{a}{2} \mathbf{e}_z = \mathbf{0}$$

And the **weight W** per unit of width of the valve reads,

$$\boxed{\begin{aligned} W &= \rho v^2 \frac{2(a+b)^2}{a} + p_1 a \\ \mathbf{W} &= -W\mathbf{e}_y = -\left(\rho v^2 \frac{2(a+b)^2}{a} + p_1 a \right) \mathbf{e}_y \end{aligned}}$$

Assignment 6.4

The *equilibrium of forces at the valve* yields,

$$\mathbf{R}_{f/valve} + \mathbf{W} + \mathbf{R}' = -\rho v^2 \frac{(a+b)^2}{a}\left(\frac{1}{\alpha}\mathbf{e}_x - \mathbf{e}_y\right) + p_1 a \mathbf{e}_y$$

$$-\rho v^2 \frac{2(a+b)^2}{a}\mathbf{e}_y - p_1 a \mathbf{e}_y$$

$$+\mathbf{R}' = \mathbf{0}$$

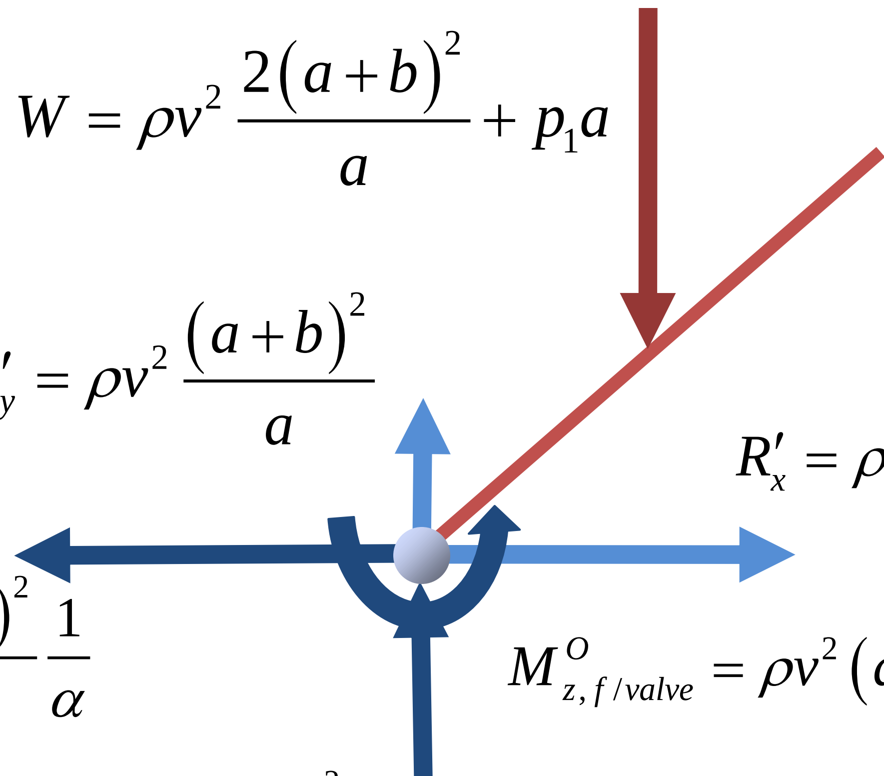
And the **reaction R'** at the point O reads,

$$\boxed{\mathbf{R}' = \rho v^2 \frac{(a+b)^2}{a}\left(\mathbf{e}_y + \frac{1}{\alpha}\mathbf{e}_x\right)}$$

Assignment 6.4

Equilibrium of the valve

$$W = \rho v^2 \frac{2\left(a+b\right)^2}{a} + p_1 a$$



$$R'_y = \rho v^2 \frac{\left(a+b\right)^2}{a}$$

$$R'_x = \rho v^2 \frac{\left(a+b\right)^2}{a} \frac{1}{\alpha}$$

$$R_{x,f/valve} = \rho v^2 \frac{\left(a+b\right)^2}{a} \frac{1}{\alpha}$$

$$M^O_{z,f/valve} = \rho v^2 \left(a+b\right)^2 + p_1 \frac{a^2}{2}$$

$$R_{y,f/valve} = \rho v^2 \frac{\left(a+b\right)^2}{a} + p_1 a$$

External Thermal Power

External Thermal Power: Global Spatial Form

The *global spatial form* of the **external thermal power** is defined as,

$$
\begin{aligned}
Q_{ext}(t) &= \int_{\Omega} \rho(\mathbf{x},t)\, r(\mathbf{x},t)\, dv - \int_{\partial\Omega} \mathbf{q}(\mathbf{x},t) \cdot \mathbf{n}\, ds \\
&= \int_{\Omega} \rho(\mathbf{x},t)\, r(\mathbf{x},t)\, dv - \int_{\Omega} \operatorname{div} \mathbf{q}(\mathbf{x},t)\, dv
\end{aligned}
$$

where $r(\mathbf{x},t)$ is the *heat source per unit of mass* and $\mathbf{q}(\mathbf{x},t)$ is the *non-convective heat flux per unit of spatial surface*, both of them given in spatial description.

External Thermal Power

External Thermal Power: Global Material Form

Using the *conservation of mass* yields,

$$\int_{\Omega} \rho(\mathbf{x},t)\, r(\mathbf{x},t)\, dv = \int_{\Omega_0} \rho_0(\mathbf{X})\, R(\mathbf{X},t)\, dV$$

Using *Nanson's formula*, the heat flux through the spatial boundary of the continuum body may be written as,

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\, ds = \int_{\partial\Omega_0} \mathbf{q} \cdot J\, \mathbf{F}^{-T} \mathbf{N}\, dS := \int_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N}\, dS$$

where $\mathbf{Q}$ is the *nominal heat flux,* i.e. heat flux per unit of material surface, *given* by,

$$\mathbf{Q} = J\, \mathbf{F}^{-1} \mathbf{q}, \quad Q_A = J F_{Aa}^{-1} q_a$$

External Thermal Power

External Thermal Power: Global Material Form

The *global material form* of the **external thermal power** takes the form,

$$
\begin{aligned}
Q_{ext}(t) &= \int_{\Omega_0} \rho_0(\mathbf{X})\, R(\mathbf{X},t)\, dV - \int_{\partial\Omega_0} \mathbf{Q}(\mathbf{X},t)\cdot\mathbf{N}\, dS \\
&= \int_{\Omega_0} \rho_0(\mathbf{X})\, R(\mathbf{X},t)\, dV - \int_{\Omega_0} \mathrm{DIV}\,\mathbf{Q}(\mathbf{X},t)\, dV
\end{aligned}
$$

where $R(\mathbf{X},t)$ is the *heat source per unit of mass* and $\mathbf{Q}(\mathbf{X},t)$ is the *nominal heat flux* or *non-convective heat flux per unit of material surface*, both of them given in material description.

Total External Power

Total External Power: Global Spatial Form

The *global material form* of the **total external power**, i.e. *mechanical plus thermal external power*, may be written as,

$$
\begin{aligned}
\mathcal{P}_{ext}(t) + Q_{ext}(t) &= \int_{\Omega} \rho\, \mathbf{b} \cdot \mathbf{v}\, dv + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v}\, ds \\
&\quad + \int_{\Omega} \rho r\, dv - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\, ds \\
&= \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho \left\| \mathbf{v} \right\|^2 dv + \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}\, dv \\
&\quad + \int_{\Omega} \rho r dv - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\, ds
\end{aligned}
$$

Total External Power

Total External Power: Global Material Form (I)

The *global material form* of the **total thermal power**, i.e. *mechanical plus thermal external power*, may be written as,

$$
\begin{aligned}
\mathcal{P}_{ext}(t) + Q_{ext}(t) &= \int_{\Omega_0} \rho_0 \, \mathbf{b} \cdot \mathbf{v} \, dV + \int_{\partial\Omega_0} \mathbf{T} \cdot \mathbf{v} \, dS \\
&\quad + \int_{\Omega_0} \rho_0 r \, dV - \int_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N} \, dS \\
&= \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 \left\| \mathbf{v} \right\|^2 dV + \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d} \, dV \\
&\quad \int_{\Omega_0} \rho_0 r dV - \int_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N} \, dS
\end{aligned}
$$

Total External Power

Total External Power: Global Material Form (II)

The *global material form* of the **total thermal power**, i.e. *mechanical plus thermal external power*, may be written as,

$$\mathcal{P}_{ext}(t) + Q_{ext}(t) = \int_{\Omega_0} \rho_0 \, \mathbf{b} \cdot \mathbf{v} \, dV + \int_{\partial\Omega_0} \mathbf{T} \cdot \mathbf{v} \, dS$$

$$+ \int_{\Omega_0} \rho_0 r \, dV - \int_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N} \, dS$$

$$= \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 \left\| \mathbf{v} \right\|^2 dV + \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}} \, dV$$

$$\int_{\Omega_0} \rho_0 r dV - \int_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N} \, dS$$

Total External Power

Total External Power: Global Material Form (III)

The *global material form* of the **total thermal power**, i.e. *mechanical plus thermal external power*, may be written as,

$$\mathcal{P}_{ext}\left(t\right)+Q_{ext}\left(t\right)=\int_{\Omega_0}\rho_0\,\mathbf{b}\cdot\mathbf{v}\,dV+\int_{\partial\Omega_0}\mathbf{T}\cdot\mathbf{v}\,dS$$

$$+\int_{\Omega_0}\rho_0 r\,dV-\int_{\partial\Omega_0}\mathbf{Q}\cdot\mathbf{N}\,dS$$

$$=\frac{d}{dt}\int_{\Omega_0}\frac{1}{2}\rho_0\left\|\mathbf{v}\right\|^2 dV+\int_{\Omega_0}\mathbf{S}:\dot{\mathbf{E}}\,dV$$

$$\int_{\Omega_0}\rho_0 r dV-\int_{\partial\Omega_0}\mathbf{Q}\cdot\mathbf{N}\,dS$$

First Law of Thermodynamics

First Law of Thermodynamics

First Postulate. There exist a state function called *total energy*, denoted as $\mathcal{E}(t)$, such that its material time derivative is equal to the *total external power* supplied to the system, i.e. the *external mechanical* plus *thermal power*,

$$\frac{d}{dt}\mathcal{E}(t) := P_{ext}(t) + Q_{ext}(t)$$

First Law of Thermodynamics

First Law of Thermodynamics

Second Postulate. There exist a state function called *internal energy*, denoted as $\mathcal{U}(t)$, which is an *extensive* property, i.e. there exist a *specific internal energy* or *internal energy per unit of mass*, denoted as $e = e(\mathbf{x},t) = E(\mathbf{X},t)$, such that,

$$\mathcal{U}(t) := \int_{\Omega} \rho(\mathbf{x},t)\, e(\mathbf{x},t)\, dv = \int_{\Omega_0} \rho_0(\mathbf{X})\, E(\mathbf{X},t)\, dV$$

First Law of Thermodynamics

First Law of Thermodynamics

Ther **first law of thermodynamics** states that the material time derivative of the *total energy* is equal to sum of the material time derivative of the *kinetic energy* and the material time derivative of the *internal energy*,

$$\boxed{\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}\mathcal{K}(t) + \frac{d}{dt}\mathcal{U}(t)}$$

First Law of Thermodynamics

Using the *first postulate* and the *mechanical energy balance*, the *first law of thermodynamics* reads,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\mathcal{K}(t) + \frac{d}{dt}\mathcal{U}(t) \\ &= P_{ext}(t) + Q_{ext}(t) \\ &= \frac{d}{dt}\mathcal{K}(t) + \mathcal{P}_{int}(t) + Q_{ext}(t)\end{aligned}$$

yielding the following **internal energy balance** law,

$$\boxed{\frac{d}{dt}\mathcal{U}(t) = \mathcal{P}_{int}(t) + Q_{ext}(t)}$$

Energy Balance

Energy Balance Law: Global Spatial Form

The **internal energy balance** law in *global spatial form* may be written as,

$$\frac{d}{dt}\mathcal{U}(t) = \mathcal{P}_{int}(t) + Q_{ext}(t)$$

$$\frac{d}{dt}\int_{\Omega} \rho\, e\, dv = \int_{\Omega} \rho\, \dot{e}\, dv$$

$$= \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}\, dv + \int_{\Omega} \rho\, r\, dv - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\, ds$$

Energy Balance

Energy Balance Law: Local Spatial Form

Using the divergence theorem, the **internal energy balance** law in *global spatial form* may be written as,

$$\frac{d}{dt}\int_{\Omega}\rho\, e\, dv = \int_{\Omega}\rho\,\dot{e}\, dv$$

$$= \int_{\Omega}\boldsymbol{\sigma}:\mathbf{d}\, dv + \int_{\Omega}\rho\, r\, dv - \int_{\partial\Omega}\mathbf{q}\cdot\mathbf{n}\, ds$$

$$= \int_{\Omega}\boldsymbol{\sigma}:\mathbf{d}\, dv + \int_{\Omega}\rho\, r\, dv - \int_{\Omega}\operatorname{div}\mathbf{q}\, dv$$

Localizing, the *local spatial form* of the **energy balance law** reads,

$$\boxed{\rho\,\dot{e} = \boldsymbol{\sigma}:\mathbf{d} + \rho\, r - \operatorname{div}\mathbf{q}}$$

Energy Balance

Energy Balance Law: Global Material Form

The **internal energy balance** law in *global material form* may be written as,

$$\frac{d}{dt}\mathcal{U}(t) = \mathcal{P}_{int}(t) + Q_{ext}(t)$$

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_0} \rho_0\, e\, dV &= \int_{\Omega_0} \rho_0\, \dot{e}\, dV \\
&= \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\partial\Omega_0} \mathbf{Q}\cdot\mathbf{N}\, dS \\
&= \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\partial\Omega_0} \mathbf{Q}\cdot\mathbf{N}\, dS \\
&= \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\partial\Omega_0} \mathbf{Q}\cdot\mathbf{N}\, dS
\end{aligned}$$

Energy Balance

Energy Balance Law: Global Material Form

Using the divergence theorem, the **internal energy balance** law in *global material form* may be written as,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_0} \rho_0\, e\, dV &= \int_{\Omega_0} \rho_0\, \dot{e}\, dV \\ &= \int_{\Omega_0} \boldsymbol{\tau} : \mathbf{d}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\Omega_0} \mathrm{DIV}\,\mathbf{Q}\, dV \\ &= \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\Omega_0} \mathrm{DIV}\,\mathbf{Q}\, dV \\ &= \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\, dV + \int_{\Omega_0} \rho_0\, r\, dV - \int_{\Omega_0} \mathrm{DIV}\,\mathbf{Q}\, dV\end{aligned}$$

Energy Balance

Energy Balance Law: Local Material Form

Localizing, the **internal energy balance** law in *local material form* may be written as,

$$\boxed{\begin{aligned} \rho_0 \, \dot{e} &= \boldsymbol{\tau} : \mathbf{d} + \rho_0 \, r - \mathrm{DIV}\, \mathbf{Q} \\ &= \mathbf{P} : \dot{\mathbf{F}} + \rho_0 \, r - \mathrm{DIV}\, \mathbf{Q} \\ &= \mathbf{S} : \dot{\mathbf{E}} + \rho_0 \, r - \mathrm{DIV}\, \mathbf{Q} \end{aligned}}$$

Note that the *local material form* of the energy balance equation could have been also obtained from the *local spatial form* using,

$$\rho_0 = J \rho, \quad \mathrm{DIV}\, \mathbf{Q} = J \,\mathrm{div}\, \mathbf{q}, \quad \boldsymbol{\tau} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = J \boldsymbol{\sigma} : \mathbf{d}$$

Energy Balance

Local Spatial Form

$$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \operatorname{div} \mathbf{q}$$

Local Material Forms

$$\begin{aligned} \rho_0 \dot{e} &= \boldsymbol{\tau} : \mathbf{d} + \rho_0 r - \mathrm{DIV}\, \mathbf{Q} \\ &= \mathbf{P} : \dot{\mathbf{F}} + \rho_0 r - \mathrm{DIV}\, \mathbf{Q} \\ &= \mathbf{S} : \dot{\mathbf{E}} + \rho_0 r - \mathrm{DIV}\, \mathbf{Q} \end{aligned}$$

Second Law of Thermodynamics

Second Law of Thermodynamics

First Postulate. There exist a state function called *absolute temperature*, denoted as $\theta = \theta(\mathbf{x},t) = \Theta(\mathbf{X},t)$, which is always a *positive* scalar-valued function.

$$\theta = \theta(\mathbf{x},t) = \Theta(\mathbf{X},t) > 0$$

Second Postulate. There exist a state function called *entropy*, denoted as $\mathcal{H}(t)$, which is an *extensive* property, i.e. there exist a *specific entropy* or *entropy per unit of mass*, denoted as $\eta = \eta(\mathbf{x},t) = \Xi(\mathbf{X},t)$, such that,

$$\mathcal{H}(t) = \int_{\Omega} \rho(\mathbf{x},t)\,\eta(\mathbf{x},t)\,dv = \int_{\Omega_0} \rho_0(\mathbf{X})\,\Xi(\mathbf{X},t)\,dV$$

Second Law of Thermodynamics

Second Law of Thermodynamics

The *global spatial form* of the **second law of thermodynamics** states that any *admissible* thermodynamic process has to satisfy the following inequality,

$$\frac{d}{dt}\mathcal{H}(t) \geq \int_{\Omega} \frac{1}{\theta} \rho r dv - \int_{\partial\Omega} \frac{1}{\theta} \mathbf{q} \cdot \mathbf{n} ds$$

$$= \int_{\Omega_0} \frac{1}{\theta} \rho_0 r dV - \int_{\partial\Omega_0} \frac{1}{\theta} \mathbf{Q} \cdot \mathbf{N} dS$$

Second Law of Thermodynamics

Second Law of Thermodynamics

Admissible thermodynamic processes may be classified as *reversible* and *irreversible* processes.

A thermodynamic process is said to be **reversible** if the following condition holds,

$$\frac{d}{dt}\mathcal{H}(t) = \int_{\Omega} \frac{1}{\theta} \rho r dv - \int_{\partial\Omega} \frac{1}{\theta} \mathbf{q} \cdot \mathbf{n} ds = \int_{\Omega_0} \frac{1}{\theta} \rho_0 r dV - \int_{\partial\Omega_0} \frac{1}{\theta} \mathbf{Q} \cdot \mathbf{N} dS$$

A thermodynamic process is said to be **irreversible** if the following condition holds,

$$\frac{d}{dt}\mathcal{H}(t) > \int_{\Omega} \frac{1}{\theta} \rho r dv - \int_{\partial\Omega} \frac{1}{\theta} \mathbf{q} \cdot \mathbf{n} ds = \int_{\Omega_0} \frac{1}{\theta} \rho_0 r dV - \int_{\partial\Omega_0} \frac{1}{\theta} \mathbf{Q} \cdot \mathbf{N} dS$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Global Spatial Form

The *global spatial form* of the **second law of thermodynamics** may be written as,

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t) &= \frac{d}{dt}\int_{\Omega} \rho\eta dv \\ &= \int_{\Omega} \rho\dot{\eta} dv \\ &\geq \int_{\Omega} \frac{1}{\theta}\rho r dv - \int_{\partial\Omega} \frac{1}{\theta}\mathbf{q}\cdot\mathbf{n} ds \end{aligned}$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Global Spatial Form

Using the divergence theorem, the *global spatial form* of the **second law of thermodynamics** may be written as,

$$\frac{d}{dt}\int_{\Omega}\rho\eta dv = \int_{\Omega}\rho\dot{\eta}dv$$

$$\geq \int_{\Omega}\frac{1}{\theta}\rho r dv - \int_{\partial\Omega}\frac{1}{\theta}\mathbf{q}\cdot\mathbf{n}ds$$

$$= \int_{\Omega}\frac{1}{\theta}\rho r dv - \int_{\Omega}\mathrm{div}\left(\frac{1}{\theta}\mathbf{q}\right)dv$$

$$= \int_{\Omega}\frac{1}{\theta}\rho r dv - \int_{\Omega}\frac{1}{\theta}\mathrm{div}\,\mathbf{q}dv + \int_{\Omega}\frac{1}{\theta^2}\mathbf{q}\cdot\mathrm{grad}\,\theta dv$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Local Spatial Form

Localizing, the *local spatial form* of the **second law of thermodynamics** may be written as,

$$\rho\dot{\eta} \geq \frac{1}{\theta}\rho r - \frac{1}{\theta}\operatorname{div}\mathbf{q} + \frac{1}{\theta^2}\mathbf{q}\cdot\operatorname{grad}\theta$$

Multiplying by the absolute temperature yields,

$$\rho\theta\dot{\eta} \geq \rho r - \operatorname{div}\mathbf{q} + \frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta$$

Second Law of Thermodynamics

Clausius-Duhem Inequality: Local Spatial Form

The *local spatial form* of the **Clausius-Duhem** inequality states that the *dissipation rate per unit of spatial volume* is a non-negative scalar-valued quantity and it may be written as,

$$\mathcal{D} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} - \frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta \geq 0$$

A *stronger* assumption is usually introduced yielding,

$$\mathcal{D} := \mathcal{D}_{int} + \mathcal{D}_{cond} = \underbrace{\rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q}}_{\mathcal{D}_{int}\geq 0} \underbrace{- \frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta}_{\mathcal{D}_{cond}\geq 0} \geq 0$$

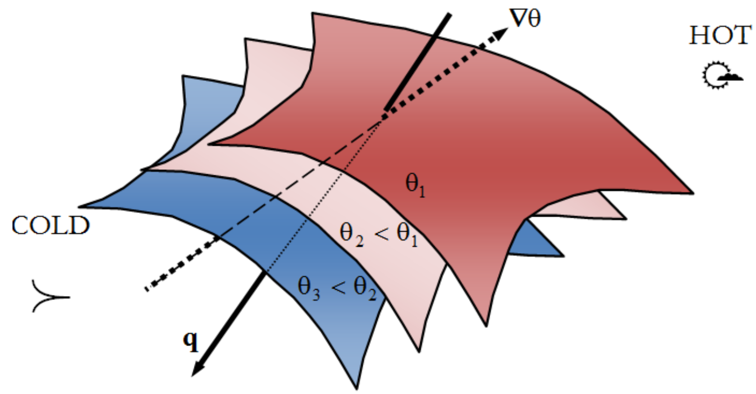
$$\mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} \geq 0, \quad \mathcal{D}_{cond} := -\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta \geq 0$$

Second Law of Thermodynamics

Heat Conduction Inequality: Local Spatial Form

The *local spatial form* of the **heat conduction** inequality states that the *projection of the heat flux per unit of spatial surface* on the direction of the *spatial gradient of the temperature* is a *non-positive* scalar-valued quantity, i.e. heat flux takes place from the hot to the cold and not the other way around,

$$\mathcal{D}_{cond} := -\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta \geq 0 \quad \Rightarrow \quad \mathbf{q}\cdot\operatorname{grad}\theta \leq 0$$

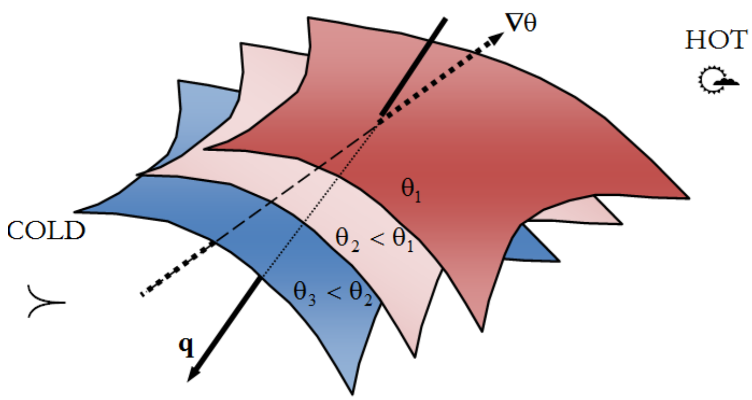


Second Law of Thermodynamics

Heat Conduction Inequality: Local Spatial Form

Using *Fourier*'s law for heat conduction for an *isotropic* continuum medium, the second law of thermodynamics yields the following *restriction* on the admissible values of the *spatial thermal conductivity parameter*,

$$\mathcal{D}_{cond} := -\frac{1}{\theta}\mathbf{q}\cdot \operatorname{grad}\theta \geq 0 \quad \Rightarrow \quad k \operatorname{grad}\theta \cdot \operatorname{grad}\theta \geq 0 \quad \Rightarrow \quad k \geq 0$$



Second Law of Thermodynamics

Clausius-Planck Inequality: Local Spatial Form

The *local spatial form* of the **Clausius-Planck** inequality states that the *internal dissipation rate per unit of spatial volume* is a *non-negative* scalar-valued quantity and it may be written as,

$$\boxed{\mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} \geq 0}$$

The Clausius-Planck inequality for *reversible* and *irreversible* processes, respectively, takes the form,

$$\mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} = 0$$

$$\mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} > 0$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Spatial Form

Using the *local spatial forms* of the *internal energy balance* equation and the *Clausius-Planck* inequality given by,

$$\left.\begin{aligned} \rho\,\dot{e} &= \boldsymbol{\sigma} : \mathbf{d} + \rho\, r - \operatorname{div} \mathbf{q} \\ \mathcal{D}_{int} &:= \rho\theta\dot{\eta} - \rho r + \operatorname{div} \mathbf{q} \geq 0 \end{aligned}\right\}$$

$$\rho\, r - \operatorname{div} \mathbf{q} = \rho\,\dot{e} - \boldsymbol{\sigma} : \mathbf{d}$$

the *local spatial form* of the **Clausius-Planck inequality** may be written in terms of the *internal energy per unit of mass* as,

$$\boxed{\mathcal{D}_{int} := \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{e} - \theta\dot{\eta}\right) \geq 0}$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Spatial Form

Introducing the *free energy per unit of mass*, denoted as ψ, defined as,

$$\psi := e - \theta\eta$$

the *local spatial form* of the **Clausius-Planck** inequality may be written in terms of the *free energy per unit of mass* as,

$$\boxed{\mathcal{D}_{int} := \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{\psi} + \eta\dot{\theta}\right) \geq 0}$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Global Material Form

The *global material form* of the **second law of thermodynamics** states that,

$$\frac{d}{dt}\mathcal{H}(t) = \frac{d}{dt}\int_{\Omega_0} \rho_0 \eta dV$$

$$= \int_{\Omega} \rho_0 \dot{\eta} dV$$

$$\geq \int_{\Omega_0} \frac{1}{\theta} \rho_0 r \, dV - \int_{\partial\Omega_0} \frac{1}{\theta} \mathbf{Q} \cdot \mathbf{N} \, dS$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Global Material Form

Using the divergence theorem, the *global material form* of the **second law of thermodynamics** may be written as,

$$\frac{d}{dt}\int_{\Omega_0} \rho_0 \eta dV = \int_{\Omega} \rho_0 \dot{\eta} dV$$

$$\geq \int_{\Omega_0} \frac{1}{\theta}\rho_0 r dV - \int_{\partial\Omega_0} \frac{1}{\theta}\mathbf{Q}\cdot\mathbf{N}\, dS$$

$$= \int_{\Omega_0} \frac{1}{\theta}\rho_0 r dV - \int_{\Omega_0} \mathrm{DIV}\left(\frac{1}{\theta}\mathbf{Q}\right) dV$$

$$= \int_{\Omega_0} \frac{1}{\theta}\rho_0 r dV - \int_{\Omega_0} \frac{1}{\theta}\mathrm{DIV}\,\mathbf{Q} dV + \int_{\Omega_0} \frac{1}{\theta^2}\mathbf{Q}\cdot\mathrm{GRAD}\,\theta dV$$

Second Law of Thermodynamics

Second Law of Thermodynamics: Local Material Form

Localizing, the *local material form* of the **second law of thermodynamics** may be written as,

$$\rho_0 \dot{\eta} \geq \frac{1}{\theta} \rho_0 r - \frac{1}{\theta} \mathrm{DIV}\,\mathbf{Q} + \frac{1}{\theta^2} \mathbf{Q} \cdot \mathrm{GRAD}\,\theta$$

Multiplying by the absolute temperature yields,

$$\rho_0 \theta \dot{\eta} \geq \rho_0 r - \mathrm{DIV}\,\mathbf{Q} + \frac{1}{\theta} \mathbf{Q} \cdot \mathrm{GRAD}\,\theta$$

Second Law of Thermodynamics

Clausius-Duhem Inequality: Local Material Form

The *local material form* of the **Clausius-Duhem** inequality states that the *dissipation rate per unit of material volume* is a non-negative scalar-valued quantity and it may be written as,

$$\mathcal{D}_0 := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} - \frac{1}{\theta}\mathbf{Q} \cdot \mathrm{GRAD}\,\theta \geq 0$$

A *stronger* assumption is usually considered yielding,

$$\mathcal{D}_0 := \mathcal{D}_{0_{int}} + \mathcal{D}_{0_{cond}} = \underbrace{\rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q}}_{\mathcal{D}_{0_{int}} \geq 0} \underbrace{- \frac{1}{\theta}\mathbf{Q} \cdot \mathrm{GRAD}\,\theta}_{\mathcal{D}_{0_{cond}} \geq 0} \geq 0$$

$$\mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} \geq 0, \quad \mathcal{D}_{0_{cond}} := - \frac{1}{\theta}\mathbf{Q} \cdot \mathrm{GRAD}\,\theta \geq 0$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Material Form

The *local material form* of the **Clausius-Planck** inequality states that the *internal dissipation rate per unit of material volume* is a non-negative scalar-valued quantity and it may be written as,

$$\boxed{\mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} \geq 0}$$

The Clausius-Planck inequality for *reversible* and *irreversible* processes, respectively, takes the form,

$$\mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} = 0$$

$$\mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} > 0$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Material Form

Using the *local materal forms* of the *internal energy balance* equation and the *Clausius-Planck* inequality given by,

$$\left.\begin{aligned} \rho_0\,\dot{e} &= \boldsymbol{\tau}:\mathbf{d}+\rho_0\,r-\mathrm{DIV}\,\mathbf{Q} \\ &= \mathbf{P}:\dot{\mathbf{F}}+\rho_0\,r-\mathrm{DIV}\,\mathbf{Q} \\ &= \mathbf{S}:\dot{\mathbf{E}}+\rho_0\,r-\mathrm{DIV}\,\mathbf{Q} \\ \mathcal{D}_{0_{int}} &:= \rho_0\theta\dot{\eta}-\rho_0 r+\mathrm{DIV}\,\mathbf{Q}\geq 0 \end{aligned}\right\}$$

$$\begin{aligned} \rho_0\,r-\mathrm{DIV}\,\mathbf{Q} &= \rho_0\,\dot{e}-\boldsymbol{\tau}:\mathbf{d} \\ &= \rho_0\,\dot{e}-\mathbf{P}:\dot{\mathbf{F}} \\ &= \rho_0\,\dot{e}-\mathbf{S}:\dot{\mathbf{E}} \end{aligned}$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Material Form

The *local material form* of the **Clausius-Planck inequality** may be written in terms of the *internal energy per unit of mass* as,

$$\begin{aligned}\mathcal{D}_{0_{int}} &:= \boldsymbol{\tau} : \mathbf{d} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \\ &= \mathbf{P} : \dot{\mathbf{F}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \\ &= \mathbf{S} : \dot{\mathbf{E}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \geq 0\end{aligned}$$

Second Law of Thermodynamics

Clausius-Planck Inequality: Local Spatial Form

Introducing the *free energy per unit of mass*, denoted as ψ, defined as,

$$\psi := e - \theta\eta$$

the *local materal forms* of the **Clausius-Planck** inequality may be written in terms of the *free energy per unit of mass* as,

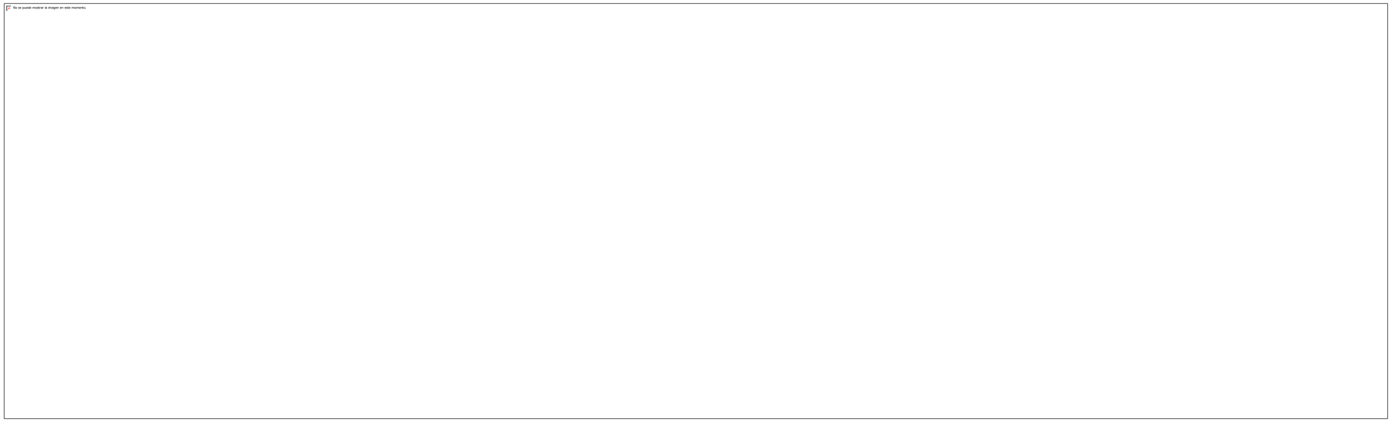
$$\boxed{\begin{aligned} \mathcal{D}_{0_{int}} &:= \boldsymbol{\tau} : \mathbf{d} - \rho_0 \left(\dot{\psi} + \eta \dot{\theta} \right) \\ &= \mathbf{P} : \dot{\mathbf{F}} - \rho_0 \left(\dot{\psi} + \eta \dot{\theta} \right) \\ &= \mathbf{S} : \dot{\mathbf{E}} - \rho_0 \left(\dot{\psi} + \eta \dot{\theta} \right) \geq 0 \end{aligned}}$$

Clausius-Planck Inequality

Local Spatial Forms

$$\mathcal{D}_{int} := \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{e} - \theta\dot{\eta}\right) = \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{\psi} + \eta\dot{\theta}\right) \geq 0$$

Local Material Forms



Adiabatic Process

Adiabatic Process

A thermodynamic process is said to be **adiabatic** if the *net heat transfer* to or from the continuum body is *zero*.

The *internal dissipation rate per unit of spatial volume* for an **adiabatic process** may be written as,

The *stress power per unit of spatial volume* for an **adiabatic process** is equal to the *material time derivative of the internal energy per unit of spatial volume*,

$$\mathcal{D}_{int} := \rho\,\theta\dot{\eta} = \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{e} - \theta\dot{\eta}\right) \geq 0 \quad \Rightarrow \quad \boldsymbol{\sigma} : \mathbf{d} = \rho\dot{e}$$

Adiabatic Process

Adiabatic Process

A thermodynamic process is said to be **adiabatic** if the *net heat transfer* to or from the continuum body is *zero*.

The *internal dissipation rate per unit of material volume* for an **adiabatic process** may be written as,

$$r = 0,\ \mathrm{DIV}\,\mathbf{Q} = 0 \quad \Rightarrow \quad \mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} = \rho_0 \theta \dot{\eta} \geq 0$$

The *stress power per unit of spatial volume* for an **adiabatic process** is equal to the *material time derivative of the internal energy per unit of spatial volume,*

$$\mathcal{D}_{0_{int}} := \rho_0\, \theta \dot{\eta} = \boldsymbol{\tau} : \mathbf{d} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \geq 0 \quad \Rightarrow \quad \boldsymbol{\tau} : \mathbf{d} = \rho_0 \dot{e}$$

Adiabatic Process

Adiabatic Process

A thermodynamic process is said to be **adiabatic** if the *net heat transfer* to or from the continuum body is *zero*.

The *internal dissipation rate per unit of material volume* for an **adiabatic process** may be written as,

$$r = 0,\ \mathrm{DIV}\,\mathbf{Q} = 0 \quad \Rightarrow \quad \mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} = \rho_0 \theta \dot{\eta} \geq 0$$

The *stress power per unit of spatial volume* for an **adiabatic process** is equal to the *material time derivative of the internal energy per unit of spatial volume*,

$$\mathcal{D}_{0_{int}} := \rho_0\, \theta \dot{\eta} = \mathbf{P} : \dot{\mathbf{F}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \geq 0 \quad \Rightarrow \quad \mathbf{P} : \dot{\mathbf{F}} = \rho_0 \dot{e}$$

Adiabatic Process

Adiabatic Process

A thermodynamic process is said to be **adiabatic** if the *net heat transfer* to or from the continuum body is *zero*.

The *internal dissipation rate per unit of material volume* for an **adiabatic process** may be written as,

$$r = 0,\ \mathrm{DIV}\,\mathbf{Q} = 0 \quad \Rightarrow \quad \mathcal{D}_{0_{int}} := \rho_0 \theta \dot{\eta} - \rho_0 r + \mathrm{DIV}\,\mathbf{Q} = \rho_0 \theta \dot{\eta} \geq 0$$

The *stress power per unit of spatial volume* for an **adiabatic process** is equal to the *material time derivative of the internal energy per unit of spatial volume,*

$$\mathcal{D}_{0_{int}} := \rho_0\, \theta \dot{\eta} = \mathbf{S} : \dot{\mathbf{E}} - \rho_0 \left(\dot{e} - \theta \dot{\eta} \right) \geq 0 \quad \Rightarrow \quad \mathbf{S} : \dot{\mathbf{E}} = \rho_0 \dot{e}$$

Isentropic Process

Isentropic Process

A thermodynamic process is said to be **isentropic** if it takes place at *constant entropy*.

The *internal dissipation rate per unit of spatial volume* for an **isentropic process** may be written as,

$$\dot{\eta} = 0 \quad \Rightarrow \quad \mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div}\mathbf{q} = -\rho r + \operatorname{div}\mathbf{q} \geq 0$$

The *stress power per unit of spatial volume* for an **isentropic process** may be written as,

$$\mathcal{D}_{int} := -\rho r + \operatorname{div}\mathbf{q} = \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{e} - \theta\dot{\eta}\right) \geq 0 \quad \Rightarrow$$

$$\boldsymbol{\sigma} : \mathbf{d} = \rho\left(\dot{e} - \theta\dot{\eta}\right) - \rho r + \operatorname{div}\mathbf{q}$$

Isentropic and Adiabatic Process

Isentropic and Adiabatic Process

A thermodynamic process is said to be **isentropic** and **adiabatic** if it takes place at *constant entropy* and the *net heat flux* to or from the continuum body is *zero*.

The *internal dissipation rate per unit of spatial volume* for an **isentropic** and **adiabatic process** is *zero* and, then, the process is **reversible.**

$$\dot{\eta} = 0, \quad r = 0, \quad \operatorname{div} \mathbf{q} = 0 \quad \Rightarrow \quad \mathcal{D}_{int} := \rho\theta\dot{\eta} - \rho r + \operatorname{div} \mathbf{q} = 0$$

The *stress power per unit of spatial volume* for an **isentropic** and **adiabatic process** is equal to the *material time derivative of the internal energy per unit of spatial volume,*

$$\mathcal{D}_{int} := \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{e} - \theta\dot{\eta}\right) = 0 \quad \Rightarrow \quad \boldsymbol{\sigma} : \mathbf{d} = \rho\,\dot{e}$$

Isentropic or Adiabatic Reversible Process

Isentropic or Adiabatic Reversible Process

If a thermodynamic process is *adiabatic* and *reversible*, then the process is also *isentropic*,

$$\mathcal{D}_{int} = 0, \quad r = 0, \quad \operatorname{div} \mathbf{q} = 0 \quad \Rightarrow \quad \mathcal{D}_{int} := \rho\theta\dot{\eta} - \cancel{\rho r} + \cancel{\operatorname{div} \mathbf{q}} = 0$$

If a thermodynamic process is *isentropic* and *reversible*, then the process is also *adiabatic*,

$$\dot{\eta} = 0, \quad \mathcal{D}_{int} = 0 \quad \Rightarrow \quad \mathcal{D}_{int} := \cancel{\rho\theta\dot{\eta}} - \rho r + \operatorname{div} \mathbf{q} = 0$$

Adiabatic and *isentropic* processes are identical for the case in which both of them are *reversible*.

Isothermal Process

Isothermal Process

A thermodynamic process is said to be **isothermal** if it takes place at *constant temperature*.

The *internal dissipation rate per unit of spatial volume* for an **isentropic** and **adiabatic process** is *zero*,

$$\dot{\theta} = 0 \quad \Rightarrow \quad \mathcal{D}_{int} := \boldsymbol{\sigma} : \mathbf{d} - \rho\left(\dot{\psi} + \eta\dot{\theta}\right) = \boldsymbol{\sigma} : \mathbf{d} - \rho\dot{\psi} \geq 0$$

Governing Equations

Governing Equations: Spatial Form

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Balance of energy**

1 $$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \operatorname{div} \mathbf{q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho \theta \dot{\eta} - \rho r + \operatorname{div} \mathbf{q} \geq 0, \quad \mathcal{D}_{con} := -\mathbf{q} \cdot \operatorname{grad} \theta \geq 0$$

1 1

Governing Equations

Constitutive Equations: Spatial Form

- **Thermo-mechanical constitutive equations**

6 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{v}, \theta, \pi)$$ 1

1 $$\eta = \eta(\mathbf{v}, \theta, \pi)$$

- **Thermal constitutive equation**. Fourier's law

3 $$\mathbf{q} = \mathbf{q}(\mathbf{v}, \theta) = -\mathbf{k}(\mathbf{v}, \theta)\, \mathrm{grad}\, \theta$$

- **State equations**

1 $$e = e(\mathbf{v}, \theta)$$

1 $$\pi = \pi(\rho, \theta)$$

Governing Equations

Governing Equations: Material Form (I)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\,\mathbf{P} + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum.** Symmetry restriction 1st P-K

3 $$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$$

- **Balance of energy**

1 $$\rho_0\,\dot{e} = \mathbf{P} : \dot{\mathbf{F}} + \rho_0\,r - \mathrm{DIV}\,\mathbf{Q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{0_{int}} := \rho_0\,\theta\,\dot{\eta} - \rho_0\,r + \mathrm{DIV}\,\mathbf{Q} \geq 0, \quad \mathcal{D}_{0_{cond}} := -\mathbf{Q}\cdot\mathrm{GRAD}\,\theta \geq 0$$

1 1

Governing Equations

Constitutive Equations: Material Form (I)

- **Thermo-mechanical constitutive equations**

6 $$\mathbf{P}\mathbf{F}^{T} = \boldsymbol{\tau}\left(\mathbf{v},\theta,\pi\right)$$ 1

1 $$\eta = \eta\left(\mathbf{v},\theta,\pi\right)$$

- **Thermal constitutive equation.** Material Fourier's law

3 $$\mathbf{Q} = \mathbf{Q}\left(\mathbf{v},\theta\right) = -\mathbf{K}\left(\mathbf{v},\theta\right)\mathrm{GRAD}\,\theta$$

- **State equations**

1 $$e = e\left(\mathbf{v},\theta\right)$$

1 $$\pi = \pi\left(\rho,\theta\right)$$

Governing Equations

Governing Equations: Material Form (II)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\boldsymbol{\tau}\mathbf{F}^{-T}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum.** Symmetry of Kirchhoff stress

3 $$\boldsymbol{\tau} = \boldsymbol{\tau}^T$$

- **Balance of energy**

1 $$\rho_0\,\dot{e} = \boldsymbol{\tau} : \mathbf{d} + \rho_0\,r - \mathrm{DIV}\,\mathbf{Q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{0_{int}} := \rho_0\,\theta\,\dot{\eta} - \rho_0\,r + \mathrm{DIV}\,\mathbf{Q} \geq 0, \quad \mathcal{D}_{0_{cond}} := -\mathbf{Q}\cdot\mathrm{GRAD}\,\theta \geq 0$$

1 1

Governing Equations

Constitutive Equations: Material Form (II)

- **Thermo-mechanical constitutive equations**

6 $$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{v}, \theta, \pi)$$ 1

1 $$\eta = \eta(\mathbf{v}, \theta, \pi)$$

- **Thermal constitutive equation**. Material Fourier's law

3 $$\mathbf{Q} = \mathbf{Q}(\mathbf{v}, \theta) = -\mathbf{K}(\mathbf{v}, \theta)\,\mathrm{GRAD}\,\theta$$

- **State equations**

1 $$e = e(\mathbf{v}, \theta)$$

1 $$\pi = \pi(\rho, \theta)$$

Governing Equations

Governing Equations: Material Form (III)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\mathbf{FS}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum.** Symmetry of 2nd P-K stress

3 $$\mathbf{S} = \mathbf{S}^T$$

- **Balance of energy**

1 $$\rho_0\,\dot{e} = \mathbf{S} : \dot{\mathbf{E}} + \rho_0\,r - \mathrm{DIV}\,\mathbf{Q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{0_{int}} := \rho_0\,\theta\,\dot{\eta} - \rho_0\,r + \mathrm{DIV}\,\mathbf{Q} \geq 0, \quad \mathcal{D}_{0_{cond}} := -\mathbf{Q}\cdot\mathrm{GRAD}\,\theta \geq 0$$

1 1

Governing Equations

Constitutive Equations: Material Form (III)

- **Thermo-mechanical constitutive equations**

6 $$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{v}, \theta, \pi)$$ 1

1 $$\eta = \eta(\mathbf{v}, \theta, \pi)$$

- **Thermal constitutive equation**. Material Fourier's law

3 $$\mathbf{Q} = \mathbf{Q}(\mathbf{v}, \theta) = -\mathbf{K}(\mathbf{v}, \theta)\,\mathrm{GRAD}\,\theta$$

- **State equations**

1 $$e = e(\mathbf{v}, \theta)$$

1 $$\pi = \pi(\rho, \theta)$$

Governing Equations

Mechanical Problem: Spatial Form

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Mechanical constitutive equation**

6 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{v})$$

Governing Equations

Mechanical Problem: Spatial Form

- **Conservation of mass.** Mass continuity

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy's first motion

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 6

- **Mechanical constitutive equation**

6 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{v})$$

Governing Equations

Mechanical Problem: Material Form (I)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum**. Cauchy's first motion

3 $$\text{DIV}\,\mathbf{P} + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum**. Symmetry of Kirchhoff stress

3 $$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$$

- **Mechanical constitutive equation**

6 $$\mathbf{P}\mathbf{F}^T = \boldsymbol{\tau}\left(\mathbf{v}\right)$$

Governing Equations

Mechanical Problem: Material Form (II)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\boldsymbol{\tau}\mathbf{F}^{-T}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum.** Symmetry of Kirchhoff stress

3 $$\boldsymbol{\tau} = \boldsymbol{\tau}^T$$

- **Mechanical constitutive equation**

6 $$\boldsymbol{\tau} = \boldsymbol{\tau}\left(\mathbf{v}\right)$$

Governing Equations

Mechanical Problem: Material Form (II)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\boldsymbol{\tau}\mathbf{F}^{-T}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 6 3

- **Mechanical constitutive equation**

6 $$\boldsymbol{\tau} = \boldsymbol{\tau}\left(\mathbf{v}\right)$$

Governing Equations

Mechanical Problem: Material Form (III)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\mathbf{FS}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 9 3

- **Balance of angular momentum.** Symmetry of Kirchhoff stress

3 $$\mathbf{S} = \mathbf{S}^T$$

- **Mechanical constitutive equation**

6 $$\mathbf{S} = \mathbf{S}\left(\mathbf{v}\right)$$

Governing Equations

Mechanical Problem: Material Form (III)

- **Conservation of mass.** Mass continuity

1 $$\rho J = \rho_0$$ 1

- **Balance of linear momentum.** Cauchy's first motion

3 $$\mathrm{DIV}\left(\mathbf{FS}\right) + \rho_0\,\mathbf{b} = \rho_0\dot{\mathbf{v}}$$ 6 3

- **Mechanical constitutive equation**

6 $$\mathbf{S} = \mathbf{S}\left(\mathbf{v}\right)$$